Anosov Diffeomorphisms

Recall: A $C^1$ diffeo $f: M \to M$ is Anosov if

\[ \exists C > 0, 0 < \mu < 1, \ \text{a splitting} \]

\[ TM = E^u \oplus E^s \]

(\text{\textit{SDF indicates invariance under } } Df)

Such that $\forall x \in M \ \forall n \geq 0$:

- $v \in E^s(x) \Rightarrow \| Df^n(x)v\| \leq \mu^n \|v\|
- v \in E^u(x) \Rightarrow \| Df^n(x)v\| \leq \mu^{-n} \|v\|$
Remarks

1. \( f: M \rightarrow \mathbb{R} \) is Anosov \iff \exists n_0 \geq 0 \text{ and disjoint comfields } C^{u}, C^{s} \subseteq TM \text{ such that:}

\[ \forall \epsilon > 0, \forall v \in C^{u}(x) \Rightarrow \| Df^{n_0} x \| \geq 2 \| v \| \]

\[ \forall \epsilon > 0, \forall v \in C^{s}(x) \Rightarrow \| Df^{-n_0} x \| \geq 2 \| v \| \]

\[ Df^{n_0} (C^{u}(x)) \subseteq C^{u}(f(x)) \]

\[ Df^{-n_0} (C^{s}(x)) \subseteq C^{s}(f(x)) \]

This implies:
(a) The splitting \( TM = E^u \oplus E^s \) for an Anosov diffeomorphism is unique and continuous.

(b) \{Anosov diffeomorphisms\} is open in the \( C^1 \) topology: if \( f: \mathbb{R} \to \mathbb{R} \) is Anosov, then \( \exists \varepsilon > 0 \) s.t. \( \forall \text{diff} \ g: \mathbb{R} \to \mathbb{R} \), \( d_{C^1}(f, g) < \varepsilon \implies g \) is Anosov.

(2) Anosov diffeos
are structurally stable
i.e. \( \forall f \text{ Anosov}, \exists \varepsilon > 0 \)
\[ \forall (f, g) \quad (g_f, g) < \varepsilon \implies \exists \text{ homeo } h. \quad M \to M \]
Such that
\[ g = h f h^{-1} \]

**Example:** Define, for \( \varepsilon \in \mathbb{R} \),
\[ f_{\varepsilon} : \mathbb{T}^2 \to \mathbb{T}^2 \]
by:
\[ f_{\varepsilon} (x, y) = (2x + y, x + y + \varepsilon \sin(2\pi x)) \]
then let sufficiently small
\[ \Rightarrow f_{\varepsilon} \text{ is Anosov,} \]
and \( \exists ! \text{ } \exists \Psi^3 \text{ s.t. } \\
\Psi^3 = h^{-1} \Phi h \).

(3) The bundles \( E^u \) \( \text{it's} \) are uniquely integrable: \( \mathcal{F} \) foliations \( \mathcal{W}^u \), \( \mathcal{W}^s \) of \( M \) s.t. \\
\( \forall x \in M \):

- The leaves \( \mathcal{W}^s(x) \), \( \mathcal{W}^u(x) \) of \( \mathcal{W}^s \), \( \mathcal{W}^u \) (resp.) through \( x \) are immersed \( C^1 \) submanifolds.
(C if \ f \ is \ C^r).

\begin{align*}
& T_x W^s(x) = E^s(x) \\
& T_x W^u(x) = E^u(x)
\end{align*}

The leaves \( W^s(x) \) and \( W^u(x) \) are dynamically defined:

\begin{align*}
\forall y \in W^s(x) \iff d(f^n(x), f^n(y)) \to 0 \\
\forall y \in W^u(x) \iff d(f^{-n}(x), f^{-n}(y)) \to 0
\end{align*}

Note: this implies unique integrability of \( E^u, E^s \).
The foliations $\mathcal{F}_s, \mathcal{F}_u$ have limited transverse regularity if $(d_1 = \dim E^s, d_2 = \dim E^u)$.

$\phi : (-1, 1) \times (-1, 1) \rightarrow M$ is a foliation chart for $\mathcal{F}_s$ (i.e., with the property that $\phi((-1, 1) \times \{y\}) \subseteq \mathcal{F}_s(\phi(0, y))$ for all $y \in (-1, 1)^{d_2}$).
Then in general, \( \Phi \) is not \( C^1 \). But \( \Phi \) can be chosen so that:

\[ \forall y \in (-1, 1)^d, \quad \text{the map} \]

\[ (-1, 1)^d \rightarrow M \]

\[ x \rightarrow \Phi(x, y) \]

is \( C^r \), uniformly in \( y \) and \( x \), with \( \Phi \) being \( \text{Hölder continuous} \).

We will discuss later a further property that can be imposed on
The charts $\tilde{\phi}$: absolute continuity

$\star$ if $\dim M = 2$, or more generally $\dim E^u = 1$, then the charts for $\mathcal{H}^s$ may be chosen $C^1$.

Theorem (Hopf, Anosov): If $f: M \to M$ is a $C^2$ Anosov diffeomorphism and $f_* m = m$, for some volume $m$ on $M$, then
$f$ is ergodic with respect to $\mu$.

More generally, (even if $f$ does not preserve a volume $\mu$), we have:

$$U \cup \delta = \{ \emptyset, \mathbb{M} \},$$

modulo zero-sets for volume.
Proof. The main new concept is the following:

Def. A foliation $\mathcal{F}$ is absolutely continuous if, for every local plaque family $\mathcal{F}$ of $\mathcal{F}(x) : x \in V^3$ for $\mathcal{F}$, if $V_1$ and $V_2$ are any 2 smooth disks transverse to the plaques of $\mathcal{F}$.
If \( f \) is absolutely continuous, then

\[
\dim \tilde{\mathcal{H}} = \dim \tilde{\mathcal{L}} = \text{codim} f
\]

the local holonomy map

\[
h : \tilde{\mathcal{H}} \to \tilde{\mathcal{L}}
\]

is absolutely continuous, i.e., it sends sets of Lebesgue measure 0 in \( \tilde{\mathcal{H}} \) to sets of Lebesgue measure 0 in \( \tilde{\mathcal{L}} \).

**Prop** If \( f \) is absolutely continuous, then...
If $\mathcal{F}$ is leafwise a.e.:

$$\forall Z \leq M \text{ measurable } m(Z) = 0 \iff \text{ for m-a.e. } x \in M, Z \text{ meets } F(x) \text{ in a Lebesgue 0-set.}$$

$\mathcal{F}$ is transversely absolutely ct's: If $Z$ is any smooth transversal to $\mathcal{F}$, and $Z_x \in Z$ is any
Borel set, then:
\[ Z_x \text{ is } \sigma \Leftrightarrow m(\bigcup F(z)) = 0. \]
\[ 0 \text{-set in } \mathbb{Z}^2 \]

**Theorem (Anosov)**: The stable and unstable foliations for a $C^2$ Anosov diffeomorphism are absolutely continuous.

We complete the proof of ergodicity modulo Anosov 5.
result. Because of the Hopf argument, it suffices to show:

Suppose \( \psi : W \to \mathbb{R} \)

is measurable, and

\[ \psi^s \psi^u \text{ with:} \]

\[ \psi^s = \psi^u = \psi \text{ a.e.} \]

\[ \psi^s = \text{constant along } \mathcal{N}^s \text{ leaves} \]

\[ \psi^u = \text{constant along } \mathcal{N}^u \text{ leaves}. \]

Then \( \psi \) = constant a.e.
So let $\psi, \psi^s, \psi^n$ be given, and let 

$$G = \{ x \mid \psi = \psi^s = \psi^n \}$$

Since $N^s$ is leafwise abs. cts., $E$ full measure set $G' \subseteq G$ such that 

$$x \in G' \iff G \text{ meets } N^s(x)$$

in a set of full measure in $N^s$

Working in a local plaque neighborhood, denote by $N^s_{\text{local}}, N^n_{\text{local}}$
the plagues of $W^s$, $W^u$.

Fix an $x \in G'$, and consider the set:

$$
\bigcup_{z \in G' \cap W^s(x)} W^u_{\text{loc}}(z) = G''
$$

Then $G''$ has full
measure in the open neighborhood

\[ N = \bigcup_{z \in W^s_{\text{loc}}(x)} W^u_{\text{loc}}(z) \]

(Reason: \( W^u \) is transversely absolutely continuous)

But then \( \psi^u \) is constant on \( G^u \):

\[ z \in G^u \cap W^s(x) \]

\[ \Rightarrow \psi^s(z) = \psi^s(x) = \psi^u(x) = \psi^u(z) \]
but $w \in H^u(z) \Rightarrow$

$$y^u(w) = y^u(z) = y(x).$$

Hence

$$y^u(w) = y(x)$$

$\forall w \in G^u \Rightarrow y^u = \text{const a.e. in } V$

But $y^u = y \text{ a.e.}$

$\Rightarrow y = \text{const a.e. in } V$

$\Rightarrow y$ is locally constant

$\Rightarrow y$ is constant a.e.
Let's turn to Anosov's theorem (continuity of $\mathcal{W}^s$ & $\mathcal{W}^u$).

Here is a sketch of the proof:

Assume for simplicity that $\mathcal{W}_1$, $\mathcal{W}_2$ are...
local $N^n$ leaves. Fix a smooth $u$ normal bundle to $N$ which defines a smooth local projection $\pi_x$ onto $N^u(x)$, $\forall x \in M$. This local projection has uniformly bounded $C'$-norm.
Fix $X_0 \in \mathcal{X}$ and consider $h_n : X_0 \to \mathbb{T}^2$, $n \geq 1$, defined by:

$$h_n = \frac{1}{n} \sum_{i=0}^{n-1} f^i(x_0).$$

Claim:

$$\lim_{n \to \infty} h_n = h$$

uniformly as $n \to \infty$, where $h$ is the W"{u} holonomy map.
2: $h_n$ is a $(C^2)$ diffeo $V_n$ with uniformly bounded Jacobian. 
\[ \exists K \geq 1 \text{ such that for all } x \in V, \]
\[ \frac{1}{K} \leq \left| \text{Jac}_{h_n}(x) \right| \leq K \]

**Proposition ("Three Disks")**

Let $h_n : N_1 \to N_2$ be a sequence of $C^1$ diffeomorphisms between Riemannian manifolds $N_1, N_2$.
converging uniformly to a homeomorphism.

Suppose that the 
Jacobian of $h_n$ are 
uniformly bounded 
from above and below.
Then $h$ is absolutely 
continuous. Moreover, 
for every Borel set $A \in \mathcal{N}_1$
\[
\frac{1}{K_{N_2}} m(A) < m_{N_2}(h(A)) \leq Km(A)
\]
pf etc. to show on