Anosov Diffeomorphisms

<u>Recall</u>: A C^1 diffeo $f: M \to M$ is <u>Anosov</u> if $\exists C > 0$, $0 < \mu < 1$, & a splitting

$$TM = E^u \oplus E^s \circlearrowleft Df$$

($\circlearrowleft Df$ indicates invariance under Df)

Such that $\forall x \in M$, $\forall n \geq 0$:

$$v \in E^s(x) \Rightarrow \|D_x f^n(v)\| \leq C\mu^n \|v\|$$

$$v \in E^u(x) \Rightarrow \|D_x f^{-n}(v)\| \leq C\mu^n \|v\|$$

Remarks

① $f: M \circlearrowleft$ is Anosov

$\iff \exists\, n_0 \geq 0$ and disjoint conefields $\mathcal{C}^u, \mathcal{C}^s \subseteq TM$ such that:

- $v \in \mathcal{C}^u(x) \Rightarrow \|D_x f^{n_0} v\| \geq 2\|v\|$
- $v \in \mathcal{C}^s(x) \Rightarrow \|D_x f^{-n_0} v\| \geq 2\|v\|$
- $D_x f^{n_0}(\overline{\mathcal{C}^u(x)}) \subseteq \mathcal{C}^u(f^{n_0}(x))$
- $D_x f^{-n_0}(\overline{\mathcal{C}^s(x)}) \subseteq \mathcal{C}^s(f^{-n_0}(x))$

This implies:

ⓐ The splitting

$$TM = E^u \oplus E^s$$

for an Anosov diffeomorphism is <u>unique</u> and <u>continuous</u>

ⓑ {Anosov diffeomorphisms} is <u>open</u> in the C^1 topology:

If $f: M \to M$ is Anosov, then $\exists \varepsilon > 0$ s.t. $\forall$ diffeo $g: M \to M$, $d_{C^1}(f, g) < \varepsilon$

$\Longrightarrow$ g is Anosov.

② Anosov diffeos

are <u>structurally stable</u>

i.e. $\forall f$ Anosov, $\exists \varepsilon > 0$

s.t. $d(f,g) < \varepsilon \Rightarrow$

$\exists$ homeo $h: M \to M$

such that

$$g = hfh^{-1}.$$

<u>Example</u>: Define, for $\varepsilon \in \mathbb{R}$,

$f_\varepsilon : \mathbb{T}^2 \circlearrowleft$ by:

$$f_\varepsilon(x,y) = (2x+y, x+y+\varepsilon \sin 2\pi x).$$

then $|\varepsilon|$ sufficiently small

$\Rightarrow f_\varepsilon$ is Anosov,

and $\exists h_\varepsilon : \mathbb{T}^2 \circlearrowleft$ s.t.

$$f_\varepsilon = h_\varepsilon f_0 h_\varepsilon^{-1}.$$

③ The bundles E^u, E^s are uniquely integrable: $\exists$ foliations $\mathcal{W}^u$, $\mathcal{W}^s$ of M st $\forall x \in M$:

- The leaves $\mathcal{W}^s(x)$, $\mathcal{W}^u(x)$ of $\mathcal{W}^s$, $\mathcal{W}^u$ (resp.) through x are immersed C^1 submanifolds

(C^r if f is C^r).

- $T_x W^s(x) = E^s(x)$

  $T_x W^u(x) = E^u(x)$

- The leaves $W^s(x)$ & $W^u(x)$ are dynamically defined:

$$y \in W^s(x) \iff d(f^n(x), f^n(y)) \xrightarrow{n\to\infty} 0$$

$$y \in W^u(x) \iff d(f^{-n}(x), f^{-n}(y)) \xrightarrow{n\to\infty} 0$$

<u>Note</u>: this implies unique integrability of E^u, E^s.

④ The foliations $\mathcal{W}^s$, $\mathcal{W}^u$ have <u>limited transverse regularity</u>:

if $(d_1 = \dim E^s,\ d_2 = \dim E^u)$.

$$\varphi : (-1,1)^{d_1} \times (-1,1)^{d_2} \longrightarrow M$$

is a foliation chart for $\mathcal{W}^s$

(i.e. with the property that

$$\varphi\left((-1,1)^{d_1} \times \{y\}\right) \subseteq \mathcal{W}^s(\varphi(0,y))$$

$$\forall y \in (-1,1)^{d_2}\ )$$

then in general*, φ is <u>not</u> <u>C^1</u>. But φ can be chosen so that:

* $\forall y \in (-1,1)^{d_2}$, the map

$$(-1,1)^{d_1} \longrightarrow M$$
$$x \longmapsto \varphi(x,y)$$

is C^r, uniformly in y

and

* φ is <u>Hölder</u> continuous.

We will discuss later a further property that can be imposed on

the charts φ: absolute continuity

(★ if $\dim M = 2$, or more generally $\dim E^u = 1$, then the charts for $\mathcal{H}^s$ may be chosen C^1).

Theorem (Hopf, Anosov): If $f: M \to M$ is a C^2 Anosov diffeomorphism and $f_* m = m$, for some volume m on M, then

f is ergodic with respect to m.

More generally, (even if f does not preserve a volume m), we have:

$$\mathcal{U} \cap \mathcal{S} = \{\emptyset, M\},$$

modulo zero-sets for volume.

<u>Proof</u> The main new concept is the following:

<u>Def</u> A foliation $\mathcal{F}$ is <u>absolutely continuous</u> if, for every local plaque family $\{\mathcal{F}^{loc}(x) : x \in V\}$ for $\mathcal{F}$, if τ_1 & τ_2 are any 2 smooth disks transverse to the plaques of

of $\mathcal{F}^{loc}$, with

$$\dim \tau_1 = \dim \tau_2 = \operatorname{codim} \mathcal{F},$$

the local holonomy map

$$h : \tau_1 \to \tau_2$$

is absolutely continuous, i.e. it sends sets of Lebesgue measure 0 in τ_1 to sets of Lebesgue measure 0 in τ_2.

<u>Prop</u> If $\mathcal{F}$ is absolutely continuous, then

- $\mathcal{F}$ is <u>leafwise</u> a.c.: $\forall Z \subseteq M$ measurable, $m(Z)=0 \iff$ for m-a.e. $x \in M$, Z meets $\mathcal{F}(x)$ in a Lebesgue 0-set

- $\mathcal{F}$ is <u>transversely</u> absolutely cts: If τ is any smooth transversal to $\mathcal{F}$, and $Z_\tau \subseteq \tau$ is any

Borel set, then:

$\mathcal{Z}_{\tilde{\tau}}$ is a 0-set in $\tilde{\tau}$ $\Leftrightarrow$ $m\left(\bigcup_{z \in \mathcal{Z}_{\tilde{\tau}}} \mathcal{F}(z)\right) = 0$.

<u>Theorem</u> (Anosov) The stable and unstable foliations for a C^2-Anosov diffeomorphism are absolutely continuous.

We complete the proof of ergodicity, modulo Anosov 5

result.

Because of the Hopf argument, it suffices to show:

Suppose $\psi : M \to \mathbb{R}$ is measurable, and $\exists\, \psi^s, \psi^u$ with:

- $\psi^s = \psi^u = \psi$ a.e.
- ψ^s = constant along $\mathcal{W}^s$ leaves
- ψ^u = constant along $\mathcal{W}^u$ leaves,

then ψ = const a.e.

So let, ψ, ψ^s, ψ^u be given, and let

$$G = \{x \mid \psi = \psi^s = \psi^u\}$$

Since $\mathcal{W}^s$ is leafwise abs. cts., $\exists$ full measure set $G' \subseteq G$ such that

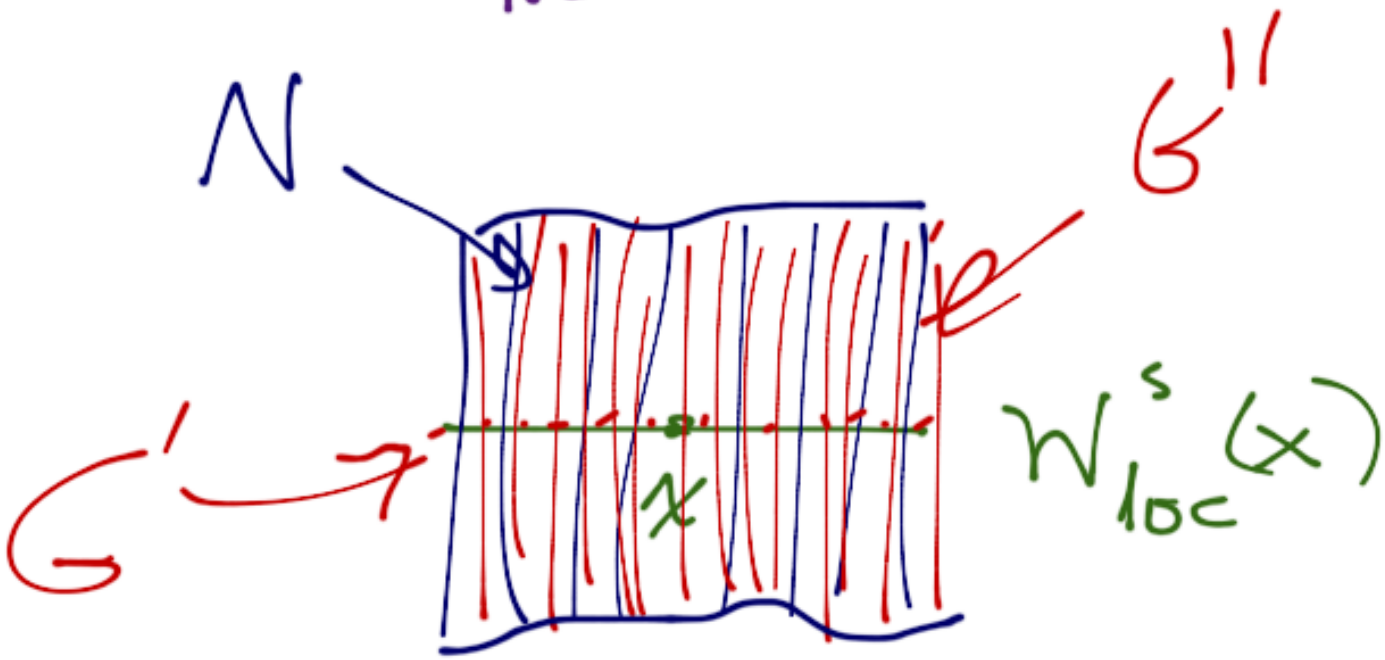
$x \in G' \Rightarrow G$ meets $\mathcal{W}^s(x)$ in a set of full measure in $\mathcal{W}^s$

Working in a local plaque neighborhood, denote by $\mathcal{W}^s_{loc}, \mathcal{W}^u_{loc}$

the plaques of $\mathcal{W}^s$, $\mathcal{W}^u$.

Fix an $x \in G'$, and consider the set:

$$\bigcup_{z \in G' \cap W^s_{loc}(x)} \mathcal{W}^u_{loc}(z) = G''$$



then G'' has full

measure in the open neighborhood

$$N = \bigcup_{z \in W^s_{loc}(x)} W^u_{loc}(z)$$

(Reason: W^u is transversely absolutely continuous)

But then ψ^u is constant on G'':

$$z \in G' \cap W^s(x)$$

$$\Rightarrow \psi^s(z) = \psi^s(x) = \psi(x)$$

$$= \psi^u(z)$$

but $w \in \mathcal{W}^u(z) \Rightarrow$

$$\psi^u(w) = \psi^u(z) = \psi(x).$$

Hence

$$\psi^u(w) = \psi(x)$$

$\forall w \in G'' \Rightarrow \psi^u = \text{const a.e. in } V$

But $\psi^u = \psi$ a.e.

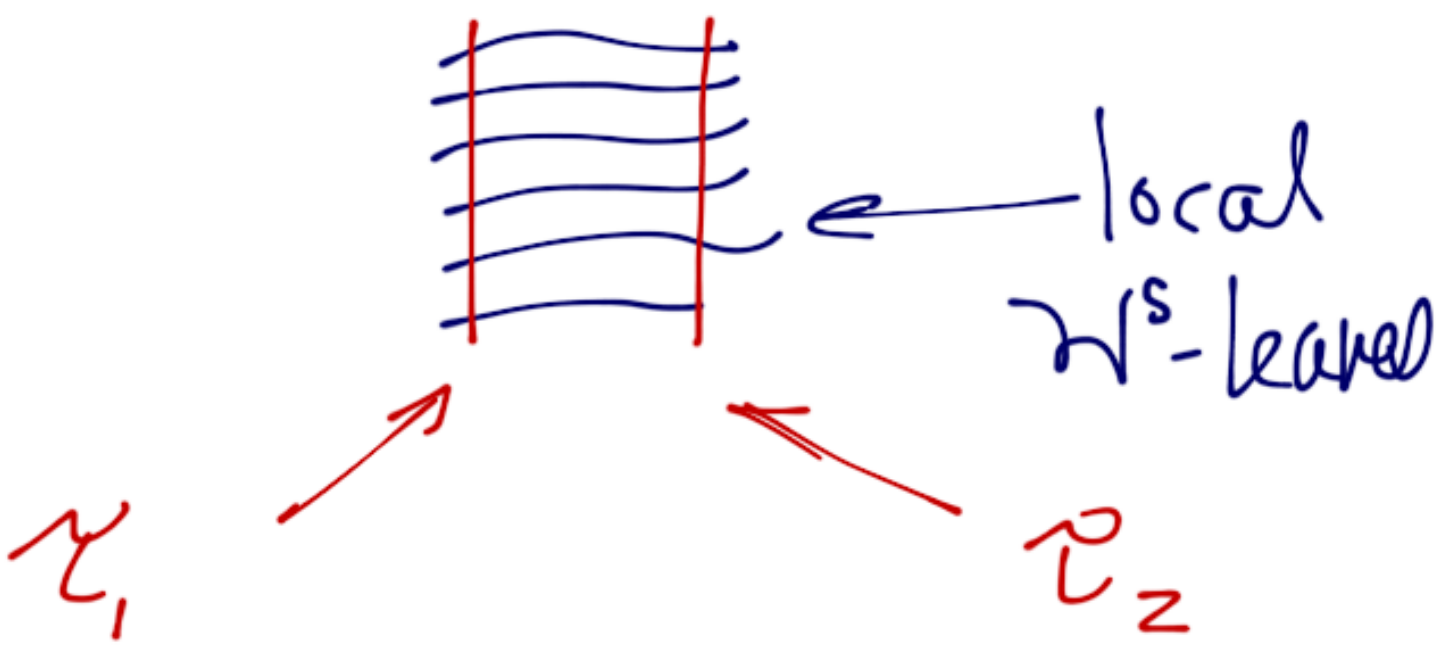
$\Rightarrow \quad \psi = \text{const a.e. in } V.$

$\Rightarrow \psi$ is locally const a.e.

$\Rightarrow \psi$ is constant a.e. $\boxtimes$

Let's turn to Anosov's theorem (continuity of $\mathcal{W}^s$ & $\mathcal{W}^u$).

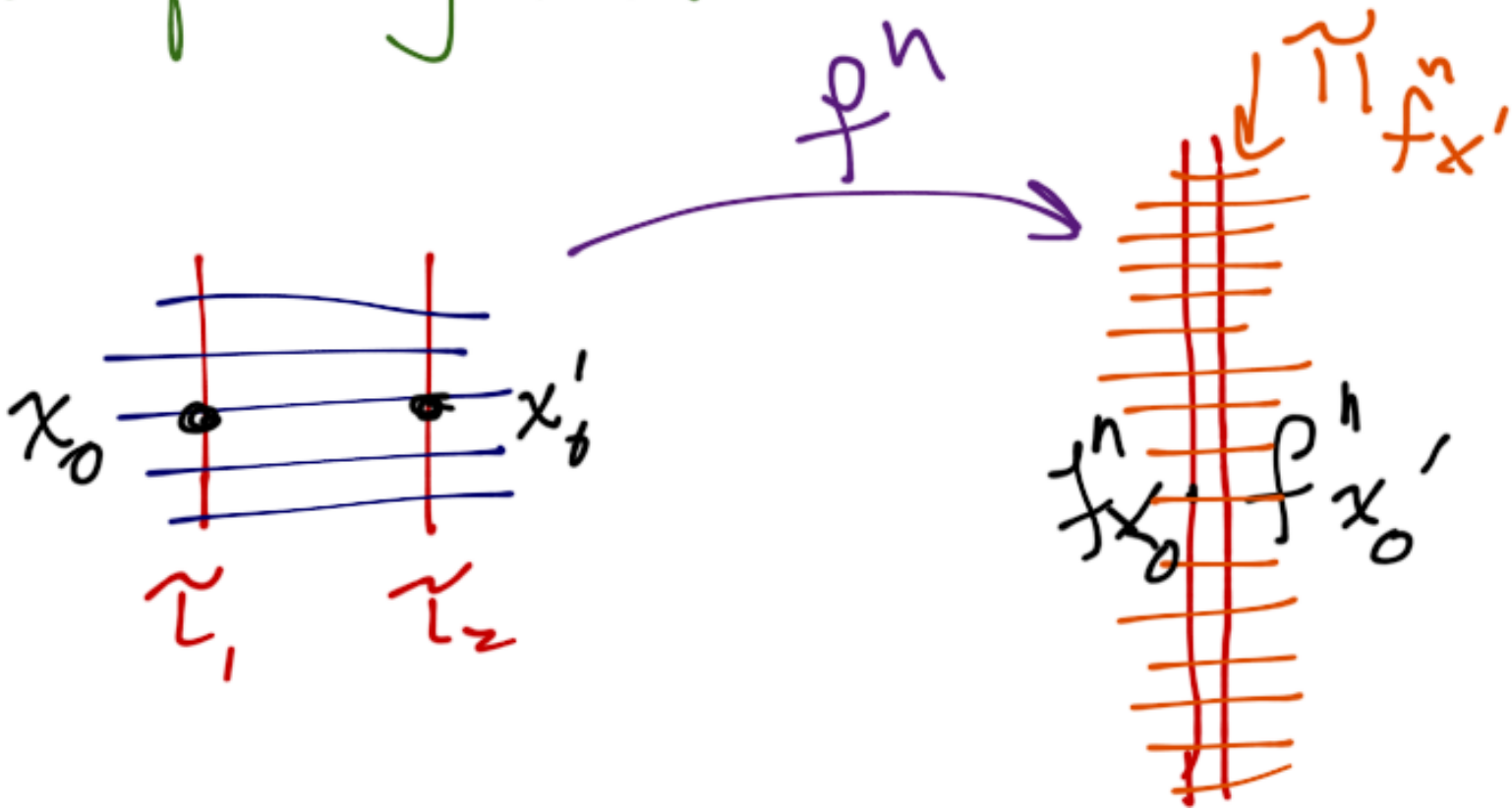
Here is a sketch of the proof:



Assume for simplicity that $\tilde{\tau}_1, \tilde{\tau}_2$ are

local $\mathcal{W}^u$ leaves.

Fix a smooth normal bundle to $\mathcal{W}^u$ which defines a smooth local projection π_x onto $\mathcal{W}^u(x)$, $\forall x \in M$. This local projection has uniformly bounded C^1-norm.



Fix $x_0' \in \mathcal{T}_2$ & consider

$$h_n : \mathcal{T}_1 \longrightarrow \mathcal{T}_2 \quad n \geq 1,$$

defined by:

$$h_n = f^{-n} \circ T_{f^n(x_0')} f^n$$

<u>Claim</u>:

1. $h_n \longrightarrow h$

uniformly as $n \to \infty$, where h is the $\mathcal{W}^u$ holonomy map.

2- h_n is a (C^2) diffeo $\forall n$ with uniformly bounded Jacobian: $\exists K \geq 1$ such that for all $x \in T_1$

$$\frac{1}{K} \leq |\mathrm{Jac}\, h_n(x)| \leq K$$

<u>Proposition</u> ("Three disks")

Let $h_n : N_1 \to N_2$ be a sequence of C^1 diffeomorphisms between Riemannian mfds N_1, N_2

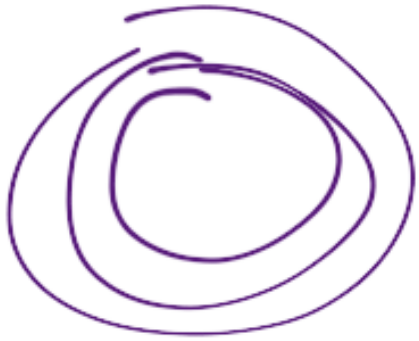
converging uniformly to a homeomorphism.

Suppose that the Jacobians of h_n are uniformly bounded from above and below.

Then h is absolutely continuous. Moreover, for every Borel set $A \subseteq N_1$

$$\frac{1}{K} m_{N_2}(A) < m_{N_2}(h(A)) \leq K m_{N_1}(A)$$

<u>Pf</u> Suff to show on disks



Given ε, $D_1 \subset D \subset D_2$

$$m_{N_2}(h_n D_1) < m_{N_2}(h(D)) \le m_{n/2}(h_n(D_2))$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_{D_1} J_n \qquad\qquad \int_{D_2} J_n$$

$$\frac{1}{K} m(D_1) \le \qquad\qquad \le K m(D_2)$$

$$\le K(1+\varepsilon) m(D)$$