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THE RIEMANN-HILBERT CORRESPONDENCE FOR UNIT F-CRYSTALS.

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Abstract. — Let $\mathbb{F}_q$ denote the finite field of order q (a power of a prime p), let X be a smooth scheme over a field k containing $\mathbb{F}_q$, and let Λ be a finite $\mathbb{F}_q$-algebra. We study the relationship between constructible Λ-sheaves on the étale site of X, and a certain class of quasi-coherent $\mathcal{O}_X \otimes_{\mathbb{F}_q} \Lambda$-modules equipped with a "unit" Frobenius structure. We show that the two corresponding derived categories are anti-equivalent as triangulated categories, and that this anti-equivalence is compatible with direct and inverse images, tensor products, and certain other operations.

We also obtain analogous results relating complexes of constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on smooth $W_n(k)$-schemes, and complexes of Berthelot's arithmetic $\mathscr{D}$-modules, equipped with a unit Frobenius.

CONTENTS

GENERAL INTRODUCTION

Let X be a smooth complex analytic space. One knows that there is an equivalence of categories between the category of local systems of $\mathbb{C}$-vector spaces on X and the category of coherent $\mathcal{O}_X$-modules equipped with an integrable connection ∇. A serious defect of this theory is that the category of local systems, and similarly that of modules with connection, is not stable under taking direct images. For example, if $f : Y \longrightarrow X$ is a proper map of smooth complex analytic spaces, then the higher direct images $R^i f_* \mathbb{C}$ are guaranteed to be a local system only over the points where f is smooth. Similarly, the connection on the relative De Rham cohomology may be singular at the points where f is not smooth. This defect is remedied by the theory of $\mathscr{D}$-modules, which shows that there is a relationship between the category of constructible sheaves, and the category of regular holonomic $\mathscr{D}$-modules. More precisely, the two corresponding derived categories are equivalent as triangulated categories, and this equivalence respects the "six operations" $f^!, f_!, f^*, f_*, \underline{RHom}^\bullet$, and $\overset{\mathbb{L}}{\otimes}$. This result, originally proved by Kashiwara and by Mebkhout with a later algebraic approach due to Beilinson and Bernstein, is known as the "Riemann-Hilbert correspondence".

The purpose of this paper is to study a certain characteristic p analogue of the above situation. Our starting point is a theorem of Nick Katz [**Ka 1**, Prop. 4.1.1]

Theorem. — *(Katz) Let k be a perfect field of characteristic p containing $\mathbb{F}_q$, where $q = p^r$. If X is a smooth scheme over $W_n(k)$, the ring of Witt vectors of k of length n, and if F_X is a lift to X of the Frobenius on $W_n(k)$, then there is an equivalence of categories between the category of locally free étale sheaves of $W_n(\mathbb{F}_q)$-modules $\mathcal{L}$, and the category of coherent, locally free $\mathcal{O}_X$-modules $\mathcal{E}$ equipped with an $\mathcal{O}_X$-linear isomorphism $(F_X^r)^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. This equivalence is realised by associating $\mathcal{E} = \mathcal{L} \otimes_{W_n(k)} \mathcal{O}_X$ to $\mathcal{L}$.*

In the context of this paper, Katz's theorem should be regarded as the analogue of the relation between local systems and vector bundles with connection. The main purpose of this volume is to extend Katz's result to a Riemann-Hilbert type correspodence, first when X is actually a smooth k-scheme (§§1–12), and then more generally, for smooth $W_n(k)$ schemes (§§13–17). Each of these two parts of the paper has its own introduction, which provides a detailed outline of its contents. The remainder of this general introduction is devoted to explaining our Riemman-Hilbert correspondence, and some of its applications, in more detail. Let us also point out that the paper [**EK 2**] provides a summary of the main results and key techniques of this volume.

To explain our results, suppose that X is a smooth $W_n(k)$-scheme with a lift of Frobenius F_X, as in Katz's theorem. We begin by introducing the notion of a *locally finitely generated unit* $\mathcal{O}_{F^r,X}$-module. If r is a fixed positive integer, then a *unit* $\mathcal{O}_{F^r,X}$-module on X is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ equipped with an isomorphism $(F_X^r)^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$. This isomorphism endows $\mathcal{M}$ with the structure of a sheaf of left modules over the sheaf of rings $\mathcal{O}_{F^r,X} = \mathcal{O}_X[F^r]$, where $\mathcal{O}_X[F^r]$ denotes the twisted polynomial ring given by the relation $F^r a = a^q F^r$, for any section a of $\mathcal{O}_X$. The unit $\mathcal{O}_{F^r,X}$-module $\mathcal{M}$ is said to be *locally finitely generated* if, locally on X, it is finitely generated as a left $\mathcal{O}_{F^r,X}$-module.

When X is a smooth k-scheme, the main result of this paper generalises Katz's theorem to an (anti-) equivalence of two triangulated categories: the derived category $D^b_c(X_{\acute{e}t})$ of bounded complexes of étale sheaves of $\mathbb{F}_q$-modules whose cohomology sheaves are constructible, and the derived category $D^b_{lfgu}(\mathcal{O}_{F^r,X})$ of bounded complexes of left $\mathcal{O}_{F^r,X}$-modules (on the Zariski site of X) whose cohomology sheaves are locally finitely generated unit $\mathcal{O}_{F^r,X}$-modules.

If we let $\pi_X : X_{\acute{e}t} \to X$ denote the natural morphism from the étale site of X to the Zariski site of X, then this anti-equivalence is given by associating

$$\mathcal{M}^\bullet = \mathrm{M}(\mathcal{F}^\bullet) = R\pi_{X*}\underline{RHom}^\bullet_{\mathbb{F}_q}(\mathcal{F}^\bullet, \mathcal{O}_X)[d_X]$$

to a complex $\mathcal{F}^\bullet$ in $D^b_c(X_{\acute{e}t})$, and associating

$$\mathcal{F}^\bullet = \mathrm{Sol}(\mathcal{M}^\bullet) = \underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}}(\pi_X^*\mathcal{M}^\bullet, \mathcal{O}_{X_{\acute{e}t}})[d_X]$$

to a complex $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathcal{O}_{F^r,X})$. (Here d_X denotes the (locally constant) dimension of X.) A comment on the choice of notation: M is intended to suggest a Dieudonné module functor, while Sol is borrowed from the theory of $\mathscr{D}$-modules.

When X is a smooth $W_n(k)$-scheme, the sheaf of rings $\mathcal{O}_{F^r,X}$ is replaced by a more intricate sheaf of rings $\mathscr{D}_{F,X}$, obtained by adjoining to $\mathcal{O}_X$ the differential operators of all orders on X, together with a "local lift of Frobenius." (When $n > 1$ we consider only the case $r = 1$.) A pleasing point is that the sheaf $\mathscr{D}_{F,X}$ that one obtains is independent of the choice of local lift, as any two lifts are congruent modulo p, and so either lift may be expressed as a polynomial combination of the other lift and appropriate differential operators. Thus (unlike in Katz's result stated above) we are not restricted to the consideration of smooth $W_n(k)$-schemes which admit a global lift of Frobenius. Another important technical result is that locally the sheaf $\mathcal{O}_X$ has a finite length resolution by free left $\mathscr{D}_{F,X}$-modules, whereas this is false if one works just with differential operators, and does not adjoin the Frobenius lifts. Thus the differential operators and lifts of Frobenius complement one another.

A sheaf of $\mathscr{D}_{F,X}$-modules $\mathcal{M}$ is called *locally finitely generated unit*, if it is quasi-coherent over $\mathcal{O}_X$, is finitely generated over $\mathscr{D}_{F,X}$ locally on X, and if the maps

$F^*\mathcal{M} \to \mathcal{M}$ defined by local lifts of Frobenius are isomorphisms. The Riemann-Hilbert correspondence then asserts an anti-equivalence between two triangulated categories. One is the category $D^b_{ctf}(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$, which is the full subcategory of the bounded derived category of étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules, consisting of complexes with constructible cohomology, and finite Tor-dimension. The other is the category $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$, which is the full subcategory of the bounded derived category of $\mathscr{D}_{F,X}$-modules, consisting of complexes with locally finitely generated unit cohomology sheaves, and finite $\mathcal{O}_X$-Tor dimension. The definition of the anti-equivalence is given by the same formulas as for $\mathcal{O}_{F,X}$-modules, except that $\mathcal{O}_{F,X}$ is replaced by $\mathscr{D}_{F,X}$ in the definition of Sol.

When $n = 1$, there is an equivalence between $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ and $D^b_{lfgu}(\mathcal{O}_{F,X})$, and one recovers the previous correspondence.

The correspondence we construct is compatible with three of Grothendieck's six operations, namely the operations $f_!$, f^{-1}, and $\overset{\mathbb{L}}{\otimes}$ on $D^b_c(X_{ét})$. (Of course, for this claim of compatibility to have any sense, one must have an *a priori* description of the corresponding triple of operators on $D^b_{lfgu}(\mathcal{O}_{F^r,X})$ and $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$. A significant part of our work is devoted to constructing such operations for left $\mathcal{O}_{F^r,X}$-modules and establishing their main properties.) This may at first seem disappointing, but in fact these are the only three which are defined in our situation. Indeed for complexes of p-torsion étale sheaves the push-forward f_* does not preserve the property of having constructible cohomology sheaves. Also one cannot define $f^!$, since there is no duality (for example because the purity theorem fails horribly). This lack of a duality is also the reason that we construct an anti-equivalence rather than an equivalence of categories.

As already remarked, when $n = 1$ we develop a somewhat more elaborate theory than in the general case. For example we show that, as a (perhaps weak) compensation for the absense of three of the six opearations, our correspondence relates two other operations which have no analogue over $\mathbb{C}$. Namely, if $q' = p^{r'}$, and $\mathbb{F}_q \subset \mathbb{F}_{q'} \subset k$, then the functor "$\mathbb{F}_{q'} \otimes_{\mathbb{F}_q} -$" converts sheaves of $\mathbb{F}_q$-vector spaces into sheaves of $\mathbb{F}_{q'}$-vector spaces. Conversely the functor "pass to the underlying $\mathbb{F}_q$-structure" converts sheaves of $\mathbb{F}_{q'}$-vector spaces into sheaves of $\mathbb{F}_q$-vector spaces. We call these operations respectively *induction* and *restriction*. There are corresponding operations for left $\mathcal{O}_{F^r,X}$-modules, namely "$\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_{F^{r'},X}} -$" and "pass to the underlying $\mathcal{O}_{F^{r'},X}$-module structure", which we also call induction and restriction. We show that our anti-equivalence interchanges the operations of induction and restriction.

Furthermore, although in the statement of Katz's theorem k denotes a perfect field, in the case when $n = 1$ we allow k to be an arbitrary field of characteristic p. A technique of inseparable descent is included in our theory, in a manner compatible with our Riemann-Hilbert correspondence.

Finally, we allow coefficients: we fix an arbitrary Noetherian $\mathbb{F}_q$-algebra Λ, and then develop a theory of $\mathcal{O}^\Lambda_{F^r,X} =: \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_{F^r,X}$-modules. If Λ is a finite $\mathbb{F}_q$-algebra we show a Riemann-Hilbert type correspondence between the triangulated category of bounded complexes of left $\mathcal{O}^\Lambda_{F^r,X}$-modules which have locally finitely generated unit cohomology sheaves and are of finite Tor-dimension as complexes of $\mathcal{O}_X \otimes_{\mathbb{F}_q}$

Λ-modules, which we denote by $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$, and the triangulated category of bounded complexes of étale Λ-sheaves which have constructible cohomology sheaves and are of finite Tor-dimension, which we denote by $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$.

The Riemann-Hilbert correspondence for $\mathscr{D}$-modules takes the usual t-structure on the derived category of regular holonomic $\mathscr{D}$-modules to an exotic t-structure on the derived category of complexes of sheaves with constructible cohomology, namely the perverse t-structure corresponding to the middle perversity. The theory of perverse t-structures is developed in [**BBD**] in the context of ℓ-adic sheaves on k-schemes (where ℓ denotes a prime different from p). Gabber [**Ga**] has extended this theory so as to define perverse t-structures on the category $D^b_c(X_{\acute{e}t})$. We show that our Riemann-Hilbert correspondence takes the usual t-structure on $D^b_{lfgu}(\mathcal{O}_{F^r,X})$ to Gabber's perverse t-structure on $D^b_c(X_{\acute{e}t})$ corresponding to the middle perversity. More, generally, if the coefficient ring Λ is a product of fields, we can identify the exotic t-structure on $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$ with the perverse t-structure of Gabber corresponding to the middle perversity. Taking the heart of this perverse t-structure gives a category of "perverse Λ-sheaves" on X. In [**EK 2**, §4] we show that these categories of perverse sheaves satisfy properties analogous to those of the usual ℓ-adic perverse sheaves of [**BBD**] (namely, there is an intermediate extension functor $j_{!*}$ for any locally closed immersion j, defined by the same formula as in [**BBD**], and any simple perverse sheaf is isomorphic to the intermediate extension of an irreducible local system placed in the appropriate degree).

It is easy to see that one can only expect to have a Riemann-Hilbert correspondence of the type we envisage when Λ is finite over $\mathbb{F}_q$, as otherwise one finds many left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules that do not correspond to étale sheaves, even in the case that X is a point. Our main motivation for allowing non-finite Λ is the hope that, even when a Riemann-Hilbert correspondence does not exist, the theory of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules may nevertheless find other applications, for example to Drinfeld modules (when $\Lambda = \mathbb{F}_q[T]$). Here we were inspired by recent work of Böckle-Pink [**BP**], where a trace formula for the L-function of a Drinfeld module is proved. After defining the L-function of a complex in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$, we prove that, when Λ is reduced, the formation of L-functions commutes with f_+. This result is closely related to the trace formula of Böckle-Pink. One consequence is a new proof of a result conjectured by Goss and proved by Taguchi-Wan [**TW**, §7], which says that the local L-function of a Drinfeld module is rational.

The proof of our trace formula is reduced by a specialisation technique to the case where Λ is a finite field. In this case the Riemann-Hilbert correspondence respects the formation of L-functions, so that the trace formula can be deduced from the corresponding formula for étale sheaves, due to Deligne [**De**, p. 116]. Interestingly, Deligne's formula ultimately depends on the so-called "Woods Hole" trace formula, which has an F-crystal flavour. (In fact, the main result of [**EK 1**] specialises to yield a proof of Deligne's formula which works directly with F-crystals.)

Our original motivation for writing this paper and its sequel was to try to generalise the results (due to Bloch, Faltings, Fontaine, Hyodo, Kato, Messing, Tsuji, ...) relating p-adic étale cohomology and crystalline cohomology of schemes over a finite extension of $\mathbb{Q}_p$ to a Riemann-Hilbert type correspondence. That such a theory should exist is strongly suggested by the work of Faltings [**Fa**], where he treats the correspondence between étale and crystalline cohomology allowing coefficients. We intended the present paper and its sequel to be a "warm-up" for this more general project, but as the reader will see, even the comparatively simple unit case has turned out to be quite technical.

After we had begun this work, we were pleasantly surprised to find that a correspondence of the type we prove here had been suggested by Lyubeznik [**Lyu**, p. 69]. (although not on the level of derived categories). The reader will observe that we are indebted to him for providing several important techniques for working with unit $\mathcal{O}_{F^r,X}$-modules, and especially for the notion of a "generator" of such a module.

We hope that our results will have applications in various directions. One application already found has been to a conjecture of Katz (stated in [**Ka 2**]) relating L-functions of p-adic lisse sheaves on varieties over finite fields of characteristic p to their cohomology with proper supports. Using the techniques of this paper, this conjecture has now been proved [**EK 1**]. There is also the possibility that the formalism developed here will have applications to the study of local cohomology on varieties in characteristic p. Indeed, in the case that X is a smooth affine variety over k, our notion of locally finitely generated unit $\mathcal{O}_{F,X}$-module coincides with the notion of F-finite module, which was introduced by Lyubeznik [**Lyu**] expressly for that purpose. Finally there is the possibility of applications to Drinfeld modules and related objects, that we already alluded to above.

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INTRODUCTION TO §§1–12: $\mathcal{O}_{F,X}$-MODULES

We now give a more detailed outline of §§1–12 of the paper. They naturally fall into three parts. The first, consisting of sections 1, 2, 3 and 4, introduces the notion of left $\mathcal{O}^{\Lambda}_{F^r}$-modules, defines the functors $f^!$ and f_+, and develops their basic properties. The approach is modelled closely on the theory of algebraic $\mathscr{D}$-modules, and as in that theory, there is a close relationship between the properties of the functors $f^!$ and f_+ and the Grothendieck-Serre duality theory of quasi-coherent sheaves.

The second part, consisting of sections 5 and 6, introduces the notions of unit and locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r}$-modules. The results of the first four sections are specialised to and further developed in this particular context. One result which is indispensable in the study of locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r}$-modules is the fact that any such module $\mathcal{M}$ on a k-scheme X admits a presentation of the form

$$0 \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M \to \mathcal{M} \to 0,$$

where M is a coherent $\mathcal{O}^{\Lambda}_X =: \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_X$-module. The idea of using such presentations to study locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules is an extension of the main technique of [**Lyu**].

Sections 7 and 8 provide a bridge between the first two parts and the third. Section 7 observes that the entire theory of sections 1 through 6 could have been developed by working on the étale site $X_{\acute{e}t}$ of a smooth k-scheme X rather than on the Zariski site. Indeed, both theories could have been developed in parallel, but we chose not to do so for expository reasons; it seemed easier to fix ideas by working on one site or the other. A key observation is that, if one restricts oneself to quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, then étale descent provides an equivalence between the Zariski and étale theories. Section 8 recalls (very briefly) the theory of constructible Λ-sheaves on $X_{\acute{e}t}$, as developed in [**SGA 4**] and [**De**].

Of the final four sections, §§9–12, the first three sections are devoted to establishing our Riemann-Hilbert correspondence in the case when Λ is finite over $\mathbb{F}_q$. In them we define the functors M and Sol, and then prove that they yield the desired equivalence of categories, that they respect the appropriate Grothendieck operations, and (if Λ is a product of fields) that the exotic t-structure that they induce on $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$

is equal to Gabber's middle perverse t-structure. Section 12 provides a discussion of L-functions attached to complexes in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$, and proves the trace formula discussed in the general introduction. It also includes the statement of the main result of [**EK 1**].

Just as in the case of the Riemann-Hilbert correspondence for $\mathscr{D}$-modules, the proof of the Riemann-Hilbert correspondence of this paper uses excision arguments to reduce to an explicit simple fact. In the case of $\mathscr{D}$-modules, one reduces to the fact that any function on a smooth complex variety with vanishing partial derivatives is locally constant. In our situation, one reduces to the fact that any section of the structure sheaf on a smooth k-scheme which is F^r-invariant is locally constant and $\mathbb{F}_q$-valued. So just as the Riemann-Hilbert correspondence for $\mathscr{D}$-modules can be regarded as a generalisation of De Rham theory, our Riemann-Hilbert correspondence can be regarded as a generalisation of Artin-Schreier theory.

In setting up a cohomological formalism of the type that we have, there are invariably a large number of compatibilities that can and should be checked between the various functors and natural transformations that one defines. Our approach to these has been as follows: those compatibilities which are essential for establishing the main results are stated and either proved or left to the reader in the case that the verifications are standard (if somewhat lengthy). In doing this our aim has been to make the arguments clear and complete, while keeping the size of the paper bounded. The construction and verification of the many other compatibilities that we have not discussed are left to the imagination of the reader.

On a note related to that of the preceding paragraph, it is important to draw attention to two foundational issues on which this paper depends. The first is that we need *some* consistent choice of sign conventions for the various homological constructions that we use. The second is that much of our theory depends (not only on the statements of, but) on the construction of Grothendieck-Serre duality, and the standard reference [**Ha 1**] for this construction is incomplete in its details. Thankfully, both issues are resolved by recent work of Brian Conrad [**Con**]; in this very detailed text, Conrad lays down a consistent set of sign conventions, and also provides an extremely close reading of [**Ha 1**], filling in many gaps and correcting various misstatements. Throughout this paper we will follow [**Con**] with regard to both its sign conventions (although these will not play any overt role) and its analysis of [**Ha 1**].

In the first appendix we have provided a summary of the results of Grothendieck-Serre duality that we use in the paper, and have developed some consequences of the general theory which are of particular importance for our purposes. In particular, we generalise the definition of the Cartier operator on top-forms to a broader context than that in which it is usually considered. In the second appendix we have given proofs of some results from homological algebra which we could not find in the literature (although they are probably standard for experts).

0. NOTATION AND CONVENTIONS

0.1. — We fix a prime $p > 0$, a positive integer r, and set $q = p^r$. Throughout the paper Λ will denote a Noetherian $\mathbb{F}_q$-algebra, and k will denote a field containing $\mathbb{F}_q$. The phrases "scheme over k" and "k-scheme" will always be taken to mean "finite type separated k-scheme", whether or not these hypotheses are explicitly mentioned.

If X is a k-scheme, we let $\mathcal{O}_X^\Lambda$ denote the sheaf of $\Lambda \otimes_{\mathbb{F}_q} k$-algebras $\Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_X$ on X. This sheaf can be thought of as the push-forward to X of the structure sheaf on the product $\operatorname{Spec} \Lambda \otimes_{\mathbb{F}_q} X$. (We do not adopt this point of view in the paper; rather we regard Λ as an auxiliary algebra of operators.)

We say that a sheaf of $\mathcal{O}_X^\Lambda$-modules on X is quasi-coherent if it is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. Such a sheaf may be regarded as the push-forward of a quasi-coherent sheaf on $\operatorname{Spec} \Lambda \otimes_{\mathbb{F}_q} X$ (although again, we will not adopt this point of view in the paper).

We say that a sheaf of $\mathcal{O}_X^\Lambda$-modules on X is coherent if it is quasi-coherent and locally finitely generated as an $\mathcal{O}_X^\Lambda$-module, or equivalently (since Λ is Noetherian) if it is locally finitely presented as an $\mathcal{O}_X^\Lambda$-module. Such a sheaf may be regarded as the push-forward of a coherent sheaf on $\operatorname{Spec} \Lambda \otimes_{\mathbb{F}_q} X$.

0.2. — If $f : Y \to X$ is a morphism of k-schemes, we let $f^\# : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ denote the morphism of sheaves of k-algebras induced by f. Tensoring this morphism with Λ over $\mathbb{F}_q$ yields a morphism of $\Lambda \otimes_{\mathbb{F}_q} k$-sheaves $f^{-1}\mathcal{O}_X^\Lambda \to \mathcal{O}_Y^\Lambda$, which we continue to denote by $f^\#$.

0.3. — We wish to explain a certain convention regarding our notation for shifts. Suppose that X is a k-scheme; then X is the disjoint union of finitely many connected components:

$$X = \coprod_{i=1}^{n} X_i$$

with each X_i a connected open subscheme of X. Let $\psi : X \to \mathbb{Z}$ be a continuous integer valued function on X. Then ψ is constant on each X_i; let ψ_i denote the value assumed by ψ on X_i.

Let $\mathcal{A}$ be a sheaf of rings on the Zariski or étale site of X, and let $\mathcal{A}_i$ denote the restriction of $\mathcal{A}$ to each X_i. We let $D(\mathcal{A})$ denote the derived category of complexes of sheaves of $\mathcal{A}$-modules on X, and $D(\mathcal{A}_i)$ denote the derived category of complexes of sheaves of $\mathcal{A}_i$-modules on X_i (for $i = 1, \ldots, n$). We wish to define the shift operator $[\psi]$ on $D(\mathcal{A})$.

There is an equivalence of categories $D(\mathcal{A}) \xrightarrow{\sim} \prod_{i=1}^{n} D(\mathcal{A}_i)$. This isomorphism identifies any object $\mathcal{M}^\bullet$ of $D(\mathcal{A})$ with the n-tuple $(\mathcal{M}_1^\bullet, \ldots, \mathcal{M}_n^\bullet)$, where each $\mathcal{M}_i^\bullet$ is the object of $D(\mathcal{A}_i)$ obtained by restricting $\mathcal{M}^\bullet$ to X_i. On the right hand side of this isomorphism we have the functor $\prod_{i=1}^{n}[\psi_i]$, which is the product of usual shift operators on each category $D(\mathcal{A}_i)$. We let $[\psi]$ denote the induced functor on $D(\mathcal{A})$ (via the preceding equivalence of categories). Thus if $\mathcal{M}^\bullet$ is an object of $D(\mathcal{A})$, the object $\mathcal{M}^\bullet[\psi]$ is defined to be the object of $D(\mathcal{A})$ corresponding (via this equivalence of categories) to the n-tuple $(\mathcal{M}_1^\bullet[\psi_1], \ldots, \mathcal{M}_n^\bullet[\psi_n])$.

We will also use some obvious variations on this notation. For example (if j is a fixed integer), we will write that $H^j(M^\bullet) = 0$ for $j > \psi$ if $H^j(\mathcal{M}_i^\bullet) = 0$ for $j > \psi_i$, and $i = 1, \ldots, n$.

One example of such a function ψ which is defined for any smooth k-scheme X is the morphism

$$d_X : x \mapsto \text{dimension of the component of } X \text{ containing } x.$$

More generally, if $f : Y \to X$ is a morphism between smooth k-schemes, then we may consider the functions $d_{Y/X} = d_Y - d_X \circ f$ and $d_{X/Y} = -d_{Y/X}$.

0.4. — We write Hom (respectively $\mathrm{Hom}^\bullet$) to denote the functor of morphisms in additive category (respectively the corresponding category of complexes), and we write $\mathrm{RHom}^\bullet$ to denote the derived functor of Hom (when this exists). When our category is the category of sheaves on a site, we write $\underline{Hom}$, $\underline{Hom}^\bullet$ and $\underline{RHom}^\bullet$ to denote the corresponding sheaf versions of these functors.

1. $\mathcal{O}^{\Lambda}_{F^r}$-MODULES

1.1. — We begin by defining the sheaf $\mathcal{O}^{\Lambda}_{F^r,X}$ on any k-scheme X.

Definition 1.1.1. — If A is a k-algebra, we define $A\,[F^r]$ to be the twisted polynomial ring over A whose multiplication is defined by $F^r a = a^q F^r$. If X is a scheme over k, we can sheafify this definition, to obtain a sheaf of rings $\mathcal{O}_X[F^r]$ on X, whose sections over an open subset U are $\mathcal{O}_X[F^r](U) = \mathcal{O}_U(U)[F^r]$. We often denote $\mathcal{O}_X[F^r]$ by $\mathcal{O}_{F^r,X}$. We set $\mathcal{O}^{\Lambda}_{F^r,X} = \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_{F^r,X}$.

Note that $\mathcal{O}^{\Lambda}_{F^r,X}$ is a sheaf of Λ-algebras on X, because $\mathbb{F}_q$ lies in the centre of $\mathcal{O}_{F^r,X}$.

1.2. — If X is a k-scheme, then the r^{th} power of the absolute Frobenius F_X on X defines a map of sheaves

$$F_X{}^{r\#} : \mathcal{O}^{\Lambda}_X \to F^r_{X*}\mathcal{O}^{\Lambda}_X = \mathcal{O}^{\Lambda}_X$$

which is given on the level of sections by the formula $\lambda \otimes a \mapsto \lambda \otimes a^q$, if $\lambda \in \Lambda$ and a is a section of $\mathcal{O}_X$.

Definition 1.2.1. — For any non-negative integer n we let $\mathcal{O}^{\Lambda(rn)}_X$ denote the sheaf $\mathcal{O}^{\Lambda}_X$ regarded as an $(\mathcal{O}^{\Lambda}_X, \mathcal{O}^{\Lambda}_X)$-bimodule in the following way: for the left $\mathcal{O}^{\Lambda}_X$-module structure, $\mathcal{O}^{\Lambda}_X$ is regarded as a module over itself in the usual way, while for the right $\mathcal{O}^{\Lambda}_X$-module structure, $\mathcal{O}^{\Lambda}_X$ is regarded as a module over itself via the morphism $(F_X{}^{r\#})^n$. Thus any section $\lambda \otimes a$ of $\mathcal{O}^{\Lambda}_X$ acts on $\mathcal{O}^{\Lambda(rn)}_X$ on the left as multiplication by $\lambda \otimes a$, and on the right as multiplication by $\lambda \otimes a^{q^n}$.

If M is an $\mathcal{O}^{\Lambda}_X$-module, then we have a natural isomorphism

$$(F^{rn}_X)^* M \xrightarrow{\sim} \mathcal{O}^{\Lambda(rn)}_X \otimes_{\mathcal{O}^{\Lambda}_X} M. \qquad (1.2.2)$$

1.3. — Since $\mathcal{O}^{\Lambda}_X$ is a sheaf of subrings of $\mathcal{O}^{\Lambda}_{F^r,X} = \mathcal{O}^{\Lambda}_X[F^r]$, we may naturally regard $\mathcal{O}^{\Lambda}_{F^r,X}$ as an $(\mathcal{O}^{\Lambda}_X, \mathcal{O}^{\Lambda}_X)$-bimodule.

Lemma 1.3.1. — *There is an isomorphism of $(\mathcal{O}_X^\Lambda, \mathcal{O}_X^\Lambda)$-bimodules*

$$\bigoplus_{n=0}^{\infty} \mathcal{O}_X^{\Lambda(rn)} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X},$$

which is given in degree n by the formula $a \mapsto aF^{rn}$, if a is a section of $\mathcal{O}_X^{\Lambda(rn)}$. Thus $\mathcal{O}^{\Lambda}_{F^r,X}$ is free as a left $\mathcal{O}_X^\Lambda$-module, and so also as a left $\mathcal{O}_X$-module. If furthermore X is smooth over k, then $\mathcal{O}^{\Lambda}_{F^r,X}$ is locally free as a right $\mathcal{O}_X^\Lambda$-module, and so also as a right $\mathcal{O}_X$-module.

Proof. — The claimed isomorphism is immediate from the definition of the ring structure of $\mathcal{O}^{\Lambda}_{F^r,X}$ and the bimodule structure of $\mathcal{O}_X^{\Lambda(rn)}$. That $\mathcal{O}^{\Lambda}_{F^r,X}$ is free as a left $\mathcal{O}_X^\Lambda$-module is now clear, and the analogous claim for the right $\mathcal{O}_X^\Lambda$-module structure follows from the fact that, when X is smooth, each $\mathcal{O}_X^{\Lambda(rn)}$ is locally free as a right $\mathcal{O}_X^\Lambda$-module (see (A.2)). □

Corollary 1.3.2. — *If M is a sheaf of $\mathcal{O}_X^\Lambda$-modules, then there is a natural isomorphism of left $\mathcal{O}_X^\Lambda$-modules*

$$\bigoplus_{n=0}^{\infty} (F_X^{rn})^* M \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X^\Lambda} M.$$

Proof. — This follows from the lemma and equation (1.2.2). □

1.4. — The natural isomorphism $\mathcal{O}_X^\Lambda \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}_X^\Lambda}(\mathcal{O}_X^\Lambda, \mathcal{O}_X^\Lambda)$ (given by a section a of $\mathcal{O}_X^\Lambda$ acting as multiplication by a) extends to a map

$$\mathcal{O}^{\Lambda}_{F^r,X} \to \underline{Hom}_\Lambda(\mathcal{O}_X^\Lambda, \mathcal{O}_X^\Lambda)$$

defined by taking F^r to the homomorphism $(F_X^r)^\#$. In future we will always regard $\mathcal{O}_X^\Lambda$ with the left $\mathcal{O}^{\Lambda}_{F^r,X}$-module structure given by this map.

This map need not be an injection (despite a remark suggesting the contrary on page 102 of [**Lyu**], where $A[F]$ is denoted by $A\{f\}$); for example if $k = \mathbb{F}_q$, and $X = \operatorname{Spec} k$, then the global section $1-F^r$ is in the kernel; if $X = \operatorname{Spec} k[x,y]/(x^2, xy)$, then the global section $x - xF^r$ is in the kernel. The following lemma clarifies the situation.

Lemma 1.4.1. — *If X is a reduced k-scheme having each irreducible component of positive dimension, then the map*

$$\mathcal{O}^{\Lambda}_{F^r,X} \to \underline{Hom}_\Lambda(\mathcal{O}_X^\Lambda, \mathcal{O}_X^\Lambda)$$

is injective.

Proof. — It is enough to prove the lemma when $r = 1$, $\Lambda = \mathbb{F}_p$, and $X = \operatorname{Spec} A$ is affine, so that A is a reduced finite type k-algebra. Let x be a closed point of X. Since each irreducible component of X is of positive dimension, there exists a non-zero divisor $a \in A$ contained in the maximal ideal $\mathfrak{m}$ of x. Indeed, otherwise $\mathfrak{m}$ is in the union of the minimal primes of A (since A is reduced, all the associated primes of A are minimal), hence is equal to one of them, which contradicts the assumption that the components of X have positive dimension.

Now suppose that $\sum_{i=0}^n a_i F^i$ is the zero endomorphism of A. Then for any positive integer m,

$$0 = \sum_{i=0}^n a_i F^i(a^m) = \sum_{i=0}^n a_i a^{mp^i}.$$

But from this equation, cancelling a power of a^m, we conclude that a_0 is divisible by arbitrarily large powers of a. Thus a_0 vanishes in a neighbourhood of x. Continuing in the same manner, we find that each a_i vanishes in a neighbourhood of x. Since x was an arbitrary closed point of X, and X is of finite type over k, we see that each a_i vanishes on X. □

1.5. — Suppose that $\mathcal{M}$ is any sheaf of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Multiplication on the left by sections of $\mathcal{O}^{\Lambda}_{F^r,X}$-yields an $\mathcal{O}^{\Lambda}_{F^r,X}$-linear, and in particular $\mathcal{O}^{\Lambda}_X$-linear, morphism

$$\mu_{\mathcal{M}} : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \to \mathcal{M}.$$

By Corollary 1.3.2 there is a natural isomorphism

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} (F_X^{rn})^* \mathcal{M},$$

and so the morphism μ_M determines, and is determined by, a series of $\mathcal{O}^{\Lambda}_X$-linear morphisms

$$\phi_{n,\mathcal{M}} : (F_X^{rn})^* \mathcal{M} = \mathcal{O}_X^{\Lambda(rn)} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \to \mathcal{M}$$

(n a non-negative integer). Explicitly, $\phi_{n,\mathcal{M}}$ is given by the formula

$$\phi_{n,\mathcal{M}} : a \otimes m \mapsto aF^{rn}(m),$$

if a is a section of $\mathcal{O}_X^{\Lambda(rn)}$ and m is a section of $\mathcal{M}$.

Lemma 1.5.1. — *Suppose that $\mathcal{M}$ is a sheaf of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and let*

$$\phi_{n,\mathcal{M}} : (F_X^{rn})^* \mathcal{M} \to \mathcal{M}$$

be the sequence of $\mathcal{O}^{\Lambda}_X$-linear morphisms constructed above. Then $\phi_{0,\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$, while for each $n > 0$, $\phi_{n,\mathcal{M}}$ is equal to the composition

$$(F_X^{rn})^* \mathcal{M} \stackrel{(F_X^{r(n-1)})^* \phi_{1,\mathcal{M}}}{\longrightarrow} (F_X^{r(n-1)})^* \mathcal{M} \stackrel{(F_X^{r(n-2)})^* \phi_{1,\mathcal{M}}}{\longrightarrow} \cdots \stackrel{F_X^{r\,*} \phi_{1,\mathcal{M}}}{\longrightarrow} F_X^{r\,*} \mathcal{M} \stackrel{\phi_{1,\mathcal{M}}}{\longrightarrow} \mathcal{M}.$$

Conversely, if M is a sheaf of $\mathcal{O}^{\Lambda}_X$-modules equipped with a morphism $\phi_1 : F_X^{r\,*} M \to M$, and one defines $\phi_0 = \mathrm{id}_M$ and, for each $n > 0$, defines ϕ_n to be the composition

$$(F_X^{rn})^* M \stackrel{(F_X{}^{r(n-1)})^* \phi_1}{\longrightarrow} (F_X^{r(n-1)})^* M \stackrel{(F_X{}^{r(n-2)})^* \phi_1}{\longrightarrow} \cdots \stackrel{F_X^{r\,*} \phi_1}{\longrightarrow} F_X^{r\,*} M \stackrel{\phi_1}{\longrightarrow} M,$$

then there is a unique left $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$ having M as underlying $\mathcal{O}^{\Lambda}_X$-module and for which $\phi_{n,\mathcal{M}} = \phi_n$ for each non-negative integer n.

Proof. — The fact that $\phi_{0,\mathcal{M}} = \mathrm{id}_X$ is equivalent to the fact that the section 1 of $\mathcal{O}^{\Lambda}_{F^r,X}$ acts identically on $\mathcal{M}$, and the composition formula for $\phi_{n,\mathcal{M}}$ $(n > 0)$ is equivalent to the associative law for the action of $\mathcal{O}^{\Lambda}_{F^r,X}$ on $\mathcal{M}$. Conversely, suppose given an $\mathcal{O}^{\Lambda}_X$-linear morphism $\phi : F^{r}_X{}^{*}M \to M$. For any section m of $\mathcal{M}$, define $F^r(m) = \phi(1 \otimes m)$. This defines the required left $\mathcal{O}^{\Lambda}_{F^r,X}$-module structure on $\mathcal{M}$. □

Definition 1.5.2. — If $\mathcal{M}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, we write

$$\phi_{\mathcal{M}} = \phi_{1,\mathcal{M}} : F^{r}_X{}^{*}\mathcal{M} \to \mathcal{M},$$

defined by $a \otimes m \mapsto aF^r(m)$, and refer to this as the structural morphism of $\mathcal{M}$.

The previous lemma shows that the left $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$ is uniquely determined by its structural morphism.

Example 1.5.3. — The structural morphism of the $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{O}^{\Lambda}_X$ is the natural isomorphism $F^{r}_X{}^{*}\mathcal{O}^{\Lambda}_X \xrightarrow{\sim} \mathcal{O}^{\Lambda}_X$.

Lemma 1.5.4. — *Suppose that $\psi : \mathcal{M} \to \mathcal{N}$ is a morphism of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Then we have $\psi \circ \phi_{\mathcal{M}} = \phi_{\mathcal{N}} \circ F^{r*}_X\psi$. Conversely, any morphism ψ of $\mathcal{O}^{\Lambda}_X$-modules for which this formula holds is a morphism of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules.*

Proof. — This is immediate. □

Definition 1.6. — We say that a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is quasi-coherent if its underlying $\mathcal{O}^{\Lambda}_X$-module is quasi-coherent. We let $\mu(X,\Lambda)$ denote the abelian category of quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, $D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X})$ denote the derived category of complexes of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, $D^{\bullet}(\mu(X,\Lambda))$ the derived category of complexes of quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and $D^{\bullet}_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ denote the full triangulated subcategory of $D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X})$ consisting of complexes whose cohomology sheaves are quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. (Here $\bullet$ denotes any of usual boundedness conditions $+$, $-$, b, or $\emptyset$.)

Theorem 1.6.1. — *The morphism $D^b(\mu(X,\Lambda)) \to D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ is an equivalence of triangulated categories.*

Proof. — This follows from Bernstein's theorem [**Bo**, VI 2.10]. □

Lemma 1.6.2. — *(i) If X is a k-scheme, any left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is the quotient of a flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-module (which is in particular also a flat $\mathcal{O}^{\Lambda}_X$-module). Thus any object of $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ may be represented by a bounded above complex of flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules (and so in particular of flat $\mathcal{O}^{\Lambda}_X$-modules).*

(ii) If X is in addition either smooth or quasi-projective as a k-scheme, then any quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is the quotient of a locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-module. Thus any object of $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ may be represented by a bounded above complex of locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules.

Proof. — The construction of [**Ha 1**, II 1.2] allows us to write any left $\mathcal{O}^{\Lambda}_{F^r,X}$-module as the quotient of a flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-module. The existence of flat resolutions for objects of $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ now follows by standard homological algebra. Also, since $\mathcal{O}^{\Lambda}_{F^r,X}$ is free as a left $\mathcal{O}^{\Lambda}_X$-module, we see that a flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is in particular a flat left $\mathcal{O}^{\Lambda}_X$-module. This proves part *(i)*, and we turn to proving part *(ii)*.

Let $\mathcal{M}$ be an object of $\mu(X,\Lambda)$. If X is either smooth or quasi-projective then there is an $\mathcal{O}_X$-linear surjection $E \to \mathcal{M}$ for some locally free $\mathcal{O}_X$-module E [**Ha 2**, Example III, 6.5.1, Excercise III, 6.8]. Thus we get a surjection $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X} E \to \mathcal{M}$, with $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X} E$ a locally free $\mathcal{O}^{\Lambda}_{F^r,X}$-module. The rest of part *(ii)* now follows via standard homological techniques from this remark together with Bernstein's theorem, since a locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is a quasi-coherent $\mathcal{O}^{\Lambda}_{F^r,X}$-module. □

Definition 1.6.3. — For any full triangulated subcategory $D \subset D^b(\mathcal{O}^{\Lambda}_{F^r,X})$ we denote by D° the full triangulated subcategory of D consisting of complexes which have finite Tor-dimension when regarded as complexes of $\mathcal{O}^{\Lambda}_X$-modules. (That this Tor-dimension is a well-defined notion follows from part *(i)* of Lemma 1.6.2.)

It is worth remarking that if X is a smooth k-scheme, and we put ourselves in the simplest case, when $\Lambda = \mathbb{F}_q$, then every complex in $D^b(\mathcal{O}_{F^r,X})$ has finite Tor-dimension as a complex of $\mathcal{O}_X$-modules (since a smooth k-scheme is in particular regular), and so the $\circ$ notation is unnecessary.

Definition 1.7. — If M is an $\mathcal{O}^{\Lambda}_X$-module, then $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, which we refer to as the left $\mathcal{O}^{\Lambda}_{F^r,X}$-module induced by M. More generally, if $\mathcal{M}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module isomorphic to a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module of the form $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M$ for some $\mathcal{O}^{\Lambda}_X$-module M, we refer to $\mathcal{M}$ as an induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-module. (Note that if $\mathcal{M} = \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M$ is induced, then $\mathcal{M}/F^r\mathcal{M} \xrightarrow{\sim} M$, and so $\mathcal{M}$ is quasi-coherent if and only if the same is true of M.)

We say that $\mathcal{M}$ is absolutely induced if it is isomorphic to an $\mathcal{O}^{\Lambda}_{F^r,X}$-module of the form $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X} M$ for some $\mathcal{O}_X$-module M.

Lemma 1.7.1. — *If X is either smooth or quasi-projective as a k-scheme, then any quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is the quotient of an absolutely induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-module of the form $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X} M$ with M a locally free $\mathcal{O}_X$-module. Thus any object of $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ may be represented by a bounded above complex of quasi-coherent absolutely induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, which may be taken to be of the form $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X} M$, with M locally free over $\mathcal{O}_X$.*

Proof. — We already saw this in the proof of part *(ii)* of Lemma 1.6.2. □

1.7.2. — For any sheaf of $\mathcal{O}^{\Lambda}_X$-modules M we have an isomorphism of functors on the category of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules

$$\mathrm{Hom}_{\mathcal{O}^{\Lambda}_X}(M,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M,-).$$

Thus the forgetful functor from $\mathcal{O}^{\Lambda}_{F^r,X}$-modules to $\mathcal{O}^{\Lambda}_{X}$-modules is right adjoint to the functor $\mathcal{O}^{\Lambda}_{F^r,X} \otimes -$, which if X is smooth over k is shown to be exact by Lemma 1.3.1. As a consequence we see that if $\mathcal{I}$ is an injective sheaf of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, then $\mathcal{I}$ is injective as a sheaf of $\mathcal{O}^{\Lambda}_{X}$-modules (provided of course that X is smooth over k).

1.7.3. — The previous paragraph provides a characterisation of the morphisms between induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Indeed, suppose that M and N are two $\mathcal{O}^{\Lambda}_{X}$-modules. Using (1.7.2) and (1.3.2) we have the formula

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M, \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} N)\\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\Lambda}_{X}}(M, \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\Lambda}_{X}}(M, \bigoplus_{n=0}^{\infty} (F^{rn}_X)^* N).\end{aligned}$$

1.8. — Let $\mathcal{M}$ be a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, equipped with its structural morphism $\phi_{\mathcal{M}} : F^{r*}_X \mathcal{M} \to \mathcal{M}$. Consider the composite

$$\mathrm{id} \oplus -\phi_{\mathcal{M}} : F^{r*}_X \mathcal{M} \to \mathcal{M} \oplus F^{r*}_X \mathcal{M} \subset \bigoplus_{n=0}^{\infty} F^{rn*}_X \mathcal{M}.$$

The isomorphism of Corollary 1.3.2 allows us to regard this as a map $F^{r*}_X \mathcal{M} \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{M}$, which then induces an $\mathcal{O}^{\Lambda}_{F^r,X}$-linear morphism

$$\imath_{\mathcal{M}} : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} F^{r*}_X \mathcal{M} \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{M}.$$

Lemma 1.8.1. — *For any left $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$, the sequence*

$$0 \longrightarrow \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} F^{r*}_X \mathcal{M} \xrightarrow{\imath_{\mathcal{M}}} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{M} \xrightarrow{\mu_{\mathcal{M}}} \mathcal{M} \longrightarrow 0$$

is short exact.

Proof. — This follows from the definition of $\imath_{\mathcal{M}}$, and the construction of $\phi_{\mathcal{M}}$ via $\mu_{\mathcal{M}}$. □

Corollary 1.8.2. — *Let X be a k-scheme, and let $\mathcal{M}^\bullet$ be in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$. Then $\mathcal{M}^\bullet$ has finite Tor-dimension as a complex of $\mathcal{O}^{\Lambda}_{X}$-modules if and only if $\mathcal{M}^\bullet$ has finite Tor-dimension as a complex of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Furthermore, any such complex $\mathcal{M}^\bullet$ has finite Tor-dimension as a complex of Λ-modules.*

Proof. — If $\mathcal{M}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module which is flat as an $\mathcal{O}^{\Lambda}_{X}$-module, then the short exact sequence of Lemma 1.8.1 provides a two-step resolution of $\mathcal{M}$ by flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and so $\mathcal{M}$ is of finite Tor-dimension as an $\mathcal{O}^{\Lambda}_{F^r,X}$-module. This prove one direction of the equivalence. For the converse direction, recall that by Lemma 1.3.1, $\mathcal{O}^{\Lambda}_{F^r,X}$ is free, and hence flat, as a left $\mathcal{O}^{\Lambda}_{X}$-module. Since $\mathcal{O}^{\Lambda}_{X} = \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_X$ is flat over Λ, the final statement about Tor-dimension over Λ follows. □

Corollary 1.8.3. — *If X is a k-scheme, then any object of $D^b(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$ may be represented by a finite length complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are flat as $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and so also as $\mathcal{O}^{\Lambda}_{X}$-modules. If X is furthermore either smooth or quasi-projective as a k-scheme, and if $\mathcal{M}^{\bullet}$ lies in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$, then the members of this complex may be taken to be quasi-coherent.*

Proof. — Let $\mathcal{M}^{\bullet}$ be an object of $D^b(\mathcal{O}_{F^r,X})^{\circ}$. Part *(i)* of Lemma 1.6.2 shows that we may represent $\mathcal{M}^{\bullet}$ via a bounded above complex $\mathcal{P}^{\bullet}$ of flat $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Since by Corollary 1.8.2, $\mathcal{P}^{\bullet}$ is of finite Tor-dimension when regarded as a complex of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, a standard argument shows that some finite truncation of $\mathcal{P}^{\bullet}$ is a complex of flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules (which are in particular also flat as $\mathcal{O}^{\Lambda}_{X}$-modules).

If X is smooth or quasi-projective over k, and if $\mathcal{M}^{\bullet}$ lies in $D^b_{qc}(\mathcal{O}_{F^r,X})^{\circ}$, then part *(ii)* of Lemma 1.6.2 shows that $\mathcal{P}^{\bullet}$ may be taken to be a complex of locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Any truncation of $\mathcal{P}^{\bullet}$ then consists of quasi-coherent $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, as required. □

Corollary 1.8.4. — *If X is a d-dimensional smooth k-scheme then any $\mathcal{O}_{F^r,X}$-module is of Tor-dimension at most $d+1$.*

Proof. — Since X is smooth over k, and hence regular, any $\mathcal{O}_X$-module is of Tor-dimension at most d. Also $\mathcal{O}_{F^r,X}$ is flat as a right $\mathcal{O}_X$-module by Lemma 1.3.1, and so any induced left $\mathcal{O}_{F^r,X}$-module has Tor-dimension at most d. Since Lemma 1.8.1 equips any left $\mathcal{O}_{F^r,X}$-module with a two-step resolution by induced left $\mathcal{O}_{F^r,X}$-modules, the corollary follows. □

1.8.5. — Let X be a k-scheme and $\mathcal{M}^{\bullet}$ be a complex in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$. We say that $\mathcal{M}^{\bullet}$ has finite locally projective dimension as a complex of sheaves of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules if the functor $\underline{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet},-)$ has finite cohomological amplitude. A standard argument via truncating a resolution of $\mathcal{M}^{\bullet}$ by free $\mathcal{O}^{\Lambda}_{F^r,X}$-modules (as in the proof of Corollary 1.8.3) shows that $\mathcal{M}^{\bullet}$ has finite locally projective dimension if and only if it may be represented (locally in general, or globally if X is either smooth or quasi-projective, so that $\mathcal{M}^{\bullet}$ has a global locally free resolution) by a finite length complex of quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are locally projective as sheaves of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, in the sense that they are direct summands of locally free sheaves of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules.

In a similar way we may define the notion of finite locally projective dimension for complexes of $\mathcal{O}^{\Lambda}_{X}$ or Λ-modules. (Note that in general the stalks of $\mathcal{O}^{\Lambda}_{X}$ are not local rings, and so the notion of locally projective is more general than that of locally free.)

Corollary 1.8.6. — *Let $\mathcal{M}^{\bullet}$ be in $D^b(\mathcal{O}^{\Lambda}_{F^r,X})$. Then $\mathcal{M}^{\bullet}$ has finite locally projective dimension as a complex of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules if and only if it has finite locally projective dimension as a complex of $\mathcal{O}^{\Lambda}_{X}$-modules. If $\mathcal{M}^{\bullet}$ has $\mathcal{O}^{\Lambda}_{X}$-coherent cohomology sheaves, then these conditions hold if and only if $\mathcal{M}^{\bullet}$ is in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$.*

Proof. — Except for the last statement, the proof is entirely analogous to that of Corollary 1.8.2, using Lemma 1.8.1. As for the final statement, only the “if” direction is not completely obvious. Suppose that $\mathcal{M}^\bullet$ is in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^\circ$, and has $\mathcal{O}^{\Lambda}_X$-coherent cohomology sheaves. Then as a complex of $\mathcal{O}^{\Lambda}_X$-sheaves, $\mathcal{M}^\bullet$ is quasi-isomorphic to a finite length complex of coherent $\mathcal{O}^{\Lambda}_X$ sheaves $\mathcal{Q}^\bullet$, having finite Tor-dimension. An argument as in Corollary 1.8.3 shows that $\mathcal{Q}^\bullet$ is isomorphic to a finite length complex of coherent flat $\mathcal{O}^{\Lambda}_X$-modules $\mathcal{P}^\bullet$. The terms of $\mathcal{P}^\bullet$ are thus locally projective, which implies that $\mathcal{M}^\bullet$ has finite locally projective dimension as a complex of $\mathcal{O}^{\Lambda}_X$-modules. The required result now follows from the first part of the corollary. □

1.9. — Suppose that $\mathcal{M}$ and $\mathcal{N}$ are left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Then the $\mathcal{O}^{\Lambda}_X$-module $\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N}$ has a natural structure of a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, which is defined by letting F act diagonally. Thus we obtain a bifunctor

$$- \otimes_{\mathcal{O}^{\Lambda}_X} - : \mathcal{O}^{\Lambda}_{F^r,X}\text{-}Mod \times \mathcal{O}^{\Lambda}_{F^r,X}\text{-}Mod \to \mathcal{O}^{\Lambda}_{F^r,X}\text{-}Mod.$$

As the tensor product of quasi-coherent $\mathcal{O}^{\Lambda}_X$-modules is quasi-coherent, this bifunctor restricts to a bifunctor

$$- \otimes_{\mathcal{O}^{\Lambda}_X} - : \mu(X,\Lambda) \times \mu(X,\Lambda) \to \mu(X,\Lambda).$$

The following results describe the structural morphism of the tensor product $\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N}$ for two left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules $\mathcal{M}$ and $\mathcal{N}$.

Lemma 1.9.1. — *If M and N are sheaves of $\mathcal{O}^{\Lambda}_X$-modules on a k-scheme X then there is a natural isomorphism*

$$F^{r\,*}_X M \otimes_{\mathcal{O}^{\Lambda}_X} F^{r\,*}_X N \xrightarrow{\sim} F^{r\,*}_X (M \otimes_{\mathcal{O}^{\Lambda}_X} N).$$

Proof. — Define a morphism on the level of sections by:

$$(a \otimes m) \otimes (b \otimes n) \mapsto ab \otimes (m \otimes n),$$

for a section $a \otimes m$ of $F^{r\,*}_X M = \mathcal{O}^{\Lambda(r)}_X \otimes_{\mathcal{O}^{\Lambda}_X} M$ and a section $b \otimes n$ of $F^{r\,*}_X N = \mathcal{O}^{\Lambda(r)}_X \otimes_{\mathcal{O}^{\Lambda}_X} N$. Then it is immediate that this provides the required isomorphism. (This is of course just a special case of a general property of $\otimes$.) □

Lemma 1.9.2. — *If $\mathcal{M}$ and $\mathcal{N}$ are left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules then the following diagram, in which the top arrow is the isomorphism of the Lemma 1.9.1, commutes:*

$$\begin{array}{ccc} F^{r\,*}_X \mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} F^{r\,*}_X \mathcal{N} & \xrightarrow{\sim} & F^{r\,*}_X(\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N}) \\ & \searrow^{\phi_{\mathcal{M}} \otimes \phi_{\mathcal{N}}} & \downarrow{\scriptstyle \phi_{\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N}}} \\ & & \mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N}. \end{array}$$

Proof. — This is immediate. □

Lemma 1.9.3. — *If $\mathcal{M}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, then the natural isomorphism of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules $\mathcal{O}^{\Lambda}_X \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \xrightarrow{\sim} \mathcal{M}$, defined by $a \otimes m \mapsto am$ for any sections a and m of $\mathcal{O}^{\Lambda}_X$ and $\mathcal{M}$ respectively, is an isomorphism of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules.*

Proof. — This follows from the equations $F^r am = a^q F^r m = (F^r a)(F^r m)$. □

1.9.4. — Part *(i)* of Lemma 1.6.2 allows us to define the left derived functor of $-\otimes_{\mathcal{O}^{\Lambda}_X} -$. We obtain bifunctors

$$-\overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} - : D^-(\mathcal{O}^{\Lambda}_{F^r,X}) \times D^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-(\mathcal{O}^{\Lambda}_{F^r,X})$$

and

$$-\overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} - : D^b(\mathcal{O}^{\Lambda}_{F^r,X}) \times D^b(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ} \to D^b(\mathcal{O}^{\Lambda}_{F^r,X}),$$

exact in each variable, which restrict to bifunctors

$$-\overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} - : D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \times D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$$

and

$$-\overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} - : D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \times D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ} \to D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}),$$

again exact in each variable (by arguing locally on affine patches of X, and appealing to part *(ii)* of Lemma 1.6.2).

1.9.5. — If $\mathcal{M}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, we may form the tensor product of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules $\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}$. The right multiplication of $\mathcal{O}^{\Lambda}_{F^r,X}$ on itself equips $\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}$ with a right $\mathcal{O}^{\Lambda}_{F^r,X}$-module structure, making it a $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodule. Thus if $\mathcal{N}$ is a second left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, we may form the tensor product $(\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N}$, which is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module.

Lemma 1.9.6. — *There is a natural isomorphism of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules*

$$(\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}.$$

Proof. — This is immediate. □

1.10. — We will need a variant of the induced module construction, which will yield bimodules for a certain pair of rings. We will need this construction in a relative setting, so we let $f : Y \to X$ be a morphism of smooth k-schemes. We put ourselves in the situation of the r^{th} relative Frobenius diagram of Y over X, as discussed in (A.2):

$$\begin{array}{ccccc} Y & \xrightarrow{F^{(r)}_{Y/X}} & Y^{(r)} & \xrightarrow{F^{r\prime}_X} & Y \\ & \searrow^{f} & \downarrow{\scriptstyle f^{(r)}} & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{F^r_X} & X. \end{array}$$

The underlying topological space of $Y^{(r)}$ is equal to that of Y, and $F^{(r)}_{Y/X}$ and $F^r_X{}'$ induce the identity map on the level of topological spaces. As for the structure sheaves, we have $\mathcal{O}_{Y^{(r)}} = f^{-1}\mathcal{O}^{(r)}_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$, and the morphisms $F^{(r)}_{Y/X}$ and $F^r_X{}'$ induce respectively the morphism $\mathcal{O}_{Y^{(r)}} \to \mathcal{O}_Y$ defined by $a \otimes b \mapsto ab^q$, and the morphism $\mathcal{O}_Y \to \mathcal{O}_{Y^{(r)}}$ defined by $b \mapsto 1 \otimes b$. (Here a denotes a section of $f^{-1}\mathcal{O}^{(r)}_X$, and b a section of $\mathcal{O}_Y$.)

We have $f^{-1}\mathcal{O}^{\Lambda}_X \subset f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} = f^{-1}\mathcal{O}^{\Lambda}_X[F^r]$. If M is a sheaf of $\mathcal{O}_Y$-modules, then we may regard it as a sheaf of $(f^{-1}\mathcal{O}_X, \mathcal{O}_Y)$-bimodules (where the $f^{-1}\mathcal{O}_X$ action is via the map $f^{\#} : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ corresponding to the morphism f), and so the tensor product $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} M$ is naturally a sheaf of $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_Y)$-bimodules.

Proposition-Definition 1.10.1. — *Let M be a sheaf of $\mathcal{O}^{\Lambda}_Y$-modules equipped with a map $\psi : F^{(r)}_{Y/X*}M \to F^{r\prime *}_X M$ of $\mathcal{O}^{\Lambda}_{Y^{(r)}}$-modules. Then the natural $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_Y)$-bimodule structure on the tensor product $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$ may be canonically extended to an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule structure, which we refer to as the bimodule structure induced by ψ.*

Proof. — Since $F^{(r)}_{Y/X}$ is the identity on underlying topological spaces, we may identify M and $F^{(r)}_{Y/X*}M$ as sheaves of abelian groups, and so regard ψ as a morphism (of sheaves of abelian groups) $M \to F^{r\prime *}_X M$. If we follow this map with the composite

$$\begin{aligned} F^{r\prime *}_X M \xrightarrow{\sim} f^{-1}\mathcal{O}^{(r)}_X \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M \xrightarrow{\sim} f^{-1}\mathcal{O}_X F^r \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M \\ \subset f^{-1}\mathcal{O}^{\Lambda}_X[F^r] \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M = f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M, \end{aligned}$$

we obtain a morphism of sheaves on Y,

$$\sigma : M \to f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M,$$

such that for any sections b of $\mathcal{O}^{\Lambda}_Y$ and η of M,

$$\sigma((F^r_Y)^{\#}(b)\eta) = b\sigma(\eta).$$

We now define a right action of $\mathcal{O}^{\Lambda}_{F^r,Y}$ on $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$ as follows: the morphism induced by right multiplication by F^r is

$$\begin{aligned} f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M \overset{\mathrm{id}\otimes\sigma}{\to} f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M \\ \to f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M, \end{aligned}$$

where the second morphism is that induced by the ring structure of $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$.

It is easily checked that this does indeed give $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$ the structure of a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module which extends its $\mathcal{O}^{\Lambda}_Y$-module structure. From the construction, and the associativity of multiplication in $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$, it is immediate that $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$ becomes an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule when endowed with a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure in this way. □

Lemma 1.10.2. — *Suppose that we have a pair of $\mathcal{O}^{\Lambda}_Y$-modules M and M' equipped with morphisms $\psi : F^{(r)}_{Y/X*}M \to F^{r'*}_X M$ and $\psi' : F^{(r)}_{Y/X*}M' \to F^{r'*}_X M'$ of $\mathcal{O}_{Y^{\Lambda(r)}}$-modules, and a morphism of $\mathcal{O}^{\Lambda}_Y$-modules $\theta : M \to M'$ such that the square*

$$\begin{array}{ccc} F^{(r)}_{Y/X*}M & \xrightarrow{\psi} & F^{r'*}_X M \\ \big\downarrow {\scriptstyle F^{(r)}_{Y/X*}\theta} & & \big\downarrow {\scriptstyle F^{r'*}_X\theta} \\ F^{(r)}_{Y/X*}M' & \xrightarrow{\psi} & F^{r'*}_X M' \end{array}$$

commutes. Then the induced morphism

$$\mathrm{id}\otimes\theta : f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M \to f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M'$$

is a morphism of $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodules, when both source and target are equipped with their induced bimodule structures.

Proof. — This is immediate from the construction. □

Remark 1.10.3. — The map $f^{\#} : f^{-1}\mathcal{O}^{\Lambda}_X \to \mathcal{O}^{\Lambda}_Y$ corresponding to the morphism f induces a map $f^{-1}\mathcal{O}^{\Lambda}_X[F^r] \to \mathcal{O}^{\Lambda}_Y[F^r]$. Thus in the situation (1.10.1), we may restrict scalars on the right via this map of rings to obtain an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structure on $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$. We refer to the resulting structure as the induced $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structure on $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$.

1.10.4. — We can apply the induced bimodule construction in the particular case when we have a single smooth k-scheme X, and f is the identity $\mathrm{id}_X : X \to X$. In this situation we find that if M is a sheaf of $\mathcal{O}^{\Lambda}_X$-modules equipped with a morphism $\beta : M \to F^{r*}_X M$, the induced module $\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{\mathcal{O}^{\Lambda}_X} M$ is naturally an $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodule.

Note that in this situation the endomorphism of $\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{\mathcal{O}^{\Lambda}_X} M$ (regarded as a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module) corresponding to right multiplication by F^r is the map of induced modules corresponding to the morphism $\beta : M \to F^{r*}_X M \subset \bigoplus_{n=0}^{\infty} F^{rn*}_X M$ via the discussion of (1.7.3).

1.10.5. — Let us now return to the relative setting of (1.10.1). There is a straightforward but important fact concerning this situation that we will need.

Suppose that M is an $\mathcal{O}^{\Lambda}_Y$-module equipped with $\psi : F^{(r)}_{Y/X*}M \to F^{r'*}_X M$. We have maps

$$f_*M \xrightarrow{\sim} f^{(r)}_* F^{(r)}_{Y/X*}M \stackrel{f^{(r)}_*\psi}{\to} f^{(r)}_* F^{r'}_X M \xrightarrow{\sim} F^{r*}_X f_*M,$$

where the last isomorphism is given by flat base-change applied to F^r_X. (Recall that X is assumed to be smooth over k, so that F^r_X is flat.) The composite is a map $f_*M \to F^{r*}_X f_*M$, which induces on $\mathcal{O}^{\Lambda}_{F^r,X}\otimes_{\mathcal{O}^{\Lambda}_X} f_*M$ an $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structure.

On the other hand, we may equip the tensor product $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M$ with its induced $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule structure, and underlying this is the induced $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structure. Thus

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} f_*M \xrightarrow{\sim} f_*(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} M)$$

(the isomorphism holding because X is smooth, so that by Lemma 1.3.1 $\mathcal{O}^{\Lambda}_{F^r,X}$ is flat as a right $\mathcal{O}^{\Lambda}_X$-module) is equipped with an $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structure via this construction.

Lemma 1.10.6. — *In the situation of (1.10.5), the two $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structures defined on $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} f_*M$ are in fact equal.*

Proof. — This is simply a matter of chasing through definitions. □

1.10.7. — There are two remaining observations regarding induced bimodules that we wish to make. As in (1.10.4), we take f to be the identity map from a smooth k-scheme X to itself, and we consider the particular case when $\mathcal{M}$ is an $\mathcal{O}^{\Lambda}_{F^r,X}$-module whose structural morphism $\phi_{\mathcal{M}}$ is an isomorphism. In this case, we may take β to be $\phi^{-1}_{\mathcal{M}^{\bullet}} : M \to F^{r*}_X M$. Then $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}$ is equipped with its induced bimodule structure. On the other hand, we may form the tensor product of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules $\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}$, which by the discussion of (1.9.5) is equipped with the structure of an $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodule.

Lemma 1.10.8. — *In the situation of (1.10.7), there is a natural isomorphism of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules*

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}.$$

Proof. — Write $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} F^{rn*}_X \mathcal{M}$ (via Corollary 1.3.2) and $\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X} \xrightarrow{\sim} \mathcal{M}[F^r] = \bigoplus_{n=0}^{\infty} \mathcal{M}F^{rn}$. Define the isomorphism to be the direct sum of the isomorphisms

$$F^{rn*}_X \mathcal{M} \overset{\phi_{n,\mathcal{M}}}{\xrightarrow{\sim}} \mathcal{M} \overset{\cdot F^{rn}}{\xrightarrow{\sim}} \mathcal{M}F^{rn}.$$

One easily checks that this is indeed a map of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules. □

Lemma 1.10.9. — *In the situation of (1.10.7), if $\mathcal{N}^{\bullet}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, then the diagram*

$$\begin{array}{ccc}
\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} (\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N}) & \xrightarrow{\sim} & (\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}) \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N} \\
 & & \downarrow \\
\Big\downarrow {\scriptstyle \mu_{\mathcal{M}\otimes\mathcal{N}}} & & (\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N} \\
 & & {\scriptstyle (1.10.8)} \downarrow \wr \\
\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{N} & \underset{(1.9.6)}{\xleftarrow{\sim}} & (\mathcal{M} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{O}^{\Lambda}_{F^r,X}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N}
\end{array}$$

commutes.

Proof. — This is easily checked. □

1.11. — It will be important in our later arguments to be able to make a change of ground field in the theory of $\mathcal{O}^{\Lambda}_{F^r}$-modules, and so we now explain the details of this operation. Let k' be an extension field of k. If X is a k-scheme, then $X' = k' \otimes_k X$ is a k'-scheme. If $\mathcal{M}$ is an $\mathcal{O}^{\Lambda}_{F^r,X}$-module, then $k' \otimes_k \mathcal{M}$ is naturally an $\mathcal{O}^{\Lambda}_{F^r,X'}$-module (with F^r acting via $a \otimes m \mapsto a^q \otimes F^r \cdot m$). This restricts to a functor $k' \otimes_k - : \mu(X, \Lambda) \to \mu(X', \Lambda)$. The functor $k' \otimes_k -$ is exact (since k' is flat over k) and so induces conservative functors (which hence reflect isomorphisms, since a map in a triangulated category is an isomorphism exactly when its cone is zero)

$$k' \otimes_k - : D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X})^* \to D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X'})^*$$

(where $\bullet$ is any of b, $+$, $-$, or $\emptyset$, and $*$ is either $\circ$ or $\emptyset$) which restrict to functors

$$k' \otimes_k - : D^{\bullet}_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^* \to D^{\bullet}_{qc}(\mathcal{O}^{\Lambda}_{F^r,X'})^*.$$

These functors take (in an obvious sense) $- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} -$ to $- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X'}} -$.

1.12. — If X is a k-scheme, we let $D^{\bullet}(X, \Lambda)$ denote the derived category of complexes of Λ-sheaves on X; here $\bullet$ can be any one of $+$, $-$, b or $\emptyset$. If $\mathcal{M}^{\bullet}$ is in $D^{-}(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^{\bullet}$ is in $D^{+}(\mathcal{O}^{\Lambda}_{F^r,X})$, then we may form $\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}, \mathcal{N}^{\bullet})$, which is an object of $D^{+}(X, \Lambda)$. (Note that the constant sheaf Λ is contained in the centre of $\mathcal{O}^{\Lambda}_{F^r,X}$.)

Such sheaves will be at the basis of our later definition of the Riemann-Hilbert correspondence, so their functoriality is of great interest to us.

Lemma 1.12.1. — *Suppose that $\mathcal{M}^{\bullet}_1$ and $\mathcal{M}^{\bullet}_2$, are in $D^{-}_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$, and that $\mathcal{N}^{\bullet}_1$ and $\mathcal{N}^{\bullet}_2$ are in $D^{b}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$. Suppose furthermore that either both $\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_1, \mathcal{N}^{\bullet}_1)$ and $\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_2, \mathcal{N}^{\bullet}_2)$ are bounded above, or that $\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_2, \mathcal{N}^{\bullet}_2)$ is bounded and of finite Tor-dimension as a sheaf of Λ-modules. (Note that in either case, the derived tensor product $\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_1, \mathcal{N}^{\bullet}_1) \overset{\mathbb{L}}{\otimes}_{\Lambda} \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_2, \mathcal{N}^{\bullet}_2)$ is defined.) Then there is a natural transformation*

$$\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_1, \mathcal{N}^{\bullet}_1) \overset{\mathbb{L}}{\otimes}_{\Lambda} \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_2, \mathcal{N}^{\bullet}_2) \longrightarrow \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_1 \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} \mathcal{M}^{\bullet}_2, \mathcal{N}^{\bullet}_1 \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} \mathcal{N}^{\bullet}_2).$$

This natural transformation is compatible with change of ground field.

Proof. — Using Lemma 1.6.2 and Corollary 1.8.3 we may assume that both $\mathcal{M}^{\bullet}_1$ and $\mathcal{M}^{\bullet}_2$ are bounded above complexes of locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and that $\mathcal{N}^{\bullet}_1$ and $\mathcal{N}^{\bullet}_2$ are bounded complexes of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are $\mathcal{O}^{\Lambda}_X$-flat (and hence Λ-flat). Then $\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_i, \mathcal{N}^{\bullet}_i)$ computes $\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet}_i, \mathcal{N}^{\bullet}_i)$ $(i = 1, 2)$.

Let $\mathcal{P}^\bullet$ be a bounded above complex of flat $\mathcal{O}_{F^r,X}^{\Lambda}$-modules that maps quasi-isomorphically to $\underline{Hom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_2^\bullet, \mathcal{N}_2^\bullet)$. Then we have a morphism

$$\begin{aligned}
&\underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_1^\bullet, \mathcal{N}_1^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_2^\bullet, \mathcal{N}_2^\bullet)\\
&\overset{(1)}{=} \underline{Hom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_1^\bullet, \mathcal{N}_1^\bullet) \otimes_\Lambda \mathcal{P}^\bullet\\
&\longrightarrow \underline{Hom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_1^\bullet, \mathcal{N}_1^\bullet) \otimes_\Lambda \underline{Hom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_2^\bullet, \mathcal{N}_2^\bullet)\\
&\longrightarrow \underline{Hom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_1^\bullet \otimes_{\mathcal{O}_X^{\Lambda}} \mathcal{M}_2^\bullet, \mathcal{N}_1^\bullet \otimes_{\mathcal{O}_X^{\Lambda}} \mathcal{N}_2^\bullet)\\
&= \underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}_1^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X^{\Lambda}} \mathcal{M}_2^\bullet, \mathcal{N}_1^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X^{\Lambda}} \mathcal{N}_2^\bullet),
\end{aligned}$$

where the equality (1) holds because either the left hand factor in the tensor product of the first line is bounded above, or else the right hand factor is of bounded Tor-dimension, so that $\mathcal{P}^\bullet$ can be chosen to be bounded. This is the required transformation, and it is clearly compatible with change of ground field. □

Lemma 1.12.2. — *Let k' be an extension field of k, let X be a k-scheme, let X' denote the base-change of X over k', and let $p_X : X' \to X$ denote the natural morphism. Then for any objects $\mathcal{M}^\bullet$ of $D^-(\mathcal{O}_{F^r,X}^{\Lambda})$ and $\mathcal{N}^\bullet$ of $D^+(\mathcal{O}_{F^r,X}^{\Lambda})$, there is a natural morphism*

$$p_X^{-1}\underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}_{F^r,X'}^{\Lambda}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet).$$

Proof. — Let $\mathcal{M}^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism, with $\mathcal{I}^\bullet$ a bounded below complex of injective $\mathcal{O}_{F^r,X}^{\Lambda}$-modules. Since $k' \otimes_k -$ is exact, $k' \otimes_k \mathcal{M}^\bullet \to k' \otimes_k \mathcal{I}^\bullet$ is again a quasi-isomorphism. Let $k' \otimes_k \mathcal{I}^\bullet \to \mathcal{I}'^\bullet$ be a quasi-isomorphism with $\mathcal{I}'^\bullet$ a bounded below complex of injective $\mathcal{O}_{F^r,X'}^{\Lambda}$-modules. Then we have

$$\begin{aligned}
p_X^{-1}\underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) &= p_X^{-1}\underline{Hom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\\
&\longrightarrow \underline{Hom}^\bullet_{\mathcal{O}_{F^r,X'}^{\Lambda}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{I}^\bullet)\\
&\longrightarrow \underline{Hom}^\bullet_{\mathcal{O}_{F^r,X'}^{\Lambda}}(k' \otimes_k \mathcal{M}^\bullet, \mathcal{I}'^\bullet)\\
&\xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}_{F^r,X'}^{\Lambda}}(k' \otimes_k \mathcal{M}^\bullet, \mathcal{I}'^\bullet).
\end{aligned}$$

This is the required natural transformation. □

Remark 1.12.3. — If $\mathcal{M}^\bullet$ is a complex in $D^-_{qc}(\mathcal{O}_{F^r,X}^{\Lambda})$, and if X is smooth as a k-scheme, then we may also compute the natural transformation of the preceding lemma by replacing $\mathcal{M}^\bullet$ by a bounded above locally free resolution (which exists by part *(ii)* of Lemma 1.6.2), rather than by replacing $\mathcal{N}^\bullet$ by a bounded below injective resolution. This will prove useful in the proof of Proposition 6.10.1 below.

1.12.4. — Taking global sections of the morphism of Lemma 1.12.2 induces a morphism

$$\mathrm{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \mathrm{RHom}^\bullet_{\mathcal{O}_{F^r,X'}^{\Lambda}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet).$$

It is clear from the construction that the corresponding morphism

$$\mathrm{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet[i]) \to \mathrm{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet[i])$$

induced between the i^{th} cohomology groups of these complexes is just the morphism of Hom's induced by the functor $k' \otimes_k -$.

1.12.5. — One particular case of Lemma 1.12.2 that will concern us later is when k' is a purely inseparable extension of k. In this situation, the morphism p_X of the lemma induces a homeomorphism of the underlying topological spaces. If we use this to identify X' with X as a topological space, the natural transformation of the lemma reduces to a morphism

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet).$$

1.13. — Let X be a k-scheme, and suppose that Λ' is a Noetherian Λ-algebra. We have natural isomorphisms

$$\Lambda' \otimes_\Lambda \mathcal{O}^{\Lambda}_X \xrightarrow{\sim} \mathcal{O}^{\Lambda'}_X \tag{1.13.1}$$

and

$$\Lambda' \otimes_\Lambda \mathcal{O}^{\Lambda}_{F^r,X} \xrightarrow{\sim} \mathcal{O}^{\Lambda'}_{F^r,X}. \tag{1.13.2}$$

Thus the functor $\mathcal{M} \mapsto \Lambda' \otimes_\Lambda \mathcal{M}$ takes $\mathcal{O}^{\Lambda}_{F^r,X}$-modules to $\mathcal{O}^{\Lambda'}_{F^r,X}$-modules. This induces a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-(\mathcal{O}^{\Lambda'}_{F^r,X}),$$

which restricts to a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-_{qc}(\mathcal{O}^{\Lambda'}_{F^r,X}).$$

Although $\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda -$ does not in general preserve boundedness, it does restrict to a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^b(\mathcal{O}^{\Lambda}_{F^r,X})^\circ \to D^b(\mathcal{O}^{\Lambda'}_{F^r,X})^\circ,$$

and also (by working locally on a cover of X by open affine subschemes, and applying Corollary 1.8.3 to the members of the covering), a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^\circ \to D^b_{qc}(\mathcal{O}^{\Lambda'}_{F^r,X})^\circ.$$

1.13.3. — The isomorphism 1.13.1 shows that there are natural isomorphisms of bifunctors

$$\Lambda' \otimes_\Lambda (- \otimes_{\mathcal{O}^{\Lambda}_X} -) \xrightarrow{\sim} (\Lambda' \otimes_\Lambda -) \otimes_{\mathcal{O}^{\Lambda'}_X} (\Lambda' \otimes_\Lambda -)$$

and

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda (- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} -) \xrightarrow{\sim} (\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda -) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda'}_X} (\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda -).$$

Lemma 1.13.4. — *Let X be a smooth or quasi-projective k-scheme, let $\mathcal{M}^\bullet$ be a complex in $D^-_{qc}(\mathcal{O}^\Lambda_{F^r,X})$, let $\mathcal{N}^\bullet$ be a complex in $D^b(\mathcal{O}^\Lambda_{F^r,X})^\circ$, and suppose that $\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$ is a bounded complex of Λ-sheaves. Then there is a natural transformation*

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,X}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet, \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet).$$

(That the two sides of this map are defined follows from our various assumptions.) This transformation is compatible with the change of field morphism of Lemma 1.12.2 (in an obvious sense).

Proof. — The existence of the natural transformation follows from Proposition B.1.1 if we take $\mathcal{A}$ to be $\mathbb{F}_q$, $\mathcal{A}'$ to be Λ, $\mathcal{A}''$ to be Λ' and $\mathcal{B}$ to be $\mathcal{O}_{F^r,X}$. We leave the checking of its compatibility with the change of field morphism as an exercise for the reader. □

1.14. — Thus far we have fixed $q = p^r$. Suppose now that r' is a multiple of r, write $q' = p^{r'}$, so that $\mathbb{F}_q \subset \mathbb{F}_{q'}$, and write $\Lambda' = \Lambda \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$. Assume furthermore that $\mathbb{F}_{q'} \subset k$. Then we may form the sheaves of rings $\mathcal{O}^{\Lambda'}_X$ and $\mathcal{O}^{\Lambda'}_{F^{r'},X}$, and we have

$$\mathcal{O}^{\Lambda'}_X = \Lambda' \otimes_{\mathbb{F}_{q'}} \mathcal{O}_X \xrightarrow{\sim} \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_X = \mathcal{O}^\Lambda_X,$$

as well as

$$\mathcal{O}^{\Lambda'}_{F^r,X} = \Lambda' \otimes_{\mathbb{F}'_q} \mathcal{O}_{F^r,X} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_{F^{r'},X} \subset \mathcal{O}^\Lambda_{F^r,X}.$$

Lemma 1.14.1. — *There is an isomorphism of left $\mathcal{O}^{\Lambda'}_{F^{r'},X}$-modules*

$$\mathcal{O}^\Lambda_{F^r,X} \xrightarrow{\sim} \bigoplus_{n=0}^{(r'/r)-1} \mathcal{O}^{(rn)}_X \otimes_{\mathcal{O}_X} \mathcal{O}^{\Lambda'}_{F^{r'},X}.$$

In particular, if X is a smooth k-scheme then $\mathcal{O}^{\Lambda'}_{F^r,X}$ is locally free as a left $\mathcal{O}^{\Lambda'}_{F^{r'},X}$-module.

Proof. — The stated isomorphism follows immediately from Lemma 1.3.1, and its analogue with r' in place of r. If X is assumed smooth then $\mathcal{O}^{(rn)}_X$ is locally free as a right $\mathcal{O}_X$-module for every n, and the second claim follows. □

1.14.2. — We let $\mathrm{Res}^{q'}_q : D^\bullet(\mathcal{O}^\Lambda_{F^r,X}) \to D^\bullet(\mathcal{O}^{\Lambda'}_{F^{r'},X})$ denote the functor defined by regarding an object of the source as an object of the target $D(\mathcal{O}^{\Lambda'}_{F^{r'},X})$. (Here $\bullet$ may assume any of its possible values.) We refer to these functors collectively as "restriction". We remark that they obviously have zero cohomological amplitude. Also, since $\mathcal{O}^{\Lambda'}_X$ and $\mathcal{O}^\Lambda_X$ are isomorphic, they restrict to a functor $\mathrm{Res}^{q'}_q : D^b(\mathcal{O}^\Lambda_{F^r,X})^\circ \to D^b(\mathcal{O}^{\Lambda'}_{F^{r'},X})^\circ$.

Suppose in particular that $\mathcal{M}$ is a left $\mathcal{O}^\Lambda_{F^r,X}$-module, with structural morphism $\phi_\mathcal{M}$. Then it is immediate that the structural morphism of $\mathrm{Res}^{q'}_q\mathcal{M}$ is equal to the morphism denoted $\phi_{(r'/r),\mathcal{M}}$ in section (1.5). Lemma 1.5.1 shows that this map is equal to the composite $F^{(r'-r)*}_X\phi_\mathcal{M} \circ \cdots \circ \phi_\mathcal{M}$.

1.14.3. — Suppose that X is smooth over k. We let $\mathrm{Ind}^{q}_{q'} : D^{\bullet}(\mathcal{O}^{\Lambda'}_{F^{r'},X}) \to D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X})$ denote the functor obtained obtained by tensoring on the left by $\mathcal{O}^{\Lambda}_{F^r,X}$ over $\mathcal{O}^{\Lambda'}_{F^{r'},X}$. (Here $\bullet$ can take on any of its possible values.) Lemma 1.14.1 shows that this functor is well-defined for any value of $\bullet$, that it has zero cohomological amplitude, and that it restricts to a functor

$$\mathrm{Ind}^{q}_{q'} : D^b(\mathcal{O}^{\Lambda'}_{F^{r'},X})^{\circ} \to D^b(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}.$$

We refer to these functors collectively as "induction".

Let $\mathcal{M}$ be a left $\mathcal{O}^{\Lambda'}_{F^{r'},X}$-module, with structural morphism $\phi_{\mathcal{M}} : F^{r'*}_X\mathcal{M} \to \mathcal{M}$. Lemma 1.5.1 shows that there is an isomorphism of $\mathcal{O}^{\Lambda}_X$-modules

$$\mathrm{Ind}^{q}_{q'}\mathcal{M} \xrightarrow{\sim} \bigoplus_{n=0}^{(r'/r)-1} F^{n*}_X\mathcal{M},$$

and that in terms of this isomorphism, the structural morphism of $\mathrm{Ind}^{q}_{q'}\mathcal{M}$ is equal to the map

$$F^{r*}_X\Big(\bigoplus_{n=0}^{(r'/r)-1} F^{rn*}_X\mathcal{M}\Big) \xrightarrow{\sim} \bigoplus_{n=1}^{(r'/r)} F^{rn*}_X\mathcal{M}$$
$$\xrightarrow{\mathrm{id}\oplus\cdots\oplus\mathrm{id}\oplus\phi_{\mathcal{M}}} \Big(\bigoplus_{n=1}^{(r'/r)} F^{rn*}_X\mathcal{M}\Big) \oplus \mathcal{M} \xrightarrow{\sim} \bigoplus_{n=0}^{(r'/r)} F^{rn*}_X\mathcal{M}.$$

1.14.4. — Since $\mathcal{O}^{\Lambda}_X$ and $\mathcal{O}^{\Lambda'}_X$ are isomorphic sheaves of rings, it is obvious that the bifunctors $\mathrm{Res}^{q'}_q(-\otimes_{\mathcal{O}^{\Lambda}_X}-)$ and $\mathrm{Res}^{q'}_q(-)\otimes_{\mathcal{O}^{\Lambda'}_X}\mathrm{Res}^{q'}_q(-)$ are naturally isomorphic. On the other hand, induction is not compatible with tensor products. Similar observations apply to the corresponding derived tensor products.

2. PULL-BACKS OF $\mathcal{O}^{\Lambda}_{F^r}$-MODULES

2.1. — Suppose that $f : Y \to X$ is a morphism of smooth k-schemes. We will define a morphism of triangulated categories

$$f^! : D^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-(\mathcal{O}^{\Lambda}_{F^r,Y})$$

called "pull-back by f." (With regard to notation, we note that if $\mathcal{M}^\bullet$ is a complex of quasi-coherent $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, then the underlying complex of $\mathcal{O}_Y$-modules $f^!\mathcal{M}^\bullet$ will *not* in general coincide with the complex of $\mathcal{O}_Y$-modules which is usually denoted by $f^!\mathcal{M}^\bullet$ in the duality theory of quasi-coherent sheaves [**Ha 1**]. In fact, temporarily denoting the latter functor by '$f^!$', on the level of complexes of $\mathcal{O}_Y$-modules there will be isomorphisms

$$f^!\mathcal{M}^\bullet \xrightarrow{\sim} \mathbb{L}f^*\mathcal{M}^\bullet[d_{Y/X}] \xrightarrow{\sim} \omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} `f^!\text{'}\mathcal{M}^\bullet.$$

In the following $f^!$ will denote the $\mathcal{O}^{\Lambda}_{F^r}$-module pull-back that we define in this section, except when we explicitly state otherwise.)

2.2. — Let $f^\# : f^{-1}\mathcal{O}^{\Lambda}_X \to \mathcal{O}^{\Lambda}_Y$ denote the morphism of sheaves of rings on Y arising from the morphism of schemes f. Then (as was already observed in remark 1.10.3) $f^\#$ induces a morphism of sheaves of rings

$$f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} = f^{-1}\mathcal{O}^{\Lambda}_X[F^r] \to \mathcal{O}^{\Lambda}_Y[F^r] = \mathcal{O}^{\Lambda}_{F^r,Y}.$$

We use this morphism to make $\mathcal{O}^{\Lambda}_{F^r,Y}$ a right $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$-module.

Definition 2.2.1. — $\mathcal{O}^{\Lambda}_{F^r,Y}$ can be regarded as an $(\mathcal{O}^{\Lambda}_{F^r,Y}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodule, via its standard left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure, and the right $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$-structure constructed above. We denote this bimodule by $\mathcal{O}^{\Lambda}_{F^r,Y\to X}$.

Lemma 2.2.2. — *If $f : Y \to X$ is an morphism of smooth k-schemes then the bimodule $\mathcal{O}^{\Lambda}_{F^r,Y\to X}$ has finite Tor-dimension as a right $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$-module.*

Proof. — By construction we see that $\mathcal{O}^{\Lambda}_{F^r,Y\to X}$ is isomorphic as a right $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$-module to $\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_{F^r,X}$. Thus the lemma follows from the fact that since

X is smooth, the stalks of $\mathcal{O}_X$, and hence of $f^{-1}\mathcal{O}_X$, are regular, implying that $\mathcal{O}_Y$ is of finite Tor-dimension as a right $f^{-1}\mathcal{O}_X$-module. □

2.3. — We now define the pull-back functor.

Definition 2.3.1. — Let $f : Y \to X$ be a morphism of smooth k-schemes. We define $f^! : D(\mathcal{O}^{\Lambda}_{F^r,X}) \to D(\mathcal{O}^{\Lambda}_{F^r,Y})$ as follows:

$$f^!\mathcal{M}^\bullet = \mathcal{O}^{\Lambda}_{F^r,Y\to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}].$$

That this functor is well-defined follows from Lemma 2.2.2, which shows furthermore that it has finite cohomological amplitude.

Lemma 2.3.2. — *(i) The functor $f^!$ preserves the property of having quasi-coherent cohomology sheaves, and so restricts to a functor $f^! : D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y})$.*

(ii) The functor $f^!$ preserves the property of being of finite Tor-dimension, and so restricts to a functor $f^! : D^b(\mathcal{O}^{\Lambda}_{F^r,X})^\circ \to D^b(\mathcal{O}^{\Lambda}_{F^r,Y})^\circ$.

Proof. — We have the following formula for computing $f^!$ on the underlying $\mathcal{O}^{\Lambda}$-modules or underlying $\mathcal{O}$-modules. Namely, if $\mathcal{M}^\bullet$ lies in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ then

$$\begin{aligned}(2.3.3)\quad f^!\mathcal{M}^\bullet &= \mathcal{O}^{\Lambda}_{F^r,Y\to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}] \\ &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}] \xrightarrow{\sim} \mathcal{O}_{Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_{X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}].\end{aligned}$$

Together with part *(ii)* of Lemma 1.6.2 (which shows an object $\mathcal{M}^\bullet$ of $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ may be resolved by a complex of flat quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules), formula (2.3.3) shows that $f^!$ preserves the property of having quasi-coherent cohomology sheaves, proving part *(i)*. It also shows that if $\mathcal{M}^\bullet$ is a bounded complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are flat as $\mathcal{O}^{\Lambda}_{X}$-modules, then $f^!\mathcal{M}^\bullet$, regarded as a complex of $\mathcal{O}^{\Lambda}_{Y}$-modules, is represented by the complex $\mathcal{O}^{\Lambda}_{Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}]$, which is a bounded complex of flat $\mathcal{O}^{\Lambda}_{Y}$-modules. Thus we have established part *(ii)*. □

Proposition 2.4. — *Let $f : Y \to X$ and $g : Z \to Y$ be two morphisms of smooth k-schemes. Then the functors $g^!f^!$ and $(fg)^!$ from $D(\mathcal{O}^{\Lambda}_{F^r,X})$ to $D(\mathcal{O}^{\Lambda}_{F^r,Z})$ are naturally isomorphic.*

Proof. — We let $\mathcal{M}^\bullet$ be a complex lying in $D(\mathcal{O}^{\Lambda}_{F^r,X})$. Using the canonical flat resolution of [**Ha 1**, II 1.2], we obtain a resolution of $\mathcal{M}^\bullet$ by flat $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, which are in particular flat as $\mathcal{O}_X$-modules. This resolution may then be used to compute the $\overset{\mathbb{L}}{\otimes}$ appearing in the definition of $f^!$ and $(fg)^!$, even though it may be unbounded (since the stalks of $\mathcal{O}_X$ are regular of bounded dimension).

Replacing $\mathcal{M}^\bullet$ by its flat resolution, we then compute $f^!\mathcal{M}^\bullet = \mathcal{O}^{\Lambda}_{Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}]$, and the right side of this equation is a complex of $\mathcal{O}^{\Lambda}_{F,Y}$-modules whose members are flat as $\mathcal{O}^{\Lambda}_{Y}$-modules. Thus we may use the right hand side to compute $g^! f^! \mathcal{M}^\bullet$, and we see that

$$\begin{aligned} g^! f^! \mathcal{M}^\bullet &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{Z} \otimes_{g^{-1}\mathcal{O}^{\Lambda}_{Y}} g^{-1}(\mathcal{O}^{\Lambda}_{Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}])[d_{Z/Y}] \\ &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{Z} \otimes_{g^{-1}\mathcal{O}^{\Lambda}_{Y}} g^{-1}\mathcal{O}^{\Lambda}_{Y} \otimes_{g^{-1}f^{-1}\mathcal{O}^{\Lambda}_{X}} g^{-1}f^{-1}\mathcal{M}^\bullet[d_Z] \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{Z} \otimes_{g^{-1}f^{-1}\mathcal{O}^{\Lambda}_{X}} g^{-1}f^{-1}\mathcal{M}^\bullet[d_Z] \\ &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{Z} \otimes_{(fg)^{-1}\mathcal{O}^{\Lambda}_{X}} (fg)^{-1}\mathcal{M}^\bullet[d_Z] \xrightarrow{\sim} (fg)^! \mathcal{M}^\bullet. \end{aligned}$$

□

Proposition 2.5. — *Let $f : Y \to X$ be a morphism of smooth k-schemes. If $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are objects of $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ then there is a natural isomorphism*

$$f^!(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X}} \mathcal{N}^\bullet) \to f^!\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{Y}} f^!\mathcal{N}^\bullet[d_{X/Y}].$$

Proof. — We may suppose that $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are both bounded above complexes of $\mathcal{O}^{\Lambda}_{X}$-flat modules. Then we compute

$$\begin{aligned} f^!(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X}} \mathcal{N}^\bullet) &= f^!(\mathcal{M}^\bullet \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{N}^\bullet) = \mathcal{O}^{\Lambda}_{Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}(\mathcal{M}^\bullet \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{N}^\bullet)[d_{Y/X}] \\ &\xrightarrow{\sim} (\mathcal{O}^{\Lambda}_{Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}]) \otimes_{\mathcal{O}^{\Lambda}_{Y}} (\mathcal{O}^{\Lambda}_{Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} f^{-1}\mathcal{N}^\bullet[d_{Y/X}])[d_{X/Y}] \\ &= f^!\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{Y}} f^!\mathcal{N}^\bullet[d_{X/Y}]. \end{aligned}$$

This proves the proposition. □

Proposition 2.6. — *Let $f : Y \to X$ be a morphism of smooth k-schemes. If $\mathcal{M}^\bullet$ is in $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^\bullet$ is in $D^+(\mathcal{O}^{\Lambda}_{F^r,X})$, then there is a natural transformation*

$$f^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(f^!\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet).$$

If f is an open immersion, this natural transformation is in fact an isomorphism.

Proof. — We may assume that $\mathcal{M}^\bullet$ is a bounded above complex of locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and that $\mathcal{N}^\bullet$ is a bounded below complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are flat as $\mathcal{O}_X$-modules (using the fact that the stalks of $\mathcal{O}_X$ are regular). Then we obtain the natural transformation

$$\begin{aligned} &f^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \\ &= f^{-1}\underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \\ &\longrightarrow \underline{Hom}^\bullet_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}}(f^{-1}\mathcal{M}^\bullet, f^{-1}\mathcal{N}^\bullet) \\ &\longrightarrow \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(\mathcal{O}^{\Lambda}_{F^r,Y\to X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{M}^\bullet, \mathcal{O}^{\Lambda}_{F^r,Y\to X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{N}^\bullet) \\ &\xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(f^!\mathcal{M}^\bullet[d_{X/Y}], f^!\mathcal{N}^\bullet[d_{X/Y}]) \\ &\xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(f^!\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet), \end{aligned}$$

where the second-to-last isomorphism holds because each of $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ is a complex of $\mathcal{O}_X$-flat modules (and so can be used to compute $\overset{\mathbb{L}}{\otimes}$ appearing in the definition of

$f^!$), and because $\mathcal{O}^{\Lambda}_{F^r,Y\to X}\otimes_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{M}^\bullet$ is furthermore a complex of locally free $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules. If f is an open immersion, then it is immediate that all the arrows in this composite are isomorphisms. □

Remark 2.6.1. — Taking global sections, the morphism of Proposition 2.6 yields a morphism

$$\mathrm{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)\to \mathrm{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(f^!\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet).$$

One sees from the construction that the corresponding morphism

$$\mathrm{Hom}_{D^b(\mathcal{O}^{\Lambda}_{F^r,X})}(\mathcal{M}^\bullet,\mathcal{N}^\bullet[i])\to \mathrm{Hom}_{D^b(\mathcal{O}^{\Lambda}_{F^r,Y})}(f^!\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet[i])$$

induced between the i^{th} cohomology groups of these complexes is just the morphism of Hom's induced by the functor $f^!$.

Proposition 2.7. — *Let $f : Y \to X$ be a morphism of smooth k-schemes, let k'/k be a field extension, and let $f' : Y' \to X'$ be the base-change of f over k'. Then the diagram*

$$\begin{array}{ccc} D(\mathcal{O}^{\Lambda}_{F^r,X}) & \xrightarrow{k'\otimes_k -} & D(\mathcal{O}^{\Lambda}_{F^r,X'}) \\ \downarrow{\scriptstyle f^!} & & \downarrow{\scriptstyle f'^!} \\ D(\mathcal{O}^{\Lambda}_{F^r,Y}) & \xrightarrow{k'\otimes_k -} & D(\mathcal{O}_{F^r,Y'}) \end{array}$$

commutes up to natural isomorphism, in a manner compatible with the natural isomorphisms of Propositions 2.4, 2.5 and 2.6.

Proof. — This is a consequence of the standard compatibilities satisfied by tensor products. □

Proposition 2.8. — *Let $f : Y \to X$ be a map of k-schemes, and let Λ' be a Noetherian Λ-algebra. If $\mathcal{M}^\bullet$ is in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$, then there is a natural isomorphism*

$$\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} f^!\mathcal{M}^\bullet \xrightarrow{\sim} f^!(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{M}^\bullet).$$

Proof. — We may assume that $\mathcal{M}^\bullet$ is a bounded above complex of flat $\mathcal{O}^{\Lambda}_X$-modules, which in particular are flat Λ-modules. The proposition then follows directly from the formula for $f^!$. □

2.8.1. — Suppose that $f : Y \to X$ is a morphism of smooth k-schemes, that $\mathcal{M}^\bullet$ is an object of $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$, that $\mathcal{N}^\bullet$ is an object of $D^b(\mathcal{O}^{\Lambda}_{F^r,X})^\circ$, and that both $\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$ and $\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(f^!\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet)$ are bounded complexes. In this situation we may form the following diagram:

$$\begin{array}{ccc}
\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} f^{-1}\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet},\mathcal{N}^{\bullet}) & \xrightarrow[(2.6)]{} & \Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(f^!\mathcal{M}^{\bullet}, f^!\mathcal{N}^{\bullet}) \\
\downarrow \wr & & \downarrow (1.13.4) \\
f^{-1}(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet},\mathcal{N}^{\bullet})) & & \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda'}_{F^r,X}}(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} f^!\mathcal{M}^{\bullet}, \Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} f^!\mathcal{N}^{\bullet}) \\
\downarrow (1.13.4) & & \downarrow (2.8)\ \wr \\
f^{-1}\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda'}_{F^r,X}}(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{M}^{\bullet}, \Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{N}^{\bullet}) & \xrightarrow[(2.6)]{} & \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda'}_{F^r,X}}(f^!(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{M}^{\bullet}), f^!(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{N}^{\bullet})).
\end{array}$$

We leave to the reader the tedious but straightforward task of checking that it commutes.

Proposition 2.9. — *Let r' be a multiple of r, let $q' = p^{r'}$, assume that $\mathbb{F}_{q'} \subset k$, and write $\Lambda' = \mathbb{F}_{q'} \otimes_{\mathbb{F}_q} \Lambda$. Then if $f : Y \to X$ is a morphism of smooth k-schemes, the diagrams*

$$\begin{array}{ccc}
D(\mathcal{O}^{\Lambda}_{F^r,X}) & \xrightarrow{f^!} & D(\mathcal{O}^{\Lambda}_{F^r,Y}) \\
\downarrow \mathrm{Res}^{q'}_{q} & & \downarrow \mathrm{Res}^{q'}_{q} \\
D(\mathcal{O}^{\Lambda'}_{F^{r'},X}) & \xrightarrow{f^!} & D(\mathcal{O}^{\Lambda'}_{F^{r'},Y})
\end{array}$$

and

$$\begin{array}{ccc}
D(\mathcal{O}^{\Lambda'}_{F^{r'},X}) & \xrightarrow{f^!} & D(\mathcal{O}^{\Lambda'}_{F^{r'},Y}) \\
\downarrow \mathrm{Ind}^{q}_{q'} & & \downarrow \mathrm{Ind}^{q}_{q'} \\
D(\mathcal{O}^{\Lambda}_{F^r,X}) & \xrightarrow{f^!} & D(\mathcal{O}^{\Lambda}_{F^r,Y})
\end{array}$$

both commute up to natural transformation.

Proof. — The first claim follows from the fact that there is a natural isomorphism of $(\mathcal{O}^{\Lambda'}_{F^{r'},Y}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodules $\mathcal{O}^{\Lambda'}_{F^{r'},Y\to X} \otimes_{f^{-1}\mathcal{O}^{\Lambda'}_{F^{r'},X}} f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,Y\to X}$, while the second claim follows from the fact that there is a natural isomorphism of $(\mathcal{O}^{\Lambda}_{F^r,Y}, f^{-1}\mathcal{O}^{\Lambda'}_{F^{r'},X})$-bimodules $\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda'}_{F^{r'},Y}} \mathcal{O}^{\Lambda'}_{F^{r'},Y\to X} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,Y\to X}$. □

2.10. — In this subsection we study the pull-back by a closed immersion $f : Y \to X$ of smooth k-schemes in more detail. Denote by $I \subset \mathcal{O}_X$ the ideal sheaf of Y. We put ourselves in the context of the r^{th} Frobenius diagram of f. (See the appendix for a discussion of this diagram, as well as a discussion of our conventions concerning duality of quasi-coherent sheaves, which will be used below.)

Lemma 2.10.1. — *If N is a sheaf of $\mathcal{O}_Y^{\Lambda}$-modules equipped with a morphism*

$$(F_X^{r\prime *}N)[I] \to N,$$

then $\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} N$ is naturally equipped with the structure of a left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module.

Proof. — It is explained in (A.2.2) that the relative Cartier operator $\mathcal{C}^{(r)}_{Y/X}$ induces an isomorphism

$$\omega_{Y/X} \otimes_{\mathcal{O}_Y} F_Y^{r*}(\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} N) \xrightarrow{\sim} (F_X^{r\prime *}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} \omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} N))[I] \xrightarrow{\sim} (F_X^{r\prime *}N)[I].$$

Composing with the given morphism $(F_X^{r\prime *}N)[I] \to N$ and then tensoring on the left by $\omega_{Y/X}^{-1}$, we obtain a morphism $F_Y^{*}(\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} N) \to \omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} N$, which by Lemma 1.5.1 is the structural isomorphism of a unique left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure on $\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} N$. □

Corollary 2.10.2. — *If $\mathcal{M}$ is a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module then $\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} \mathcal{M}[I]$ is naturally equipped with the structure of a left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module.*

Proof. — Base-change via the flat map F_X^r yields a natural isomorphism

$$F_X^{r\prime *}(\mathcal{M}[I]) \xrightarrow{\sim} (F_X^{r*}\mathcal{M})[I^{(r)}],$$

where $I^{(r)} = F_X^{r*}I$. Using the structural morphism $F_X^{r*}\mathcal{M} \to \mathcal{M}$ we obtain a map $(F_X^{r*}\mathcal{M})[I^{(r)}] \to \mathcal{M}[I^{(r)}]$. Composing these two, we get a map $F_X^{r\prime *}(\mathcal{M}[I]) \to \mathcal{M}[I^{(r)}]$, and hence a map $(F_X^{r\prime *}(\mathcal{M}[I]))[I] \to (\mathcal{M}[I^{(r)}])[I] = \mathcal{M}[I]$. The previous lemma now applies. □

2.10.3. — Since $\mathcal{O}^{\Lambda}_{F^r,X}$ is flat as a right $\mathcal{O}_X$-module (by Lemma 1.3.1, as we are assuming that X is smooth over k), any injective left $\mathcal{O}^{\Lambda}_{F^r,X}$-module is also an injective $\mathcal{O}_X$-module. Thus, by computing with resolutions by injective left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, we may use Corollary 2.10.2 to define the derived functor

$$\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} \underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) : D^+(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^+(\mathcal{O}^{\Lambda}_{F^r,Y}).$$

Since $\underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -)$ has cohomological amplitude $d_{X/Y}$ $(< \infty)$, as one sees by computing locally with Koszul complexes, and since $M \mapsto M[I]$ takes quasi-coherent $\mathcal{O}_X^{\Lambda}$-modules to quasi-coherent $\mathcal{O}_Y^{\Lambda}$-modules, we see that this functor restricts to functors

$$\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} \underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) : D^b(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^b(\mathcal{O}^{\Lambda}_{F^r,Y}),$$

$$\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} \underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) : D^+_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^+_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y}),$$

and

$$\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} \underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) : D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y}).$$

Proposition 2.10.4. — *There is a natural isomorphism of functors*

$$\omega_{Y/X}^{-1} \otimes_{\mathcal{O}_Y} \underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) \xrightarrow{\sim} f^!.$$

On the level of $\mathcal{O}_Y^{\Lambda}$-modules this is the (twist by $\omega_{Y/X}^{-1}$ of the) isomorphism (A.1.4).

Proof. — It suffices to construct a natural isomorphism

$$\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{Ext}^{d_{X/Y}}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) \xrightarrow{\sim} H^{d_{X/Y}}(f^!) \tag{2.10.5}$$

The desired isomorphism will then follow by homological algebra ([**Ha 1**, I 7.4], and [**Con**, 2.1]).

There is an isomorphism on the level of $\mathcal{O}^{\Lambda}_Y$-modules $H^{d_{X/Y}}(f^!) \xrightarrow{\sim} f^*$, and the fundamental local isomorphism (see the discussion and references in (A.1.3)) yields (2.10.5) on the level of $\mathcal{O}_Y$-modules, and hence also on the level of $\mathcal{O}^{\Lambda}_Y$-modules by functoriality. We will show that this is compatible with the $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structures on each side. This involves unwinding all the relevant constructions.

One thing that makes this slightly awkward is that the functor $F^!_{Y/X}$, which is the derived functor of the functor $M \mapsto M[I]$ on $\mathcal{O}_{Y^{(r)}}$-modules appearing in the statement of Lemma 2.10.1 and the construction of Corollary 2.10.2, usually has infinite cohomological amplitude. This makes the edge morphism

$$\underline{Ext}^{d_{X/Y}}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -) \to \underline{Ext}^{d_{X/Y^{(r)}}}_{\mathcal{O}_X}(f^{(r)}_*\mathcal{O}_{Y^{(r)}}, -)$$

discussed in (A.2.2) a little difficult to analyse.

We can get around this problem by the following device. If $\mathcal{M}$ is flat as a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module (and so also flat as an $\mathcal{O}_X$-module) then

$$\underline{Ext}^{i}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}) = \underline{Ext}^{i}_{\mathcal{O}_X}(f^{(r)}_*\mathcal{O}_{Y^{(r)}}, \mathcal{M}) = 0$$

if $i \neq d_{Y/X}$, and so the above edge map in this case becomes an isomorphism

$$\underline{Ext}^{d_{X/Y}}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}) \xrightarrow{\sim} \underline{Ext}^{d_{X/Y^{(r)}}}_{\mathcal{O}_X}(f^{(r)}_*\mathcal{O}_{Y^{(r)}}, \mathcal{M})[I] \tag{2.10.6}$$

Since any left $\mathcal{O}^{\Lambda}_{F^r,X}$-module may be written as the quotient of a flat $\mathcal{O}^{\Lambda}_{F^r,X}$-module, and since the functors appearing on either side of (2.10.5) are right exact functors of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, it suffices to verify that the isomorphism of (2.10.5) respects the $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structures on each side in the case that $\mathcal{M}$ is a flat left $\mathcal{O}^{\Lambda}_{F^r,X}$-module. We assume that $\mathcal{M}$ is of this form from now on.

Let $a_1, \cdots, a_s$ be a regular sequence which (locally) generates I. We have the following commutative diagram, in which the horizontal arrows are provided by the fundamental local isomorphism, and in which the vertical arrows are constructed using base-change by the flat isomorphism F^r_X, the structural morphism $F^*_X\mathcal{M} \to \mathcal{M}$, and the inverse of the isomorphism (2.10.6) (the explicit descriptions of the right hand arrows being explained in (A.2.2); in particular, the morphism labelled (1) is given by the formula $a_1^{q-1}\cdots a_s^{q-1}(da_1^q\wedge\cdots\wedge da_s^q)^{-1}\otimes m\mapsto(da_1\wedge\cdots\wedge da_s)^{-1}\otimes m)$:

$$\begin{array}{ccc}
(F_X^{r\prime *}\underline{Ext}_{\mathcal{O}_X}^{d_{X/Y}}(f_*\mathcal{O}_Y,\mathcal{M}))[I] & \xrightarrow{\sim} & (F_X^{r\prime *}(\omega_{Y/X}\otimes_{\mathcal{O}_Y} f^*\mathcal{M}))[I] \\
 & & \downarrow\wr \\
\downarrow\wr & & (F_X^{r\prime *}\omega_{Y/X}\otimes_{\mathcal{O}_{Y^{(r)}}} F_X^{r\prime *}\mathcal{M})[I] \\
 & & \downarrow\wr \\
\underline{Ext}_{\mathcal{O}_X}^{d_{X/Y^{(r)}}}(f_*^{(r)}\mathcal{O}_{Y^{(r)}},F_X^{r*}\mathcal{M}))[I] & \xrightarrow{\sim} & (\omega_{Y^{(r)}/X}\otimes_{\mathcal{O}_{Y^{(r)}}} f^{(r)*}F_X^{r*}\mathcal{M})[I] \\
\downarrow & & \downarrow \\
\underline{Ext}_{\mathcal{O}_X}^{d_{X/Y^{(r)}}}(f_*^{(r)}\mathcal{O}_{Y^{(r)}},\mathcal{M}))[I] & \xrightarrow{\sim} & (\omega_{Y^{(r)}/X}\otimes_{\mathcal{O}_{Y^{(r)}}} f^{(r)*}\mathcal{M})[I] \\
 & & \| \\
\downarrow\wr & & a_1^{q-1}\cdots a_s^{q-1}\omega_{Y^{(r)}/X}\otimes_{\mathcal{O}_{Y^{(r)}}} f^{(r)*}\mathcal{M} \\
 & & \downarrow{\scriptstyle (1)} \\
\underline{Ext}_{\mathcal{O}_X}^{d_{X/Y}}(f_*\mathcal{O}_Y,\mathcal{M}) & \xrightarrow{\sim} & \omega_{Y/X}\otimes_{\mathcal{O}_Y} f^*\mathcal{M}.
\end{array}$$

If one applies the construction of Lemma 2.10.1 to the composite of the left hand arrows in this diagram, one obtains the $\mathcal{O}_{F^r,Y}^{\Lambda}$-module structure defined on the left hand side of (2.10.5).

On the other hand, recalling from (A.2.2) the explicit formula for the relative Cartier operator

$$\mathcal{C}_{Y/X}^{(r)}:(da_1^q\wedge\cdots\wedge da_s^q)^{-1}\to a_1^{q-1}\cdots a_s^{q-1}(da_1^q\wedge\cdots\wedge da_s^q)^{-1},$$

we find that applying the construction of Lemma 2.10.1 to the composite of the right hand arrows of this diagram yields the structural morphism

$$F_Y^*f^*\mathcal{M}\xrightarrow{\sim}f^*F_X^*\mathcal{M}\xrightarrow{f^*\phi_{\mathcal{M}}}f^*\mathcal{M},$$

which is exactly the structural morphism of $H^{d_{Y/X}}(f^!\mathcal{M})$. Thus (2.10.5) is an isomorphism of $\mathcal{O}_{F^r,Y}^{\Lambda}$-modules, and the proof of the proposition is complete. □

Remark 2.10.7. — An alternative description of the isomorphism of Proposition 2.10.4, via residual complexes, is given in the course of proving Proposition 4.5.3.

2.11. — If X is a smooth k-scheme, then we may consider the r^{th} relative Frobenius diagram of X over k (see A.2)

$$\begin{array}{ccccc} X & \xrightarrow{F^{(r)}_{X/k}} & X^{(r)} & \xrightarrow{F^{r\prime}_k} & X \\ & \searrow & \downarrow & & \downarrow \\ & & \operatorname{Spec} k & \xrightarrow{F^r_k} & \operatorname{Spec} k. \end{array}$$

In this diagram, $F^{(r)}_{X/k}$ is a finite flat morphism of smooth k-schemes. Thus

$$F^{(r)!}_{X/k} : D^-(\mathcal{O}^{\Lambda}_{F^r,X^{(r)}}) \to D^-(\mathcal{O}^{\Lambda}_{F^r,X})$$

is simply pull-back via $F^{(r)}_{X/k}$ on the level of $\mathcal{O}^{\Lambda}_{X^{(r)}}$-modules, and if we compose this with the functor from $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ to $D^-(\mathcal{O}_{F^r,X^{(r)}})$ defined as base-change via the q^{th}-power map from k to itself, we obtain a functor which we denote

$$F^{r*}_X : D^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-(\mathcal{O}^{\Lambda}_{F^r,X}),$$

which on the level of $\mathcal{O}^{\Lambda}_X$-modules is just the usual pull-back by F^r_X. (We are forced to describe this functor in this slightly round-about manner, because by restricting ourselves to the context of k-schemes and morphisms of k-schemes, we have not defined a pull-back functor for morphisms such as F^r_X which are not k-linear.) In fact, since all the tensor products considered in the definition of the functor F^{r*}_X on $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ are taken with flat objects, this functor extends to a functor on $D(\mathcal{O}^{\Lambda}_{F^r,X})$ (that we continue to denote by F^{r*}_X), which preserves $D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X})^*$, with $\bullet$ being any one of $+$, $-$, b or $\emptyset$, and $*$ being either $\circ$ or $\emptyset$.

If $\mathcal{M}^{\bullet}$ is a complex in $D(\mathcal{O}^{\Lambda}_{F^r,X})$, then the structural morphisms of each member of $\mathcal{M}^{\bullet}$ yield a morphism

$$\phi_{\mathcal{M}^{\bullet}} : F^{r*}_X \mathcal{M}^{\bullet} \to \mathcal{M}^{\bullet}$$

in $D(\mathcal{O}^{\Lambda}_{F^r,X})$, which we refer to as the structural morphism of $\mathcal{M}^{\bullet}$. Note that since F^r_X is a flat morphism, formation of the structural morphism commutes with the passage to cohomology, in the sense that we have a commutative diagram of natural transformations

(2.11.1)
$$\begin{array}{ccc} F^{r*}_X H^i(\mathcal{M}^{\bullet}) & \xrightarrow{\phi_{H^i(\mathcal{M}^{\bullet})}} & H^i(\mathcal{M}^{\bullet}) \\ \downarrow\wr & \nearrow_{H^i(\phi_{\mathcal{M}^{\bullet}})} & \\ H^i(F^{r*}_X \mathcal{M}^{\bullet}) & & \end{array} .$$

2.11.2. — Let $\mathcal{M}^\bullet$ be a complex in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^\bullet$ a complex in $D^+(\mathcal{O}^{\Lambda}_{F^r,X})$. Then we get a commutative diagram of complexes of sheaves on X:

$$\begin{array}{ccc}
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, F^{r*}_X\mathcal{N}^\bullet) & \longrightarrow & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X\mathcal{M}^\bullet, F^{r*}_X\mathcal{N}^\bullet) \\
\downarrow & \nearrow & \downarrow \\
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) & \longrightarrow & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X\mathcal{M}^\bullet, \mathcal{N}^\bullet),
\end{array}$$

in which the horizontal arrows are induced by $\phi_{\mathcal{M}^\bullet} : F^{r*}_X\mathcal{M}^\bullet \to \mathcal{M}^\bullet$, the vertical arrows are induced by $\phi_{\mathcal{N}^\bullet} : F^{r*}_X\mathcal{N}^\bullet \to \mathcal{N}^\bullet$, and the diagonal arrow is induced by the functor F^{r*}_X (by combining Lemma 1.12.2, remark 1.12.5 and Proposition 2.6).

2.11.3. — Suppose that k' is a field extension of k, and let X' (respectively $X^{(r)\prime}$) denote the base-change of X (respectively $X^{(r)}$) over k'. Then $X^{(r)'}$ is naturally identified with $X'^{(r)}$ (the base-change of X' over k' via the r^{th} power of the Frobenius endomorphism of k'), and under this identification, $F_{X'/k'}$ is identified with the base-change of $F_{X/k}$. Thus for any complex $\mathcal{M}^\bullet$ in $D^b(\mathcal{O}^{\Lambda}_{F^r,X})$ there is a natural isomorphism

$$k' \otimes_k F^{r*}_X\mathcal{M}^\bullet \xrightarrow{\sim} F^{r*}_{X'}(k' \otimes_k \mathcal{M}^\bullet).$$

As in Lemma 1.12.2, let p_X denote the natural map $X' \to X$. Then for any pair of complexes $\mathcal{M}^\bullet$ in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^\bullet$ in $D^+(\mathcal{O}^{\Lambda}_{F^r,X})$ there is a commutative diagram (in which we have written $(-)_{k'}$ to denote $k' \otimes_k -$)

$$\begin{array}{ccc}
p_X^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) & \xrightarrow{p_X^{-1}F^{r*}_X} & p_X^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X\mathcal{M}^\bullet, F^{r*}_X\mathcal{N}^\bullet) \\
 & & \downarrow{\scriptstyle k'\otimes_k -} \\
\downarrow{\scriptstyle k'\otimes_k -} & & p_X^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}((F^{r*}_X\mathcal{M}^\bullet)_{k'}, (F^{r*}_X\mathcal{N}^\bullet)_{k'}) \\
 & & \downarrow{\scriptstyle \sim} \\
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(\mathcal{M}^\bullet_{k'}, \mathcal{N}^\bullet_{k'}) & \xrightarrow{F^{r*}_{X'}} & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(F^{r*}_{X'}(\mathcal{M}^\bullet_{k'}), F^{r*}_{X'}(\mathcal{N}^\bullet_{k'}))
\end{array}$$

(in which the vertical morphisms are those provided by Lemma 1.12.2).

Recall from remark 1.12.5 that if k' is a purely inseparable algebraic extension of k then we may use p_X to identify X' and X as topological spaces, and that having done this, we omit p_X^{-1} from the notation. The following Lemma studies a particular case of this situation.

Lemma 2.11.4. — *Let k' be a (necessarily purely inseparable extension) of k such that $(k')^q$ is contained in k. Then for any two complexes $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ as above there is a natural morphism*

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(\mathcal{M}^\bullet_{k'}, \mathcal{N}^\bullet_{k'}) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X\mathcal{M}^\bullet, F^{r*}_X\mathcal{N}^\bullet)$$

such that the diagram

$$\begin{array}{ccc}
\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet},\mathcal{N}^{\bullet}) & \longrightarrow & \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X\mathcal{M}^{\bullet},F^{r*}_X\mathcal{N}^{\bullet}) \\
\downarrow & \nearrow & \downarrow \\
\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(\mathcal{M}^{\bullet}_{k'},\mathcal{N}^{\bullet}_{k'}) & \longrightarrow & \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(F^{r*}_{X'}(\mathcal{M}^{\bullet}_{k'}),F^{r*}_{X'}(\mathcal{N}^{\bullet}_{k'}))
\end{array}$$

(in which the outer edges of the diagram are provided by the discussion of (2.11.3)) commutes.

Proof. — Since k'^q is contained in k, we have a sequence of morphisms $k \longrightarrow k' \overset{a\mapsto a^q}{\longrightarrow} k$. Considering the corresponding maps on Spec 's allows us to factor the endomorphism F^r_k of $\operatorname{Spec} k$ as a composite $\operatorname{Spec} k \to \operatorname{Spec} k' \to \operatorname{Spec} k$. Since F^{r*}_X is computed by first base-changing via the Frobenius endomorphism of k, and then pulling-back via $F^{(r)}_{X/k}$, we see that it may equally well computed by first base-changing from k', then base-changing from k' to k (via the q^{th} power map), and then pulling-back via $F^{(r)}_{X/k}$. Thus (combining Lemma 1.12.2, remark 1.12.5 and Proposition 2.6) we get a commutative triangle

$$\begin{array}{ccc}
\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^{\bullet},\mathcal{N}^{\bullet}) & \longrightarrow & \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X\mathcal{M}^{\bullet},F^{r*}_X\mathcal{N}^{\bullet}) \\
\downarrow & \nearrow & \\
\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(\mathcal{M}^{\bullet}_{k'},\mathcal{N}^{\bullet}_{k'}) & &
\end{array}$$

whose diagonal arrow is the desired map. This triangle is exactly the upper portion of the diagram appearing in the statement of the lemma, and so we conclude that that portion of the diagram commutes. That the lower portion also commutes is easily checked. □

3. PUSH-FORWARDS OF $\mathcal{O}^{\Lambda}_{F^r}$-MODULES

3.1. — The object of this section is to define the push-forward

$$f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$$

for any morphism $f : Y \to X$ of smooth k-schemes.

3.2. — As observed in remark (1.10.3), the morphism $f^{\#} : f^{-1}\mathcal{O}^{\Lambda}_X \to \mathcal{O}^{\Lambda}_Y$ induces a morphism $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} = f^{-1}\mathcal{O}^{\Lambda}_X[F^r] \to \mathcal{O}^{\Lambda}_Y[F^r] = \mathcal{O}^{\Lambda}_{F^r,Y}$. Thus if $\mathcal{M}^{\bullet}$ is any bounded below complex of left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module, its total push-forward $Rf_*\mathcal{M}$ is naturally a complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. At first sight this may seem like a reasonable definition of push-forward for $\mathcal{O}^{\Lambda}_{F^r}$-modules. However, it turns out that this naive definition is not the appropriate one, and we are led to a more involved construction, which we now describe.

3.3. — We begin by associating an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule to any morphism $f : Y \to X$ of smooth k-schemes.

Proposition-Definition 3.3.1. — *If $f : Y \to X$ is a morphism of smooth k-schemes, so that $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}$ is naturally an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_Y)$-bimodule, then the right $\mathcal{O}^{\Lambda}_Y$-module structure extends to a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure in such a way that $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}$ becomes an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule. We denote this bimodule by $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$.*

Proof. — The discussion of the r^{th} relative Frobenius diagram for the morphism f and the r^{th} relative Cartier operator in (A.2) shows that the latter is a morphism

$$\mathcal{C}^{(r)}_{Y/X} : F^{(r)}_{Y/X*}\omega_{Y/X} \to F^{r\,\prime *}_X \omega_{Y/X}.$$

Via the induced bimodule construction of (1.10) (applied to $id_{\Lambda} \otimes_{\mathbb{F}_q} \mathcal{C}^{(r)}_{Y/X}$), we find that $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}$ is indeed endowed with the structure of an $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule. □

Lemma 3.3.2. — *If $f: Y \to X$ is a morphism of smooth k-schemes then the functor $D^-(\mathcal{O}_{F^r,Y}) \to D^-(f^{-1}\mathcal{O}_{F^r,X})$ defined by*

$$\mathcal{M}^\bullet \mapsto \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_{F^r,Y}} \mathcal{M}^\bullet$$

has finite cohomological amplitude.

Proof. — First note that (as is immediate from the construction), there is an isomorphism

$$\Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_{F^r,X\leftarrow Y} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}.$$

By definition, we also have an isomorphism $\mathcal{O}^{\Lambda}_{F^r,Y} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_{F^r,Y}$. Thus, if we forget the auxiliary Λ-module structure on an object $\mathcal{M}^\bullet$ of $D^-(\mathcal{O}_{F^r,Y})$, we find an isomorphism of complexes of $f^{-1}\mathcal{O}_{F^r,X}$-modules

$$\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M} \xrightarrow{\sim} \mathcal{O}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F^r,Y}} \mathcal{M}.$$

Thus it suffices to prove the lemma in the case that $\Lambda = \mathbb{F}_q$. In this case, the lemma follows from Corollary 1.8.4, as we see by computing the $\overset{\mathbb{L}}{\otimes}$ using a flat resolution of the second variable. □

Lemma 3.3.3. — *If $f: Y \to X$ and $g: Z \to Y$ are morphisms of smooth k-schemes, then there is a natural isomorphism of $((fg)^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Z})$-bimodules,*

$$g^{-1}\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{g^{-1}\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{O}^{\Lambda}_{F^r,Y\leftarrow Z} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Z}.$$

Proof. — We compute

$$\begin{aligned}
&g^{-1}\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{g^{-1}\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{O}^{\Lambda}_{F^r,Y\leftarrow Z} \\
&\quad \xrightarrow{\sim} g^{-1}(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{g^{-1}\mathcal{O}^{\Lambda}_{F^r,Y}} g^{-1}\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{g^{-1}\mathcal{O}_Y} \omega_{Z/Y} \\
&\qquad \xrightarrow{\sim} (fg)^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{(fg)^{-1}\mathcal{O}_X} g^{-1}\omega_{Y/X} \otimes_{g^{-1}\mathcal{O}_Y} \omega_{Z/Y} \\
&\qquad\qquad \xrightarrow{\sim} (fg)^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{(fg)^{-1}\mathcal{O}_X} \omega_{Z/X} \xrightarrow{\sim} \mathcal{O}_{F^r,X\leftarrow Z}.
\end{aligned}$$

It is clear that all the natural isomorphisms preserve the left $(fg)^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$-module structure. That they also preserve the right $\mathcal{O}^{\Lambda}_{F^r,Z}$-module structure follows from the formula for the relative Cartier operator of the composite fg in terms of the relative Cartier operator of the morphisms f and g (A.2.3 *(iii)*) □

Lemma 3.3.4. — *If $X = \operatorname{Spec} A$ is a smooth affine k-scheme, and a is an element of A such that $Y = V(a)$ is also smooth, then $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ is faithfully flat as a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module.*

Proof. — In the situation of the lemma, the invertible sheaf $\omega_{Y/X}$ is freely generated by the section $(da)^{-1}$ over $\mathcal{O}_Y = \mathcal{O}_X/a$. Also $\mathcal{O}_{Y^{(r)}} = \mathcal{O}_X/a^q$, and the Cartier operator is the morphism $b(da)^{-1} \mapsto ba^{q-1}F^r(da)^{-1}$ (A.2.3 *(iv)*), where the right hand side is viewed as element of $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$. Thus

$$\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} = f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} \omega_{Y/X} \xrightarrow{\sim} f^{-1}\mathcal{O}^{\Lambda}_X[F^r] \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} (\mathcal{O}^{\Lambda}_X/a)(da)^{-1} \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathcal{O}^{\Lambda}_X/a^{q^n} F^{rn}\,(da)^{-1},$$

with the right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure on $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ defined by

$$bF^{rn}\,(da)^{-1} \overset{\cdot F^r}{\mapsto} ba^{q^n(q-1)}F^{rn+r}\,(da)^{-1}.$$

For each $i \geq 0$, set

$$\mathcal{M}_i = \bigoplus_{n=0}^{\infty} a^{\max(0,q^n-i)}\mathcal{O}^{\Lambda}_X/a^{q^n} F^{rn}\,(da)^{-1}.$$

Then each $\mathcal{M}_i$ is a $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,Y}, \mathcal{O}^{\Lambda}_{F^r,Y})$-submodule of $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$, and $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} = \varinjlim_i \mathcal{M}_i$. We claim that each $\mathcal{M}_i$ is free as a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module. Once we prove this, the lemma will follow, since a direct limit of free right $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules is a flat right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module.

It will suffice to show that each quotient $\mathcal{M}_i/\mathcal{M}_{i+1}$ is free. Let m be the least integer greater than or equal to $\log_q i$. Then

$$\begin{aligned}\mathcal{M}_i/\mathcal{M}_{i+1} &= \bigoplus_{n=0}^{\infty} a^{\max(0,q^n-i)}\mathcal{O}^{\Lambda}_X/a^{\max(0,q^n-i+1)} F^{rn}\,(da)^{-1}\\ &= \bigoplus_{n=m}^{\infty} a^{q^n-i}\mathcal{O}^{\Lambda}_X/a^{q^n-i+1}F^{rn}\,(da)^{-1},\end{aligned}$$

and this is a free right $\mathcal{O}^{\Lambda}_{F^r,Y} = (\mathcal{O}^{\Lambda}_X/a)[F^r]$-module of rank one, freely generated by F^{rm}. This completes the proof of the lemma. □

Corollary 3.3.6. — *If $f : Y \to X$ is an immersion of smooth k-schemes, then $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ is flat as a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module.*

Proof. — We may factor f as the composition of an open immersion and a closed immersion, and it suffices to prove the corollary for each kind of immersion separately, by Lemma 3.3.3. For closed immersions, the result follows from Lemmas 3.3.3 and 3.3.4, since any closed immersion can be factored locally into a composite of closed immersions of the form considered in Lemma 3.3.4. On the other hand, the result is obvious for open immersions, since in this case $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ is isomorphic to $\mathcal{O}^{\Lambda}_{F^r,Y}$. □

Definition 3.4. — Let $f : Y \to X$ be a morphism of smooth k-schemes. We let

$$f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$$

denote the functor which sends a complex $\mathcal{M}^{\bullet}$ in $D(\mathcal{O}^{\Lambda}_{F^r,Y})$ to the complex

$$f_+\mathcal{M}^{\bullet} = Rf_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^{\bullet}).$$

That this functor is well-defined follows from Lemma 3.3.2 together with Grothendieck's theorem showing that Rf_* has cohomological amplitude at most d_Y [**Ha 2**, III 2.7]. Indeed, taken together these results show that f_+ has finite cohomological amplitude.

Remark 3.4.1. — Note that if $f : Y \to X$ is a closed immersion, then by Corollary 3.3.5, the sheaf $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ is faithfully flat as a right $\mathcal{O}^{\Lambda}_{F^r,Y}$-module, while the functor f_* is fully faithful and exact. Thus in this case, there is no need to consider derived functors, and we see that the functor f_+ is of zero cohomological amplitude, and fully faithful.

3.5. — In this section we show that f_+ restricts to a functor $f_+ : D_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$.

Lemma 3.5.1. — *If $f : Y \to X$ is a morphism of smooth k-schemes, then any induced left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module is acyclic for the right-exact functor $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} -$.*

Proof. — Let M be an $\mathcal{O}^{\Lambda}_{Y}$-module, and $\mathcal{M} = \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_{Y}} M$ the corresponding induced left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module. By Lemma 1.3.1, $\mathcal{O}^{\Lambda}_{F^r,X}$ is locally free as a right $\mathcal{O}^{\Lambda}_{X}$-module. Thus $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$ is locally free as a right $f^{-1}\mathcal{O}^{\Lambda}_{X}$-module. Since $\omega_{Y/X}$ is an invertible sheaf on Y, we conclude that

$$\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} = f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}$$

is locally free as an $\mathcal{O}^{\Lambda}_{Y}$-module. We now compute

$$\begin{aligned}\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M} &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{O}^{\Lambda}_{F^r,Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{Y}} M \\ &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{Y}} M \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{Y}} M,\end{aligned}$$

proving that $\mathcal{M}$ is acyclic for $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} -$. □

Lemma 3.5.2. — *If $f : Y \to X$ is a morphism of smooth k-schemes and M is any $\mathcal{O}^{\Lambda}_{Y}$-module, then the natural map*

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} Rf_*M \to Rf_*(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{X}} M)$$

is an isomorphism.

Proof. — By Lemma 1.3.1, $\mathcal{O}^{\Lambda}_{F^r,X} = \bigoplus_{n=0}^{\infty}(F^{rn}_X)^*\mathcal{O}^{\Lambda}_{X}$ is a locally free right $\mathcal{O}^{\Lambda}_{X}$-module. The lemma is thus a special case of the projection formula. □

We are now ready to prove that f_+ preserves the property of having quasi-coherent cohomology modules.

Theorem 3.5.3. — *Let $f : Y \to X$ be a morphism of smooth k-schemes. Then the functor f_+ restricts to functors*

$$f_+ : D_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$$

and

$$f_+ : D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y})^{\circ} \to D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}.$$

Proof. — We begin with the first claim. A spectral sequence argument (taking into account the fact that f_+ has finite cohomological amplitude) shows that it is enough to prove this for a single quasi-coherent $\mathcal{O}^{\Lambda}_{F^r,Y}$-module $\mathcal{M}$. The two-step resolution of $\mathcal{M}$ provided by Lemma 1.8.1 shows that in fact it suffices to prove the theorem for an induced quasi-coherent $\mathcal{O}^{\Lambda}_{F^r,Y}$-module $\mathcal{M} = \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M$, where M is a quasi-coherent $\mathcal{O}^{\Lambda}_Y$-module.

Lemmas 3.5.1 and 3.5.2 shows that

$$\begin{aligned} H^j(f_+)\mathcal{M} &= Rf^j_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}) \\ &= Rf^j_*((f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M) \\ &= Rf^j_*(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} (\omega_{Y/X} \otimes_{\mathcal{O}_Y} M)) \\ &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf^j_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M). \end{aligned}$$

Since $\omega_{Y/X} \otimes_{\mathcal{O}_Y} M$ is a quasi-coherent $\mathcal{O}^{\Lambda}_Y$-module, we see that $Rf^j_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M)$ is a quasi-coherent $\mathcal{O}^{\Lambda}_X$-module. Thus f_+M does indeed have quasi-coherent cohomology sheaves, as we wanted to show.

We now turn to the second claim. Lemmas 1.6.2 and 1.8.1 show that any object in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y})^{\circ}$ has a resolution by a bounded complex of left $\mathcal{O}_{F^r,Y}$-modules which are induced from quasi-coherent flat $\mathcal{O}^{\Lambda}_Y$-modules. Thus it will suffice to show that $f_+\mathcal{M}$ lies in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$ whenever $\mathcal{M} = \mathcal{O}_{F^r,Y} \otimes_{\mathcal{O}_Y} M$ is induced from a quasi-coherent flat $\mathcal{O}^{\Lambda}_Y$-module M.

In this case we compute that

$$\begin{aligned} f_+\mathcal{M}^{\bullet} = Rf_*(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} (\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M)) \\ \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M). \end{aligned}$$

Since M is a quasi-coherent flat $\mathcal{O}^{\Lambda}_Y$-module, the same is true of $\omega_{Y/X} \otimes_{\mathcal{O}_Y} M$. Thus we are reduced to proving that if N is a quasi-coherent flat $\mathcal{O}^{\Lambda}_Y$-module, then Rf_*N is a complex of $\mathcal{O}^{\Lambda}_X$-modules of finite Tor-dimension. However, this is standard. □

3.6. — The proof of Theorem 3.5.3 shows that induced modules play an important role in computing push-forwards. For this reason it will be useful to answer the following question: suppose that $\beta' : \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M \to \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} N$ is a morphism of induced left $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules. How then can one describe the induced morphism

$$\begin{aligned} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \xrightarrow{\sim} f_+(\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M) \\ \xrightarrow{f_+\beta'} f_+(\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} N) \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N)\,? \end{aligned}$$

As observed in (1.7.3), to give the morphism β' is equivalent to giving a morphism

$$\beta : M \to \bigoplus_{n=0}^{\infty} (F^{rn}_Y)^* N$$

of $\mathcal{O}^{\Lambda}_Y$-modules. Since f_+ commutes with direct sums of morphisms, we may deal with each summand individually. Our question is then answered by the following result:

Proposition 3.6.1. — *Let $f : Y \to X$ be a morphism of smooth k-schemes. Suppose given $\mathcal{O}^{\Lambda}_Y$-modules M and N and a morphism*

$$\beta : M \to (F^{rn}_Y)^* N$$

for some n, and let $\beta' : \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M \to \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} N$ be the corresponding morphism of induced left $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules (via (1.7.3)).

Then β gives rise to a natural morphism

$$\gamma : Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to (F^{rn}_X)^* Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N),$$

such that, if

$$\gamma' : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N)$$

denotes the corresponding morphism of induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, then under the natural identifications

$$f_+(\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M) \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M)$$

and

$$f_+(\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} N) \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N)$$

(inverse to the isomorphisms provided by Lemma 3.5.2), the morphism γ' coincides with the morphism $f_+\beta'$.

Proof. — We will construct the morphism γ, but leave it to the reader to check the asserted equality $\gamma' = f_+\beta'$, since it consists simply of chasing through the definitions. We use the notation and terminology of the rn^{th} Frobenius diagram of Y over X. (See (A.2)). In particular, the rn^{th} relative Cartier operator is an $\mathcal{O}_{Y^{(rn)}}$-linear morphism $F^{(rn)}_{Y/X*}\omega_{Y/X} \to (F^{rn}_X)'^*\omega_{Y/X}$. Combining this with the morphism β and the projection formula for the affine morphism $F^{(rn)}_{Y/X}$ we obtain the following sequence of morphisms:

$$\begin{aligned} F^{(rn)}_{Y/X*}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) &\longrightarrow F^{(rn)}_{Y/X*}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} (F^{rn}_Y)^* N) \\ &\xrightarrow{\sim} F^{(rn)}_{Y/X*}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} (F^{(rn)}_{Y/X})^*(F^{rn}_X)'^* N) \xrightarrow{\sim} F^{(rn)}_{Y/X*}\omega_{Y/X} \otimes_{\mathcal{O}_{Y^{(rn)}}} (F^{rn}_X)'^* N \\ &\longrightarrow (F^{rn}_X)'^*\omega_{Y/X} \otimes_{\mathcal{O}_{Y^{(rn)}}} (F^{rn}_X)'^* N \xrightarrow{\sim} (F^{rn}_X)'^*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N). \end{aligned}$$

Applying $Rf^{(rn)}_*$ to the source and target of the composite of this sequence of morphisms, and applying base-change for the flat map F^{rn}_X, yields the morphism

$$\begin{aligned} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) &\xrightarrow{\sim} Rf^{(rn)}_* F^{(rn)}_{Y/X*}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \\ &\longrightarrow Rf^{(rn)}_*(F^{rn}_X)'^*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N) \xrightarrow{\sim} (F^{rn}_X)^* Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N). \end{aligned}$$

This is the required morphism γ. □

Proposition 3.7. — *Suppose that $g : Z \to Y$ and $f : Y \to X$ are two morphisms of smooth k-schemes. Then there is a natural isomorphism*

$$f_+g_+ \xrightarrow{\sim} (fg)_+.$$

Proof. — Lemma 3.5.5 provides an isomorphism

$$\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Z} \xrightarrow{\sim} g^{-1}\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{O}^{\Lambda}_{F^r,Y\leftarrow Z}.$$

So for any object $\mathcal{M}^\bullet$ of $D^b(\mathcal{O}^{\Lambda}_{F^r,Z})$ we have

$$\begin{aligned}(fg)_+\mathcal{M}^\bullet &= R(fg)_*(\mathcal{O}_{F^r,X\leftarrow Z} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Z}} \mathcal{M}^\bullet)\\ &\xrightarrow{\sim} Rf_*Rg_*(g^{-1}\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{O}^{\Lambda}_{F^r,Y\leftarrow Z} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Z}} \mathcal{M}^\bullet)\\ &\xrightarrow{\sim} Rf_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} Rg_*(\mathcal{O}_{F^r,Y\leftarrow Z} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Z}} \mathcal{M}^\bullet)) = f_+g_+\mathcal{M}^\bullet.\end{aligned}$$

Thus the proposition is proved. □

Proposition 3.8. — *Let $f : Y \to X$ be a morphism of smooth k-schemes and $g : U \to X$ be an open immersion. Denote by $g' : f^{-1}(U) \to Y$ the base-change of f by g. Then there is a natural isomorphism of functors $g^! f_+ \xrightarrow{\sim} (f|_{f^{-1}(U)})_+(g')^!$.*

Proof. — This follows from the fact that pull-back by an open immersion consists simply of restricting to the source of the immersion, that the formation of Rf_* is local on the base, and that the construction of $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ is also local on the base. □

Proposition 3.9. — *Let $f : Y \to X$ be a morphism of smooth k-schemes, let k'/k be a field extension, and let $f' : Y' \to X'$ be the base-change of f over k'. Then the diagram*

$$\begin{array}{ccc} D^b(\mathcal{O}^{\Lambda}_{F^r,Y}) & \xrightarrow{k'\otimes_k -} & D^b(\mathcal{O}_{F^r,Y'}) \\ \Big\downarrow{\scriptstyle f_+} & & \Big\downarrow{\scriptstyle f'_+} \\ D^b(\mathcal{O}^{\Lambda}_{F^r,X}) & \xrightarrow{k'\otimes_k -} & D^b(\mathcal{O}^{\Lambda}_{F^r,X'}) \end{array}$$

commutes up to natural isomorphism, in a manner compatible with the natural isomorphisms of Propositions 3.7 and 3.8.

Proof. — This follows from the standard compatibilities satisfied by tensor products, together with flat base-change for the total derived push-forward of quasi-coherent modules (applied to the base-change from k' to k) and the natural isomorphism $k' \otimes_k \omega_{Y/X} \xrightarrow{\sim} \omega_{Y'/X'}$. □

Proposition 3.10. — *Let $f : Y \to X$ be a map of smooth k-schemes, and let Λ' be a Noetherian Λ-algebra. If $\mathcal{M}^\bullet$ is in $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y})$, then there is a natural isomorphism*

$$\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} f_+\mathcal{M}^\bullet \xrightarrow{\sim} f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{M}^\bullet).$$

Proof. — Using the formulas

$$\mathcal{O}^{\Lambda'}_{F^r,X\leftarrow Y} = \Lambda' \otimes_{\Lambda} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} = \Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$$

and

$$\mathcal{O}^{\Lambda'}_{F^r,Y} = \Lambda' \otimes_{\Lambda} \mathcal{O}^{\Lambda}_{F^r,Y} = \Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{O}^{\Lambda}_{F^r,Y},$$

we compute that

$$f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet) = Rf_*(\mathcal{O}^{\Lambda'}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda'}_{F^r,Y}} (\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet))$$
$$\xrightarrow{\sim} Rf_*(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda (\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet)) \xrightarrow{\sim}$$
$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda Rf_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet) \xrightarrow{\sim} \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+\mathcal{M}^\bullet,$$

as required. Here the existence of the second-to-last isomorphism follows from Proposition B.1.3, taking $\mathcal{A}$ to be $\mathbb{F}_q$, $\mathcal{A}'$ to be Λ, $\mathcal{A}''$ to be Λ', and $\mathcal{C}$ to be $\mathcal{O}_{F^r,X}$. □

Proposition 3.11. — *Let r' be a multiple of r, let $q' = p^{r'}$, assume that $\mathbb{F}_{q'} \subset k$, and write $\Lambda' = \mathbb{F}_{q'} \otimes_{\mathbb{F}_q} \Lambda$. Then if $f : Y \to X$ is a morphism of smooth k-schemes, the diagrams*

$$\begin{array}{ccc} D(\mathcal{O}^{\Lambda}_{F^r,Y}) & \xrightarrow{f_+} & D(\mathcal{O}^{\Lambda}_{F^r,X}) \\ \downarrow{\scriptstyle \mathrm{Res}^{q'}_q} & & \downarrow{\scriptstyle \mathrm{Res}^{q'}_q} \\ D(\mathcal{O}^{\Lambda'}_{F^{r'},Y}) & \xrightarrow{f_+} & D(\mathcal{O}^{\Lambda'}_{F^{r'},X}) \end{array}$$

and

$$\begin{array}{ccc} D(\mathcal{O}^{\Lambda'}_{F^{r'},Y}) & \xrightarrow{f_+} & D(\mathcal{O}^{\Lambda'}_{F^{r'},X}) \\ \downarrow{\scriptstyle \mathrm{Ind}^{q}_{q'}} & & \downarrow{\scriptstyle \mathrm{Ind}^{q}_{q'}} \\ D(\mathcal{O}^{\Lambda}_{F^r,Y}) & \xrightarrow{f_+} & D(\mathcal{O}^{\Lambda}_{F^r,X}) \end{array}$$

both commute up to natural transformation.

Proof. — There is an evident natural isomorphism of $(f^{-1}\mathcal{O}^{\Lambda'}_{F^{r'},X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodules

$$\mathcal{O}^{\Lambda'}_{F^{r'},X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda'}_{F^{r'},Y}} \mathcal{O}^{\Lambda}_{F^r,Y} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y},$$

and this implies the first claim. Similarly, there is a natural isomorphism of $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda'}_{F^{r'},Y})$-bimodules

$$f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda'}_{F^{r'},X}} \mathcal{O}^{\Lambda'}_{F^{r'},X\leftarrow Y} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y},$$

and the second claim follows from this together with the projection formula. □

4. RELATIONS BETWEEN f_+ AND $f^!$

4.1. — In this section we prove a projection formula relating f_+ and $f^!$ for any morphism $f : Y \to X$ of smooth k-schemes, as well as an adjointness formula relating f_+ and $f^!$ in the cases when f is proper or an open immersion.

We begin with the projection formula, which is straightforward to prove.

Proposition 4.2. — *If $f : Y \to X$ is a morphism of smooth k-schemes, let $\mathcal{M}^\bullet$ be an object of $D^-(\mathcal{O}^\Lambda_{F^r,Y})$ and $\mathcal{N}^\bullet$ be an object of $D^-_{qc}(\mathcal{O}^\Lambda_{F^r,X})$. Then there is a natural isomorphism*

$$f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_Y} f^!\mathcal{N}^\bullet) \xrightarrow{\sim} f_+\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_X} \mathcal{N}^\bullet.$$

Proof. — We may assume that $\mathcal{N}^\bullet$ is a bounded above complex of locally free left $\mathcal{O}^\Lambda_{F^r,X}$-modules. Then $f^!\mathcal{N}^\bullet$ is represented by

$$\mathcal{O}^\Lambda_Y \otimes_{f^{-1}\mathcal{O}^\Lambda_X} f^{-1}\mathcal{N}^\bullet[d_{Y/X}].$$

Observe also that this is a bounded above complex of locally free left $\mathcal{O}^\Lambda_{F^r,Y}$-modules. Thus

$$\begin{aligned}
&f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_Y} f^!\mathcal{N}^\bullet)\\
&\xrightarrow{\sim} Rf_*(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} (\mathcal{M}^\bullet \otimes_{\mathcal{O}^\Lambda_Y} \mathcal{O}^\Lambda_Y \otimes_{f^{-1}\mathcal{O}^\Lambda_X} f^{-1}\mathcal{N}^\bullet[d_{Y/X}]))\\
&\xrightarrow{\sim} Rf_*(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet \otimes_{f^{-1}\mathcal{O}^\Lambda_X} f^{-1}\mathcal{N}^\bullet[d_{Y/X}])\\
&\overset{(1)}{\xrightarrow{\sim}} Rf_*(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet[d_{Y/X}]) \otimes_{\mathcal{O}^\Lambda_X} \mathcal{N}^\bullet\\
&\xrightarrow{\sim} f_+\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_X} \mathcal{N}^\bullet.
\end{aligned}$$

(Isomorphism (1) follows from the projection formula applied to the complex of locally free $\mathcal{O}^\Lambda_X$-modules $\mathcal{N}^\bullet$.) This proves the proposition. □

4.3. — We now prove the adjointness between f_+ and $f^!$ in the rather simple case of an open immersion. In fact it will follow immediately from the fact that in this case one can describe f_+ and $f^!$ in terms of the usual pull-back and push-forward of sheaves.

Lemma 4.3.1. — *If $f : Y \to X$ is an open immersion of smooth k-schemes, then the functor*

$$f^! : D(\mathcal{O}^{\Lambda}_{F^r,X}) \to D(\mathcal{O}^{\Lambda}_{F^r,Y})$$

is naturally isomorphic to the functor f^{-1}, while the functor

$$f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$$

is naturally isomorphic to the functor Rf_*. Thus if $\mathcal{M}^\bullet$ is any object of $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^\bullet$ any object of $D^+(\mathcal{O}^{\Lambda}_{F^r,Y})$, then there is a natural isomorphism of objects in $D^+(X,\Lambda)$:

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet) \xrightarrow{\sim} Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(f^!\mathcal{M}^\bullet, \mathcal{N}^\bullet).$$

In particular (take degree zero cohomology of the right derived functor of global sections) the functor f_+ is right adjoint to the functor $f^!$, and the resulting natural transformation

$$f^! f_+ \to \mathrm{id}$$

is an isomorphism of functors.

Proof. — The descriptions of f_+ and $f^!$ follow directly from the definitions, since when f is an open immersion, $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} = \mathcal{O}^{\Lambda}_{F^r,Y}$ and $\omega_{Y/X} = f^{-1}\mathcal{O}_X = \mathcal{O}_Y$. The consequent adjointness is a standard property of the functors Rf_* and f^{-1}. □

4.4. — We now turn to the theorem which expresses the adjointness of the morphisms f_+ and $f^!$ in the sense of derived categories, for the case of a proper morphism:

Theorem 4.4.1. — (*i*) *Let $f : Y \to X$ be a proper morphism of smooth k-schemes. If $\mathcal{M}^\bullet$ is any object of $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y})$ and $\mathcal{N}^\bullet$ any object of $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$, then there is a natural isomorphism of objects in $D^+(X,\Lambda)$:*

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet).$$

In particular (take degree zero cohomology of the right derived functor of global sections) the functor f_+ is left adjoint to the functor $f^!$.

(*ii*) *Let $f : Y \to X$ be a proper morphism of smooth k-schemes and $g : U \to X$ be an open immersion. Form the cartesian diagram*

$$\begin{array}{ccc} V = g^{-1}(U) & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & X. \end{array}$$

Then for any objects $\mathcal{M}^\bullet$ of $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y})$ and $\mathcal{N}^\bullet$ of $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ the diagram of natural isomorphisms

$$\begin{array}{ccc}
g^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(f_+\mathcal{M}^\bullet,\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{\sim} & g^{-1}Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(\mathcal{M}^\bullet,f^!\mathcal{N}^\bullet) \\
\downarrow\sim & & \downarrow\sim \\
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,U}}(g^{-1}f_+\mathcal{M}^\bullet,g^{-1}\mathcal{N}^\bullet) & & Rf'_*g'^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(\mathcal{M}^\bullet,f^!\mathcal{N}^\bullet) \\
\text{(4.3.1)}\downarrow\sim & & \downarrow\sim \\
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,U}}(g^!f_+\mathcal{M}^\bullet,g^!\mathcal{N}^\bullet) & & Rf'_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,V}}(g'^{-1}\mathcal{M}^\bullet,g'^{-1}f^!\mathcal{N}^\bullet) \\
\text{(3.8)}\downarrow\sim & & \text{(4.3.1)}\downarrow\sim \\
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,U}}(f'_+g'^!\mathcal{M}^\bullet,g^!\mathcal{N}^\bullet) & & Rf'_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,V}}(g'^!\mathcal{M}^\bullet,g'^!f^!\mathcal{N}^\bullet) \\
& \searrow^{\sim}_{\text{part (i)}} & \text{(2.4)}\downarrow\sim \\
& & Rf'_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,V}}(g'^!\mathcal{M}^\bullet,f'^!g^!\mathcal{N}^\bullet)
\end{array}$$

(in which the isomorphisms are labelled by the result which gives rise to them) commutes. In other words, the adjointness of (i) is local on the base.

(*iii*) *Let $f : Y \to X$ and $g : Z \to Y$ be proper morphisms of smooth k-schemes. Then for any objects $\mathcal{M}^\bullet$ of $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,Z})$ and $\mathcal{N}^\bullet$ of $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ the diagram of natural isomorphisms*

$$\begin{array}{ccc}
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}((fg)_+\mathcal{M}^\bullet,\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{\sim} & R(fg)_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Z}}(\mathcal{M}^\bullet,(fg)^!\mathcal{N}^\bullet) \\
\text{(3.7)}\downarrow\sim & & \\
\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(f_+g_+\mathcal{M}^\bullet,\mathcal{N}^\bullet) & & \text{(2.4)}\downarrow\sim \\
\text{part (i)}\downarrow\sim & & \\
Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(g_+\mathcal{M}^\bullet,f^!\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{\sim} & Rf_*Rg_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Z}}(\mathcal{M}^\bullet,g^!f^!\mathcal{N}^\bullet)
\end{array}$$

(in which the isomorphisms are labelled by the results which give rise to them) commutes. In other words, the adjointness of (i) is compatible with compositions.

(*iv*) *Let $f : Y \to X$ be a proper morphism of smooth k-schemes, and let Λ' be a Noetherian k-algebra. Then for any objects $\mathcal{M}^\bullet$ of $D^b_{qc}(\mathcal{O}_{F^r,Y})$ and $\mathcal{N}^\bullet$ of $D^b_{qc}(\mathcal{O}_{F^r,X})^\circ$ with the property that*

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,Y}}(\mathcal{M}^\bullet,f^!\mathcal{N}^\bullet)$$

and

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(f_+\mathcal{M}^\bullet,\mathcal{N}^\bullet)$$

are both bounded, the diagram

$$\begin{array}{ccc}
\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) \\
\downarrow \wr & & \downarrow{\scriptstyle (1.13.4)} \\
Rf_*(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet)) & & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,X}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+\mathcal{M}^\bullet, \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet) \\
{\scriptstyle (1.13.4)}\downarrow & & {\scriptstyle (3.10)}\downarrow \wr \\
Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,Y}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet, \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet) & & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,X}}(f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet), \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet) \\
{\scriptstyle (2.8)}\downarrow \wr & & \\
Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,Y}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet, f^!(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet)) & &
\end{array}$$

$$Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,Y}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet, f^!(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet)) \xrightarrow{\text{part (i)}} \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,X}}(f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet), \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet)$$

(in which the isomorphisms are labelled by the results which give rise to them) commutes. In other words, the adjointness of (i) is compatible with change of coefficient ring.

We begin by proving some necessary preliminary results.

Proposition 4.4.2. — (*i*) *Let $f : Y \to X$ be a morphism of smooth k-schemes, and let $\mathcal{M}^\bullet$ be an object of $D^-_{qc}(\mathcal{O}^\Lambda_{F^r,Y})$ and $\mathcal{N}^\bullet$ be an object of $D^+(\mathcal{O}^\Lambda_{F^r,Y})$. Then there is a natural transformation in the derived category of sheaves of Λ-modules*

$$Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet).$$

(*ii*) *Let $f : Y \to X$ be a morphism of smooth k-schemes and $g : U \to X$ be an open immersion. Form the cartesian diagram*

$$\begin{array}{ccc}
V = g^{-1}(U) & \xrightarrow{g'} & Y \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
U & \xrightarrow{g} & X.
\end{array}$$

Then for any object $\mathcal{M}^\bullet$ of $D^-_{qc}(\mathcal{O}_{F^r,Y})$ and $\mathcal{N}^\bullet$ of $D^+(\mathcal{O}^\Lambda_{F^r,Y})$ the diagram of natural transformations

$$
\begin{array}{ccc}
g^{-1}Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & g^{-1}\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet) \\
\downarrow \wr & & \downarrow \wr \\
Rf'_*g'^{-1}\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) & & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,U}}(g^{-1}f_+\mathcal{M}^\bullet, g^{-1}f_+\mathcal{N}^\bullet) \\
\downarrow \wr & & \text{(4.3.1)} \downarrow \wr \\
Rf'_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,V}}(g'^{-1}\mathcal{M}^\bullet, g'^{-1}\mathcal{N}^\bullet) & & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,U}}(g^!f_+\mathcal{M}^\bullet, g^!f_+\mathcal{N}^\bullet) \\
\text{(4.3.1)} \downarrow \wr & & \text{(3.8)} \downarrow \wr \\
Rf'_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,V}}(g'^!\mathcal{M}^\bullet, g'^!\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,U}}(f'_+g'^!\mathcal{M}^\bullet, f'_+g'^!\mathcal{N}^\bullet)
\end{array}
$$

(in which the morphisms are labelled the results which give rise to them) commutes. In other words, the natural transformation of (i) is local on the base.

(*iii*) *Let $f : Y \to X$ and $g : Z \to Y$ be morphisms of smooth k-schemes. Then for any object $\mathcal{M}^\bullet$ of $D^-_{qc}(\mathcal{O}^\Lambda_{F^r,Z})$ and $\mathcal{N}^\bullet$ of $D^+(\mathcal{O}^\Lambda_{F^r,Z})$ the diagram of natural transformations*

$$
\begin{array}{ccc}
R(fg)_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Z}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}((fg)_+\mathcal{M}^\bullet, (fg)_+\mathcal{N}^\bullet) \\
\downarrow \wr & & \\
Rf_*Rg_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Z}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) & & \text{(3.7)} \downarrow \wr \\
\text{part (i)} \downarrow & & \\
Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(g_+\mathcal{M}^\bullet, g_+\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+g_+\mathcal{M}^\bullet, f_+g_+\mathcal{N}^\bullet)
\end{array}
$$

(in which the morphisms are labelled the results which give rise to them) commutes. In other words, the natural transformation of (i) is compatible with compositions.

(*iv*) *Let $f : Y \to X$ be a morphism of smooth k-schemes, let $\mathcal{M}^\bullet$ be an object of $D^-_{qc}(\mathcal{O}_{F^r,Y})$, let $\mathcal{N}^\bullet$ be an object of $D^b(\mathcal{O}_{F^r,Y})^\circ$, and suppose that*

$$\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$$

and

$$\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet)$$

are both bounded. Then for any Noetherian Λ*-algebra* Λ' *there is a commutative diagram*

$$\begin{array}{ccc}
\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet) \\
\text{(B.1.3)}\big\downarrow\wr & & \text{(1.13.4)}\big\downarrow \\
Rf_*(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)) & & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,X}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+\mathcal{M}^\bullet, \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+\mathcal{N}^\bullet) \\
\text{(1.13.4)}\big\downarrow & & \text{(3.10)}\big\downarrow\wr \\
Rf_*\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,Y}}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet, \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{} & \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^r,X}}(f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet), f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet))
\end{array}$$

(in which the morphisms are labelled the results which give rise to them; note that in the indicated application of Proposition B.1.3, one should take $\mathcal{A} = \mathcal{B} = \mathbb{F}_q$, $\mathcal{A}' = \Lambda$, *and* $\mathcal{A}'' = \Lambda'$*). In other words, the natural transformation of (i) is compatible with respect to change of coefficient ring.*

Proof. — Let us begin by proving *(i)*. We may and do assume (via Lemma 1.7.1 and Lemma 3.3.2) that $\mathcal{M}^\bullet$ is a complex bounded above of locally free left $\mathcal{O}^\Lambda_{F^r,Y}$-modules and that $\mathcal{N}^\bullet$ is a complex of left $\mathcal{O}^\Lambda_{F^r,Y}$-modules which are acyclic for the left-exact functor $\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^\Lambda_{F^r,Y}} -$ (which exists by Lemma 3.3.2; note that such a complex may be used to compute the corresponding derived functor, even if it is not bounded above, because this derived functor is of finite cohomological amplitude). Then we have natural morphisms of complexes of Λ-sheaves

$$\begin{aligned}
\text{(4.4.3)}\quad & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) = \underline{Hom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) \\
& \overset{f\mapsto \mathrm{id}\otimes f}{\longrightarrow} \underline{Hom}^\bullet_{f^{-1}\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{N}^\bullet) \\
& = \underline{Hom}^\bullet_{f^{-1}\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{N}^\bullet) \\
& \longrightarrow \underline{RHom}^\bullet_{f^{-1}\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{N}^\bullet).
\end{aligned}$$

Let us denote the composite morphism by ϕ.

Proposition B.2 yields a morphism in the derived category of Λ-sheaves

$$\begin{aligned}
\text{(4.4.4)}\quad & \psi : Rf_*\underline{RHom}^\bullet_{f^{-1}\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{N}^\bullet) \\
& \longrightarrow \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(Rf_*(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} \mathcal{M}^\bullet), Rf_*(M'^\bullet, Rf_*\mathcal{N}'^\bullet)) \\
& \qquad\qquad \overset{\sim}{\longrightarrow} \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet).
\end{aligned}$$

(Take $\mathcal{A}$ to be Λ and $\mathcal{B}$ to be $\mathcal{O}^\Lambda_{F^r,X}$.) We take $\psi \circ Rf_*\phi$ for the required morphism.

Part *(ii)* follows immediately from the fact that the construction is local on the base. We leave the verification of parts *(iii)* and *(iv)* (which is standard and tedious) to the reader. In the case of part *(iv)*, one should take into account the commutative diagram of (B.3). □

4.4.5. — In the following discussion we will have occasion to consider complexes of sheaves of $(\mathcal{O}^{\Lambda}_{F^r}, \mathcal{O}^{\Lambda}_{F^r})$-bimodules on smooth k-schemes. We will show that the derived categories of bounded complexes of such bimodules are stable under push-forwards.

Suppose that $f : Y \to X$ is a morphism of smooth k-schemes, and let $\mathcal{M}^\bullet$ be a bounded complex of $(\mathcal{O}^{\Lambda}_{F^r,Y}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodules. Lemma 3.3.2 shows that the derived tensor product $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet$ is naturally a bounded complex of $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodules. The natural morphism

$$f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} = f^{-1}\mathcal{O}^{\Lambda}_{X}[F^r] \to \mathcal{O}^{\Lambda}_{Y}[F^r] = \mathcal{O}^{\Lambda}_{F^r,Y}$$

allows us to restrict scalars on the right from $\mathcal{O}^{\Lambda}_{F^r,Y}$ to $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$ and thus to regard $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet$ as a complex of $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodules. Now if we resolve this complex by a complex of flasque $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodules, we find that $f_+\mathcal{M}^\bullet = Rf_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet)$ is naturally a bounded complex of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules.

For example, $\mathcal{O}^{\Lambda}_{F^r,Y}$ is naturally an $(\mathcal{O}^{\Lambda}_{F^r,Y}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule, and the discussion of the preceding paragraph shows that $f_+\mathcal{O}^{\Lambda}_{F^r,Y}$ is naturally an object of the bounded derived category of complexes of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules. If we forget the right $\mathcal{O}^{\Lambda}_{F^r,X}$-module structure, then since $\mathcal{O}^{\Lambda}_{F^r,Y}$ is induced by $\mathcal{O}^{\Lambda}_{Y}$ we see as in the proof of 3.5.3 that $f_+\mathcal{O}^{\Lambda}_{F^r,Y} \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} Rf_*\omega_{Y/X}$.

4.4.6. — If $\mathcal{M}^\bullet$ is a bounded above complex of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules, then since any object in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ has a resolution by a bounded above complex of flat $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, we may define the derived functor

$$\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,X}} - : D^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^-(\mathcal{O}^{\Lambda}_{F^r,X}).$$

The following lemma is a variant for bimodules of Proposition 4.2.

Lemma 4.4.7. — (*i*) *Let $f : Y \to X$ be a morphism of smooth k-schemes and let $\mathcal{M}^\bullet$ be a bounded above complex of $(\mathcal{O}^{\Lambda}_{F^r,Y}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodules. If $\mathcal{N}^\bullet$ is an object of $D^-_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$, then there is a natural isomorphism of objects of $D^b(\mathcal{O}^{\Lambda}_{F^r,X})$:*

$$f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^!\mathcal{N}^\bullet) \xrightarrow{\sim} f_+\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N}^\bullet.$$

(*ii*) *Let $f : Y \to X$ be a morphism of smooth k-schemes and let $g : U \to X$ be an open immersion. Form the fibre product*

$$\begin{array}{ccc} V & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & X. \end{array}$$

Then for any bounded above complex $\mathcal{M}^\bullet$ of $(\mathcal{O}^\Lambda_{F^r,Y}, \mathcal{O}^\Lambda_{F^r,Y})$-bimodules and any object $\mathcal{N}^\bullet$ of $D^-_{qc}(\mathcal{O}^\Lambda_{F^r,X})$, the diagram of isomorphisms

$$
\begin{array}{ccc}
g^!(f_+\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,X}} \mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{\sim} & g^! f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} f^!\mathcal{N}^\bullet) \\
\text{(2.6)} \downarrow \sim & & \text{(3.8)} \downarrow \sim \\
g^! f_+\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,U}} g^!\mathcal{N}^\bullet & & f'_+ g'^!(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} f^!\mathcal{N}^\bullet) \\
\text{(3.8)} \downarrow \sim & & \text{(2.6)} \downarrow \sim \\
f'_+ g'^!\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,U}} g^!\mathcal{N}^\bullet & & f'_+(g'^!\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,V}} g'^! f^!\mathcal{N}^\bullet) \\
\text{part (i)} \downarrow \sim & \nearrow^{\sim}_{(2.4)} & \\
f'_+(g'^!\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,V}} f'^! g^!\mathcal{N}^\bullet) & &
\end{array}
$$

(in which the isomorphisms are labelled by the results which give rise to them) commutes. In other words, the natural isomorphism of (i) is compatible with localisation on the base.

(*iii*) *Let $f : Y \to X$ and $g : Z \to Y$ be morphisms of smooth k-schemes. Then for any bounded above complex $M^\bullet$ of $(\mathcal{O}^\Lambda_{F^r,Z}, \mathcal{O}^\Lambda_{F^r,Z})$-bimodules and any object $\mathcal{N}^\bullet$ of $D^-_{qc}(\mathcal{O}^\Lambda_{F^r,X})$, the diagram of natural isomorphisms*

$$
\begin{array}{ccc}
(fg)_+\mathcal{M}^\bullet[d_{Z/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,X}} \mathcal{N}^\bullet & \xrightarrow[\text{part (i)}]{\sim} & (fg)_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Z}} (fg)^!\mathcal{N}^\bullet) \\
\text{(3.7)} \downarrow \sim & & \\
f_+ g_+\mathcal{M}^\bullet[d_{Z/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,X}} \mathcal{N}^\bullet & & \text{(2.4), (3.7)} \downarrow \sim \\
\text{part (i)} \downarrow \sim & & \\
f_+(g_+\mathcal{M}^\bullet[d_{Z/Y}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Y}} f^!\mathcal{N}^\bullet) & \xrightarrow[\text{part (i)}]{\sim} & f_+ g_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,Z}} g^! f^!\mathcal{N}^\bullet)
\end{array}
$$

(in which the isomorphisms are labelled by the results which give rise to them) commutes. In other words, the natural isomorphism of (i) is compatible with composition.

(iv) For any Noetherian Λ-algebra Λ' there is a commutative diagram

$$\begin{array}{ccc}
f_+((\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda'}_{F^r,Y}} f^!(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet)) & \xrightarrow[\text{part (i)}]{\sim} & f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet)[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda'}_{F^r,X}} (\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{N}^\bullet) \\
{\scriptstyle (2.8)}\big\downarrow\wr & & {\scriptstyle (1.13.2)}\big\downarrow\wr \\
f_+((\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda'}_{F^r,Y}} (\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f^!\mathcal{N}^\bullet)) & & f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N}^\bullet \\
{\scriptstyle (1.13.2)}\big\downarrow\wr & & {\scriptstyle (3.10)}\big\downarrow\wr \\
f_+((\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^!\mathcal{N}^\bullet & & (\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{N}^\bullet \\
{\scriptstyle (3.10)}\big\downarrow\wr & \nearrow_{\text{part (i)}}^{\sim} & \\
\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^!\mathcal{N}^\bullet) & &
\end{array}$$

(in which the isomorphisms are labelled by the results which give rise to them) commutes. In other words, the natural isomorphism of (i) is compatible with change of coefficient ring.

Proof. — To prove part *(i)* we may replace $\mathcal{N}^\bullet$ by a bounded above complex of locally free $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Let $\mathcal{P}^\bullet$ be a bounded above resolution of $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$ by $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules which are flat as right $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules. Then

$$f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^!\mathcal{N}^\bullet) = Rf_*(\mathcal{P}^\bullet \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{N}^\bullet[d_{Y/X}]).$$

Since Rf_* has finite cohomological amplitude, we can apply the projection formula to f and the complex of locally free $\mathcal{O}^{\Lambda}_{F^r,X}$-modules $\mathcal{N}^\bullet$ to obtain an isomorphism

$$\begin{aligned} f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^!\mathcal{N}^\bullet) &\xrightarrow{\sim} Rf_*(\mathcal{P}^\bullet \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} \mathcal{M}^\bullet) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N}^\bullet[d_{Y/X}] \\ &= f_+\mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{N}^\bullet. \end{aligned}$$

This proves the formula of part *(i)*.

The commutative diagram of part *(ii)* follows immediately from the fact that the construction of the isomorphism of part *(i)* is local on the base (since the isomorphism provided by the projection formula is local on the base).

The proof of parts *(iii)* and *(iv)* are straightforward but tedious, and so we leave them for the reader to verify. □

4.4.8. — Suppose that $f : Y \to X$ is a proper morphism of smooth k-schemes. Then (as discussed in (A.1)) Grothendieck-Serre duality defines a natural morphism of complexes of $\mathcal{O}^{\Lambda}_X$-modules

$$Rf_*\omega_{Y/X}[d_{Y/X}] \to \mathcal{O}^{\Lambda}_X,$$

which after tensoring on the left by $\mathcal{O}^{\Lambda}_{F^r,X}$ over $\mathcal{O}^{\Lambda}_{X}$ yields a morphism of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules

$$f_+\mathcal{O}^{\Lambda}_{F^r,Y}[d_{Y/X}] \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*\omega_{Y/X}[d_{Y/X}] \to \mathcal{O}^{\Lambda}_{F^r,X}.$$

Now both the source and target of this morphism are naturally objects of the derived category of complexes of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules, and we have the following proposition.

Proposition 4.4.9. — *(i) Let $f : Y \to X$ be a proper morphism of smooth k-schemes. Then there is a natural morphism*

$$tr_{F^r,f} : f_+\mathcal{O}^{\Lambda}_{F^r,Y}[d_{Y/X}] \to \mathcal{O}^{\Lambda}_{F^r,X}$$

in the derived category of complexes of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules, which after forgetting the right $\mathcal{O}^{\Lambda}_{F^r,X}$-modules structure reduces to the morphism

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*\omega_{Y/X}[d_{Y/X}] \to \mathcal{O}^{\Lambda}_{F^r,X}$$

constructed via Grothendieck-Serre duality.

(ii) Let $f : Y \to X$ be a proper morphism of smooth k-schemes and let $g : U \to X$ be an open immersion. Form the fibre product

$$\begin{array}{ccc} V & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & X. \end{array}$$

Then the diagram

$$\begin{array}{ccc} g^!f_+\mathcal{O}^{\Lambda}_{F^r,Y}[d_{Y/X}] & \xrightarrow{\text{part (i)}} & g^!\mathcal{O}^{\Lambda}_{F^r,X} \\ {\scriptstyle (3.8)}\downarrow\wr & \nearrow{\scriptstyle \text{part (i)}} & \\ f'_+g'^!\mathcal{O}^{\Lambda}_{F^r,Y}[d_{Y/X}] = f'_+\mathcal{O}^{\Lambda}_{F^r,V}[d_{V/U}] & & \end{array}$$

(in which the morphisms are labelled by the result which gives rise to them) commutes. In other words, the construction of tr_{F^r} is compatible with localisation on the base.

(iii) Let $f : Y \to X$ and $g : Z \to Y$ be proper morphisms of smooth k-schemes. Then the diagram

$$\begin{array}{ccc} (fg)_+\mathcal{O}^{\Lambda}_{F^r,Z}[d_{Z/X}] & \xrightarrow{\text{part (i)}} & \mathcal{O}^{\Lambda}_{F^r,X} \\ {\scriptstyle (3.7)}\downarrow\wr & & \uparrow{\scriptstyle \text{part (i)}} \\ f_+g_+\mathcal{O}^{\Lambda}_{F^r,Z}[d_{Z/Y}][d_{Y/X}] & \xrightarrow{\text{part (i)}} & f_+\mathcal{O}^{\Lambda}_{F^r,Y}[d_{Y/X}] \end{array}$$

(in which the morphisms are labelled by the result which gives rise to them) commutes. In other words, the formation of tr_{F^r} is compatible with composition.

(iv) For any Noetherian Λ*-algebra* Λ*, there is a commutative diagram*

$$\begin{array}{ccc}
\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda f_+\mathcal{O}^{\Lambda}_{F^r,Y}[d_{Y/X}] & \xrightarrow[\text{part (i)}]{} & \Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{O}^{\Lambda}_{F^r,X} \\
{\scriptstyle (3.10)}\big\downarrow \wr & & {\scriptstyle (1.13.2)}\big\downarrow \wr \\
f_+(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{O}^{\Lambda}_{F^r,Y})[d_{Y/X}] & & \mathcal{O}^{\Lambda'}_{F^r,X} \\
\big\downarrow \wr & \nearrow_{\text{part (i)}} & \\
f_+\mathcal{O}^{\Lambda'}_{F^r,Y}[d_{Y/X}] & &
\end{array}$$

(in which the morphisms are labelled by the result which gives rise to them). In other words, the formation of tr_{F^r} *is compatible with change of coefficient ring.*

Proof. — Note that it suffices to construct the morphism tr_{F^r} of the proposition in the case that $\Lambda = \mathbb{F}_q$; the map for arbitrary Λ can then be obtained from that for $\mathbb{F}_q$ by tensoring through with Λ over $\mathbb{F}_q$. This will guarantee that part *(iv)* holds true, and will affords us minor simplifications in the discussion to follow. Thus we assume for the remainder of the proof that $\Lambda = \mathbb{F}_q$.

We begin by proving part *(i)* of the proposition. Let $h : V \to U$ denote an arbitrary morphism of finite type k-schemes. Let us suspend our usual notational convention concerning $h^!$, and use this to denote (not $\mathcal{O}_{F^r}$-module pull-back, but rather) the functor which is normally so denoted in the theory of Grothendieck-Serre duality (as explained in [**Ha 1**] and [**Con**]). We use the theory of residual and pointwise dualising complexes developed in [**Ha 1**, VI, VII]. In particular, recall that there is a functor $h^\Delta : \mathrm{Res}(U) \to \mathrm{Res}(V)$ from residual complexes on U to residual complexes on V which realises on the level of complexes (of quasi-coherent, injective $\mathcal{O}_U$-modules) the functor $h^!$ restricted to pointwise dualising complexes on U. Following [**Ha 1**], for a pointwise dualising complex $\mathcal{M}^\bullet$, we denote by $E^\bullet(\mathcal{M}^\bullet)$ the unique residual complex which realises it. It is pointed out in [**Con**] that $E^\bullet$ is in fact not functorial for arbitrary morphisms between pointwise dualising complexes, and hence that h^Δ and the trace map tr_h are not functorial for arbitrary morphisms between residual complexes. However, Conrad shows that they are functorial for *isomorphisms* of pointwise dualising complexes, and this is all that we need for the argument we are going to make.

Now let us return to the situation of the proposition, and consider the r^{th} relative Frobenius diagram of f (see (A.2)). Since f is proper, the same is true of $f^{(r)}$. As explained in (A.2), the morphism $F^{(r)}_{Y/X}$ is finite, and so also proper. Thus the functors tr_f, $tr_{f^{(r)}}$ and $tr_{F^{(r)}_{Y/X}}$ all yield morphisms of complexes.

Note that the morphism F^r_X is *residually stable*, meaning (following [**Ha 1**]) that it is flat and integral, with Gorenstein fibres. In fact, since X is smooth the fibres are even local complete intersections. It follows that the base-change $F^{r\prime}_X$ is also residually stable. Thus $F^{r*}_X(E^\bullet(\mathcal{O}_X))$ is a residual complex, by [**Ha 1**, VI 5.3]. It provides a resolution of the pointwise dualising complex $F^{r*}_X\mathcal{O}_X$, and [**Con**, Lemma 3.2.1] shows that the canonical isomorphism of pointwise dualising complexes $F_X^{r*}\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X$ is induced by a canonical isomorphism $E^\bullet(\mathcal{O}_X) \xrightarrow{\sim} F_X^{r*}(E^\bullet(\mathcal{O}_X))$. Also, pulling back the residual complex $f^\triangle E^\bullet(\mathcal{O}_X)$ by the residually stable morphism $F_X^{r\prime}$, we see that $F_X^{r\prime *} f^\triangle E^\bullet(\mathcal{O}_X)$ is a residual complex.

Consider the following diagram of morphisms of complexes of quasi-coherent $\mathcal{O}_{Y^{(r)}}$-modules:

(4.4.10)
$$\begin{array}{ccc}
F^{(r)}_{Y/X*} f^\triangle E^\bullet(\mathcal{O}_X) & & \\
\downarrow \wr & & \\
F^{(r)}_{Y/X*} F^{(r)\triangle}_{Y/X} f^{(r)\triangle} E^\bullet(\mathcal{O}_X) & \xrightarrow{tr_{F^{(r)}_{Y/X}} f^{(r)\triangle}E^\bullet(\mathcal{O}_X)} & f^{(r)\triangle} E^\bullet(\mathcal{O}_X) \\
\downarrow \wr & & \downarrow \wr \\
F^{(r)}_{Y/X*} F^{(r)\triangle}_{Y/X} f^{(r)\triangle} F_X^{r*} E^\bullet(\mathcal{O}_X) & \xrightarrow{tr_{F^{(r)}_{Y/X}} f^{(r)\triangle}F_X^{r*}E^\bullet(\mathcal{O}_X)} & f^{(r)\triangle} F_X^{r*} E^\bullet(\mathcal{O}_X) \\
\downarrow \wr & & \downarrow \wr \\
F^{(r)}_{Y/X*} F^{(r)\triangle}_{Y/X} F_X^{r\prime *} f^\triangle E^\bullet(\mathcal{O}_X) & \xrightarrow{tr_{F^{(r)}_{Y/X}} F_X^{r\prime *}f^\triangle E^\bullet(\mathcal{O}_X)} & F_X^{r\prime *} f^\triangle E^\bullet(\mathcal{O}_X).
\end{array}$$

The top-most vertical arrow in this diagram is deduced from [**Ha 1**, VI 3.1] (compare [**Con**, 3.2]). The upper square is defined using the isomorphism $E^\bullet(\mathcal{O}_X) \xrightarrow{\sim} F_X^{r*}E^\bullet(\mathcal{O}_X)$ constructed above; that it commutes follows from the fact that $tr_{F^{(r)}_{Y/X}}$ is functorial for isomorphism of residual complexes. The lower square is constructed using residually stable base-change to obtain an isomorphism $f^{(r)\triangle}F_X^{r*}E^\bullet(\mathcal{O}_X) \xrightarrow{\sim} F_X^{r\prime *} f^\triangle E^\bullet(\mathcal{O}_X)$ [**Ha 1**, 5.5], and then applying functoriality of $tr_{F^{(r)}_{Y/X}}$ to this isomorphism.

If we trace the maps in (4.4.10) through from the upper left to the lower right, we obtain a morphism $F^{(r)}_{Y/X*} f^\triangle E^\bullet(\mathcal{O}_X) \to F_X^{r\prime *} f^\triangle E^\bullet(\mathcal{O}_X)$, which (one sees after comparing with the definition given in (A.2)) provides a realisation on the level of residual complexes of the relative Cartier operator $\mathcal{C}^{(r)}_{Y/X} : F^{(r)}_{Y/X*}\omega_{Y/X} \to F_X^{r\prime *}\omega_{Y/X}$, shifted by $d_{Y/X}$.

Applying the induced bimodule construction of Proposition-Definition 1.10.1, we may endow the tensor product $f^{-1}\mathcal{O}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} f^\triangle E^\bullet(\mathcal{O}_X)$ with the structure of a complex of $(f^{-1}\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,Y})$-bimodules, and (since $f^{-1}\mathcal{O}_{F^r,X}$ is locally free as a right $f^{-1}\mathcal{O}_X$-module) this complex provides a flasque resolution of the $(f^{-1}\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,Y})$-bimodule $\mathcal{O}_{F^r,X\leftarrow Y}[d_{Y/X}] = f^{-1}\mathcal{O}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}[d_{Y/X}]$. Thus we may apply f_* to this resolution to compute the complex of $(\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,X})$-bimodules $f_+\mathcal{O}_{F^r,Y}$.

In order to continue our construction, we apply $f^{(r)}_*$ to diagram (4.4.10), and then embed the result in the following larger commutative diagram (in which we have abbreviated $E^\bullet(\mathcal{O}_X)$ to $E^\bullet$):

(4.4.11)
$$
\begin{array}{ccccc}
f_*f^{\triangle}E^{\bullet} & & \xrightarrow{\qquad\qquad tr_f E^{\bullet} \qquad\qquad} & & E^{\bullet} \\
\downarrow{\scriptstyle\sim} & & & & \| \\
f_*^{(r)}F^{(r)}_{Y/X*}f^{\triangle}E^{\bullet} & & & & \| \\
\downarrow{\scriptstyle\sim} & & & & \| \\
f_*^{(r)}F^{(r)}_{Y/X*}F^{(r)\triangle}_{Y/X}f^{(r)\triangle}E^{\bullet} & \xrightarrow{f_*^{(r)}tr_{F^{(r)}_{Y/X}}\, f^{(r)\triangle}E^{\bullet}} & f_*^{(r)}f^{(r)\triangle}E^{\bullet} & \xrightarrow{tr_{f^{(r)}}E^{\bullet}} & E^{\bullet} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
f_*^{(r)}F^{(r)}_{Y/X*}F^{(r)\triangle}_{Y/X}f^{(r)\triangle}F_X^{r*}E^{\bullet} & \xrightarrow{f_*^{(r)}tr_{F^{(r)}_{Y/X}}\, f^{(r)\triangle}F_X^{r*}E^{\bullet}} & f_*^{(r)}f^{(r)\triangle}F_X^{r*}E^{\bullet} & \xrightarrow{tr_{f^{(r)}}} & F_X^{r*}E^{\bullet} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \| \\
f_*^{(r)}F^{(r)}_{Y/X*}F^{(r)\triangle}_{Y/X}F_X^{r\prime*}f^{\triangle}E^{\bullet} & \xrightarrow{f_*^{(r)}tr_{F^{(r)}_{Y/X}}\, F_X^{r\prime*}f^{\triangle}E^{\bullet}} & f_*^{(r)}F_X^{r\prime*}f^{\triangle}E^{\bullet} & & \| \\
& & \downarrow{\scriptstyle\sim} & & \| \\
& & F_X^{r*}f_*f^{\triangle}E^{\bullet} & \xrightarrow{F_X^{r*}tr_f} & F_X^{r*}E^{\bullet}.
\end{array}
$$

The top-most rectangle of this diagram is constructed via [**Ha 1**, VI 4.2, TRA 1)]. The centre right rectangle is constructed by applying functoriality of tr_f to the isomorphism $E^{\bullet}(\mathcal{O}_X) \xrightarrow{\sim} F_X^{r*}E^{\bullet}(\mathcal{O}_X)$. The lower right rectangle is constructed using residually stable base-change, and its commutativity follows from [**Ha 1**, VI 5.6]. The remainder of the diagram is obtained by applying $f_*^{(r)}$ to (4.4.10).

From this diagram we can extract the following commutative rectangle:

$$
\begin{array}{ccc}
f_*f^{\triangle}E^{\bullet}(\mathcal{O}_X) & \xrightarrow{tr_f E^{\bullet}(\mathcal{O}_X)} & E^{\bullet}(\mathcal{O}_X) \\
\downarrow & & \downarrow \\
F_X^{r*}f_*f^{\triangle}E^{\bullet}(\mathcal{O}_X) & \xrightarrow{F_X^{r*}tr_f E^{\bullet}(\mathcal{O}_X)} & F_X^{r*}E^{\bullet}(\mathcal{O}_X).
\end{array}
$$

By (1.10.4) and Lemma 1.10.2 we see that each of $\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} f_*f^{\triangle}E^{\bullet}(\mathcal{O}_X)$ and $\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} E^{\bullet}(\mathcal{O}_X)$ is endowed with the structure of an $(\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,X})$-bimodule, and that $tr_f E^{\bullet}(\mathcal{O}_X)$ induces a morphism of bimodules

$$tr_{F^r,f} : \mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} f_*f^{\triangle}E^{\bullet}(\mathcal{O}_X) \to \mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} E^{\bullet}(\mathcal{O}_X).$$

Furthermore, (1.10.5) shows that the bimodule structure that we have induced on $\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} f_*f^{\triangle}E^{\bullet}(\mathcal{O}_X)$ is identical to the bimodule structure obtained by pushing forward the $(f^{-1}\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,Y})$-bimodule structure on $f^{-1}\mathcal{O}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} f^{(r)\triangle}E^{\bullet}(\mathcal{O}_X)$, and so $\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} f_*f^{\triangle}E^{\bullet}(\mathcal{O}_X)$ is a complex of $(\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,X})$-bimodules which represents $f_+\mathcal{O}_{F^r,Y}$. On the other hand, since $E^{\bullet}(\mathcal{O}_X)$ is a resolution of $\mathcal{O}_X$, and since $\mathcal{O}_{F^r,X}$ is flat as a right $\mathcal{O}_X$-module, we see that $\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} E^\bullet(\mathcal{O}_X)$ is a resolution of $\mathcal{O}_{F^r,X}$ as a complex of $(\mathcal{O}_{F^r,X}, \mathcal{O}_{F^r,X})$-bimodules. Putting this all together, we see that we have indeed found the desired morphism

$$tr_{F^r,f} : f_+\mathcal{O}_{F^r,Y}[d_{Y/X}] \to \mathcal{O}_{F^r,X}.$$

Since all of our constructions may be localised on X, part *(ii)* of the proposition is immediate. The proof of part *(iii)* relies on the compatibility of the trace maps of coherent duality with composition [**Ha 1**, VI 4.2]. Granting this, the necessary verification is standard but tedious, and is left to the reader. □

4.4.12. — *Proof of 4.4.1.* — Let $f : Y \to X$ be a proper morphism of smooth k-schemes. Let $\mathcal{M}^\bullet$ be be an object of $D^b_{qc}(\mathcal{O}^\Lambda_{F^r,Y})$ and let $\mathcal{N}^\bullet$ be an object of $D^b_{qc}(\mathcal{O}^\Lambda_{F^r,X})$. Combining the results of parts *(i)* of Propositions 4.4.2 and 4.4.9 and Lemma 4.4.7 we obtain a natural transformation

$$\begin{aligned} Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet) &\longrightarrow \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+f^!\mathcal{N}^\bullet) \\ &\xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{O}^\Lambda_{F^r,Y}[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,X}} \mathcal{N}^\bullet) \\ &\longrightarrow \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{F^r,X}} \mathcal{N}^\bullet) \\ &= \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet). \end{aligned}$$

It remains to show that this is an isomorphism.

For this, we replace $\mathcal{M}^\bullet$ by a complex, bounded above, of induced quasi-coherent left $\mathcal{O}^\Lambda_{F^r,Y}$-modules, and by the usual spectral sequence argument, it suffices to verify that the natural transformation is indeed an isomorphism for the case of a single absolutely induced quasi-coherent left $\mathcal{O}^\Lambda_{F^r,Y}$-module $\mathcal{M} = \mathcal{O}^\Lambda_{F^r,Y} \otimes_{\mathcal{O}_Y} M$ (where M is a quasi-coherent $\mathcal{O}_Y$-module).

In this case

$$f_+\mathcal{M} = \mathcal{O}^\Lambda_{F^r,X} \otimes_{\mathcal{O}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M),$$

and so there are natural isomorphisms

$$\underline{RHom}^\bullet_{\mathcal{O}_Y}(M, f^!\mathcal{N}^\bullet) \xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y}}(\mathcal{M}, f^!\mathcal{N}^\bullet)$$

and

$$\underline{RHom}^\bullet_{\mathcal{O}_X}(Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M), \mathcal{N}^\bullet) \xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}, \mathcal{N}^\bullet).$$

Finally we have natural maps

$$\begin{aligned} &Rf_*\underline{RHom}^\bullet_{\mathcal{O}_Y}(M, f^!\mathcal{N}^\bullet) \\ &\quad\longrightarrow \underline{RHom}^\bullet_{\mathcal{O}_X}(Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M), Rf_*\omega_{Y/X}[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{N}^\bullet) \\ &\qquad\longrightarrow \underline{RHom}^\bullet_{\mathcal{O}_X}(Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M), \mathcal{N}^\bullet), \end{aligned}$$

and the composite is an isomorphism by Grothendieck-Serre duality. (Note that here we are using the "explicit trace map" form of duality, rather than that based on dualising complexes, and that we are also applying duality to the (not necessarily coherent) quasi-coherent complex $\mathcal{N}^\bullet$. The justification for this is provided in (A.1).) These isomorphisms fit into the following commutative diagram:

$$\begin{array}{ccc} Rf_*\underline{RHom}^\bullet_{\mathcal{O}_X}(M, f^!\mathcal{N}^\bullet) & \xrightarrow{\sim} & Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(\mathcal{M}, f^!\mathcal{N}^\bullet) \\ \sim\downarrow & & \downarrow \\ \underline{RHom}^\bullet_{\mathcal{O}_X}(Rf_*(\omega_{Y/X}\otimes_{\mathcal{O}_Y} M), \mathcal{N}^\bullet) & \xrightarrow{\sim} & \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X}}(f_+\mathcal{M}, \mathcal{N}^\bullet). \end{array}$$

So the right hand vertical arrow is an isomorphism, and part *(i)* of the theorem is proved.

Parts *(ii)*, *(iii)* and *(iv)* of the theorem follow from the construction of the adjunction map, and the corresponding parts of Propositions 4.4.2 and 4.4.9 and Lemma 4.4.7. □

4.4.13. — It will be useful to make Proposition 4.4.9 explicit in the case when $f : Y \to X$ is the closed immersion of a smooth divisor into a smooth k-scheme X.

If a is a local equation for Y, we can represent the commutative rectangle

$$\begin{array}{ccc} f_*f^\triangle\mathcal{O}_X & \xrightarrow{tr_f E^\bullet(\mathcal{O}_X)} & E^\bullet(\mathcal{O}_X) \\ \downarrow & & \downarrow\sim \\ F^{r*}_X f_* f^\triangle E^\bullet(\mathcal{O}_X) & \xrightarrow{F^{r*}_X tr_f E^\bullet(\mathcal{O}_X)} & F^{r*}_X E^\bullet(\mathcal{O}_X) \end{array}$$

in a more concrete fashion via Koszul complexes. Following the conventions of [**Con**, 1.3], the resulting diagram (in which the upper left complex is $K^\bullet(\mathbf{a}, \mathcal{O}_X)$ and the lower left complex is $K^\bullet(\mathbf{a}^q, \mathcal{O}_X)$) is

$$\begin{array}{ccc} (\mathcal{O}_X \xrightarrow{\cdot a} \mathcal{O}_X) & \longrightarrow & (\mathcal{O}_X \longrightarrow 0) \\ \| \qquad\qquad \downarrow{\cdot a^{q-1}} & & \| \\ (\mathcal{O}_X \xrightarrow{\cdot a^q} \mathcal{O}_X) & \longrightarrow & (\mathcal{O}_X \longrightarrow 0). \end{array}$$

Applying the induced bimodule construction of 1.10 we obtain a morphism of complexes of $(\mathcal{O}^\Lambda_{F^r,X}, \mathcal{O}^\Lambda_{F^r,X})$-bimodules

$$(\mathcal{O}^\Lambda_{F^r,X} \to \mathcal{O}^{\Lambda\prime}_{F^r,X}) \to \mathcal{O}^\Lambda_{F^r,X}$$

representing the morphism $f_+\mathcal{O}^\Lambda_{F^r,Y}[d_{Y/X}] \to \mathcal{O}^\Lambda_{F^r,X}$; here $\mathcal{O}^{\Lambda\prime}_{F^r,X}$ denotes $\mathcal{O}^\Lambda_{F^r,X}$ with its usual left $\mathcal{O}^\Lambda_{F^r,X}$-module structure, but with its right module structure defined by

$$F^{rn} \cdot F^r = a^{q^n(q-1)} F^{r(n+1)}.$$

The differential $\mathcal{O}^\Lambda_{F^r,X} \to \mathcal{O}^{\Lambda\prime}_{F^r,X}$ is defined by $F^{rn} \mapsto a^{q^n} F^{rn}$.

It will be useful to note for our applications that (since $\mathcal{O}^{\Lambda\prime}_{F^r,X}$ is isomorphic to $\mathcal{O}^\Lambda_{F^r,X}$ as a right $\mathcal{O}^\Lambda_X$-module) both $\mathcal{O}^\Lambda_{F^r,X}$ and $\mathcal{O}^{\Lambda\prime}_{F^r,X}$ are acyclic with respect to tensor product on the right over $\mathcal{O}^\Lambda_{F^r,X}$ by induced left $\mathcal{O}^\Lambda_{F^r,X}$-modules.

4.5. — If $f : Y \to X$ is a proper morphism of smooth k-schemes, then Theorem 4.4.1 yields an adjunction morphism $f_+f^!\mathcal{M}^\bullet \to \mathcal{M}^\bullet$. If f is a closed immersion then the source of this arrow has the following rather explicit description (if one combines Corollary 3.3.5 with Proposition 2.10.4): it is equal to

$$f_*((f^{-1}\mathcal{O}^\Lambda_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{\mathcal{O}^\Lambda_{F^r,Y}} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet))).$$

There is an obvious morphism in the derived category,

$$\begin{aligned}(4.5.1)\quad & f_*((f^{-1}\mathcal{O}^\Lambda_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{\mathcal{O}^\Lambda_Y} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet))) \\ & \xrightarrow{\sim} f_*(f^{-1}\mathcal{O}^\Lambda_{F^r,X} \otimes_{f^{-1}\mathcal{O}^\Lambda_X} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet)) \\ & \overset{(1)}{\xrightarrow{\sim}} \mathcal{O}^\Lambda_{F^r,X} \otimes_{\mathcal{O}^\Lambda_X} f_*\underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet) \xrightarrow{(2)} \mathcal{M}^\bullet,\end{aligned}$$

in which the isomorphism (1) is given by the projection formula, and the morphism (2) is obtained by resolving $\mathcal{M}^\bullet$ by a complex of injective left $\mathcal{O}^\Lambda_{F^r,X}$-modules $\mathcal{I}^\bullet$, and then defining (2) to be the morphism

$$\mathcal{O}^\Lambda_{F^r,X} \otimes_{\mathcal{O}^\Lambda_X} f_*(\mathcal{I}^\bullet[I]) \longrightarrow \mathcal{O}^\Lambda_{F^r,X} \otimes_{\mathcal{O}^\Lambda_X} \mathcal{I}^\bullet \xrightarrow{\mu_{\mathcal{I}^\bullet}} \mathcal{I}^\bullet.$$

There is also a natural morphism

$$\begin{aligned}(4.5.2)\quad & f_*((f^{-1}\mathcal{O}^\Lambda_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{\mathcal{O}^\Lambda_Y} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet))) \\ & \longrightarrow f_*((f^{-1}\mathcal{O}^\Lambda_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{\mathcal{O}^\Lambda_{F^r,Y}} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet))),\end{aligned}$$

given by contracting the tensor product over $\mathcal{O}^\Lambda_Y$ to a tensor product over $\mathcal{O}^\Lambda_{F^r,Y}$.

Proposition 4.5.3. — *Let $f : Y \to X$ be a closed immersion of smooth k-schemes, and let $\mathcal{M}^\bullet$ be a complex in $D^b_{qc}(\mathcal{O}^\Lambda_{F^r,X})$. Then the diagram*

$$\begin{array}{ccc} f_*(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^\Lambda_Y} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet))) & \xrightarrow{(4.5.1)} & \mathcal{M}^\bullet \\ \Big\downarrow{\scriptstyle (4.5.2)} & & \Big\uparrow{\scriptstyle adj.} \\ f_*(\mathcal{O}^\Lambda_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^\Lambda_{F^r,Y}} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet))) & = & f_+f^!\mathcal{M}^\bullet \end{array}$$

commutes.

Proof. — To prove this one has to reconcile the isomorphism of Proposition 2.10.4 with the proof of Proposition 4.4.9. Let $E^\bullet = E^\bullet(\mathcal{O}_X)$ be the residual complex resolving $\mathcal{O}_X$. Since X is regular of dimension d_X, $E^\bullet$ is a complex of length d_X. Then $E^\bullet[I]$ *is* the complex $f^\triangle E^\bullet$, and the map $tr_f E^\bullet$ is simply the inclusion $f_*(E^\bullet[I]) \to E^\bullet$. The fundamental local isomorphism (discussed in (A.1.3)) shows that $E^\bullet[I]$ is a left resolution of $\omega_{Y/X}[d_{Y/X}]$.

The isomorphism $\mathcal{O}_X \xrightarrow{\sim} F^{r*}_X\mathcal{O}_X$ induces an isomorphism of complexes $E^\bullet \xrightarrow{\sim} F^{r*}_X E^\bullet$. Thus the complex $\mathcal{O}^\Lambda_{F^r,X} \otimes_{\mathcal{O}_X} E^\bullet$ is a complex of induced $(\mathcal{O}^\Lambda_{F^r,X}, (\mathcal{O}^\Lambda_{F^r,X})$-bimodules, and the quasi-isomorphism $\mathcal{O}_X \to E^\bullet$ induces a quasi-isomorphism

$$\mathcal{O}^\Lambda_{F^r,X} \to \mathcal{O}^\Lambda_{F^r,X} \otimes_{\mathcal{O}_X} E^\bullet$$

of $(\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,X})$-bimodules.

Let us recall the construction of the r^{th} relative Cartier operator in this context, following the proof of Proposition 4.4.9. The isomorphism $E^\bullet \xrightarrow{\sim} F_X^{r*}E^\bullet$ induces an isomorphism

$$E^\bullet[I^{(r)}] \xrightarrow{\sim} (F_X^{r*}E^\bullet)[I^{(r)}] \xrightarrow{\sim} F_X^{r\prime*}(E^\bullet[I]).$$

Composing this with the map $F^{(r)}_{Y/X*}E^\bullet[I] \to E^\bullet[I^{(r)}]$ we obtain the map

$$F^{(r)}_{Y/X*}E^\bullet[I] \to F_X^{r\prime*}(E^\bullet[I])$$

which realises the relative Cartier operator (shifted by $d_{Y/X}$) on the level of residual complexes. This morphism gives $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} E^\bullet[I]$ the structure of a complex of induced $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodules, resolving $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}[d_{Y/X}]$.

There is one fact which we did not take notice of in the proof of 4.4.9, but that is useful to note here. Namely, the inverse of the above isomorphism $E^\bullet \xrightarrow{\sim} F_X^{r*}E^\bullet$ provides a structural morphism for $E^\bullet$, making it a complex of left $\mathcal{O}_{F^r,X}$-modules which resolves $\mathcal{O}_X$ as a left $\mathcal{O}_{F^r,X}$-module. (This a generalisation to the non-affine case of the observation of [Lyu, ex. 1.2 (b")].) The construction of Corollary 2.10.2 then applies to give $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} E^\bullet[I]$ the structure of a complex of left $\mathcal{O}_{F^r,Y}$-modules. This complex is a left resolution of $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \omega_{Y/X}[d_{Y/X}] \xrightarrow{\sim} \mathcal{O}_Y[d_{Y/X}]$. It follows from the construction of 2.10.2 that the augmentation

$$\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} E^\bullet[I] \to \mathcal{O}_Y[d_{Y/X}] \tag{4.5.4}$$

is a quasi-isomorphism of complexes of left $\mathcal{O}_{F^r,Y}$-modules.

We may and do assume that $\mathcal{M}^\bullet$ is a bounded above complex of locally free left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules (which are then also locally free as $\mathcal{O}_X$-modules), so that $f^!\mathcal{M}^\bullet$ is computed by $f^*\mathcal{M}^\bullet[d_{Y/X}]$. We begin with an explicit description of the adjunction map $f_+f^!\mathcal{M}^\bullet \to \mathcal{M}^\bullet$, following the proof of Theorem 4.4.1. Namely, it is represented by the morphism

$$\begin{aligned}
&f_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^*\mathcal{M}^\bullet[d_{Y/X}]) \\
\xleftarrow{q.i.}\ &f_*((f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} E^\bullet[I][d_{X/Y}]) \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} f^*\mathcal{M}^\bullet[d_{Y/X}]) \\
\xrightarrow{\sim}\ &(\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}_X} f_*(E^\bullet[I])) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{M}^\bullet \\
\longrightarrow\ &(\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} E^\bullet) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{M}^\bullet \\
\xleftarrow{q.i.}\ &\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} \mathcal{M}^\bullet \\
\xrightarrow{\sim}\ &\mathcal{M}^\bullet
\end{aligned} \tag{4.5.5}$$

in the derived category. (Here the morphisms labelled $q.i.$ are quasi-isomorphisms, and so may be inverted in the derived category.)

Since the members of the complex $\mathcal{M}^\bullet$ are locally free as $\mathcal{O}_X$-modules, the complex $E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet$ is a resolution of $\mathcal{M}^\bullet$ by a bounded complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules (here we are using the left $\mathcal{O}_{F^r,X}$-module structures on the complexes $E^\bullet$ and $\mathcal{M}$ to put a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module structure on the tensor product) which are injective as $\mathcal{O}_X$-modules. (To see this, note that X is Noetherian, so that a direct limit of injective $\mathcal{O}_X$-modules is injective and the property of being injective as an $\mathcal{O}_X$-module may be checked locally.) Thus

$$\underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet) \xrightarrow{\sim} (E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I],$$

and so the construction of Corollary 2.10.2 yields an isomorphism of complexes of left $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules

$$\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M}^\bullet) \xrightarrow{\sim} \omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} (E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I].$$

We now have an explicit representation of the morphism (4.5.1) as the composite

$$\begin{aligned}
&f_*(\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_Y} (\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} (E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I]))\\
&\xrightarrow{\sim} f_*(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} (E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I])\\
&\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} f_*((E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I])\\
&\longrightarrow \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} (E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)\\
&\xrightarrow{\mu_{E^\bullet \otimes \mathcal{M}^\bullet}} E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet\\
&\xleftarrow{q.i.} \mathcal{M}^\bullet
\end{aligned} \tag{4.5.6}$$

in the derived category. (Again, the map labelled $q.i.$ is a quasi-isomorphism, and so may be inverted in the derived category.)

Again using the fact that the members of $\mathcal{M}^\bullet$ are locally free as $\mathcal{O}_X$-modules, we see that there is an isomorphism of $\mathcal{O}_Y$-modules

$$(E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I] \xrightarrow{\sim} E^\bullet[I][d_{X/Y}] \otimes_{\mathcal{O}_Y} f^*\mathcal{M}^\bullet[d_{Y/X}],$$

which induces an isomorphism

$$\omega^{-1}_{Y/X}(E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I] \xrightarrow{\sim} (\omega^{-1}_{Y/X} \otimes E^\bullet[I][d_{X/Y}]) \otimes_{\mathcal{O}_Y} f^*\mathcal{M}^\bullet[d_{Y/X}]. \tag{4.5.7}$$

We may give the source of (4.5.7) the left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure provided by Corollary 2.10.2, and the target of (4.5.7) the left $\mathcal{O}^{\Lambda}_{F^r,Y}$-module structure obtained by applying Corollary 2.10.2 to $E^\bullet$, and then tensoring the resulting complex of left $\mathcal{O}_{F^r,Y}$-modules over $\mathcal{O}_Y$ with the complex of left $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules $f^*\mathcal{M}^\bullet[d_{Y/X}]$. It is then easy to see that (4.5.7) is an isomorphism of left $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules. We may compose it with (4.5.4) (tensored by $f^*\mathcal{M}^\bullet$) to obtain a quasi-isomorphism of complexes of left $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules

$$\omega^{-1}_{Y/X}(E^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)[I] \to f^*\mathcal{M}^\bullet[d_{Y/X}] \tag{4.5.8}$$

which is a realisation on the level of complexes of the isomorphism of Proposition 2.10.4 (as one sees by examining the construction of [**Ha 1**, I 7.4] and [**Con**, 2.1]).

To prove the proposition, we embed (4.5.5) and (4.5.6) into diagram (4.5.9), which is displayed at the end of this section. The commutativity of this diagram then implies the proposition. The only part of (4.5.9) whose commutativity is not clear is the lower left quadrilateral, whose commutativity follows from Lemma 1.10.9. □

The proof of [Lyu, prop. 3.1] involves, at least implicitly, the natural transformation (4.5.1). Thus Proposition 4.5.3 plays a role in showing that our constructions generalise those of Lyubeznik.

Remark 4.6. — Let us note that all the constructions of this section are compatible with change of ground field via an algebraic extension k'/k, in the obvious sense. Indeed, the only subtle point is the construction of proposition 4.4.9, which involves base-change of the trace map and residual complexes. However, an algebraic field extension is residually stable, and base-change via residually stable morphisms is compatible with the formation of residual complexes and the trace map.

The constructions are also compatible with change of ring from Λ to Λ', if Λ' is a Noetherian Λ-algebra. Indeed the key point is that this should be true for the trace map, and in that case it is built into the construction.

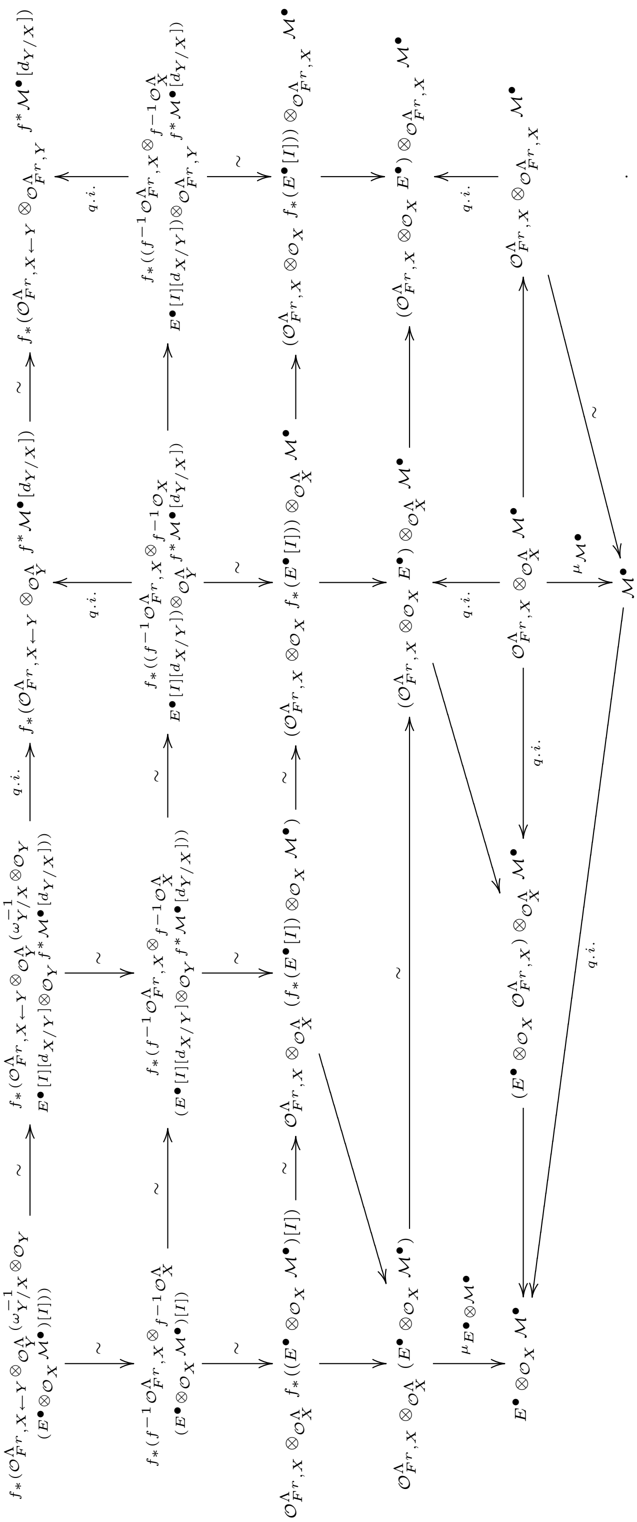


Diagram (4.5.9)

5. UNIT $\mathcal{O}^{\Lambda}_{F^r}$-MODULES

Definition 5.1. — A unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module on a smooth k-scheme X is a quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$ for which the structural morphism $\phi_{\mathcal{M}} : F_X^{r\,*}\mathcal{M} \to \mathcal{M}$ is an isomorphism of $\mathcal{O}^{\Lambda}_X$-modules.

Lemma 5.2. — *Let X be a smooth k-scheme, and suppose that*

$$\mathcal{M}_\bullet = \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4 \to \mathcal{M}_5$$

is an exact sequence of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, and that $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$, and $\mathcal{M}_5$ are unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. Then $\mathcal{M}_3$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module.

Proof. — Since the $\mathcal{M}_i$ are quasi-coherent for $i \neq 3$, we conclude that $\mathcal{M}_3$ is quasi-coherent. Since X is smooth, $F_X^{r\,*}$ is exact, and we get a morphism of exact sequences $F_X^{r*}\mathcal{M}_\bullet \longrightarrow \mathcal{M}_\bullet$, so that the result follows from the five lemma. □

5.3. — The notion of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, when $\Lambda = \mathbb{F}_q$, $r = 1$ and $X = \operatorname{Spec} A$ is an affine scheme, is identical to that of F-module as defined in [**Lyu**]. Lyubeznik gives a method for constructing F-modules via a generating morphism, which immediately generalises to the case of general r and an arbitrary smooth k-scheme X.

Construction-Definition 5.3.1. — Suppose that M is a quasi-coherent $\mathcal{O}^{\Lambda}_X$-module equipped with a morphism of $\mathcal{O}^{\Lambda}_X$-modules

$$\beta : M \to F_X^{r\,*}M.$$

We define $\mathcal{M}$ to be the direct limit of the direct system

$$M \xrightarrow{\beta} F_X^{r\,*}M \xrightarrow{F_X^{r\,*}\beta} (F_X^{2r})^*M \xrightarrow{(F_X^{2r})^*\beta} \cdots \xrightarrow{(F_X^{rn-r})^*\beta} (F_X^{rn})^*M \xrightarrow{(F_X^{rn})^*\beta} \cdots .$$

Since pull-back commutes with direct limits, $F_X^{r*}\mathcal{M}$ is the direct limit of the system

$$F_X^{r*}M \xrightarrow{F_X^{r*}\beta} (F_X^{2r})^*M \xrightarrow{(F_X^{2r})^*\beta} \cdots \xrightarrow{(F_X^{rn-r})^*\beta} (F_X^{rn})^*M \xrightarrow{(F_X^{rn})^*\beta} \cdots ,$$

and so is naturally identified with $\mathcal{M}$. This identification $F_X^{r\,*}\mathcal{M} = \mathcal{M}$ gives $\mathcal{M}$ the structure of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module (by Lemma 1.5.1).

We say that $\mathcal{M}$ is the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module generated by the morphism $\beta : M \to F_X^{r\,*}M$. More generally, if $\mathcal{N}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module isomorphic to $\mathcal{M}$, we will say that $\mathcal{N}$ is generated by β, or that β is a generator of $\mathcal{N}$.

5.3.2. — If $\mathcal{M}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, let β denote the inverse to the structural morphism:

$$\beta = \phi_{\mathcal{M}}^{-1} : \mathcal{M} \to F_X^{*}\mathcal{M}.$$

Then the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module generated by β is naturally isomorphic to $\mathcal{M}$. Thus any unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module has at least one generator.

The following result gives an alternative description of the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module generated by a morphism $\beta : M \to F_X^{r\,*}M$:

Proposition 5.3.3. — *Let M be an $\mathcal{O}^{\Lambda}_{X}$-module equipped with an $\mathcal{O}^{\Lambda}_{X}$-linear map $\beta : M \to F_X^{r\,*}M$. Let β' be the corresponding morphism of induced modules*

$$\beta' : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M$$

(in the sense of (1.7.3)); equivalently, β' can be described as right multiplication by F^r on $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M$, when this tensor product is given its induced bimodule structure via the discussion of (1.10.4). Let $\mathcal{M}$ be the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module generated by β. Then $\mathcal{M}$ sits in the following short exact sequence:

$$0 \longrightarrow \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M \stackrel{1-\beta'}{\longrightarrow} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M \longrightarrow \mathcal{M} \longrightarrow 0.$$

Proof. — The natural map of left $\mathcal{O}^{\Lambda}_{X}$-modules $M \to \mathcal{M}$ induces a map of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M \to \mathcal{M}$, which is surjective by definition (see 1.5).

Furthermore, from the definition of the map β' and the construction of the direct limit we now see that $\mathcal{M}$ is presented as

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M \stackrel{1-\beta'}{\to} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} M \to \mathcal{M} \to 0.$$

It remains to see that the first arrow is injective. But this is clear, since β' has image in $\mathcal{O}^{\Lambda}_{F^r,X}F^r \otimes_{\mathcal{O}^{\Lambda}_{X}} M$, so that any element in the kernel of $1-\beta'$ lies in the intersection $\bigcap_{n=0}^{\infty} \mathcal{O}^{\Lambda}_{F^r,X}F^{rn} \otimes_{\mathcal{O}^{\Lambda}_{X}} M = 0$. ☐

5.3.4. — As already noted, if $\mathcal{M}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module then we can in particular take $\beta = \phi_{\mathcal{M}}^{-1} : \mathcal{M} \to F_X^{r*}\mathcal{M}$ to be a generator of $\mathcal{M}$. We remark that with this choice of generator the short exact sequences constructed by Proposition 5.3.3 and Lemma 1.8.1 are essentially the same; namely, they fit into the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} F_X^{*r}\mathcal{M} & \xrightarrow{\imath_{\mathcal{M}}} & \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{M} & \xrightarrow{\mu_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}\otimes\phi_{\mathcal{M}}}\downarrow\wr & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{M} & \xrightarrow{1-\beta'} & \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{X}} \mathcal{M} & \xrightarrow{\mu_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0,
\end{array}$$

whose commutativity follows immediately from the definitions of the maps involved.

5.3.5. — We suppose that X is a smooth affine k-scheme, and that M is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module. We will construct a particular kind of free resolution of $\mathcal{M}$, which will be useful on several occasions.

Let $\beta : M \to F_X^{r\,*}M$ be a generator of $\mathcal{M}$, and let $P^\bullet$ be a resolution of M by free $\mathcal{O}^{\Lambda}_X$-modules. Then $F_X^{r*}P^\bullet$ is a free resolution of $F_X^{r*}M$, and we may lift β to a morphism

$$\beta^\bullet : P^\bullet \to F_X^{r*}P^\bullet.$$

¿From this we follow Proposition 5.3.3 and construct a double complex

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} P^\bullet \xrightarrow{1-\beta^{\bullet\prime}} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} P^\bullet$$

whose associated total complex is a free resolution of the complex

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M \xrightarrow{1-\beta'} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M,$$

which is in turn a resolution of $\mathcal{M}$.

It is also useful to note that we may form the complex $\mathcal{P}^\bullet$ of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules generated by the complex of generators $\beta^\bullet$, and that this complex again resolves $\mathcal{M}^\bullet$. Since $\mathcal{P}^i$ is a direct limit of the pull-back of P^i by powers of F_X^r for each integer i, we see furthermore that each $\mathcal{P}^i$ is a flat $\mathcal{O}^{\Lambda}_X$-module.

5.4. — We define $\mu_u(X,\Lambda)$ to be the full subcategory of $\mu(X,\Lambda)$ consisting of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. We define $D^\bullet_u(\mathcal{O}^{\Lambda}_{F^r,X})^*$ to be the full triangulated subcategory of $D^\bullet(\mathcal{O}^{\Lambda}_{F^r,X})^*$ consisting of complexes whose cohomology sheaves are unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. (Here $\bullet$ denotes one of $+$, $-$, b, or $\emptyset$, and $*$ denotes one of $\circ$ or $\emptyset$.)

Note that Lemma 5.2 shows that $\mu_u(X,\Lambda)$ is a thick subcategory of $\mu(X,\Lambda)$, so that $D^\bullet_u(\mathcal{O}^{\Lambda}_{F^r,X})^*$ is indeed a triangulated subcategory of $D^\bullet(\mathcal{O}^{\Lambda}_{F^r,X})^*$.

5.4.1. — A complex $\mathcal{M}^\bullet$ of $D(\mathcal{O}^{\Lambda}_{F^r,X})$ belongs to $D_u(\mathcal{O}^{\Lambda}_{F^r,X})^*$ if and only if the structural isomorphism $\phi_{\mathcal{M}^\bullet} : F^{r*}\mathcal{M}^\bullet \to \mathcal{M}^\bullet$ is an isomorphism of objects in $D(\mathcal{O}^{\Lambda}_{F^r,X})$.

Proof. — The complex $\mathcal{M}^\bullet$ belongs to $D_u(\mathcal{O}^{\Lambda}_{F^r,X})$ if and only if each $\phi_{H^i(\mathcal{M}^\bullet)}$ is an isomorphism. Now (2.11.1) shows that this is the case exactly if each $H^i(\phi_{\mathcal{M}^\bullet})$ is an isomorphism. Thus each $\phi_{H^i(\mathcal{M}^\bullet)}$ is an isomorphism if and only if $\phi_{\mathcal{M}^\bullet}$ is an isomorphism in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$. □

5.5. — It is an immediate consequence of Lemma 1.9.2 that if $\mathcal{M}$ and $\mathcal{N}$ are unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules then $\mathcal{M}\otimes_{\mathcal{O}^{\Lambda}_X}\mathcal{N}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module. The following lemma makes an even stronger observation:

Lemma 5.5.1. — *Suppose that $\beta : M \to F_X^{r\,*}M$ generates the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$ and that $\gamma : N \to F_X^{r\,*}N$ generates the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{N}$. Then for each $i \geq 0$, the $\mathcal{O}^{\Lambda}_X$-module $\mathrm{Tor}^i_{\mathcal{O}^{\Lambda}_X}(\mathcal{M},\mathcal{N})$ has the structure of a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module generated by the morphism*

$$H^i(\beta \overset{\mathbb{L}}{\otimes} \gamma) : \mathrm{Tor}^i_{\mathcal{O}^{\Lambda}_X}(M,N) \to \mathrm{Tor}^i_{\mathcal{O}^{\Lambda}_X}(F_X^{r*}M, F_X^{r*}N) \xrightarrow{\sim} F_X^{r*}\mathrm{Tor}^i_{\mathcal{O}^{\Lambda}_X}(M,N).$$

Proof. — As $-\otimes_{\mathcal{O}_X^{\Lambda}} -$ commutes both with direct limits and with $(F_X^{rn})^*$, and both these functors are exact, $\mathrm{Tor}^i_{\mathcal{O}_X^{\Lambda}}$ also commutes with these functors for every $i \geq 0$. The lemma follows. □

Corollary 5.5.2. — *The functor $-\otimes_{\mathcal{O}_X^{\Lambda}} -$ of section 1.9.4 restricts to a functor*

$$-\otimes_{\mathcal{O}_X^{\Lambda}} - : D_u^-(\mathcal{O}^{\Lambda}_{F^r,X}) \times D_u^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_u^-(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — A standard spectral sequence argument shows that this is a consequence of Lemma 5.5.1. □

Proposition 5.6. — *Let X be a smooth k-scheme, and Λ' a Noetherian Λ-algebra. Then $\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} -$ restricts to a functor*

$$\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} - : D_u^-(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_u^-(\mathcal{O}^{\Lambda'}_{F^r,X}),$$

and so also to a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} - : D_u^b(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ} \to D_u^b(\mathcal{O}^{\Lambda'}_{F^r,X})^{\circ}.$$

Proof. — Note that once we prove the first claim, the second immediately follows. If $\mathcal{M}^{\bullet}$ lies in $D_u^-(\mathcal{O}^{\Lambda}_{F^r,X})$, then $\Lambda' \overset{\mathbb{L}}{\otimes} \mathcal{M}^{\bullet}$ lies in $D^-(\mathcal{O}^{\Lambda'}_{F^r,X})$. We must show that in addition $\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{M}^{\bullet}$ has unit cohomology sheaves.

We may check this locally, and hence may assume that X is affine. The usual spectral sequence argument also shows that it suffices to consider the case when $\mathcal{M}^{\bullet}$ is a single unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$. We apply the construction of (5.3.5), and deduce that we may assume that $\mathcal{M}$ is generated by a map $\beta : M \to F_X^{r*}M$ for which the $\mathcal{O}_X^{\Lambda}$-module M is free. In this case we see that $\Lambda' \otimes_{\Lambda} M$ is the unit $\mathcal{O}^{\Lambda'}_{F^r,X}$-module generated by the map

$$\mathrm{id}_{\Lambda'} \otimes \beta : \Lambda' \otimes_{\Lambda} M \to \Lambda' \otimes_{\Lambda} F_X^{r*} M \xrightarrow{\sim} F_X^{r*}(\Lambda' \otimes_{\Lambda} M).$$

This proves the Proposition. □

Proposition 5.7. — *Let r' be a multiple of r, let $q' = p^{r'}$, assume that $\mathbb{F}_{q'} \subset k$, and write $\Lambda' = \mathbb{F}_{q'} \otimes_{\mathbb{F}_q} \Lambda$. Then for any smooth k-scheme X, restriction and induction (as defined in section 1.14) induce functors*

$$\mathrm{Res}_q^{q'} : D_u(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_u(\mathcal{O}^{\Lambda'}_{F^{r'},X})$$

and

$$\mathrm{Ind}_{q'}^{q} : D_u(\mathcal{O}^{\Lambda'}_{F^{r'},X}) \to D_u(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — Since restriction (respectively induction) has zero cohomological amplitude, it suffices to establish the claim of the proposition in the case of a single unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module (respectively of a single unit $\mathcal{O}^{\Lambda'}_{F^{r'},X}$-module) $\mathcal{M}$. In this case, the result follows from the fact that $\phi_{\mathcal{M}}$ is an isomorphism, together with the prescription for constructing the structural morphism of $\mathrm{Res}_q^{q'}\mathcal{M}$ (respectively $\mathrm{Ind}_{q'}^{q}\mathcal{M}$) from that of $\mathcal{M}$ given in (1.14.2) (respectively (1.14.3)). □

5.7.1. — It will be convenient to note (and is easily seen) that if $\beta : M \to F_X^{r*} M$ generates the unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$, then the composite

$$M \xrightarrow{\beta} F_X^{r*} M \xrightarrow{F_X^{r*}\beta} \cdots \xrightarrow{F_X^{(r'-r)*}\beta} F_X^{r'*} M$$

generates $\mathrm{Res}_q^{q'} \mathcal{M}$. Similarly, if $\gamma : N \to F_X^{r'*} N$ generates the unit $\mathcal{O}^{\Lambda'}_{F^{r'},X}$-module $\mathcal{N}$, then the map

$$\begin{aligned} N \oplus F_X^{r*} N \oplus \cdots \oplus F_X^{(r'-r)*} N &\xrightarrow{\gamma \oplus \mathrm{id} \oplus \cdots \oplus \mathrm{id}} F_X^{(r')*} N \oplus F_X^{r*} \oplus \cdots \oplus F_X^{(r'-r)*} N \\ &\xrightarrow{\sim} F_X^{r*}(N \oplus F_X^{r*} N \oplus \cdots \oplus F_X^{(r-r')*N}) \end{aligned}$$

generates $\mathrm{Ind}_{q'}^{q} \mathcal{N}$.

Lemma 5.7.2. — *There is a projection formula relating induction and restriction for unit modules. Namely, the bifunctors* $\mathrm{Ind}_{q'}^{q}(- \otimes_{\mathcal{O}_X^{\Lambda'}} \mathrm{Res}_q^{q'}(-))$ *and* $\mathrm{Ind}_{q'}^{q}(-) \otimes_{\mathcal{O}_X^{\Lambda}} -$ *on* $\mu_u(X, \Lambda') \times \mu_u(X, \Lambda)$ *are naturally isomorphic.*

Proof. — Let $\mathcal{M}^\bullet$ be an object of $\mu_u(X, \Lambda')$ and $\mathcal{N}^\bullet$ be an object of $\mu_u(X, \Lambda)$. Then Lemma 1.14.1 shows that $\mathrm{Ind}_{q'}^{q}(\mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda'}} \mathrm{Res}_q^{q'} \mathcal{N})$ is isomorphic to

$$\bigoplus_{n=0}^{(r'/r)-1} F_X^{rn*}(\mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda'}} \mathcal{N}) \xrightarrow{\sim} \bigoplus_{n=0}^{(r'/r)-1} F_X^{rn*} \mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda'}} F_X^{rn*} \mathcal{N},$$

while $\mathrm{Ind}_{q'}^{q} \mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda}} \mathcal{N}$ is isomorphic to

$$\left(\bigoplus_{n=0}^{(r'/r)-1} F_X^{rn*} \mathcal{M} \right) \otimes_{\mathcal{O}_X^{\Lambda'}} \mathcal{N} \xrightarrow{\sim} \bigoplus_{n=0}^{(r'/r)-1} F_X^{rn*} \mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda'}} \mathcal{N}.$$

We have the following natural isomorphism between these two objects of $\mu_u(X, \Lambda)$,

$$\bigoplus_{n=0}^{(r'/r)-1} F_X^{rn*} \mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda'}} F_X^{rn*} \mathcal{N} \xrightarrow{\mathrm{id} \otimes \mathrm{id} \oplus \mathrm{id} \otimes \phi_{1,\mathcal{M}} \cdots \oplus \mathrm{id} \otimes \phi_{(r'/r)-1,\mathcal{M}}} \bigoplus_{n=0}^{(r'/r)-1} F_X^{rn*} \mathcal{M} \otimes_{\mathcal{O}_X^{\Lambda'}} \mathcal{N}$$

(where we are using the notation $\phi_{n,\mathcal{N}}$ introduced in section (1.5)), which one easily checks to be compatible with the structural morphisms of source and target (using the description of these structural morphisms provided by (1.9.2) and (1.14.3)). □

Theorem 5.8. — *Suppose that $f : Y \to X$ is a morphism of smooth k-schemes.*

(*i*) *The functor $f^! : D(\mathcal{O}^{\Lambda}_{F^r,X}) \to D(\mathcal{O}^{\Lambda}_{F^r,Y})$ restricts to a functor (which we denote by the same symbol)*

$$f^! : D_u(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_u(\mathcal{O}^{\Lambda}_{F^r,Y}).$$

(*ii*) *The functor $f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$ restricts to a functor (which we denote by the same symbol)*

$$f_+ : D_u(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_u(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — We prove part *(i)* first. Let $\mathcal{M}^{\bullet}$ be a complex in $D(\mathcal{O}^{\Lambda}_{F^r,X})$ whose cohomology sheaves are unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. We must show that the cohomology sheaves of $f^!\mathcal{M}^{\bullet}$ are unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules.

We begin by considering the case of a single unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$. We must show that $f^!\mathcal{M}$ has unit cohomology sheaves. This can be verified locally, and so after replacing X by an open affine subset and Y be the inverse image under f of this subset we may assume that X is affine.

We apply the construction of (5.3.5) to $\mathcal{M}$, and so obtain a double complex

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} P^{\bullet} \overset{1-\beta^{\bullet\prime}}{\longrightarrow} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} P^{\bullet}$$

whose associated total complex is a free resolution of $\mathcal{M}$.

Thus $f^!\mathcal{M}$ is (up to shifting) the total complex associated to the double complex

$$\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} f^*P^{\bullet} \overset{1-(f^*\beta)'}{\longrightarrow} \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} f^*P^{\bullet}.$$

There is a spectral sequence converging to the cohomology sheaves of this total complex, whose E_1 terms are the horizontal cohomology groups of this double complex, which by Proposition 5.3.3 are unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules. By repeated applications of Lemma 5.2 we see that the E_{∞} terms are again unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules, and so finally we see that the same is true of the cohomology sheaves of $f^!\mathcal{M}$.

This proves the proposition in the case of a single unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$. Now by a standard spectral sequence argument, taking into account Lemma 5.2, as well as the fact that $f^!$ is of finite cohomological amplitude by Lemma 2.3.2, we deduce part *(i)* of the proposition for arbitrary complexes in $D_u(X)$.

We turn to proving part *(ii)*. To begin with, suppose that $\mathcal{M}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module, with generator $\beta : M \to F^{r\,*}_Y M$, for some $\mathcal{O}^{\Lambda}_Y$-module M. (As in the case of part *(i)*, it will be useful to give the proof having made an arbitrary choice of generator.) Then $\mathcal{M}$ has the resolution

$$\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M \overset{1-\beta'}{\to} \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M,$$

which by Lemma 3.5.1 is acyclic for the functor $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y} \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} -$.

Thus

$$f_+\mathcal{M} = Rf_*((f^{-1}\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,Y}} (\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M \overset{1-\beta'}{\to} \mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{\mathcal{O}^{\Lambda}_Y} M)).$$

Now using this description of $f_+\mathcal{M}$, the cohomology sheaves of $f_+\mathcal{M}$ may be computed by a spectral sequence whose E_1 terms are equal to (using Lemma 3.5.2)

$$E_1^{i,j} = R^j f_*(f^{-1}\mathcal{O}_{F^r,X} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_X} (\omega_{Y/X} \otimes_{\mathcal{O}_Y} M)) = \mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_X} R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M),$$

for $i = -1$ or $i = 0$, and vanish for all other values of i. We wish to compute the boundary map $d_1^{-1,j}$.

This boundary map is equal to $H^j(f_+)(1-\beta') = 1 - H^j(f_+)\beta'$. Now Proposition 3.6.1 shows that β induces a map

$$\gamma : Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to F^{r*}_X Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M),$$

such that $f_+\beta'$ is equal to the corresponding map of induced complexes

$$\gamma' : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M).$$

Let us write

$$\gamma^j = H^j(\gamma) : R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to {F^r_X}^* R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M)$$

(since F^r_X is flat, the pull-back ${F^r_X}^*$ commutes with taking cohomology) and

$$\gamma'^j = H^j(\gamma') : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M).$$

Then γ'^j is the morphism of induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules corresponding to the morphism γ^j, and we see that

$$d_1^{-1,j} = 1 - \gamma'^j : \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} Rf^j(\omega_{Y/X} \otimes_{\mathcal{O}^{\Lambda}_Y} M).$$

Thus Proposition 5.3.3 shows that $E_2^{-1,j}$ (which is the kernel of $d_1^{-1,j}$) vanishes, and that $E_2^{0,j}$ (which is the cokernel of $d_1^{-1,j}$) is the unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module generated by γ^j.

Thus the spectral sequence collapses to a single column at the E_2 stage, so that $H^j(f_+)\mathcal{M} = E_2^{0,j}$ is the unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module generated by γ^j. This concludes the proof that $f_+\mathcal{M} \in D^b_u(\mathcal{O}^{\Lambda}_{F^r,X})$.

Now a standard spectral sequence argument, taking into account Lemma 5.2 together with the fact that f_+ is of finite cohomological amplitude, establishes part *(ii)* of the proposition. □

5.9. — If $f : Y \to X$ is a closed immersion of smooth k-schemes then we can give an alternative proof of part *(i)* of Theorem 5.8 which is related to the description of $f^!$ provided by Proposition 2.10.4. In the following discussion we let I denote the ideal sheaf cutting out Y in X.

We suppose that $\mathcal{M}$ is a single unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, generated by the morphism $\beta : M \to F^{r*}_X M$. Let $J^\bullet$ be a right resolution of $\mathcal{M}^\bullet$ by injective $\mathcal{O}^{\Lambda}_X$-modules. Then (since F^r_X is residually stable) we see that $F^{r*}_X J^\bullet$ is an injective resolution of $F^{r*}_X M$. The morphism β can thus be extended to a morphism

$$\beta^\bullet : J^\bullet \to F^{r*}_X J^\bullet.$$

Let $\mathcal{J}^\bullet$ denote the complex of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules generated by the morphisms $\beta^\bullet$. Then $\mathcal{J}^\bullet$ is constructed as a direct limit of the modules $F^{rn*}_X J^\bullet$. Since each of these is injective as an $\mathcal{O}^{\Lambda}_X$-module, and since X is Noetherian, we see that $\mathcal{J}^\bullet$ is a complex of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are injective as $\mathcal{O}^{\Lambda}_X$-modules. Since taking direct limits is exact, we see that $\mathcal{J}^\bullet$ provides a right resolution of $\mathcal{M}^\bullet$.

Being $\mathcal{O}^{\Lambda}_X$-injective, each $\mathcal{J}^\bullet$ is acyclic for the functor $\underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -)$. Thus we may use the resolution $\mathcal{J}^\bullet$ of $\mathcal{M}$ to compute $f^!\mathcal{M}$, under the guise of $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \underline{RHom}^\bullet_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{M})$, as allowed by Proposition 2.10.4. Thus to see that $f^!\mathcal{M}$ lies in $D^b_u(\mathcal{O}^{\Lambda}_{F^r,Y})$, it suffices to show that each of the $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{J}^\bullet[I]$ is a unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module. This is a consequence of the following result.

Lemma 5.9.1. — *Suppose that $\mathcal{M}^\bullet$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, generated by a morphism $\gamma : M \to F_X^{r*}M$. Then γ induces in a natural way a morphism*

$$\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} M[I] \to F_Y^{r*}(\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} M[I]),$$

and the unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module generated by this morphism is naturally isomorphic to the $\mathcal{O}^{\Lambda}_{F^r,Y}$-module $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M}[I]$ constructed via Corollary 2.10.2. In particular, this latter module is a unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module. (Of course, this last claim also follows from Theorem 5.8, part (i), and Proposition 2.10.4.)

Proof. — The morphism γ induces a map

$$M[I^{(r)}] \to (F_X^{r*}M)[I^{(r)}] \xrightarrow{\sim} F_X^{r'*}(M[I]),$$

and consequently a morphism

$$M[I] \to F_X^{r*}(M[I])[I].$$

Combining this with the inverse of the isomorphism

$$\omega_{Y/X} \otimes_{\mathcal{O}_Y} F_Y^{r*}(\omega^{-1}_{Y/X} \otimes M[I]) \xrightarrow{\sim} F_X^{r\prime *}(M[I])$$

constructed at the end of (A.2.2) (taking the N of that discussion to be $\omega^{-1}_{Y/X} \otimes M[I]$), we obtain a morphism

$$M[I] \to \omega_{Y/X} \otimes_{\mathcal{O}_Y} F_Y^{r*}(\omega^{-1}_{Y/X} \otimes M[I]),$$

and consequently a morphism

$$\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} M[I] \to F_Y^{r*}(\omega^{-1}_{Y/X} \otimes M[I]).$$

Tracing through the construction of Corollary 2.10.2 and (hence) that of Lemma 2.10.1, one verifies that this morphism does indeed generate $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M}[I]$. □

5.9.2. — This recipe for passing from a generator of $\mathcal{M}$ to a generator of $\omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M}[I] = H^0(f^!)\mathcal{M}$ is found in the proof of [Lyu, prop. 3.1] (although it is not expressed in this language).

5.10. — The following theorem, which is the analogue for unit $\mathcal{O}^{\Lambda}_{F^r}$-modules of Kashiwara's theorem in the theory of $\mathscr{D}$-modules, generalises [Lyu, prop. 3.1]. To see the connection between our approach and that of Lyubeznik, one should refer to the description of $f^!$ provided by Proposition 2.10.4 and the discussion of section (5.9), as well as the description of the adjunction morphism $f_+f^! \to \mathrm{id}$ provided by Proposition 4.5.3.

Theorem 5.10.1. — *Let $f : Y \to X$ be a closed immersion of smooth k-schemes. If $\mathcal{M}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module supported on Y, then the adjunction map $f_+f^!\mathcal{M} \to \mathcal{M}$ is an isomorphism. As a consequence, we see that $H^0(f^!)\mathcal{M} \xrightarrow{\sim} f^!\mathcal{M}$. The functors f_+ and $f^!$ induce an equivalence of categories between the category of unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules, and the category of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules supported on Y.*

Proof. — Since by remark 3.4.1 f_+ has zero cohomological amplitude, we see that if the stated isomorphism holds, then $f^!\mathcal{M}$ must be a single module in degree zero, which proves the second claim. The final claim then follows because f_+ is faithful by the remark in 3.4.1.

The isomorphism of the theorem may be verified locally, and so we may assume that X is affine and that Y is cut out by a regular sequence $(a_1, \ldots, a_s)$ with the property that each of the subschemes $V(a_1, \ldots, a_i)$ of X is smooth over k, for $1 \leq i \leq s$. Then we may factor the closed immersion f as a product

$$Y = V(a_1, \ldots, a_s) \xrightarrow{f_s} V(a_1, \ldots, a_{s-1}) \xrightarrow{f_{s-1}} \cdots \xrightarrow{f_2} V(a_1) \xrightarrow{f_1} X,$$

with each f_i being the closed immersion of a divisor. The adjunction morphism has the factorisation

$$\begin{aligned} f_+f^!\mathcal{M} &= f_{1+} \cdots f_{s+} f_s^! \cdots f_1^! \mathcal{M} \\ &\to f_{1+} \cdots f_{s-1\,+} f_{s-1}^! \cdots f_1^! \mathcal{M} \to \cdots \to f_{1+} f_1^! \mathcal{M} \to \mathcal{M}, \end{aligned}$$

Since each $f_{i-1}^! \cdots f_1^! \mathcal{M}$ is a unit $\mathcal{O}_{F^r, V(a_1, \ldots, a_{i-1})}$-module which is supported on $V(a_1, \ldots, a_i)$, we see that to prove that

$$f_+f^!\mathcal{M} \to \mathcal{M}$$

is an isomorphism, it also suffices to consider the case of $s = 1$. Thus for the remainder of the proof we assume that $Y = V(a)$ is a smooth divisor on X, and that $\mathcal{M}$ is a unit $\mathcal{O}_{F^r,X}^{\Lambda}$-module supported on Y.

To prove that the adjunction map is an isomorphism, we factor it as follows:

$$f_+f^!\mathcal{M} \xrightarrow{\sim} f_+\mathcal{O}_{F^r,Y}^{\Lambda}[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F^r,X}^{\Lambda}} \mathcal{M} \xrightarrow{tr_{F^r,f} \overset{\mathbb{L}}{\otimes} \mathrm{id}_{\mathcal{M}}} \mathcal{O}_{F^r,X}^{\Lambda} \otimes_{\mathcal{O}_{F^r,X}^{\Lambda}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}.$$

(Compare the proof of Theorem 4.4.1.) We will use the explicit formula of (4.4.13) to show that

$$f_+\mathcal{O}_{F^r,Y}^{\Lambda}[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F^r,X}^{\Lambda}} \mathcal{M} \to \mathcal{M} \tag{5.10.2}$$

is an isomorphism.

Since $\mathcal{M}$ is supported on Y, every section of $\mathcal{M}$ is annihilated by some power of a, and so $\mathcal{M} = \bigcup_{n=0}^{\infty} \mathcal{M}[a^{q^n}]$. Base-change by the flat morphism F_X^{rn*} shows that

$$F_X^{rn*}(\mathcal{M}[a]) \xrightarrow{\sim} (F_X^{rn*}\mathcal{M})[a^{q^n}].$$

Composing with $\phi_{\mathcal{M}}$ we find that $F_X^{rn*}(\mathcal{M}[a]) \xrightarrow{\sim} \mathcal{M}[a^{q^n}]$. Thus $\mathcal{M}$ is generated by the inclusion

$$\mathcal{M}[a] \to \mathcal{M}[a^q] \xrightarrow{\sim} F_X^{r*}(\mathcal{M}[a]),$$

and we have the consequent presentation

$$0 \to \mathcal{O}_{F^r,X}^{\Lambda} \otimes_{\mathcal{O}_X^{\Lambda}} \mathcal{M}[a] \to \mathcal{O}_{F^r,X}^{\Lambda} \otimes_{\mathcal{O}_X^{\Lambda}} \mathcal{M}[a] \to \mathcal{M} \to 0$$

of $\mathcal{M}$.

We now compute (5.10.2) using (4.4.13) and this presentation. The source of (5.10.2) is represented by the tensor product

$$(\mathcal{O}^{\Lambda}_{F^r,X} \to \mathcal{O}^{\Lambda\prime}_{F^r,X}) \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} (\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a] \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a]).$$

(Here we are using the observation of (4.4.13) that the modules $\mathcal{O}^{\Lambda}_{F^r,X}$ and $\mathcal{O}^{\Lambda\prime}_{F^r,X}$ are acyclic with respect to tensor product by induced left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules.) To prove that (5.10.2) is an isomorphism, it suffices to show that

$$\mathcal{O}^{\Lambda\prime}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_{F^r,X}} (\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a] \to \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a]) \xrightarrow{\sim} (\mathcal{O}^{\Lambda\prime}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a] \to \mathcal{O}^{\Lambda\prime}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a])$$

is an acyclic complex. Let d denote the differential of this complex.

Since $\mathcal{O}^{\Lambda\prime}_{F^r,X}$ is isomorphic to $\mathcal{O}^{\Lambda}_{F^r,X}$ as a right $\mathcal{O}^{\Lambda}_X$-module, we see by (1.3.2) that

$$\mathcal{O}^{\Lambda\prime}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M}[a] \xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} \mathcal{M} \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathcal{M}[a^{q^n}],$$

Using the explicit description of the right $\mathcal{O}^{\Lambda}_{F^r,X}$-module structure of $\mathcal{O}^{\Lambda\prime}_{F^r,X}$, we see that d restricted to $\mathcal{M}[a^{q^n}]$ is given by

$$\mathcal{M}[a^{q^n}] \xrightarrow{m \mapsto m \oplus a^{q^n(q-1)}m} \mathcal{M}[a^{q^n}] \oplus \mathcal{M}[a^{q^{n+1}}] \subset \bigoplus_{n=0}^{\infty} \mathcal{M}[a^{q^n}].$$

Since $a^{q^n(q-1)}$ annihilates $\mathcal{M}[a^{q^n}]$, this map is simply the tautological injection of $\mathcal{M}[a^{q^n}]$ into $\bigoplus_{n=0}^{\infty} \mathcal{M}[a^{q^n}]$. This shows that d is just the identity map, and in particular an isomorphism, and so completes the proof of the theorem. □

We have the following corollary of this result:

Corollary 5.10.3. — *If $f : Y \to X$ is a closed immersion of smooth k-schemes, and $\mathcal{M}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, then the adjunction morphism*

$$f_+ H^0(f^!)\mathcal{M} \to \mathcal{M}$$

is an injection, and identifies $f_+H^0(f^!)\mathcal{M}$ with the subsheaf $\Gamma_Y(\mathcal{M})$ of $\mathcal{M}$ consisting of sections of $\mathcal{M}$ supported on Y.

Proof. — Since $\Gamma_Y(\mathcal{M}) = \bigcup_{n=1}^{\infty} \mathcal{M}[I^n]$, and $F^{r*}(\mathcal{M}[I^n]) \subset \mathcal{M}[I^{qn}]$, one sees that $\Gamma_Y(\mathcal{M})$ is a sheaf of left $\mathcal{O}^{\Lambda}_{F^r,X}$-submodules of $\mathcal{M}$. Furthermore, the structural isomorphism $\phi_{\mathcal{M}}$ induces an isomorphism $F^{r\,*}_X(\Gamma_Y(\mathcal{M})) \xrightarrow{\sim} \Gamma_Y(F^{r\,*}_X(\mathcal{M})) \xrightarrow{\sim} \Gamma_Y(\mathcal{M})$. Thus $\Gamma_Y(\mathcal{M})$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module.

Now $\Gamma_Y(\mathcal{M})[I] = \mathcal{M}[I]$, and so one concludes by proposition 2.10.4 that the natural morphism $H^0(f^!)\Gamma_Y(\mathcal{M}) \to H^0(f^!)\mathcal{M}$ is an isomorphism. Thus

$$f_+H^0(f^!)\mathcal{M} \xrightarrow{\sim} f_+H^0(f^!)\Gamma_Y(\mathcal{M}) \xrightarrow{\sim} \Gamma_Y(\mathcal{M}),$$

where the second isomorphism is given by Theorem 5.10.1. □

5.11. — We will also deduce an analogue of Theorem 5.10.1 for the derived category. First, some definitions:

Definition 5.11.1. — If X is a k-scheme, and if $\mathcal{M}^\bullet$ is an object of $D(\mathcal{O}^{\Lambda}_{F^r,X})$, the support of $\mathcal{M}^\bullet$ is defined to be the Zariski closure of the union of the support of the cohomology sheaves $H^i(\mathcal{M}^\bullet)$, for every i.

Definition 5.11.2. — If Y is a closed subset of a smooth k-scheme X, we define $D_{u,Y}(\mathcal{O}^{\Lambda}_{F^r,X})$ to be the full triangulated subcategory of $D_u(\mathcal{O}^{\Lambda}_{F^r,X})$ consisting of objects whose support is contained in Y.

Corollary 5.11.3. — *Let $f : Y \to X$ be a closed immersion of smooth k-schemes. Then the essential image of the functor*

$$f_+ : D_u(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_u(\mathcal{O}^{\Lambda}_{F^r,X})$$

is equal to $D_{u,Y}(\mathcal{O}^{\Lambda}_{F^r,X})$.

Furthermore, f_+ induces an equivalence of categories between $D_u(\mathcal{O}^{\Lambda}_{F^r,Y})$ and $D_{u,Y}(\mathcal{O}^{\Lambda}_{F^r,X})$, with $f^!$ providing a quasi-inverse.

Proof. — This follows from Theorem 5.10.1 by a standard argument, bearing in mind that $f^!$ and f_+ are both of finite cohomological amplitude. (In fact, f_+ is even of zero cohomological amplitude, since f is a closed immersion.) $\square$

5.11.4. — Recall that if Y is a closed subset of X then the functor $R\Gamma_Y$ is defined as the total right derived functor of the functor Γ_Y of sections with support in Y. If $\mathcal{M}^\bullet$ is a bounded complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, then $R\Gamma_Y(\mathcal{M}^\bullet)$ is naturally a complex of left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, since we may compute $R\Gamma_Y$ by taking a resolution of $\mathcal{M}^\bullet$ by injective left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules (1.3.1). By a result of Grothendieck, $R\Gamma_Y$ has finite cohomological amplitude. Thus $R\Gamma_Y(\mathcal{M}^\bullet)$ has only finitely many non-vanishing cohomology sheaves and so is an object of $D^b(\mathcal{O}^{\Lambda}_{F^r,X})$.

The following result generalises [Lyu, e.g. 1.2 (b)], and provides an analogue of Corollary 5.10.2 for the derived category.

Proposition 5.11.5. — *If $f : Y \to X$ is a closed immersion of a k-scheme Y into a smooth k-scheme X and $\mathcal{M}^\bullet$ is an object of $D^+_u(\mathcal{O}^{\Lambda}_{F^r,X})$, then $R\Gamma_Y(\mathcal{M}^\bullet)$ is an object of $D^+_{u,Y}(\mathcal{O}^{\Lambda}_{F^r,X})$. If furthermore Y is smooth k-scheme, then there is a natural isomorphism*

$$R\Gamma_Y(\mathcal{M}^\bullet) \xrightarrow{\sim} f_+f^!M^\bullet$$

of objects in $D^+_{u,Y}(\mathcal{O}^{\Lambda}_{F^r,X})$.

Proof. — Let $j : X \setminus Y \to X$ be the open immersion of the complement of Y. Then we have the distinguished triangle of objects of $D^+(\mathcal{O}^{\Lambda}_{F^r,X})$

$$\cdots \to R\Gamma_Y(\mathcal{M}^\bullet) \to \mathcal{M}^\bullet \to j_+j^!\mathcal{M}^\bullet \to \cdots .$$

Since $\mathcal{M}^\bullet$ and $j_+j^!\mathcal{M}^\bullet$ are both objects of $D^+_u(\mathcal{O}^{\Lambda}_{F^r,X})$ we see that the same is true of $R\Gamma_Y(\mathcal{M}^\bullet)$. By construction, it has support in Y.

If furthermore Y is smooth as a k-scheme then we may apply $f^!$ to the above distinguished triangle to obtain the distinguished triangle

$$\cdots \to f^!R\Gamma_Y(\mathcal{M}^\bullet) \to f^!\mathcal{M}^\bullet \to 0 \to \cdots$$

of objects in $D^+_u(\mathcal{O}^{\Lambda}_{F^r,Y})$. Thus $f^! R\Gamma_Y(\mathcal{M}^\bullet) \xrightarrow{\sim} f^!\mathcal{M}^\bullet$. As $R\Gamma_Y(\mathcal{M}^\bullet)$ is an object of $D^+_{u,Y}(\mathcal{O}^{\Lambda}_{F^r,X})$, Corollary 5.11.3 gives

$$R\Gamma_Y(\mathcal{M}) \xrightarrow{\sim} f_+ f^! R\Gamma_Y(\mathcal{M}^\bullet) \xrightarrow{\sim} f_+ f^! \mathcal{M}^\bullet.$$

□

Example 5.11.6. — Consider the case when Y is smooth and $\mathcal{M}^\bullet = \mathcal{O}^{\Lambda}_X$. Then $f^!\mathcal{O}^{\Lambda}_X = \mathcal{O}^{\Lambda}_Y[d_{Y/X}]$. Thus we see that in the situation of the preceding proposition

$$f_+\mathcal{O}^{\Lambda}_Y \xrightarrow{\sim} R\Gamma_Y(\mathcal{O}^{\Lambda}_X)[d_{X/Y}].$$

5.12. — We present one final application of Theorem 5.10.1.

Proposition 5.12.1. — *Proposition 5.12.1 Let X be a smooth k-scheme, and let $\mathcal{M}^\bullet$ be an object of $D^+_u(\mathcal{O}^{\Lambda}_{F^r,X})$ with support contained in the closed subset Z of X. Let U be any open subscheme of Z which (given its reduced induced structure) is smooth as a k-scheme, and let $f : U \to X$ denote the immersion of U into X. Then there is a natural morphism*

$$\mathcal{M}^\bullet \to f_+ f^! \mathcal{M}^\bullet,$$

whose cone is supported on $Z \setminus U$.

Proof. — Let $V = X \setminus (Z \setminus U)$. Then U is closed in V and V is open in X. Let $i : U \to V$ and $j : V \to X$ denote the corresponding immersions, so that $f = ji$. Then by Lemma 4.3.1 there is the natural morphism of adjunction

$$\mathcal{M}^\bullet \to j_+ j^! \mathcal{M}^\bullet. \quad (5.12.2)$$

Now since $\mathcal{M}^\bullet$ is supported on Z, we see that $j^!\mathcal{M}^\bullet$ is supported on $Z \cap V = U$. Thus by Corollary 5.11.3 we see that

$$j^!\mathcal{M}^\bullet \xrightarrow{\sim} i_+ i^! j^! \mathcal{M}^\bullet. \quad (5.12.3)$$

This induces an isomorphism

$$j_+ j^! \mathcal{M}^\bullet \xrightarrow{\sim} j_+ i_+ i^! j^! \mathcal{M}^\bullet = f_+ f^! \mathcal{M}^\bullet.$$

Composing with the morphism (5.12.2) yields a morphism

$$\psi : \mathcal{M}^\bullet \to j_+ i_+ i^! j^! \mathcal{M}^\bullet = f_+ f^! \mathcal{M}^\bullet.$$

Applying $j^!$ to ψ recovers the isomorphism (5.12.3), and so the cone of ψ is supported on $X \setminus V = Z \setminus U$. This proves the proposition. □

5.13. — Let X be a smooth k-scheme. If k' is an extension field of k, and X' is the base-change of X over k', then base-change of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules via k' induces functors

$$k' \otimes_k - : \mu_u(X, \Lambda) \to \mu_u(X', \Lambda)$$

and

$$k' \otimes_k - : D_u(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_u(\mathcal{O}^{\Lambda}_{F^r,X'}).$$

A particular case of interest is that in which k' is a purely inseparable algebraic extension of k.

Proposition 5.13.1. — *Let k' be a (necessarily purely inseparable algebraic) extension of k such $(k')^{q^n}$ is contained in k for some n, let X be a smooth k-scheme, and let $X' = k' \times_k X$ be the base-change of X over k'.*

(*i*) *For any pair of complexes $\mathcal{M}^\bullet$ in $D_u^-(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^\bullet$ in $D_u^+(\mathcal{O}^{\Lambda}_{F^r,X})$, the natural morphism*

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet)$$

of (1.12.5) is an isomorphism.

(*ii*) *The functor $k' \otimes_k - : \mu_u(X, \Lambda) \to \mu_u(X', \Lambda)$ induces an equivalence of categories.*

As a consequence of (i) and (ii) we deduce that

$$k' \otimes_k - : D^b_u(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^b_u(\mathcal{O}^{\Lambda}_{F^r,X'})$$

induces an equivalence of categories.

Proof. — By considering successive changes of ground field and induction on n, we may reduce to the case that $(k')^q \subset k$.

Since $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, we deduce from the commutative diagram of (2.11.2) that the morphism

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(F^{r*}_X \mathcal{M}^\bullet, F^{r*}_X \mathcal{N}^\bullet)$$

is an isomorphism in $D^+(X)$, and similarly for the morphism

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(F^{r*}_{X'}(k' \otimes_k \mathcal{M}^\bullet), F^{r*}_{X'}(k' \otimes_k \mathcal{N}^\bullet)).$$

Now in the commutative diagram of Lemma 2.11.4 we see that the two horizontal arrows are isomorphisms. This is sufficient to conclude that the left-hand vertical arrow is an isomorphism, proving *(i)*.

Since *(i)* shows, in particular, that the functor of *(ii)* is fully faithful, to establish *(ii)* it remains to show that any object $\mathcal{M}'$ of $\mu_u(X', \Lambda)$ is isomorphic to $k' \otimes_k \mathcal{M}$ for some object $\mathcal{M}$ of $\mu_u(X, \Lambda)$. As in the proof of *(i)*, we may assume that $(k')^q \subset k$. Via this inclusion, we obtain a diagram

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k' \end{array}$$

such that the composition of the top two horizontal arrows is $F^r_{X'}$ and the composition of the bottom two horizontal arrows is $F^r_{k'}$. If we define $\mathcal{M} = k \otimes_{k'} \mathcal{M}'$ (where k is regarded as a k'-algebra via the inclusion $(k')^q \subset k$) then $\mathcal{M}$ is an object of $\mu_u(X, \Lambda)$ and $k' \otimes_k \mathcal{M} = F^r_{X'}{}^* \mathcal{M}'$. But the structural morphism $\phi_{\mathcal{M}'}$ provides an isomorphism between the $\mathcal{O}^{\Lambda}_{F^r,X'}$-module $F^r_{X'}{}^* \mathcal{M}'$ and $\mathcal{M}'$. Thus $\mathcal{M}'$ is isomorphic to $k' \otimes_k \mathcal{M}$. □

6. LOCALLY FINITELY GENERATED UNIT $\mathcal{O}^{\Lambda}_{F^r}$-MODULES

6.1. — In [**Lyu**] there is defined a notion of F-finite module, which in our terminology is a unit $\mathcal{O}_{F,X}$-module over an affine scheme X which can be generated by a morphism $\beta : M \to F^*M$ with M a coherent $\mathcal{O}^{\Lambda}_X$-module. Our first aim in this section is to show that a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module admits such a generator if and only if $\mathcal{M}$ is locally finitely generated as an $\mathcal{O}^{\Lambda}_{F^r,X}$-module. We first state a definition, which is an adaptation of a notion of [**Lyu**].

Definition 6.1.1. — A root of a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$ on a k-scheme X is an injective morphism $\beta : M \to F^{r*}_X M$, with M a coherent $\mathcal{O}^{\Lambda}_X$-module, which generates $\mathcal{M}$.

Remark 6.1.2. — By definition, if a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module has a root, then it has a coherent generator. Since coherent $\mathcal{O}^{\Lambda}_X$-modules are (by definition) locally finitely generated as $\mathcal{O}^{\Lambda}_X$-modules, we see from Proposition 5.3.3 that any unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module which has a coherent generator is locally finitely generated as an $\mathcal{O}^{\Lambda}_{F^r,X}$-module. The following result completes this chain of implications, by showing that all three conditions are equivalent:

Theorem 6.1.3. — *Let X be a smooth k-scheme. If $\mathcal{M}$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module which is locally finitely generated (as a left $\mathcal{O}^{\Lambda}_{F^r,X}$-module), then $\mathcal{M}$ has a root.*

Proof. — The quasi-coherent module $\mathcal{M}$ is equal to the direct limit of its $\mathcal{O}^{\Lambda}_X$-coherent submodules. Since $\mathcal{M}$ is locally finitely generated as an $\mathcal{O}^{\Lambda}_{F^r,X}$-module, there exists an $\mathcal{O}^{\Lambda}_X$-coherent submodule $M \subset \mathcal{M}$ which generates $\mathcal{M}$. The morphism $\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M \to \mathcal{M}$ is then surjective.

Pulling back by F^{r*}_X, we see that $F^r_X{}^*(\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M) \to F^r_X{}^*\mathcal{M}$ is surjective. By Corollary 1.3.2,

$$F^r_X{}^*(\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} M) = F^r_X{}^*(\bigoplus_{n=0}^{\infty}(F^{rn}_X)^*M) = \bigoplus_{n=1}^{\infty}(F^{rn}_X)^*M.$$

Thus we have a surjection $\bigoplus_{n=1}^{\infty}(F^{rn}_X)^*M \to F^{r*}_X\mathcal{M}$.

Consider the isomorphism $\phi^{-1}_{\mathcal{M}} : \mathcal{M} \to F^{r*}_X \mathcal{M}$. Since M is coherent, its image under $\phi^{-1}_{\mathcal{M}}$ is contained in the sum of the images of $(F^{rn}_X)^* M$ for finitely many values of n, say $1 \leq n \leq N$.

Let $M' = \sum_{n=0}^{N-1} \phi_{n,\mathcal{M}} F^{rn*}_X M \subset \mathcal{M}$. Then we see that

$$\phi_{\mathcal{M}}(F^{r*}_X M') = \sum_{n=1}^{N} \phi_{n,\mathcal{M}}(F^{rn*}_X M) \supset M + \sum_{n=1}^{N-1} \phi_{n,m}(F^{r*}_X M)$$
$$= \sum_{n=0}^{N-1} \phi_{n,m}(F^{r*}_X M) = M'.$$

(To see the inclusion, note that the inclusion of the second summand is obvious, while the inclusion of the first summand follows from the choice of N.)

Define β to be the restriction of $\phi^{-1}_{\mathcal{M}}$ to M'. We see from the preceding calculation that $\beta : M' \to F^{r*}_X M'$. This morphism generates $\mathcal{M}$ by construction, and is injective with coherent domain, again by construction. This proves the theorem. □

6.2. — The following results generalise some of the more elementary results of [Lyu, §2] (once one makes the translation from locally finitely generated unit $\mathcal{O}_F$-modules to modules admitting a coherent generator).

Lemma 6.2.1. — *If*

$$0 \to \mathcal{M} \to \mathcal{M} \to \mathcal{M}'' \to 0$$

is a short exact sequence of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules on a smooth k-scheme X, then $\mathcal{M}$ is locally finitely generated if and only if both $\mathcal{M}'$ and $\mathcal{M}''$ are locally finitely generated.

Proof. — The proof of this result follows that of the proof of [Lyu, thm. 2.8]. The key ingredient in this argument is the construction of a root of a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$, and this is provided by Theorem 6.1.3. □

Remark 6.2.2. — The conclusion of Lemma 6.2.1 is false without the unit hypothesis. For example, if $A = k[x]$ then the ring $A[F]$ is not left Noetherian (consider the left ideal generated by the elements xF^n for $n \geq 1$), and so submodules of finitely generated left $A[F]$-modules need not be finitely generated in general.

Corollary 6.2.3. — *If*

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4 \to \mathcal{M}_5$$

is an exact sequence of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$ and $\mathcal{M}_5$ are locally finitely generated unit, then $\mathcal{M}_3$ is locally finitely generated unit.

Proof. — It follows from Lemma 5.2 that $\mathcal{M}_3$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module. Let $\mathcal{M}'$ be the cokernel of the morphism $\mathcal{M}_1 \to \mathcal{M}_2$. Then by Lemma 5.2, $\mathcal{M}'$ is a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, and so by the preceding lemma, $\mathcal{M}'$ is locally finitely generated. A similar argument shows that the kernel $\mathcal{M}''$ of $\mathcal{M}_4 \to \mathcal{M}_5$ is locally finitely generated. Since $\mathcal{M}_3$ is an extension of $\mathcal{M}'$ by $\mathcal{M}''$, we see by the preceding lemma that $\mathcal{M}_3$ is locally finitely generated. □

Definition 6.3. — We let $\mu_{lfgu}(X,\Lambda)$ denote the full subcategory of $\mu(X,\Lambda)$ consisting of locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. We let $D^{\bullet}_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^*$ denote the full triangulated subcategory of $D^{\bullet}_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})^*$ consisting of complexes whose cohomology sheaves are locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. (Here $\bullet$ denotes one of $+$, $-$, b, or $\emptyset$, and $*$ denotes one of $\circ$ or $\emptyset$.)

It follows by the preceding corollary that $\mu_{lfgu}(X,\Lambda)$ is a thick subcategory of $\mu(X,\Lambda)$, so that the full subcategory $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^*$ of $D^b(\mathcal{O}^{\Lambda}_{F^r,X})$ is indeed triangulated. It is for the category $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^*$ that we will eventually prove our Riemann-Hilbert correspondence, in the case when Λ is a finite ring.

Lemma 6.4. — *If $\mathcal{M}$ and $\mathcal{N}$ are locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, then for every $i \geq 0$, the $\mathcal{O}^{\Lambda}_X$-module $\mathrm{Tor}^i_{\mathcal{O}^{\Lambda}_X}(\mathcal{M},\mathcal{N})$ has a natural structure of a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module. In particular this holds for the product $\mathcal{M}\otimes_{\mathcal{O}^{\Lambda}_X}\mathcal{N}$.*

Proof. — This follows immediately from Theorem 6.1.3, Lemma 5.5.1 and the fact that for $i \geq 0$, the bifunctor $\mathrm{Tor}^i_{\mathcal{O}^{\Lambda}_X}(-,-)$ takes (pairs of) coherent $\mathcal{O}^{\Lambda}_X$-modules to coherent $\mathcal{O}^{\Lambda}_X$-modules. □

Corollary 6.4.1. — *Let X be a smooth k-scheme. Then the functor*

$$-\overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} - : D^-(\mathcal{O}^{\Lambda}_{F^r,X})\times D^-(\mathcal{O}^{\Lambda}_{F^r,X})\to D^-(\mathcal{O}^{\Lambda}_{F^r,X})$$

restricts to a functor

$$-\overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_X} - : D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})\times D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})\to D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — If $\mathcal{M}^{\bullet}$ and $\mathcal{N}^{\bullet}$ are two objects in $D^-(\mathcal{O}^{\Lambda}_{F^r,X})$ then there is a convergent spectral sequence

$$H^p(H^q(\mathcal{M}^{\bullet})\overset{\mathbb{L}}{\otimes}\mathcal{N}^{\bullet}) \implies H^{p+q}(\mathcal{M}^{\bullet}\overset{\mathbb{L}}{\otimes}\mathcal{N}^{\bullet}).$$

Thus Corollary 6.2.3 shows that it suffices to prove the proposition in the case that $\mathcal{M}^{\bullet}$ is a single object $\mathcal{M}$ of $\mu_{lfgu}(X,\Lambda)$. Interchanging the roles of $\mathcal{M}^{\bullet}$ and $\mathcal{N}^{\bullet}$ allows us to similarly assume that $\mathcal{N}^{\bullet}$ is a single object $\mathcal{N}$ of $\mu_{lfgu}(X,\Lambda)$. The result now follows from Lemma 6.4. □

Proposition 6.5. — *Let X be a smooth k-scheme, and let Λ' be a Noetherian Λ-algebra. If $\mathcal{M}^{\bullet}$ is in $D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ then $\mathcal{M}^{\bullet}\overset{\mathbb{L}}{\otimes}_{\Lambda}\Lambda'$ is in $D^-_{lfgu}(\mathcal{O}^{\Lambda'}_{F^r,X})$.*

Proof. — First let $\mathcal{M}$ be any locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, and $\beta : M\to F^{r*}_X M$ a root for $\mathcal{M}$. Then $\mathrm{Tor}^i_{\Lambda}(\mathcal{M},\Lambda')$ has a canonical structure of a unit $\mathcal{O}^{\Lambda'}_{F^r,X}$-module, generated by

$$\mathrm{Tor}^i_{\Lambda}(M,\Lambda')\overset{\mathrm{Tor}^i_{\Lambda}(\beta,\Lambda')}{\longrightarrow}\mathrm{Tor}^i_{\Lambda}(F^{r*}_X M,\Lambda') = F^{r*}_X\mathrm{Tor}^i_{\Lambda}(M,\Lambda').$$

Now, since M is a coherent $\mathcal{O}^{\Lambda}_X$-module, taking a local resolution of M by finite flat $\mathcal{O}^{\Lambda}_X$-modules shows that $\mathrm{Tor}^i_{\Lambda}(M,\Lambda')$ is a finitely generated $\mathcal{O}^{\Lambda'}_X$-module (since Λ' is Noetherian!). This shows that $\mathrm{Tor}^i_{\Lambda}(\mathcal{M},\Lambda')$ is locally finitely generated.

Now consider $\mathcal{M}^{\bullet}$ as in the proposition. We have a convergent Künneth spectral sequence

$$\mathrm{Tor}^p_{\Lambda}(H^q(\mathcal{M}^{\bullet}),\Lambda') \implies H^{q-p}(\mathcal{M}^{\bullet} \overset{\mathbb{L}}{\otimes}_{\Lambda} \Lambda').$$

We have already seen that the expression on the left is a locally finitely generated $\mathcal{O}^{\Lambda'}_X$-module, so Corollary 6.2.3 implies that the expression on the right is also. This completes the proof of the proposition. □

Proposition 6.6. — *Let r' be a multiple of r, let $q' = p^{r'}$, assume that $\mathbb{F}_{q'} \subset k$, and write $\Lambda' = \mathbb{F}_{q'} \otimes_{\mathbb{F}_q} \Lambda$. Then for any smooth k-scheme X, restriction and induction (as defined in section 1.14) induce functors*

$$\mathrm{Res}^{q'}_q : D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_{lfgu}(\mathcal{O}^{\Lambda'}_{F^{r'},X})$$

and

$$\mathrm{Ind}^{q}_{q'} : D_{lfgu}(\mathcal{O}^{\Lambda'}_{F^{r'},X}) \to D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — Since restriction (respectively induction) has zero cohomological amplitude, it suffices to establish the claim of the proposition in the case of a single locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module (respectively $\mathcal{O}^{\Lambda'}_{F^{r'},X}$-module) $\mathcal{M}$. In this case, the result follows by observing that the prescription of section (5.7.1) shows how to construct a coherent generator for $\mathrm{Res}^{q'}_q\mathcal{M}$ (respectively $\mathrm{Ind}^q_{q'}\mathcal{M}$) from a coherent generator for $\mathcal{M}$. □

Proposition 6.7. — *If $f : Y \to X$ is a morphism of smooth k-schemes, then the functor $f^! : D(\mathcal{O}^{\Lambda}_{F^r,X}) \to D(\mathcal{O}^{\Lambda}_{F^r,Y})$ restricts to a functor (which we denote by the same symbol)*

$$f^! : D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y}).$$

Proof. — Let $\mathcal{M}^{\bullet}$ be a complex in $D(\mathcal{O}^{\Lambda}_{F^r,X})$ whose cohomology sheaves are locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. We must show that the same is true of the cohomology sheaves of $f^!\mathcal{M}^{\bullet}$.

If we consider the proof of part *(i)* of Theorem 5.8, and note that Corollary 6.2.3 provides the analogue in the context of locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules of Lemma 5.2, we see that it is enough to prove the proposition in the case of a single locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$. We must show that $f^!\mathcal{M}$ has locally finitely generated cohomology sheaves.

Since this can be verified locally, after replacing X by an open affine subset and Y by the inverse image under f of this subset, we may assume that X is affine.

By Theorem 6.1.3 we may choose a generator $\beta : M \to F^{r\,*}_X M$ of $\mathcal{M}$ with M a coherent $\mathcal{O}^{\Lambda}_X$-module, and so applying the construction of (5.3.5), we choose a free resolution $P^{\bullet}$ of M. Since M is coherent, we may choose each P^j to be free of finite rank. We then choose a lift of β to a morphism $\beta^\bullet : P^\bullet \to F_X^{r*}P^\bullet$ and construct the double complex

$$\mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} P^\bullet \xrightarrow{1-\beta^{\bullet\prime}} \mathcal{O}^{\Lambda}_{F^r,X} \otimes_{\mathcal{O}^{\Lambda}_X} P^\bullet, \tag{6.7.1}$$

which as in the proof of 5.8 *(i)* gives rise to a spectral sequence computing the cohomology sheaves of $f^!\mathcal{M}$. The E_1 terms of the spectral sequence are the horizontal cohomology groups of the double complex obtained from (6.7.1) by pulling back by f, and tensoring by $\mathcal{O}^{\Lambda}_{F^r,Y} \otimes_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}}$. These are locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules, since the P^j are of finite rank. By repeated applications of Corollary 6.2.3 we see that the E_∞ terms are again locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules, and so finally the cohomology sheaves of $f^!\mathcal{M}$ are locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-modules. This completes the proof of the proposition. □

6.8. — The analogous result for f_+ is also true, but the proof is more involved.

Proposition 6.8.1. — *Let $f : Y \to X$ be an open immersion of smooth k-schemes such that the complement of the image of f is a divisor on X. If $\mathcal{M}$ is a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module, then $H^0(f_+)\mathcal{M} = f_+\mathcal{M}$ is a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module.*

Proof. — By part *(ii)* of Theorem 5.8 we see that $f_+\mathcal{M}$ is an object of $D^b_u(\mathcal{O}^{\Lambda}_{F^r,X})$. However, since f is the open immersion of the complement of a divisor, the higher derived direct images of f vanish when applied to quasi-coherent $\mathcal{O}^{\Lambda}_Y$-modules, and so by Lemma 4.3.1

$$f_+\mathcal{M} = Rf_*\mathcal{M} = H^0(Rf_*\mathcal{M}) = f_*\mathcal{M}$$

is in fact an object of $\mu(X, \Lambda)$.

Let $D = X \setminus Y$ be the complement of Y, and let $\mathcal{O}^{\Lambda}_X(D)$ denote the associated invertible sheaf. Let

$$\beta : M \to F_Y^{r\,*}M$$

be a generator of $\mathcal{M}$, with M a coherent $\mathcal{O}^{\Lambda}_Y$-module. Then applying f_* we obtain a map

$$f_*(\beta) : f_*M \to f_*F_Y^{r\,*}M \xrightarrow{\sim} F_X^{r\,*}f_*M.$$

(The isomorphism holds by flat base-change.) This morphism generates $f_*\mathcal{M}$, but f_*M is not coherent. To find a coherent generator, we argue as in the proof of [Lyu, prop. 2.9(b)].

Let N be a coherent submodule of f_*M such that $f^*N = M$. Then

$$f_*M = \bigcup_{n=0}^{\infty} N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes n},$$

while

$$F_X^{r\,*}(f_*M) = \bigcup_{n=0}^{\infty} F_X^{r\,*}N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes n}.$$

Since N is coherent, there is an integer n of the form $n = (q-1)m$ for some positive integer m such that $f_*(\beta)(N) \subset F_X^{r\,*}N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes n}$. Let us denote by γ' the morphism $\gamma' : N \to {F^r_X}^* N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes n}$ obtained by restricting $f_*(\beta)$ to N and by γ the morphism

$$\gamma = \gamma' \otimes \mathrm{id}_{\mathcal{O}_X(D)^{\otimes m}} : N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes m} \\ \to {F^r_X}^* N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes qm} = {F^r_X}^*(N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes m}).$$

Then γ is the generating morphism for a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, which is immediately seen to be $f_*\mathcal{M}$, since m was chosen to be positive. Since $N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes m}$ is coherent we see that $f_*\mathcal{M}$ is locally finitely generated, proving the proposition. □

Proposition 6.8.2. — *If $f : Y \to X$ is a proper morphism of smooth k-schemes, then the functor $f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$ restricts to a functor (which we denote by the same symbol)*

$$f_+ : D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — The proof of this follows the proof of part *(ii)* of Theorem 5.8. We note that Corollary 6.2.3 provides the analogue in the case of locally finitely generated unit modules of Lemma 5.2. As in that proof, we may reduce to the case of a single locally finitely generated unit module $\mathcal{M}$, generated by $\beta : M \to {F^r_Y}^* M$, with M a coherent $\mathcal{O}^{\Lambda}_Y$-module.

Then, in the notation of that proof, we see that $H^j f_+ \mathcal{M}$ is the unit module generated by

$$\gamma^j : R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M) \to {F^r_X}^* R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M).$$

Since f is proper, and $\omega_{Y/X} \otimes_{\mathcal{O}_Y} M$ is coherent, we see that $R^j f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M)$ is coherent. Thus indeed $H^j f_+ \mathcal{M}$ is a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module, and the proposition is proved. □

Corollary 6.8.3. — *If $f : Y \to X$ is an immersion of smooth k-schemes, then the functor $f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$ restricts to a functor (which we denote by the same symbol)*

$$f_+ : D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — We factor $f = f_1 f_2$, with f_1 an open immersion and f_2 a closed immersion. To prove the corollary, it suffices to prove it for open and closed immersions separately.

Suppose first that f is an open immersion, and that $\mathcal{M}$ is a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module on X. We have to show that $f_+\mathcal{M}$ has locally finitely generated unit cohomology sheaves.

We can check this by restricting $f_+ M$ to affine open subschemes of X, and so we may assume that $X = \operatorname{Spec} A$ is affine. Let Y be the complement of the closed subset $V(a_1, \ldots, a_s)$ cut out by the elements $a_1, \ldots, a_s$ of A.

By Lemma 4.3.1 we see that $f_+\mathcal{M} = Rf_*\mathcal{M}$. We may compute $Rf_*\mathcal{M}$ by using the Čech complex associated to the covering $\{X_{a_i}\}_{1 \le i \le s}$ of Y. Then $Rf_*\mathcal{M}$ is represented by a complex which in each degree is a direct sum of terms of the form $j_*\mathcal{M}|_U$, where U is the complement of a divisor in X, and $j : U \to X$ is the natural inclusion. By Proposition 6.8.1, each term in this complex is a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module, hence its cohomology sheaves have the same property.

Now by the standard spectral sequence argument, together with Corollary 6.2.3, we see that for any complex $\mathcal{M}^\bullet$ in $D^b_{lfgu}(Y)$, its push-forward $f_+\mathcal{M}$ is a complex in $D^b_{lfgu}(X)$. This completes the proof of the proposition for an open immersion.

Suppose that f is a closed immersion. Then in particular f is proper, and so this case follows from Proposition 6.8.2. □

Corollary 6.8.4. — *If $f : Y \to X$ is a morphism of smooth k-schemes, then the functor $f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$ restricts to a functor (which we denote by the same symbol)*

$$f_+ : D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y}) \to D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}).$$

Proof. — Suppose first that Y is quasi-projective, and choose an immersion $g : Y \to \mathbb{P}^n_k$, for some positive integer n. Then we may factor f as

$$Y \xrightarrow{\Gamma_f} X \times Y \xrightarrow{\mathrm{id}_X \times g} X \times \mathbb{P}^n_k \xrightarrow{p_1} X,$$

where Γ_f is the graph of f and p_1 is the projection onto the first factor.

It suffices to prove the corollary for each of the morphisms of this factorisation. Since Γ_f and $\mathrm{id}_X \times g$ are both immersions, the result follows for these two morphisms by Corollary 6.8.3. The result follows for the proper morphism p_1 by Theorem 6.8.2. This completes the proof of the corollary if Y is quasi-projective.

For the general case, note that since f_+ is of finite cohomological amplitude, it suffices to prove the corollary for objects of $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y})$. If $\mathcal{M}^\bullet$ is such a bounded complex, then by Theorem 5.8 *(ii)* we know that $f_+\mathcal{M}^\bullet$ is a bounded complex of unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules. To show that $f_+\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ we proceed by induction on the support Z of $\mathcal{M}^\bullet$. Choose a dense open affine $U \subset Y$ such that $U \cap Z$ is dense in Z, and denote by $j : U \to Y$ the natural inclusion. As the cone of the adjunction morphism $\mathcal{M}^\bullet \to j_+j^{-1}\mathcal{M}^\bullet$ is supported on $Z \setminus (Z \cap U)$, it is enough to prove the result for $j_+j^{-1}\mathcal{M}^\bullet$, by Corollary 6.2.3. Using Proposition 3.7, we see that we may replace Y by U, and $\mathcal{M}$ by $j^{-1}\mathcal{M}$. As U is affine, the result now follows from the quasi-projective case. □

6.9. — Let X be a smooth k-scheme. In this section we will show that a locally finitely generated unit $\mathcal{O}_{F^r,X}$-module has a particularly nice form on a dense open subset of X. This will be important later for calculations.

Definition 6.9.1. — We call a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module which is a coherent locally projective $\mathcal{O}^{\Lambda}_X$-module a unit (Λ, F^r)-crystal, or if $\Lambda = \mathbb{F}_q$, simply a unit F^r-crystal.

Lemma 6.9.2. — *If $f : Y \to X$ is a morphism of smooth k-schemes and $\mathcal{M}$ is a unit (Λ, F^r)-crystal on X, then $f^!\mathcal{M}[d_{X/Y}]$ is a unit (Λ, F^r)-crystal on Y.*

Proof. — If $\mathcal{M}$ is a coherent locally projective $\mathcal{O}^{\Lambda}_X$-module then

$$f^!\mathcal{M}[d_{X/Y}] \xrightarrow{\sim} \mathcal{O}_Y \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{M} \xrightarrow{\sim} \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{M}$$

is a coherent locally projective $\mathcal{O}^{\Lambda}_Y$-module. This proves the lemma. □

Proposition 6.9.3. — *If $\mathcal{M}$ is a unit $\mathcal{O}_{F^r,X}$-module on X which is coherent as an $\mathcal{O}_X$-module, then $\mathcal{M}$ is a unit F^r-crystal.*

Proof. — Suppose that $\mathcal{M}$ is a unit $\mathcal{O}_{F^r,X}$-module which is coherent as an $\mathcal{O}_X$-module. It suffices to show that the stalk of $\mathcal{M}$ at x is free for any point x of X.

Let $\mathcal{O}_{X,x}$ be the stalk of $\mathcal{O}_X$ at x, a local ring with maximal ideal $\mathfrak{m}_x$ and residue field $\kappa(x)$. Choose a surjection

$$\mathcal{O}_{X,x}^n \to \mathcal{M}_x \tag{6.9.4}$$

with n minimal (so that n equals the rank of the $\kappa(x)$-vector space $\mathcal{M}_x/\mathfrak{m}_x$); then this surjection becomes an isomorphism after reducing modulo $\mathfrak{m}_x$. Thus if we let R denote the kernel of (6.9.4) then $R \subset \mathfrak{m}_x \mathcal{O}_{X,x}^n$.

Applying F_X^{r*} to (6.9.4) yields a surjection

$$\mathcal{O}_{X,x}^n = F_X^{r*}\mathcal{O}_{X,x}^n \to F_X^{r*}\mathcal{M}_x \tag{6.9.5}$$

whose kernel is $\mathcal{O}_{X,x}R^q$ (that is, the module generated by the vectors obtained by raising the elements of R to the q^{th} power coordinate-wise).

We may find a morphism $A : \mathcal{O}_{X,x}^n \to \mathcal{O}_{X,x}^n$ (which is simply an $n \times n$ matrix with coefficients in $\mathcal{O}_{X,x}$) which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}_{X,x}^n & \xrightarrow{(6.9.5)} & F_X^{r*}\mathcal{M}_x \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \phi_{\mathcal{M}}} \\ \mathcal{O}_{X,x}^n & \xrightarrow{(6.9.4)} & \mathcal{M}_x. \end{array}$$

Upon reducing this diagram modulo $\mathfrak{m}_x$, the two horizontal arrows become isomorphisms. Since the right vertical arrow is also an isomorphism we find that A becomes an isomorphism when reduced modulo $\mathfrak{m}_x$. Thus the determinant of A is a unit modulo $\mathfrak{m}_x$, and so is a unit in the local ring $\mathcal{O}_{X,x}$, showing that A is an isomorphism.

We conclude that $R = A\mathcal{O}_{X,x}R^q$. Proceeding by induction we see that

$$R = A^{1+q+\cdots+q^m}\mathcal{O}_{X,x}R^{q^{m+1}} \subset \mathfrak{m}_x^{q^{m+1}}\mathcal{O}_{X,x}^n$$

for all integers m. This implies that $R = 0$, hence that (6.9.4) is an isomorphism, and so we conclude that $\mathcal{M}_x$ is indeed a free $\mathcal{O}_{X,x}$-module. □

Proposition 6.9.6. — *Assume that k is a perfect field. Let $\mathcal{M}$ be a locally finitely generated, unit $\mathcal{O}_{F^r,X}$-module. Let $Z \subset X$ denote a closed subset containing the support of $\mathcal{M}$. Then there exists a smooth, dense open subset $U \subset Z$ such that if $f : U \to X$ denotes the natural inclusion, then $f^!\mathcal{M}$ has $\mathcal{O}_X$-coherent cohomology.*

Proof. — Since k is perfect, we may find a dense open subscheme V of Z (given its induced reduced structure) which is smooth as a k-scheme. Let $W = X \setminus (Z \setminus V)$. Then W is an open subscheme of X and V a closed subset of W. Let g be the open immersion of W in X, and i the closed immersion of V in W. Then $g^!\mathcal{M}$ is simply the sheaf-theoretic restriction $g^{-1}\mathcal{M}$ of $\mathcal{M}$ to W, which by assumption is supported on V. Thus by Theorem 5.10.1 together with Proposition 6.7 we see that $i^!g^!\mathcal{M} = H^0(i^!g^!\mathcal{M})$ is a locally finitely generated unit $\mathcal{O}_{F^r,V}$-module, which thus admits a root, by Theorem 6.1.3. That is, we may find a generator $\beta : N \to F_V^{r*} N$ with N coherent on V and β injective. Then we may find a dense open subscheme U of V on which N is locally free of finite rank and β is an isomorphism. If j is the open immersion of U in V then we see that $j^! i^! g^! \mathcal{M} = (gij)^! \mathcal{M}$ is coherent on U. □

6.10. — Let X be a smooth k-scheme. If k' is an extension field of k, and X' is the base-change of X over k', then base-change of $\mathcal{O}^{\Lambda}_{F^r,X}$-modules via k' over k induces functors

$$k' \otimes_k - : \mu_{lfgu}(X, \Lambda) \to \mu_{lfgu}(X', \Lambda)$$

and

$$k' \otimes_k - : D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X'}).$$

Again, a case of particular interest is that in which k' is a purely inseparable algebraic extension of k.

Proposition 6.10.1. — *Let k' be a purely inseparable algebraic extension of k, let X be a smooth k-scheme, and let $X' = k' \times_k X$ be the base-change of X over k'.*

(*i*) *For any two complexes $\mathcal{M}^\bullet$ in $D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ and $\mathcal{N}^\bullet$ in $D^+_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$, the natural morphism*

$$\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{M}^\bullet, k' \otimes_k \mathcal{N}^\bullet)$$

of (1.12.5) is an isomorphism.

(*ii*) *The functor $k' \otimes_k - : \mu_{lfgu}(X, \Lambda) \to \mu_{lfgu}(X', \Lambda)$ induces an equivalence of categories.*

As a consequence of (i) and (ii) we deduce that

$$k' \otimes_k - : D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X}) \to D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X'})$$

induces an equivalence of categories.

The statement of this proposition is similar to that of Proposition 5.13.1, except that k' is allowed to be an *arbitrary* inseparable extension of k.

Proof. — To prove *(i)*, note that (since $\underline{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X}}(-,-)$ is right exact in its first variable and left exact in its second variable) it suffices to check the claim when $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are bounded complexes, and then an argument using truncation and induction on the length of a complex allows us to restrict to the case in which each of $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are objects $\mathcal{M}$ and $\mathcal{N}$ of $\mu_{lfgu}(X, \Lambda)$. We may also work locally on X, and so we may assume that X is affine.

Then, via the construction of (5.3.5) applied to a coherent generator of $\mathcal{M}$, we may find a resolution $\mathcal{P}^\bullet$ of $\mathcal{M}$ by left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules which are free of finite rank. We may use this resolution to rewrite the natural transformation of *(i)* as the map

$$\begin{aligned}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{M}, \mathcal{N}) &= \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{P}^\bullet, \mathcal{N}) \\ &\to \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{P}^\bullet, k' \otimes_k \mathcal{N}) = \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{M}, k' \otimes_k \mathcal{N}).\end{aligned}$$

(since $k' \otimes_k \mathcal{P}^{\bullet}$ is clearly a locally free resolution of $k' \otimes_k \mathcal{M}$; compare remark 1.12.3). Thus we must show that the morphism of complexes

$$\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{P}^{\bullet}, \mathcal{N}) \to \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{P}^{\bullet}, k' \otimes_k \mathcal{N})$$

is a quasi-isomorphism. Since both source and target of this morphism are bounded below, it suffices to show that the corresponding map of truncations

$$\tau_{\leq i}\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{P}^{\bullet}, \mathcal{N}) \to \tau_{\leq i}\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{P}^{\bullet}, k' \otimes_k \mathcal{N}) \tag{6.10.2}$$

is an isomorphism for each integer i. This it is what we will show.

Let k_1 denote a finite subextension of k in k', and let X_1 denote the base-change $X_1 = k_1 \times_k X$. By Proposition 5.13.1, the morphism

$$\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{P}^{\bullet}, \mathcal{N}) \to \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_1}}(k_1 \otimes_k \mathcal{P}^{\bullet}, k_1 \otimes_k \mathcal{N})$$

is an isomorphism, and so the natural map

$$\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{P}^{\bullet}, \mathcal{N}) \to \varinjlim_{k_1} \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_1}}(k_1 \otimes_k \mathcal{P}^{\bullet}, k_1 \otimes_k \mathcal{N})$$

is an isomorphism, where the direct limit is taken over all finite subextensions k_1. ¿From this we obtain the following sequence of isomorphisms:

$$\begin{aligned}
\tau_{\leq i}\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X}}(\mathcal{P}^{\bullet}, \mathcal{N}) &\overset{(1)}{\xrightarrow{\sim}} \tau_{\leq i}\varinjlim_{k_1} \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_1}}(k_1 \otimes_k \mathcal{P}^{\bullet}, k_1 \otimes_k \mathcal{N}) \\
&\overset{(2)}{\xrightarrow{\sim}} \varinjlim_{k_1} \tau_{\leq i}\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_1}}(k_1 \otimes_k \mathcal{P}^{\bullet}, k_1 \otimes_k \mathcal{N}) \\
&\xrightarrow{\sim} \varinjlim_{k_1} \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_1}}(k_1 \otimes_k \tau_{\geq -i}\mathcal{P}^{\bullet}, k_1 \otimes_k \mathcal{N}) \\
&\overset{(3)}{\xrightarrow{\sim}} \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \tau_{\geq -i}\mathcal{P}^{\bullet}, k' \otimes_k \mathcal{N}) \\
&\xrightarrow{\sim} \tau_{\geq i}\underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X'}}(k' \otimes_k \mathcal{P}^{\bullet}, \mathcal{N}).
\end{aligned}$$

(Here, isomorphism (1) is obtained by truncating, isomorphism (2) follows from the exactness of direct limits, and isomorphism (3) follows from the fact that, since $\mathcal{P}^{\bullet}$ is a bounded above complex of free left $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, $\tau_{\geq -i}\mathcal{P}^{\bullet}$ is a bounded complex of finitely presented $\mathcal{O}_{F^r,X}$-modules.) The composition of these isomorphisms is the morphism (6.10.1), which is now seen to be an isomorphism.

The proof of part *(ii)* uses a similar argument. Since *(i)* shows that the functor in question is fully faithful, we need only show that any object $\mathcal{M}'$ of $\mu_{lfgu}(X', \Lambda)$ is isomorphic to $k' \otimes_k \mathcal{M}$ for some object $\mathcal{M}$ of $\mu_{lfgu}(X, \Lambda)$. Since X is of finite type and $\mathcal{M}'$ is determined by a generator with a coherent domain, by the same direct limit argument as used above, we may first reduce to the case in which k' is finite over k, and the result then follows from 5.13.1 *(ii)*. □

7. $\mathcal{O}^{\Lambda}_{F^r}$-MODULES ON THE ÉTALE SITE

7.1. — We let $X_{\acute{e}t}$ denote the small étale site of X, and we let $\pi_X : X_{\acute{e}t} \to X$ denote the natural morphism of sites. (In this context, we write X without a subscript to indicate the Zariski site of X.) As usual, $\mathcal{O}_{X_{\acute{e}t}}$ will denote the structure sheaf of $X_{\acute{e}t}$, defined by $\mathcal{O}_{X_{\acute{e}t}}(U) = \mathcal{O}_U(U)$. The morphism π_X is naturally a morphism of locally ringed sites.

We may use π_X to pull-back the sheaf $\mathcal{O}^{\Lambda}_{F^r,X}$, and for any U étale over X, there is a natural isomorphism $\pi_X^*\mathcal{O}^{\Lambda}_{F^r,X}(U) = \mathcal{O}^{\Lambda}_{F^r,U}(U)$. Thus $\pi_X^*\mathcal{O}^{\Lambda}_{F^r,X}$ is naturally a sheaf of rings on $X_{\acute{e}t}$. (Note that this is not a formal consequence of it being the ringed site pull-back of a sheaf of rings, because $\mathcal{O}^{\Lambda}_X$ is not in the center of $\mathcal{O}^{\Lambda}_{F^r,X}$.)

Definition 7.1.1. — We define $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} = \pi_X^*\mathcal{O}^{\Lambda}_{F^r,X}$, with its natural structure as a sheaf of rings.

7.2. — The maps

$$\mathcal{O}^{\Lambda}_{F^r,U}(U) \to \mathcal{H}om_{\Lambda}(\mathcal{O}^{\Lambda}_U(U), \mathcal{O}^{\Lambda}_U(U))$$

induce a map

$$\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} \to \mathcal{H}om_{\Lambda}(\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}).$$

The following lemma shows that $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$ always acts faithfully on $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$.

Lemma 7.2.1. — *The map*

$$\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} \to \mathcal{H}om_{\Lambda}(\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}})$$

is injective.

Proof. — Let $\bar{k}$ denote the separable closure of k. It suffices to show that if $X = \mathrm{Spec}\,(A)$, then the map

$$\Lambda \otimes_{\mathbb{F}_q} A\,[F^r] \to \mathrm{Hom}_{\Lambda}(\Lambda \otimes_{\mathbb{F}_q} A \otimes_k \bar{k}, \Lambda \otimes_{\mathbb{F}_q} A \otimes_k \bar{k}) \xrightarrow{\sim} \Lambda \otimes_{\mathbb{F}_q} \mathrm{Hom}_{\mathbb{F}_q}(A \otimes_k \bar{k}, A \otimes_k \bar{k})$$

is injective, and for this it suffices to show that the map

$$A\,[F^r] \to \mathrm{Hom}_{\mathbb{F}_q}(\bar{k}, A \otimes_k \bar{k}) = A \otimes_k \mathrm{Hom}_{\mathbb{F}_q}(\bar{k}, \bar{k})$$

is injective. But this map is obtained by tensoring the injective map $k[F^r] \to \mathrm{Hom}_{\mathbb{F}_q}(\bar{k}, \bar{k})$ on the left with A over k. □

7.3. — Just as we have considered $\mathcal{O}^{\Lambda}_{F^r,X}$-modules, we will now also have to consider $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules. In particular we may consider sheaves of left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules on $X_{\acute{e}t}$ which are quasi-coherent as $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-modules. We will refer to such a sheaf as a quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-module, and let $\mu(X_{\acute{e}t}, \Lambda)$ denote the category of such sheaves.

If $\mathcal{M}$ is a quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, then $\pi_X^*\mathcal{M}$ is naturally a quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-module, and the natural map $\mathcal{M} \to \pi_{X*}\pi_X^*\mathcal{M}$ is an isomorphism. Conversely, the theory of étale descent shows that if $\mathcal{N}$ is a quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-module, then $\pi_{X*}\mathcal{N}$ is a quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X}$-module, and the natural map $\pi_X^*\pi_{X*}\mathcal{N} \to \mathcal{N}$ is an isomorphism. Thus the functors π_X^* and π_{X*} provide an equivalence of categories between $\mu(X, \Lambda)$ and $\mu(X_{\acute{e}t}, \Lambda)$.

We let $D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})$ denote the derived category of complexes of left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules satisfying the boundedness condition $\bullet$ (equal to one of $+$, $-$, b, or $\emptyset$). For any triangulated subcategory D of $D^b(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})$ we denote by D° the full triangulated subcategory consisting of complexes which have finite Tor-dimension when considered as complexes of $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-modules. We let $D^{\bullet}_{qc}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})$ denote the triangulated subcategory of $D^{\bullet}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})$ consisting of complexes whose cohomology sheaves are quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules. We let $D^b(\mu(X_{\acute{e}t}, \Lambda))$ denote the derived category of finite length complexes of quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules.

Theorem 7.3.1. — *The morphism $D^b(\mu(X_{\acute{e}t}, \Lambda)) \to D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})$ is an equivalence of triangulated categories.*

Proof. — This follows from Bernstein's theorem [**Bo**, VI 2.10]. (That Bernstein's theorem holds in the étale case is a consequence of étale descent. More precisely, the crux of the proof of Bernstein's theorem is that quasi-coherent cohomology vanishes on an affine scheme, and étale descent shows that quasi-coherent cohomology computed on the étale site agrees with quasi-coherent cohomology computed on the Zariski site.) □

7.3.2. — Since étale morphisms are flat, the ringed site pull-back π_X^* from X to $X_{\acute{e}t}$ is exact, and so we obtain a commutative diagram of morphisms of triangulated categories

$$\begin{array}{ccc} D(\mu(X,\Lambda)) & \longrightarrow & D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X}) \\ {\scriptstyle \pi_X^*}\big\downarrow & & {\scriptstyle \pi_X^*}\big\downarrow \\ D(\mu(X_{\acute{e}t},\Lambda)) & \longrightarrow & D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}) \end{array}$$

It is an immediate consequence of étale descent that the left-hand vertical arrow is an equivalence of categories. It is a slightly less obvious consequence of étale descent that the same is true for the right hand vertical arrow. (If we had restricted our attention to bounded derived categories, then this would follow from Theorems 1.6.1 and 7.3.1.)

We first note that the functor $R\pi_{X*} : D^+(\mathcal{O}^{\Lambda}_{F^r,X_{ét}}) \to D^+(\mathcal{O}^{\Lambda}_{F^r,X})$ has finite cohomological amplitude, and so extends to a functor $R\pi_{X*} : D(\mathcal{O}^{\Lambda}_{F^r,X_{ét}}) \to D(\mathcal{O}^{\Lambda}_{F^r,X})$.

Proposition 7.3.3. — *The functor $R\pi_{X*}$ restricts to a functor*

$$R\pi_{X*} : D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}}) \to D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$$

which has zero cohomological amplitude, and the natural transformations $\pi_X^* \circ R\pi_{X*} \to$ id and id $\to R\pi_{X*} \circ \pi_X^*$ are both isomorphisms. Thus $R\pi_{X*}$ is an equivalence of categories, with quasi-inverse given by π_X^*. If Λ' is a Noetherian Λ-algebra, then this equivalence of categories is compatible with the functors $\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} -$ on its source and target.

Proof. — Since $R\pi_{X*}$ has finite cohomological amplitude, one immediately reduces to checking the corresponding statements for bounded complexes. If $\mathcal{M}^\bullet$ is a complex in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$ then by Theorem 7.3.1, we may assume that $\mathcal{M}^\bullet$ is a complex in $D^b(\mu(X_{ét}, \Lambda))$. The fact that the étale cohomology of a quasi-coherent sheaf on an affine scheme vanishes shows that the higher derived direct images of π_X vanish when applied to quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-modules, and thus when applied to complexes of quasi-coherent left $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-modules. Thus $R\pi_{X*}\mathcal{M}^\bullet \xrightarrow{\sim} \pi_{X*}\mathcal{M}^\bullet$ is a bounded complex of quasi-coherent $\mathcal{O}^{\Lambda}_{F^r,X}$-modules and

$$\pi_X^* R\pi_{X*}\mathcal{M}^\bullet \xrightarrow{\sim} \pi_X^*\pi_{X*}\mathcal{M}^\bullet \xrightarrow{\sim} \mathcal{M}^\bullet,$$

the second natural isomorphism holding by étale descent. The composite of these morphisms is the adjunction morphism

$$\pi_X^* R\pi_{X*}\mathcal{M}^\bullet \to \mathcal{M}^\bullet,$$

and so we see that this morphism is indeed a natural isomorphism.

A similar argument shows that for any complex $\mathcal{M}^\bullet$ in $D^b_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$ the adjunction morphism $\mathcal{M}^\bullet \to R\pi_{X*}\pi_X^*\mathcal{M}^\bullet$ is an isomorphism. Thus we see that $R\pi_{X*}$ is an equivalence, as claimed.

The compatibility of this equivalence with $\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} -$ follows from (an obvious variation of) Proposition B.1.3. □

7.3.4. — Let us remark that all the definitions and results of section 1 carry over directly from the Zariski to the étale setting, and we will feel free to use them in either setting from now on.

7.4. — We now briefly discuss pull-back and push-forwards of $\mathcal{O}^{\Lambda}_{F^r}$-modules in the context of the étale site.

Definition 7.4.1. — If $f : Y \to X$ is a morphism of smooth k-schemes then we define the $(\mathcal{O}^{\Lambda}_{F^r,Y_{ét}}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$-bimodule $\mathcal{O}^{\Lambda}_{F^r,Y_{ét}\to X_{ét}}$ to be

$$\mathcal{O}^{\Lambda}_{F^r,Y_{ét}\to X_{ét}} = \pi_Y^*\mathcal{O}^{\Lambda}_{F^r,Y\to X},$$

where the bimodule structure is induced by the $(\mathcal{O}^{\Lambda}_{F^r,Y}, f^{-1}\mathcal{O}^{\Lambda}_{F^r,X})$-bimodule structure on $\mathcal{O}^{\Lambda}_{F^r,Y\to X}$.

Definition 7.4.2. — Let $f : Y \to X$ be a morphism of smooth connected k-schemes. We define $f^! : D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}}) \to D_{qc}(\mathcal{O}^{\Lambda}_{F^r,Y_{ét}})$ as follows:

$$f^!\mathcal{M}^\bullet = \mathcal{O}^{\Lambda}_{F^r,Y_{ét}\to X_{ét}} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X_{ét}}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}].$$

This functor is well-defined because $\mathcal{O}^{\Lambda}_{F^r,Y_{ét}\to X_{ét}}$ is of finite Tor-dimension as a right $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-module (being the pull-back by π_Y of $\mathcal{O}^{\Lambda}_{F^r,Y\to X}$, which is of finite Tor-dimension as a right $f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}$-module).

This functor satisfies analogues of all the results of section 2. We also have the following result, which shows that it is compatible with pull-back of Zariski $\mathcal{O}^{\Lambda}_{F^r}$-modules via étale descent.

Proposition 7.4.3. — *Let $f : Y \to X$ be a morphism of smooth k-schemes, and let $\pi_X : X_{ét} \to X$ and $\pi_Y : Y_{ét} \to Y$ denote the natural morphisms of sites. Then if we restrict our attention to complexes with quasi-coherent cohomology sheaves, there are natural isomorphisms $R\pi_{Y*}f^! \xrightarrow{\sim} f^!R\pi_{X*}$ and $\pi_Y^*f^! \xrightarrow{\sim} f^!\pi_X^*$.*

Proof. — Since $R\pi_{X*}$ and π_X^* (respectively $R\pi_{Y*}$ and π_Y^*) are quasi-inverse, it is enough to construct the second isomorphism. If $\mathcal{M}^\bullet$ is a complex in $D_{qc}(\mathcal{O}^{\Lambda}_{F^r,X})$, then we have (using Definition 7.4.1)

$$\begin{aligned}\pi_Y^*f^!\mathcal{M}^\bullet &= \pi_Y^*(\mathcal{O}^{\Lambda}_{F^r,Y\to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}} f^{-1}\mathcal{M}^\bullet)[d_{Y/X}]\\ &\xrightarrow{\sim} \mathcal{O}^{\Lambda}_{F^r,Y_{ét}\to X_{ét}} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}^{\Lambda}_{F^r,X_{ét}}} f^{-1}\pi_X^*\mathcal{M}^\bullet[d_{Y/X}] = f^!\pi_X^*\mathcal{M}^\bullet.\end{aligned}$$

□

Definition 7.4.4. — If $f : Y \to X$ is a morphism of smooth k-schemes then we define the $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X_{ét}}, \mathcal{O}^{\Lambda}_{F^r,Y_{ét}})$-bimodule $\mathcal{O}^{\Lambda}_{F^r,X_{ét}\leftarrow Y_{ét}}$ as

$$\mathcal{O}^{\Lambda}_{F^r,X_{ét}\leftarrow Y_{ét}} = \pi_Y^*\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y},$$

with the bimodule structure induced by the $(f^{-1}\mathcal{O}^{\Lambda}_{F^r,X}, \mathcal{O}^{\Lambda}_{F^r,Y})$-bimodule structure on $\mathcal{O}^{\Lambda}_{F^r,X\leftarrow Y}$.

Definition 7.4.5. — Let $f : Y \to X$ be a morphism of smooth k-schemes. We define

$$f_+ : D(\mathcal{O}^{\Lambda}_{F^r,Y_{ét}}) \to D(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$$

by the formula

$$f_+\mathcal{M}^\bullet = Rf_*(\mathcal{O}^{\Lambda}_{F^r,X_{ét}\leftarrow Y_{ét}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{F^r,Y_{ét}}} \mathcal{M}^\bullet).$$

That this functor is well-defined follows for the same reasons as in the case of Zariski $\mathcal{O}_{F^r}^{\Lambda}$-modules: the bimodule $\mathcal{O}_{F^r,X_{ét}\leftarrow Y_{ét}}^{\Lambda}$ is of finite Tor-dimension as a right $\mathcal{O}_{F^r,Y_{ét}}^{\Lambda}$-module (being the pull-back by π_Y of $\mathcal{O}_{F^r,X\leftarrow Y}^{\Lambda}$, which is of finite Tor-dimension as a right $\mathcal{O}_{F^r,Y}^{\Lambda}$-module), and the functor Rf_* is of finite cohomological dimension.

This functor satisfies analogues of all the results of section 3. The following result shows that it is compatible with the push-forward of Zariski $\mathcal{O}_{F^r}^{\Lambda}$-modules via étale descent.

Proposition 7.4.6. — *Let $f : Y \to X$ be a morphism of smooth k-schemes, and let $\pi_X : X_{ét} \to X$ and $\pi_Y : Y_{ét} \to Y$ denote the natural morphisms of sites. Then if we restrict our attention to complexes with quasi-coherent cohomology sheaves, there are natural isomorphisms $R\pi_{X*}f_+ = f_+R\pi_{Y*}$ and $\pi_X^*f_+ = f_+\pi_Y^*$.*

Proof. — Since $R\pi_{X*}$ and π_X^* (respectively $R\pi_{Y*}$ and π_Y^*) are quasi-inverse, it is enough to construct the first isomorphism.

If $\mathcal{M}^\bullet$ is a complex in $D_{qc}(\mathcal{O}_{F^r,Y_{ét}}^{\Lambda})$, then we have

$$
\begin{aligned}
f_+R\pi_{Y*}\mathcal{M}^\bullet =&Rf_*(\mathcal{O}_{F^r,X\leftarrow Y}^{\Lambda} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F^r,Y}^{\Lambda}} R\pi_Y{*}\mathcal{M}^\bullet)\\
&\xrightarrow{\sim} Rf_*R\pi_{Y*}(\mathcal{O}_{F_{ét},X_{ét}\leftarrow Y_{ét}}^{\Lambda} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F^r,Y_{ét}}^{\Lambda}} \mathcal{M}^\bullet)\\
&\xrightarrow{\sim} R\pi_{X*}Rf_*(\mathcal{O}_{F_{ét},X_{ét}\leftarrow Y_{ét}}^{\Lambda} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F^r,Y_{ét}}^{\Lambda}} \mathcal{M}^\bullet) = R\pi_{X*}f_+\mathcal{M}^\bullet.
\end{aligned}
$$

□

7.4.7. — Propositions 7.3.3, 7.4.3 and 7.4.6 show that the morphisms π_X^* and $R\pi_{X*}$ provide an equivalence of categories between $D_{qc}(\mathcal{O}_{F^r,X}^{\Lambda})$ and $D_{qc}(\mathcal{O}_{F^r,X_{ét}}^{\Lambda})$ which is compatible with pull-backs and push-forwards. We now observe that the obvious analogues of all the results proved for $\mathcal{O}_{F^r,X}^{\Lambda}$-modules in sections 4, 5 and 6 have obvious analogues for $\mathcal{O}_{F^r,X_{ét}}^{\Lambda}$-modules. Indeed, most of these results concern complexes with quasi-coherent cohomology, and so their analogues follow immediately by étale descent. The only exceptions are the results that deal with $\underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}^{\Lambda}}(-,-)$, since this functor produces sheaves of Λ-modules which are in general not quasi-coherent and so do not satisfy étale descent. However, the analogues of these results are also easily established by applying their Zariski topology analogues to each étale neighbourhood U of X.

8. Λ-SHEAVES ON THE ÉTALE SITE

8.1. — Recall [**SGA 4**] that a constructible sheaf of Λ-modules on the étale site of a scheme X is a sheaf of Λ-sheaves on $X_{ét}$ which restricts to a locally constant sheaf of finitely generated Λ-modules along the members of some stratification of X by locally closed subsets.

We let $D^\bullet(X_{ét}, \Lambda)$ denote the derived category of complexes of Λ-sheaves on $X_{ét}$ satisfying the boundedness condition $\bullet$ (equal to one of $+$, $-$, b, or $\emptyset$). We let $D_c^\bullet(X_{ét}, \Lambda)$ denote the full triangulated subcategory of $D^\bullet(X_{ét}, \Lambda)$ consisting of finite-length complexes of Λ-sheaves on $X_{ét}$ whose cohomology sheaves are constructible.

Finally we denote by $D_{ctf}^b(X_{ét}, \Lambda)$ denote the full triangulated subcategory of $D_c^b(X_{ét}, \Lambda)$ consisting of complexes which have finite Tor-dimension. Such complexes are represented by finite length complexes of flat constructible Λ-sheaves [**De**, Prop. 4.6, p. 93].

8.2. — If $f : Y \to X$ is a morphism of schemes, it induces morphisms

$$f^{-1} : D(X_{ét}, \Lambda) \to D(Y_{ét}, \Lambda) \text{ and } Rf_* : D^+(Y_{ét}, \Lambda) \to D^+(X_{ét}, \Lambda),$$

the first having zero cohomological amplitude, and the second having finite cohomological amplitude. If f is an immersion, we also have the extension by zero functor

$$f_! : D^+(Y_{ét}, \Lambda) \to D^+(X_{ét}, \Lambda),$$

which has zero cohomological amplitude.

If $f : Y \to X$ is an arbitrary morphism of k-schemes, let $i : Y \to \bar{Y}$ be an open immersion of Y into a proper k scheme. (Such an immersion exists by Nagata's theorem.) We may factor f into the product of an immersion and a proper morphism, as follows:

$$Y \xrightarrow{\Gamma_f} X \times Y \xrightarrow{\mathrm{id}_X \times i} X \times \bar{Y} \xrightarrow{p_1} X,$$

where Γ_f is the graph of f, which is an immersion, and p_1 is the projection onto the first factor, which is proper. We then define the functor $f_!$ as the composite

$$f_! = Rp_{1*}\,((\mathrm{id}_X \times i)\Gamma_f)_! : D(Y_{ét}, \Lambda) \to D(X_{ét}, \Lambda).$$

If $f = hj$ is any other factorisation of f into the composition of a proper map h with an immersion j then one verifies that $f_!$, as defined above, is naturally isomorphic to the functor $Rh_* j_!$. Thus the functor $f_!$ is (up to natural isomorphism) independent of the choice of factorisation of f as the composition of an immersion and a proper map. (See [**De**, p. 48]). It has finite cohomological amplitude.

If $g : Z \to Y$ is a second morphism of k-schemes, then there are natural isomorphisms

$$(fg)^{-1} = g^{-1} f^{-1} \text{ and } (fg)_! = f_! g_!.$$

8.3. — Any morphism $f : Y \to X$ pulls back a stratification on X to a stratification on Y, and f^{-1} is an exact functor which takes locally constant sheaves to locally constant sheaves. Thus f^{-1} restricts to functors (denoted by the same symbol)

$$f^{-1} : D_c(X_{\acute{e}t}, \Lambda) \to D_c(Y_{\acute{e}t}, \Lambda)$$

and

$$f^{-1} : D^b_{ctf}(X_{\acute{e}t}, \Lambda) \to D^b_{ctf}(Y_{\acute{e}t}, \Lambda).$$

If $f : Y \to X$ is an immersion, then $f_!$ is an exact functor, and takes constructible sheaves on Y to constructible sheaves on X whose stalks are zero outside Y. Thus $f_!$ restricts to functors (denoted by the same symbol)

$$f_! : D_c(Y_{\acute{e}t}, \Lambda) \to D_c(X_{\acute{e}t}, \Lambda)$$

and

$$f_! : D^b_{ctf}(Y_{\acute{e}t}, \Lambda) \to D^b_{ctf}(X_{\acute{e}t}, \Lambda).$$

If $f : Y \to X$ is a proper morphism, then it is a consequence of the proper base-change theorem that Rf_* restricts to functors (denoted by the same symbol)

$$Rf_* : D_c(Y_{\acute{e}t}, \Lambda) \to D_c(X_{\acute{e}t}, \Lambda)$$

and

$$Rf_* : D_{ctf}(Y_{\acute{e}t}, \Lambda) \to D_{ctf}(X_{\acute{e}t}, \Lambda).$$

Thus for any morphism $f : Y \to X$ of k-schemes, $f_!$ restricts to functors (denoted by the same symbol)

$$f_! : D_c(Y_{\acute{e}t}, \Lambda) \to D_c(X_{\acute{e}t}, \Lambda)$$

and

$$f_! : D_{ctf}(Y_{\acute{e}t}, \Lambda) \to D_{ctf}(X_{\acute{e}t}, \Lambda).$$

8.4. — We will frequently use the following analogue of Proposition 6.9.6: if k is perfect and $\mathcal{L}$ is a constructible Λ-sheaf on $X_{\acute{e}t}$ for some smooth k-scheme X, with support contained in a closed subset Z of X, then there exists an open subscheme U of Z (given its reduced induced structure) which is smooth as a k-scheme, and such that the restriction of $\mathcal{L}$ to U is locally constant.

8.5. — In order to apply (8.4), we will have to be able to replace k by its perfect closure if necessary. That this is legitimate follows from the topological invariance of the étale site. More precisely and more generally, let k'/k be a purely inseparable algebraic field extension. If X is any smooth k-scheme then the base-change $X' = k' \times_k X$ is a smooth k' scheme, and the morphism $X' \to X$ induces an isomorphism between

$X_{\acute{e}t}$ and $X'_{\acute{e}t}$. Thus there is induced an equivalence of categories between $D^\bullet(X_{\acute{e}t}, \Lambda)$ and $D^\bullet(X'_{\acute{e}t}, \Lambda)$ (where $\bullet$ can be any one of $+$, $-$, b or $\emptyset$), which restricts to equivalences of categories between $D_c(X_{\acute{e}t}, \Lambda)$ and $D_c(X'_{\acute{e}t}, \Lambda)$ and between $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$ and $D^b_{ctf}(X'_{\acute{e}t}, \Lambda)$.

8.6. — Let Λ' be a Noetherian Λ-algebra. Then the functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^-(X_{\acute{e}t}, \Lambda) \to D^-(X_{\acute{e}t}, \Lambda')$$

restricts to a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^-_c(X_{\acute{e}t}, \Lambda) \to D^-_c(X_{\acute{e}t}, \Lambda'),$$

and hence also to a functor

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda - : D^b_{ctf}(X_{\acute{e}t}, \Lambda) \to D^b_{ctf}(X_{\acute{e}t}, \Lambda').$$

To see this, note that any bounded above complex of sheaves with constructible cohomology has a bounded above resolution by a complex consisting of flat and constructible Λ-sheaves [**De**, Prop. 4.6, p. 93], which may then be used to compute the above $\overset{\mathbb{L}}{\otimes}$.

It is immediate that the functor $\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda -$ is compatible with f^{-1}, and that it is compatible with $f_!$ when f is an immersion. That it is compatible with Rf_* follows from (an obvious analogue of) Proposition B.1.3. (Take $\mathcal{A} = \mathcal{B} = \mathbb{F}_q$, $\mathcal{A}' = \Lambda$, $\mathcal{A}'' = \Lambda'$, and recall that Rf_* is of finite cohomological amplitude.) Thus we see that $f_!$ is compatible with $\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda -$ for any morphism f of k-schemes.

8.7. — Let r' be a multiple of r, write $q' = q^r$, so that $\mathbb{F}_q \subset \mathbb{F}_{q'}$, and let $\Lambda' = \Lambda \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$.

We let $\mathrm{Res}^q_{q'} : D^\bullet(X_{\acute{e}t}, \Lambda') \to D^\bullet(X_{\acute{e}t}, \Lambda)$ denote the functor obtained by regarding a complex of étale sheaves of Λ'-modules as a complex of étale sheaves of Λ-modules. (Here $\bullet$ can assume any one of its usual values.) We refer to this functor as "restriction". It is clear that it is of zero cohomological amplitude, preserves the properties of having constructible cohomology sheaves and of having finite Tor-dimension (since Λ' is free of finite rank over Λ), and commutes with f^{-1} and $f_!$ for any morphism f of k-schemes.

We let $\mathrm{Ind}^{q'}_q : D^\bullet(X_{\acute{e}t}, \Lambda) \to D^\bullet(X_{\acute{e}t}, \Lambda')$ denote the functor obtained by tensoring by Λ' over Λ. (Again $\bullet$ may take on any one of its usual values.) We refer to this functor as "induction". That it is well defined follows from the fact that Λ' is free of finite rank over Λ, which also implies that it is of zero cohomological dimension, preserves the properties of having constructible cohomology sheaves and of having finite Tor-dimension, and commutes with f^{-1} and $f_!$ for any morphism f of k-schemes.

It is immediate that induction is compatible with taking the tensor products of étale sheaves, and so with derived tensor products of complexes in the derived category. On the other hand, restriction is not compatible with tensor products, but does satisfy a projection formula: there is a natural isomorphism between the bifunctors $\mathrm{Res}^q_{q'}(- \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Ind}^q_{q'}(-))$ and $\mathrm{Res}^q_{q'}(-) \overset{\mathbb{L}}{\otimes}_{\Lambda'} -$.

9. THE FUNCTOR $\mathrm{Sol}_{\acute{e}t}$

9.1. — In this section we will construct a functor from the derived category of locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-modules to the derived category of constructible sheaves. Although the definition makes sense in general, many of the good properties of this functor can be proved only when Λ is a finite ring. Proposition 9.6 is a notable exception.

Definition 9.2. — Let X be a smooth k-scheme. Define

$$\mathrm{Sol}_{\acute{e}t}(-) = \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(-, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}})[d_X] : D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}) \to D^+(X_{\acute{e}t}, \Lambda).$$

Remark 9.2.1. — Note that the functor $\mathrm{Sol}_{\acute{e}t}$ is compatible with inseparable base-change, in the sense that if k' is a purely inseparable algebraic extension of k, if X is a smooth k-scheme, and if X' is the base-change of X over k', then the diagram

$$\begin{array}{ccc} D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}) & \xrightarrow[\sim]{k'\otimes_k -} & D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X'_{\acute{e}t}}) \\ \downarrow{\scriptstyle \mathrm{Sol}_{\acute{e}t}} & & \downarrow{\scriptstyle \mathrm{Sol}_{\acute{e}t}} \\ D^+(X_{\acute{e}t}, \Lambda) & \xrightarrow{\sim} & D^+(X'_{\acute{e}t}, \Lambda) \end{array}$$

commutes (by Proposition 6.10.1).

Proposition 9.3. — *Assume that Λ is finite. Let $f : Y \to X$ be a morphism of smooth k-schemes. Then the functors $f^{-1}\mathrm{Sol}_{\acute{e}t}$ and $\mathrm{Sol}_{\acute{e}t}f^!$ are naturally isomorphic, in a manner compatible with inseparable base-change.*

Proof. — If we note that $f^!\mathcal{O}^\Lambda_{X_{\acute{e}t}} = \mathcal{O}^\Lambda_{Y_{\acute{e}t}}[d_{Y/X}]$, then we may use Proposition 2.6 to define a natural transformation

$$\begin{aligned}
f^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) &= f^{-1}\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{O}^\Lambda_{X_{\acute{e}t}})[d_X] \\
&\to \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}}(f^!\mathcal{M}^\bullet, \mathcal{O}^\Lambda_{Y_{\acute{e}t}}[d_{Y/X}])[d_X] \\
&= \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}}(f^!\mathcal{M}, \mathcal{O}^\Lambda_{Y_{\acute{e}t}})[d_Y] \\
&= \mathrm{Sol}_{\acute{e}t}(f^!\mathcal{M}^\bullet).
\end{aligned}$$

We must now prove that this natural transformation is an isomorphism.

By Proposition 2.7 this morphism is compatible with inseparable base-change. Thus we may replace k by its perfect closure, and for the duration of the proof we assume that k is perfect. Secondly, we may work locally on X and Y, and so we may assume that X is affine. Thirdly, truncation and induction on the length of the complex $\mathcal{M}^\bullet$ allow us to reduce to the case of a single locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}$-module $\mathcal{M}$. This completes our initial set of reductions.

Let $\beta : M \to F_X^{r\,*}M$ be a generator of $\mathcal{M}$, with M a coherent $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-module. Using the construction of (5.3.5), we choose a resolution $P^\bullet$ of M by finite rank free $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-modules, and a lift $\beta^\bullet : P^\bullet \to F_X^{r\,*}P^\bullet$ of the morphism β. Let $\mathcal{P}^i$ be the locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}$-module generated by β^i. Then the complex $\mathcal{P}^\bullet$ is a resolution of $\mathcal{M}$, and the complex $f^!\mathcal{P}^\bullet$ is a resolution of $f^!\mathcal{M}$.

Thus there is a spectral sequence computing the cohomology sheaves of $\mathrm{Sol}_{\acute{e}t}(\mathcal{M})$ whose E_1 terms are $E_1^{i,j} = H^j(\mathrm{Sol}_{\acute{e}t}(\mathcal{P}^i))$; applying the exact functor f^{-1} yields a spectral sequence computing the cohomology sheaves of $f^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{M})$ whose E_1 terms are $E_1^{i,j} = H^j(f^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{P}^i))$. Similarly, there is a spectral sequence computing the cohomology sheaves of $H^i(\mathrm{Sol}_{\acute{e}t}(f^!\mathcal{M}))$ whose E_1 terms are $E_1^{i,j} = H^j(\mathrm{Sol}_{\acute{e}t}(f^!\mathcal{P}^i))$. The natural transformation $f^{-1}\mathrm{Sol}_{\acute{e}t} \mapsto \mathrm{Sol}_{\acute{e}t}f^!$ induces a morphism of spectral sequences. Thus it suffices to show that the morphism

$$f^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{P}^i) \to \mathrm{Sol}_{\acute{e}t}f^!\mathcal{P}^i$$

is an isomorphism for each i. Write $P = P_i$, $\mathcal{P} = \mathcal{P}^i$, and choose an isomorphism $P \xrightarrow{\sim} (\mathcal{O}^\Lambda_{X_{\acute{e}t}})^n$. The map β^i is then identified with an $n \times n$ matrix μ

$$\mu = \beta^i : (\mathcal{O}^\Lambda_{X_{\acute{e}t}})^n \to (\mathcal{O}^\Lambda_{X_{\acute{e}t}})^n.$$

The generator P of $\mathcal{P}$ induces a free presentation of $\mathcal{P}$

$$0 \longrightarrow (\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^n \xrightarrow{1-\mu F^r} (\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^n \longrightarrow \mathcal{P} \longrightarrow 0.$$

(Here, the $n \times n$ matrix μF^r is acting by right multiplication on $(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^n$.)

Similarly, $f^!\mathcal{P}$ is the unit $\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}$-module with free presentation

$$0 \longrightarrow (\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}})^n \xrightarrow{1-\mu F^r} (\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}})^n \longrightarrow f^!\mathcal{P} \longrightarrow 0,$$

placed in degree $d_{X/Y}$.

We have the isomorphisms

$$\mathrm{Sol}_{ét}(\mathcal{P}) \xrightarrow{\sim} \underline{Hom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_{ét}}}((\mathcal{O}^{\Lambda}_{F^r,X_{ét}})^n \stackrel{1-\mu F^r}{\longrightarrow} (\mathcal{O}^{\Lambda}_{F^r,X_{ét}})^n, \mathcal{O}^{\Lambda}_{X_{ét}})[d_X]$$
$$\xrightarrow{\sim} ((\mathcal{O}^{\Lambda}_{X_{ét}})^n \stackrel{1-\mu F^r}{\longrightarrow} (\mathcal{O}^{\Lambda}_{X_{ét}})^n)$$

(where in this last complex, the two members of the complex sit in degrees $-d_X$ and $-d_X+1$), and similarly

$$\mathrm{Sol}_{ét}(f^!\mathcal{P}) \xrightarrow{\sim} ((\mathcal{O}^{\Lambda}_{Y_{ét}})^n \stackrel{1-\mu F^r}{\longrightarrow} (\mathcal{O}^{\Lambda}_{Y_{ét}})^n)$$

We may regard μ as an F^r-semi-linear transformation between two free $\mathcal{O}_{X_{ét}}$-modules of rank $n[\Lambda : \mathbb{F}_q]$. Thus, to show that the natural map $f^{-1}\mathrm{Sol}_{ét}(\mathcal{P}) \to \mathrm{Sol}_{ét}(f^!\mathcal{P})$ is an isomorphism, is is enough to show that this is the case when $\mathcal{P}$ is considered as an $\mathcal{O}_{F^r,X}$-module, since Λ is a finite ring. From now on we therefore forget the Λ-module structure on $\mathcal{P}$ and assume that $\Lambda = \mathbb{F}_q$.

If (y_i) is an n-tuple of sections of $\mathcal{O}_X$ over any étale neighbourhood U of X, the equation

$$(1-\mu F^r)(x_i) = (x_i) - \mu(x_i^q) = (y_i)$$

describes a (not necessarily finite) étale cover of U, and so the morphism

$$1-\mu F^r : \mathcal{O}^n_{X_{ét}} \to \mathcal{O}^n_{X_{ét}}$$

is surjective. Thus $\mathrm{Sol}_{ét}(\mathcal{P})$ is isomorphic to the kernel of the morphism $1-\mu F$, placed in degree $-d_X$, and since f^{-1} is exact, we find that

$$f^{-1}\mathrm{Sol}_{ét}(\mathcal{P}) \xrightarrow{\sim} f^{-1}ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X].$$

A similar computation shows that $\mathrm{Sol}_{ét}(f^!\mathcal{P}) \xrightarrow{\sim} ker(\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{Y_{ét}})[d_X]$, and that the map $f^{-1}\mathrm{Sol}_{ét}(\mathcal{P}) \to \mathrm{Sol}_{ét}(f^!\mathcal{P})$ is simply the natural map

$$f^{-1}ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X] \longrightarrow ker(\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{Y_{ét}})[d_X].$$

Thus to prove the proposition, we are reduced to showing that this map is an isomorphism. We begin by studying a number of cases.

(i) f is the closed immersion of a divisor: As remarked at the beginning of the proof, we may work locally. Thus we may assume that Y is the divisor cut out by a global section a of $\mathcal{O}_{X_{ét}}$.

We may work on the level of stalks; thus we are reduced to showing that if A is a strictly Henselian local ring occurring as a stalk of a geometric point of X, μ is an $n \times n$ matrix with entries in A, and $a \in A$ lies in the maximal ideal of A, then the natural map

$$ker(A^n \stackrel{1-\mu F^r}{\longrightarrow} A^n) \longrightarrow ker((A/a)^n \stackrel{1-\mu F^r}{\longrightarrow} (A/a)^n)$$

is an isomorphism.

To see that it is injective, suppose that $(ax_i) \in aA^n$ satisfies $(1 - \mu F^r)(ax_i) = 0$. Then $(ax_i) = \mu F^r(ax_i) = a^q \mu F^r(x_i)$, and

$$\begin{aligned}(x_i) = a^{q-1}\mu F^r(x_i) &= a^{q-1}\mu F^r(a^{q-1}\mu F^r(x_i)) \\ &= a^{(q+1)(q-1)}\mu F^r \mu F^r(a^{q-1}\mu F^r(x_i)) = \dots .\end{aligned}$$

Continuing, we find that the $x_i \in A$ are divisible by arbitrarily high powers of a, and so vanish, since a lies in the maximal ideal of A. This yields the injectivity.

To see the surjectivity, suppose that $(\bar{x}_i) \in (A/a)^n$ satisfies $(1-\mu F^r)(\bar{x}_i) = 0$. Let $(x_i) \in A^n$ lift $(\bar{x}_i)$. Then $(1 - \mu F^r)(x_i) = (ax_i')$ for some $(x_i') \in A^n$. Now since A is strictly Henselian, we may find an n-tuple $(y_i) \in A^n$ solving $(1-a^{q-1}\mu F^r)(y_i) = (x_i')$. Then $x_i - ay_i = \bar{x}_i$ modulo the maximal ideal of A, and

$$(1 - \mu F^r)(x_i - ay_i) = (ax_i' - ax_i') = 0.$$

(ii) f is étale: We must show that

$$f^{-1}ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X] \longrightarrow ker(\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{Y_{ét}})[d_X]$$

is an isomorphism. But since f is étale, $f^{-1}\mathcal{O}_{X_{ét}} = \mathcal{O}_{Y_{ét}}$, and so

$$\begin{aligned}f^{-1}ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X] &= ker(f^{-1}\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} f^{-1}\mathcal{O}^n_{X_{ét}})[d_X] \\ &= ker(\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{Y_{ét}})[d_X].\end{aligned}$$

(Here we have used the exactness of f^{-1}.) Thus in this case we do indeed have an isomorphism.

(iii) f is the projection $X \times \mathbb{A}^n_k \to X$: Let s be any section of f. Then the composite

$$\begin{aligned}s^{-1}f^{-1}ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X] &\longrightarrow s^{-1}ker(\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{Y_{ét}})[d_X] \\ &\longrightarrow ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X],\end{aligned}$$

corresponds to the identity map, and the second arrow factors as a composite of embeddings of smooth divisors, and so is an isomorphism by *(i)*. Thus the first arrow is also an isomorphism. It follows that

$$f^{-1}ker(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{X_{ét}})[d_X] \longrightarrow ker(\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F^r}{\longrightarrow} \mathcal{O}^n_{Y_{ét}})[d_X]$$

is an isomorphism, as it is after pulling back by any section.

More precisely, denote the cone of this morphism by C. Any closed point of $X \times \mathbb{A}^n_k$ is in the image of an étale local section of f (since k is perfect), and so we see that C has vanishing stalks at any closed point of $X \times \mathbb{A}^n_k$. Since these elements are dense in $X \times \mathbb{A}^n_k$, we see that C does indeed vanish.

(iv) f is arbitrary: We factor $f : Y \to X$ as

$$Y \stackrel{\Gamma_f}{\longrightarrow} X \times Y \stackrel{p_1}{\longrightarrow} X.$$

Now Γ_f is a closed immersion, and so locally factors as a composite of embeddings of smooth divisors. Thus part *(i)* above shows that the proposition holds for Γ_f.

Since the morphism $Y \to \operatorname{Spec} k$ is smooth, it factors locally as an étale morphism followed by the map $\mathbb{A}^n_k \to \operatorname{Spec} k$, and so the result follows by *(ii)* and *(iii)*. □

Example 9.3.1. — If X is a smooth k-scheme, then $\mathrm{Sol}_{\acute{e}t}(\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}) = \Lambda\,[d_X]$. When Λ is finite, this result follows from Proposition 9.3, since taking $f : X \to \operatorname{Spec} k$ to be the projection to the point, one sees that it suffices to prove the result for the point, where it is immediately verified.

In general, one may verify this result directly by computing

$$\mathrm{Sol}_{\acute{e}t}(\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}) = \underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}})[d_X]$$

using the canonical resolution of $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$ by left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules:

$$0 \longrightarrow \mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} \xrightarrow{1-F^r} \mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}.$$

It then amounts to the fact that the sequence of sheaves

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{O}^{\Lambda}_{X_{\acute{e}t}} \xrightarrow{1-F^r} \mathcal{O}^{\Lambda}_{X_{\acute{e}t}} \to 0$$

is exact on $X_{\acute{e}t}$.

It will be convenient to isolate the following lemma from the proof of Proposition 9.3.

Lemma 9.3.2. — *Suppose as before that Λ is finite. Let P be a locally free $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-module, $\beta : P \to F^{r*}P$ an $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-linear map, and $\mathcal{P}$ the unit $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-module generated by β. Then $\mathcal{P}$ is acyclic for the functor $\underline{Hom}_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(-, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}})$. This holds in particular if $\mathcal{P} = P$ is a unit F-crystal.*

9.4. — The following lemma is a mild generalisation of part of [**Ka 1**, Prop. 4.1.1]. We recall the proof, which depends on a technique of Lang.

Lemma 9.4.1. — *Suppose that Λ is finite, that X is a smooth k-scheme and that $\mathcal{M}$ is a unit (Λ, F^r)-crystal on X. Then $\mathcal{M}$ is, étale locally on X, isomorphic to a left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-module of the form $L \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{\acute{e}t}}$ for some finitely generated projective Λ-module L.*

Proof. — Denote by $\mathcal{L}$ the kernel of the map $\mathcal{M} \xrightarrow{1-F^r} \mathcal{M}$. We claim that the natural map $\mathcal{L} \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{\acute{e}t}} \to \mathcal{M}$ is an isomorphism, and that $\mathcal{L}$ is a locally constant sheaf. This claim implies the lemma (and conversely it is easy to see that the lemma implies that this map is an isomorphism). Indeed, if $\mathcal{L}$ is locally constant, then, by descent, it must be a locally finitely generated and locally projective sheaf of Λ-modules, as $\mathcal{M} = \mathcal{L} \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{\acute{e}t}}$ is a coherent and locally projective sheaf of $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-modules.

To prove the claim, it is enough to check that it is true when $\mathcal{M}$ is regarded as an $\mathcal{O}_{F^r,X_{\acute{e}t}}$-module, and so we may assume that $\Lambda = \mathbb{F}_q$. In this case, we have to show that $\mathcal{M}$ is étale locally isomorphic to $\mathcal{O}^n_X$ as an $\mathcal{O}_{F^r,X}$-module for some integer n.

By assumption $\mathcal{M}$ is locally free of finite rank on X. Let U be any open set of X over which $\mathcal{M} \xrightarrow{\sim} \mathcal{O}^n_X$ (as $\mathcal{O}_X$-modules). Then the structural morphism of $\mathcal{M}$ is an isomorphism $\mathcal{O}^n_X = F^{r\,*}_X \mathcal{O}^n_X \to \mathcal{O}^n_X$, and so is given by some section A of $GL_n(\mathcal{O}_X)$ over U.

If B is a section of $GL_n(\mathcal{O}_X)$, let $B^{[q]}$ denote the section of $GL_n(\mathcal{O}_X)$ obtained by raising the entries of B to the q^{th} power. Lang has observed that the equation $B^{[q]} = BA$ describes a finite étale cover V of U. Thus over V, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_V^n & \xrightarrow{A} & \mathcal{O}_V^n \\ & {}_{B^{[q]}}\searrow & \downarrow{\scriptstyle B} \\ & & \mathcal{O}_V^n, \end{array}$$

which shows that $\mathcal{M}$ is isomorphic as a left $\mathcal{O}_{F^r,X}$-module to $\mathcal{O}_X^n$ over V. $\square$

Corollary 9.4.2. — *In the situation of the previous lemma, we have*

$$\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\mathcal{M}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}) = \mathcal{L}$$

for some locally constant étale sheaf of projective Λ-modules $\mathcal{L}$. Furthermore, the natural transformation

$$\mathcal{M} \to \underline{Hom}_{\Lambda}(\mathcal{L}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}) \xrightarrow{\sim} \underline{Hom}_{\Lambda}(\mathcal{L}, \Lambda) \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{\acute{e}t}}$$

is an isomorphism.

Proof. — Both claims may be checked locally on $X_{\acute{e}t}$, so that Lemma 9.4.1 reduces us to checking them for $\mathcal{M}$ of the form $L \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{\acute{e}t}}$ for some projective Λ-module L. Moreover, if L' is a finitely generated projective Λ-module such that $L \oplus L'$ is free, then the corollary holds for L provided it holds for $L \oplus L'$. Thus, we may assume that L is a finite free Λ-module, in which case our result follows by example 9.3.1. $\square$

Proposition 9.5. — (*i*) *Suppose that Λ is finite. If $f : Y \to X$ is an open immersion of smooth k-schemes, there is a natural isomorphism of functors*

$$f_!\mathrm{Sol}_{\acute{e}t} \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t} f_+.$$

(*ii*) *If $g : Z \to Y$ is a second open immersion of smooth k-schemes then the diagram of natural isomorphisms*

$$\begin{array}{ccc} \mathrm{Sol}_{\acute{e}t} f_+ g_+ & \xrightarrow[(3.7)]{\sim} & \mathrm{Sol}_{\acute{e}t}(fg)_+ \\ {\scriptstyle part\ (i)}\downarrow{\scriptstyle \sim} & & \\ f_!\mathrm{Sol}_{\acute{e}t} g_+ & & {\scriptstyle part\ (i)}\downarrow{\scriptstyle \sim} \\ {\scriptstyle part\ (i)}\downarrow{\scriptstyle \sim} & & \\ f_! g_! \mathrm{Sol}_{\acute{e}t} & \xrightarrow{\sim} & (fg)_!\mathrm{Sol}_{\acute{e}t} \end{array}$$

(in which the natural isomorphisms have been labelled by the results which give rise to them) commutes.

Proof. — We first construct a natural transformation $f_!\mathrm{Sol}_{\acute{e}t} \to \mathrm{Sol}_{\acute{e}t} f_+$, which will clearly satisfy the compatibility with compositions claimed by part *(ii)*. We then show that it is an isomorphism.

Recall from Lemma 4.3.1 that $f^!\mathcal{O}^{\Lambda}_{X_{\acute{e}t}} = f^{-1}\mathcal{O}^{\Lambda}_{X_{\acute{e}t}} = \mathcal{O}^{\Lambda}_{U_{\acute{e}t}}$, and that if $\mathcal{M}^\bullet$ is any complex in $D^b_{lfgu}(\mathcal{O}_{F^r,U_{\acute{e}t}})$, there is a natural isomorphism $f^!f_+\mathcal{M}^\bullet \xrightarrow{\sim} \mathcal{M}^\bullet$. Thus we obtain the natural isomorphism

$$\begin{aligned} f^{-1}\mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet) &= f^{-1}\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(f_+\mathcal{M}^\bullet, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}})[d_X] \\ &\overset{(1)}{\xrightarrow{\sim}} \underline{RHom}^\bullet_{\mathcal{O}_{F^r,U_{\acute{e}t}}}(f^!f_+\mathcal{M}^\bullet, \mathcal{O}^{\Lambda}_{U_{\acute{e}t}})[d_U] \\ &\xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}_{F^r,U_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{O}^{\Lambda}_{U_{\acute{e}t}})[d_U] = \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet), \end{aligned}$$

in which the isomorphism labelled (1) is provided by Proposition 2.6. By adjointness of $f_!$ and f^{-1}, the inverse of this isomorphism induces a morphism

$$f_!\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \to \mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet),$$

which is the required natural transformation. It is straightforward to verify that this natural transformation satisfies the claim of part *(ii)*.

Let $Z = X \setminus Y$ denote the complement of Y, endowed with the reduced induced scheme structure, and let $g : Z \to X$ be the closed immersion of Z into X. To show that the morphism $f_!\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \to \mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet)$ is an isomorphism, it suffices to show that $g^{-1}\mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet)$ is quasi-isomorphic to zero.

This can be checked locally on $X = \mathrm{Spec}\, A$, and so we may assume that X is affine, and that $Z = V(a_1, \ldots, a_s)$. Also, by truncation and induction on the length of a complex, we may reduce to the case of a single finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,Y}$-module $\mathcal{M}$. In this situation, the proof of Corollary 6.8.3 shows that $f_+\mathcal{M}$ is represented by a Čech complex whose members are finite direct sums of modules of the form $j_+\mathcal{N}$, where j is the open immersion of some open subscheme of Y of the form X_a (with a being a product of some number of the a_i's) into X, and $\mathcal{N}$ is the restriction of $\mathcal{M}$ to X_a. The cohomology sheaves of $\mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M})$ are computed by a spectral sequence whose E_1 terms are finite direct sums of the cohomology sheaves of the $\mathrm{Sol}_{\acute{e}t}(j_+\mathcal{N})$. Thus $g^{-1}\mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M})$ is computed by pulling back this spectral sequence by the exact functor g^{-1}, and so it suffices to show that each $g^{-1}\mathrm{Sol}_{\acute{e}t}(j_+\mathcal{N})$ is quasi-isomorphic to zero. Let $h : V(a) \to X$ be the closed immersion. Then $Z \subset V(a)$, and so it will in particular suffice to show that $h^{-1}\mathrm{Sol}_{\acute{e}t}(j_+\mathcal{N})$ is quasi-isomorphic to zero. Thus we need only consider the case where Y is the complement of a divisor $V(a)$ in X.

The proof of Proposition 6.8.1 shows that $f_+\mathcal{M}$ is a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module which admits a generator of the form $M \xrightarrow{\beta} F^{r\,*}_X M$, where M is a coherent $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-module, and $\beta = a\beta'$ for some $\beta' : M \to F^{r\,*}_X M$.

Let P be a resolution of M by free finite rank $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$-modules, and lift β' to

$$\beta'^\bullet : P \to F^{r\,*}_X P.$$

Define $\beta^\bullet = a\beta'^\bullet$. Let $\mathcal{P}^i$ denote the unit $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-module generated by β^i. Then $\mathcal{P}^\bullet$ is a resolution of $\mathcal{M}$, and by the usual spectral sequence argument, it suffices to show that $g^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{P}^i)$ is quasi-isomorphic to zero for each i.

Let $P^i = (\mathcal{O}^{\Lambda}_{X_{\acute{e}t}})^n$ and write

$$\mu = \beta'^i : (\mathcal{O}^{\Lambda}_{X_{\acute{e}t}})^n \to F^{r\,*}_X(\mathcal{O}^{\Lambda}_{X_{\acute{e}t}})^n = (\mathcal{O}^{\Lambda}_{X_{\acute{e}t}})^n.$$

Then, as was observed in the proof of Proposition 9.3,

$$\mathrm{Sol}_{ét}(\mathcal{P}^i) = ker((\mathcal{O}^{\Lambda}_{X_{ét}})^n \stackrel{1-a\mu F^r}{\longrightarrow} (\mathcal{O}^{\Lambda}_{X_{ét}})^n)[d_X].$$

(It is at this point that the hypothesis that Λ is finite is used.) We want to show that the stalks of this kernel vanish at the geometric points of $V(a)$. Let A be the stalk of $\mathcal{O}^{\Lambda}_{X_{ét}}$ at such a point; then a lies in the maximal ideal of A. Thus if $(x_i) \in A^n$ lies in this kernel,

$$(x_i) = a\mu F^r(x_i) = a^{q+1}\mu(F^r\mu F^r(x_i)) = \cdots ;$$

thus each x_i is divisible by arbitrarily high powers of a and hence it vanishes. This completes the proof of the proposition. □

Remark 9.5.1. — The reader should compare this result with Lemma 3.3 of [**De**, p. 120]. In that lemma Deligne takes a "dual" approach to the one taken here. Suppose that $f : Y \to X$ is an open immersion of the complement of a divisor, and that $\mathcal{M}$ is a unit F^r-crystal on Y. Let us follow the steps of the proof of Proposition 6.8.1 by letting N be a coherent extension of $\mathcal{M}$ to X. In this proof we tensor the coherent sheaf N with large positive powers of the sheaf $\mathcal{O}^{\Lambda}_X(D)$ to obtain a generator

$$\gamma : N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes m} \to F^{r*}_X(N \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)^{\otimes m})$$

for $f_+\mathcal{M}$. Rather than doing this, Deligne tensors N with a large negative power of $\mathcal{O}^{\Lambda}_X(D)$, to obtain a morphism

$$\Phi : F^{r*}_X(N \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)^{\otimes m}) \to N \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)^{\otimes m}.$$

Thus N becomes a (non-unit) $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-module, and Deligne shows (when $\Lambda = \mathbb{F}_q$) that

$$\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,X_{ét}}}(\mathcal{O}^{\Lambda}_{X_{ét}}, \pi^*_X(N \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)^{\otimes m})) = j_!\underline{RHom}^{\bullet}_{\mathcal{O}^{\Lambda}_{F^r,Y_{ét}}}(\mathcal{O}^{\Lambda}_{Y_{ét}}, \pi^*_Y\mathcal{M}).$$

The disadvantage of this approach is that there is no unique choice of the integer m, and so the $\mathcal{O}^{\Lambda}_{F^r,X}$-module $N \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)^{\otimes m}$ is not uniquely determined. From our point of view, this is analogous to the fact that there is no unique generator for a unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module. Although the generators are useful for computations, it is the locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $f_+\mathcal{M}$ which is the object canonically associated with pushing forward $\mathcal{M}$ via f.

Proposition 9.6. — *(i) Let $f : Y \to X$ be a proper morphism of smooth k-schemes. Then there is a natural isomorphism*

$$\mathrm{Sol}_{ét} f_+ \stackrel{\sim}{\longrightarrow} f_!\mathrm{Sol}_{ét}.$$

(*ii*) *If $g : Z \to Y$ is a second proper morphism of smooth k-schemes then the diagram of natural isomorphisms*

$$\begin{array}{ccc} \mathrm{Sol}_{\acute{e}t} f_+ g_+ & \xrightarrow[(3.7)]{\sim} & \mathrm{Sol}_{\acute{e}t}(fg)_+ \\ {\scriptstyle part\ (i)}\downarrow\wr & & \\ f_! \mathrm{Sol}_{\acute{e}t} g_+ & & {\scriptstyle part\ (i)}\downarrow\wr \\ {\scriptstyle part\ (i)}\downarrow\wr & & \\ f_! g_! \mathrm{Sol}_{\acute{e}t} & \xrightarrow{\sim} & (fg)_! \mathrm{Sol}_{\acute{e}t} \end{array}$$

(in which the natural isomorphisms have been labelled by the results which give rise to them) commutes.

Proof. — Since f is a proper morphism, we have $f_! = Rf_*$. Let $\mathcal{M}^\bullet$ be an object of $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,Y})$. Then using part *(i)* of Theorem 4.4.1, we compute that

$$\begin{aligned} \mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet) &= \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(f_+\mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})[d_X] \\ \xrightarrow{\sim} Rf_*\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}}&(\mathcal{M}^\bullet, f^!\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})[d_X] \xrightarrow{\sim} f_!\underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}})[d_Y] \\ &= f_!\mathrm{Sol}_{\acute{e}t}\mathcal{M}^\bullet. \end{aligned}$$

The compatibility with compositions claimed by part *(ii)* follows from the given construction, together with part *(iii)* of Theorem 4.4.1. □

9.7. — We would like to prove an analogue of Propositions 9.5 and 9.6 for an arbitrary morphism $f : Y \to X$ of smooth k-schemes. If we can factor f as a composition of open immersions and proper morphisms then we can combine the results of Propositions 9.5 and 9.6 to obtain an analogous result for the morphism f. Unfortunately, in order to prove that this isomorphism is independent of the choice of such a factorisation, our techniques require us to assume that f has a factorisation of a more restricted type.

We will say that f is *allowable* if there is a smooth k-scheme W and a factorisation $f = gh$, where $g : Y \to W$ is an immersion and $h : W \to X$ is a smooth proper morphism. The main examples of allowable morphisms that we have in mind are quasi-projective morphisms. If one had resolution of singularities in characteristic p then by combining it with Nagata's theorem on the existence of compactifications, one could show that every morphism f is allowable.

Theorem 9.7.1. — (*i*) *Suppose that Λ is finite. Let $f : Y \to X$ be an allowable morphism of smooth k-schemes. Then there is a natural isomorphism*

$$\mathrm{Sol}_{\acute{e}t} f_+ \xrightarrow{\sim} f_! \mathrm{Sol}_{\acute{e}t}.$$

(*ii*) *If $g : Z \to Y$ is a second allowable morphism of smooth k-schemes such that the composite fg is also allowable then the diagram of natural isomorphisms*

$$\begin{array}{ccc}
\mathrm{Sol}_{\acute{e}t} f_+ g_+ & \xrightarrow[\text{(3.7)}]{\sim} & \mathrm{Sol}_{\acute{e}t}(fg)_+ \\
{\scriptstyle \text{part (i)}}\Big\downarrow{\scriptstyle \sim} & & \\
f_! \mathrm{Sol}_{\acute{e}t} g_+ & & {\scriptstyle \text{part (i)}}\Big\downarrow{\scriptstyle \sim} \\
{\scriptstyle \text{part (i)}}\Big\downarrow{\scriptstyle \sim} & & \\
f_! g_! \mathrm{Sol}_{\acute{e}t} & \xrightarrow{\sim} & (fg)_! \mathrm{Sol}_{\acute{e}t}
\end{array}$$

(in which the natural isomorphisms have been labelled by the results which give rise to them) commutes.

(*iii*) *The natural transformation of (i) is compatible with inseparable base-change.*

9.7.2. — The construction of the isomorphism of part *(i)* will depend upon the factorisation of the allowable morphism f into a composition of proper morphisms and open immersions, and so not only is the compatibility with respect to compositions as stated in part *(ii)* not immediate from the construction, but it implicitly includes as a special case the independence of the construction of part *(i)* from the particular choice of such a factorisation.

Lemma 9.7.3. — *Suppose that $f : Y \to X$ is a morphism of smooth k-schemes which has two factorisations,*

$$f : Y \xrightarrow{j_1} W_1 \xrightarrow{p_1} X,$$

and

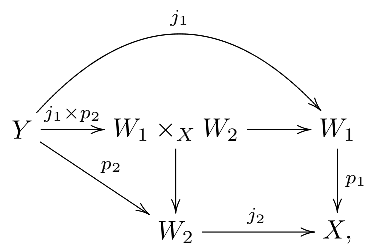
$$f : Y \xrightarrow{p_2} W_2 \xrightarrow{j_2} X,$$

in which each of W_1, W_2 is a smooth k-scheme, each of the morphisms j_1, j_2 is an open immersion, and each of the morphisms p_1, p_2 is proper. Then the diagram of natural isomorphisms

$$\begin{array}{ccccc}
p_{1+} j_{1+} \mathrm{Sol}_{\acute{e}t} & \xrightarrow[\text{(9.5)}]{\sim} & p_{1+} \mathrm{Sol}_{\acute{e}t} j_{1!} & \xrightarrow[\text{(9.6)}]{\sim} & \mathrm{Sol}_{\acute{e}t} p_{1!} j_{1!} \\
{\scriptstyle (3.7)}\Big\downarrow{\scriptstyle \sim} & & & & \Big\downarrow{\scriptstyle \sim} \\
f_+ \mathrm{Sol}_{\acute{e}t} & & & & \mathrm{Sol}_{\acute{e}t} f_! \\
{\scriptstyle (3.7)}\Big\uparrow{\scriptstyle \sim} & & & & {\scriptstyle \sim}\Big\uparrow \\
j_{2+} p_{2+} \mathrm{Sol}_{\acute{e}t} & \xrightarrow[\text{(9.6)}]{\sim} & j_{2+} \mathrm{Sol}_{\acute{e}t} p_{2!} & \xrightarrow[\text{(9.5)}]{\sim} & \mathrm{Sol}_{\acute{e}t} j_{2!} p_{2!}
\end{array}$$

(in which each isomorphism has been labelled by the result that gives rise to it) commutes.

Proof. — We may form the commutative diagram



in which the square is cartesian, and the morphism $j_1 \times p_2$ is both an open immersion and a proper morphism. Applying parts *(ii)* of Propositions 9.5 and 9.6 respectively to the upper and left-hand triangles, we see that it suffices to verify the claimed commutativity in the case that the pair of factorisations

$$f = p_1 j_1 = j_2 p_2$$

arises from a cartesian square. In this case the result follows from the constructions of Propositions 9.5 and 9.6, together with part *(ii)* of Theorem 4.4.1. □

9.7.4. — *Proof of Theorem 9.7.1.* — Note that if we have the isomorphism of part *(i)* for two morphisms f and g, Proposition 3.7 will allow us to construct such an isomorphism for the composite fg. (In other words, we can take the diagram of part *(ii)* to define the isomorphism for the composite fg.) Thus if f is allowable, and $f = f_1 \cdots f_n$ is some factorisation of f into a composite of morphisms f_i between smooth k-schemes, such that each f_i is an open immersion or a proper morphism, we may construct a particular isomorphism $\mathrm{Sol}_{ét} f_+ \xrightarrow{\sim} f_! \mathrm{Sol}_{ét}$. (At least one such factorisation exists, since we have assumed that f is allowable.) We will prove that the isomorphism so constructed is independent of choice of such a factorisation of f. This will in particular provide a proof of part *(ii)*.

We first suppose that f is a closed immersion equipped with a factorisation $f = f_1 \cdots f_n$ of f into a composite of open immersions and proper morphisms. Using induction on n, we will show that the isomorphism of part *(i)* obtained by using this factorisation of f is the same as that obtained by applying Proposition 9.6 directly to the closed immersion (and hence proper morphism) f itself.

If $n = 1$ this is a tautology, and if $n = 2$ it follows from Lemma 9.7.3. The general case now follows immediately by induction on n, since f being a closed immersion shows that each of the partial composites $f_i \cdots f_n$ $(1 \leq i \leq n)$ is also a closed immersion.

Now suppose that $f : Y \to X$ is an allowable morphism of smooth k-schemes, and that we have a factorisation

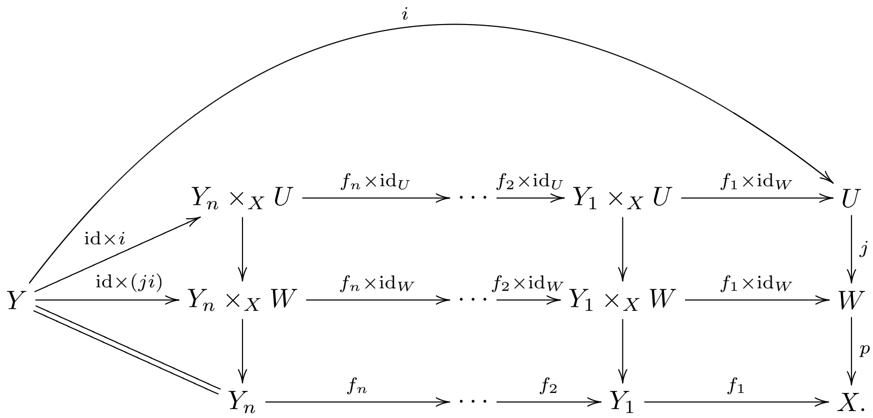
$$f : Y = Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X,$$

in which each of the morphisms f_i is either an open immersion or a proper morphism. We will show that the isomorphism induced by this factorisation is independent of the given factorisation. In order to do this, we choose a factorisation of f,

$$f : Y \xrightarrow{i} U \xrightarrow{j} W \xrightarrow{p} X,$$

in which i is a closed immersion, j is an open immersion, and p is a *smooth* proper morphism. We will prove that the isomorphism of part *(i)* obtained using the factorisation $f = f_1 \cdots f_n$ is the same as that obtained using the factorisation $f = pji$. Since this latter factorisation is independent of the factorisation $f = f_1 \cdots f_n$ with which we began, the independence will follow.

The argument depends on a consideration of the commutative diagram



(Because p is smooth, each of the k-schemes appearing in this diagram is smooth.)

The case of a closed immersion considered above shows that we get the same isomorphism whether we use the closed immersion i or the factorisation

$$Y \longrightarrow Y_n \times_X U \longrightarrow \cdots \longrightarrow Y_1 \times_X U \longrightarrow U.$$

Also in each square, opposite sides are either both open immersions or both proper morphisms. Thus Propositions 9.5 and 9.6, and Lemma 9.7.3, show that we get the same isomorphism whichever way we go around each square. Finally, the edges of each of the two left-hand triangles are either open immersions or proper maps, so that Proposition 9.6 and Lemma 9.7.3 show that we get the same isomorphism whichever way we go around each of the triangles. This completes the proof of the first two parts of the theorem. The compatibility with inseparable base-change follows from remark 4.6, which observes that the constructions of that section, and hence those of Propositions 9.5 and 9.6, and hence that of this theorem, are compatible with change of ground field. □

Proposition 9.8. — *Suppose that* Λ *is finite and let* X *be a smooth* k*-scheme. Then* $\mathrm{Sol}_{ét}$ *restricts to a functor*

$$\mathrm{Sol}_{ét} : D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r, X_{ét}}) \to D^+_c(X_{ét}, \Lambda).$$

and

$$\mathrm{Sol}_{ét} : D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r, X_{ét}})^{\circ} \to D^b_{ctf}(X_{ét}, \Lambda).$$

Proof. — First note that the compatibility of $\mathrm{Sol}_{ét}$ with inseparable base-change allows us to replace k by its perfect closure, and so for the duration of the proof we may assume that k is perfect.

Suppose $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{ét}})$. We begin by showing that $\mathrm{Sol}_{ét}(\mathcal{M}^\bullet)$ is in $D^+_c(X_{ét},\Lambda)$. We proceed by induction on the dimension of the support of $\mathcal{M}^\bullet$. Let Z be this support. Let U be a smooth dense open subscheme of Z (given its reduced induced structure), and denote by f the inclusion of U into X. Proposition 5.12.1 shows that there is a morphism $\mathcal{M}^\bullet \to f_+f^!\mathcal{M}^\bullet$ whose cone is supported on $Z \setminus U$, which has dimension strictly less than that of Z. Thus to prove the proposition for $\mathcal{M}^\bullet$, it suffices to prove it for $f_+f^!\mathcal{M}^\bullet$.

Now $\mathrm{Sol}_{ét}(f_+f^!\mathcal{M}^\bullet) = f_!\mathrm{Sol}_{ét}(f^!\mathcal{M}^\bullet)$, and $f_!$ takes $D^+_c(U_{ét},\Lambda)$ to $D^+_c(X_{ét},\Lambda)$. Thus, it suffices to show that $\mathrm{Sol}_{ét}(f^!\mathcal{M}^\bullet)$ is in $D^+_c(U_{ét},\Lambda)$. Moreover, we are free to replace U by any dense open subscheme.

Let $\mathcal{N} = H^i(f^!\mathcal{M}^\bullet)$ be a cohomology sheaf of $\mathcal{M}^\bullet$. By Proposition 6.7, $f^!\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,U_{ét}})$, and so $\mathcal{N}$ is a locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,U_{ét}}$-module. A standard dévissage implies that it suffices to show that $\mathrm{Sol}_{ét}(\mathcal{N})$ has constructible cohomology sheaves. For this, we shrink U if necessary and perform the construction of (5.3.5). That is, we choose a $\mathcal{O}^\Lambda_{U_{ét}}$-coherent generator $\beta : N \to F^{r*}N$ for $\mathcal{N}$, a left resolution $P^\bullet$ of N by finite free $\mathcal{O}^\Lambda_{U_{ét}}$-modules, and a lift of β to a map of complexes of $\mathcal{O}^\Lambda_{U_{ét}}$-modules $\beta^\bullet : P^\bullet \to F^{r*}P^\bullet$. The complex $\mathcal{P}^\bullet$ of locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,U}$-modules generated by $\beta^\bullet$ is a resolution of $\mathcal{N}$. Lemma 9.3.2 implies that

$$\mathrm{Sol}_{ét}(\mathcal{N}) \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}^\Lambda_{F^r,U_{ét}}}(\mathcal{P}^\bullet, \mathcal{O}^\Lambda_{U_{ét}}),$$

and so it suffices to show that each term $\mathrm{Sol}_{ét}(\mathcal{P}^i) = \underline{Hom}_{\mathcal{O}^\Lambda_{F^r,U_{ét}}}(\mathcal{P}^i, \mathcal{O}^\Lambda_{U_{ét}})$ is constructible. Shrinking U further if necessary, and combining Proposition 6.9.6 and the fact that Λ is finite, we may assume that $\mathcal{P}^i$ is $\mathcal{O}_{U_{ét}}$-coherent. Since $\mathcal{P}^i$ is flat over $\mathcal{O}^\Lambda_{U_{ét}}$ it must also be a unit (Λ, F^r)-crystal. The desired constructibility now follows from Corollary 9.4.2.

Now suppose that $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{ét}})^\circ$. It remains to show that $\mathrm{Sol}_{ét}(\mathcal{M}^\bullet)$ is bounded of finite Tor dimension. Again, we may proceed by induction on the support Z of $\mathcal{M}^\bullet$. Using our previous notation, note that $f_!$ takes $D^b_{ctf}(U_{ét},\Lambda)$ to $D^b_{ctf}(X_{ét},\Lambda)$ and that Lemma 2.3.2 implies that $f^!\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}_{F^r,X_{ét}})^\circ$. An argument as above shows that it is enough to prove that $\mathrm{Sol}_{ét}(f^!\mathcal{M}^\bullet)$ is in $D^b_{ctf}(U_{ét},\Lambda)$ for some dense open subset U of Z. By Proposition 6.9.6 we may choose U so that $f^!\mathcal{M}^\bullet$ has $\mathcal{O}_{U_{ét}}$-coherent cohomology sheaves. Then Corollary 1.8.6 implies that $f^!\mathcal{M}^\bullet$ has finite locally projective dimension as a complex of left $\mathcal{O}^\Lambda_{F^r,U_{ét}}$-modules, so that $f^!\mathcal{M}^\bullet$ is represented by a finite length complex of locally projective left $\mathcal{O}^\Lambda_{F^r,U_{ét}}$-modules $\mathcal{N}^\bullet$, and we have

$$\mathrm{Sol}_{ét}(f^!\mathcal{M}^\bullet) \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}^\Lambda_{F^r,U_{ét}}}(\mathcal{N}^\bullet, \mathcal{O}^\Lambda_{U_{ét}}).$$

We complete the argument by observing that the right hand of this isomorphism is a bounded complex of flat $\mathcal{O}^\Lambda_{U_{ét}}$-modules, and so of flat Λ-modules. Indeed, since locally on U each term of $\mathcal{N}^\bullet$ is a direct summand of a free left $\mathcal{O}^\Lambda_{F^r,U_{ét}}$-module, this follows from the fact that

$$\underline{Hom}_{\mathcal{O}^{\Lambda}_{F^r,U_{ét}}}((\mathcal{O}^{\Lambda}_{F^r,U_{ét}})^I, \mathcal{O}^{\Lambda}_{U_{ét}}) = \prod_{i\in I} \mathcal{O}^{\Lambda}_{U_{ét}}$$

is $\mathcal{O}^{\Lambda}_{U_{ét}}$-flat for any index set I (since U is a Noetherian scheme). □

Proposition 9.9. — *Suppose that Λ is finite, that X is a smooth k-scheme, that $\mathcal{M}^\bullet$ is a complex in $D^-_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$, and that $\mathcal{N}^\bullet$ is a complex in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})^\circ$. Then there is a natural isomorphism*

$$\mathrm{Sol}_{ét}(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{ét}(\mathcal{N}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{ét}(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{ét}}} \mathcal{N}^\bullet)[d_X]$$

in $D^b_c(X_{ét})$, which is compatible (in an obvious sense) with inseparable base-change. (Note that the $\overset{\mathbb{L}}{\otimes}$ on the left-hand side of this isomorphism is defined, since Proposition 9.8 shows that $\mathrm{Sol}_{ét}(\mathcal{N}^\bullet)$ has finite Tor-dimension over Λ.)

Proof. — If we combine Lemmas 1.12.1 and 1.9.3, then we obtain a natural transformation

$$\begin{aligned}
&\mathrm{Sol}_{ét}(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{ét}(\mathcal{N}^\bullet)\\
=&\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{ét}}}(\mathcal{M}^\bullet, \mathcal{O}^{\Lambda}_{X_{ét}})[d_X] \overset{\mathbb{L}}{\otimes}_\Lambda \underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{ét}}}(\mathcal{N}^\bullet, \mathcal{O}^{\Lambda}_{X_{ét}})[d_X]\\
\to&\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{ét}}}(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{ét}}} \mathcal{N}^\bullet, \mathcal{O}^{\Lambda}_{X_{ét}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{ét}}} \mathcal{O}^{\Lambda}_{X_{ét}})[2d_X]\\
=&\underline{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{ét}}}(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{ét}}} \mathcal{N}^\bullet, \mathcal{O}^{\Lambda}_{X_{ét}})[2d_X]\\
=&\mathrm{Sol}_{ét}(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{ét}}} \mathcal{N}^\bullet)[d_X].
\end{aligned}$$

Denote the composite of the above maps by $\chi(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$. This is the natural transformation which we will show is an isomorphism.

Suppose first that $\mathcal{M}^\bullet$ is of the form $L \otimes_\Lambda \mathcal{O}^{\Lambda}_{X_{ét}}$ for some projective Λ-module L. Then $\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{ét}}} \mathcal{N}^\bullet = L \otimes_\Lambda \mathcal{N}^\bullet$ and Corollary 9.4.2 shows that

$$\mathrm{Sol}_{ét}(\mathcal{M}^\bullet) = L^*[d_X],$$

where L^* denotes the Λ-dual of L. Thus the natural transformation constructed above reduces to the isomorphism

$$L^*[d_X] \otimes_\Lambda \mathrm{Sol}_{ét}(\mathcal{N}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{ét}(L \otimes_\Lambda \mathcal{N}^\bullet)[d_X],$$

and we are done in this case. The case when $\mathcal{M}^\bullet$ is a single unit (Λ, F^r)-crystal follows from this by Lemma 9.4.1.

We now turn to the case of an arbitrary complex $\mathcal{M}^\bullet$. Lemma 1.12.1 together with the compatibility of $\mathrm{Sol}_{ét}$ with inseparable base-change shows that $\chi(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$ is compatible with inseparable base-change. In particular this allows us to replace k by its perfect closure, and so for the remainder of the proof we may assume that k is perfect.

We proceed by induction on the dimension of the support of $\mathcal{M}^\bullet$. Let Z be this support, and $U \subset Z$ a smooth dense affine open subscheme (given its reduced induced structure). Proposition 5.12.1 now shows that there is a map $\mathcal{M}^\bullet \to f_+ f^! \mathcal{M}^\bullet$ whose support is contained in $Z \setminus U$, which is of dimension strictly less than that of Z. By our inductive assumption, we are reduced to verifying that the natural transformation

$$\mathrm{Sol}_{\acute{e}t}(f_+ f^! \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{N}^\bullet) \to \mathrm{Sol}_{\acute{e}t}(f_+ f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{X_{\acute{e}t}}} \mathcal{N}^\bullet)[d_X] \tag{9.9.1}$$

is an isomorphism.

Proposition 4.2 provides a natural isomorphism

$$f_+ f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{X_{\acute{e}t}}} \mathcal{N}^\bullet \xrightarrow{\sim} f_+(f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{U_{\acute{e}t}}} f^! \mathcal{N}^\bullet)[d_{X/U}],$$

and so we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Sol}_{\acute{e}t}(f_+ f^! \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{N}^\bullet) & \xrightarrow[(1)]{\sim} & f_! \mathrm{Sol}_{\acute{e}t}(f^! \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{N}^\bullet) \\
\downarrow{\scriptstyle (9.9.1)} & & \downarrow{\scriptstyle \sim} \\
\mathrm{Sol}_{\acute{e}t}(f_+ f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{X_{\acute{e}t}}} \mathcal{N}^\bullet)[d_X] & & f_!(\mathrm{Sol}_{\acute{e}t}(f^! \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda f^{-1} \mathrm{Sol}_{\acute{e}t}(\mathcal{N}^\bullet)) \\
{\scriptstyle (4.2)}\downarrow{\scriptstyle \sim} & & {\scriptstyle (3)}\downarrow{\scriptstyle \sim} \\
\mathrm{Sol}_{\acute{e}t}(f_+(f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{U_{\acute{e}t}}} f^! \mathcal{N}^\bullet)[d_{X/U}])[d_X] & & f_!(\mathrm{Sol}_{\acute{e}t}(f^! \mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(f^! \mathcal{N}^\bullet)) \\
\downarrow{\scriptstyle \sim} & & {\scriptstyle (4)}\downarrow \\
\mathrm{Sol}_{\acute{e}t}(f_+(f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{U_{\acute{e}t}}} f^! \mathcal{N}^\bullet))[d_U] & \xrightarrow[(2)]{\sim} & f_!(\mathrm{Sol}_{\acute{e}t}(f^! \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^\Lambda_{X_{\acute{e}t}}} f^! \mathcal{N}^\bullet))[d_U].
\end{array}$$

Here (1) and (2) are deduced from the natural isomorphism $\mathrm{Sol}_{\acute{e}t} f_+ \xrightarrow{\sim} f_! \mathrm{Sol}_{\acute{e}t}$, isomorphism (3) is deduced from the natural isomorphism $\mathrm{Sol}_{\acute{e}t} f^! \xrightarrow{\sim} f^{-1} \mathrm{Sol}_{\acute{e}t}$, and (4) is obtained by applying $f_!$ to the morphism $\chi(f^! \mathcal{M}^\bullet, f^! \mathcal{N}^\bullet)$. ¿From this diagram we see that to show that (9.9.1) is an isomorphism, we have only to show that $\chi(f^! \mathcal{M}^\bullet, f^! \mathcal{N}^\bullet)$ is an isomorphism. Let $\mathcal{M}' = H^i(f^! \mathcal{M}^\bullet)$ be one of the cohomology sheaves of $f^! \mathcal{M}^\bullet$. A standard spectral sequence argument shows that it suffices to prove that $\chi(\mathcal{M}', f^! \mathcal{N}^\bullet)$ is an isomorphism.

Since U is affine we may apply the construction of (5.3.5) to a coherent generator of $\mathcal{M}'$, and so obtain a complex $\mathcal{P}^\bullet$ of locally finitely generated unit $\mathcal{O}^\Lambda_{F^r, U_{\acute{e}t}}$-modules resolving $\mathcal{M}'$, whose members are flat as $\mathcal{O}^\Lambda_{U_{\acute{e}t}}$-modules.

We are thus reduced to proving that the natural morphism $\chi(\mathcal{P}^\bullet, f^! \mathcal{N}^\bullet)$ is an isomorphism, and a standard spectral sequence argument shows that is suffices to prove that $\chi(\mathcal{P}^j, f^! \mathcal{N}^\bullet)$ is a quasi-isomorphism for each integer j. Shrinking U further, if necessary, Proposition 6.9.6 allows us to assume that $\mathcal{P}^j$ is coherent as a $\mathcal{O}^\Lambda_{U_{\acute{e}t}}$-module. Since it is flat as a $\mathcal{O}^\Lambda_{U_{\acute{e}t}}$-module, we see that it is in fact a unit (Λ, F^r)-crystal, and so indeed $\chi(\mathcal{P}^j, f^! \mathcal{N}^\bullet)$ is an isomorphism. □

Proposition 9.10. — *We continue to assume that Λ is finite.*

(i) *Let Λ' be a Noetherian Λ-algebra and X a smooth k-scheme. If $\mathcal{M}^\bullet$ is an object of $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^\circ$ then there is a natural isomorphism*

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet),$$

which is compatible with inseparable base-change.

(

(ii) *If $f : Y \to X$ is a morphism of smooth k-schemes, and if $\mathcal{M}^\bullet$ is an object of $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^\circ$, then the following diagram of natural isomorphisms commutes:*

$$\begin{array}{ccccc}
\Lambda' \otimes_\Lambda \mathrm{Sol}_{\acute{e}t}(f^!\mathcal{M}^\bullet) & \xrightarrow[\text{(9.3)}]{\sim} & \Lambda' \otimes_\Lambda f^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) & \xrightarrow[\text{(8.6)}]{\sim} & f^{-1}(\Lambda' \otimes_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)) \\
{\scriptstyle part\ (i)}\downarrow\wr & & & & {\scriptstyle part\ (i)}\downarrow\wr \\
\mathrm{Sol}_{\acute{e}t}(\Lambda' \otimes_\Lambda f^!\mathcal{M}^\bullet) & \xrightarrow[\text{(2.8)}]{\sim} & \mathrm{Sol}_{\acute{e}t}(f^!(\Lambda' \otimes_\Lambda \mathcal{M}^\bullet)) & \xrightarrow[\text{(9.3)}]{\sim} & f^{-1}(\mathrm{Sol}_{\acute{e}t}(\Lambda' \otimes_\Lambda \mathcal{M}^\bullet)).
\end{array}$$

(iii) *If $f : Y \to X$ is an allowable morphism of smooth k-schemes, and if $\mathcal{M}^\bullet$ is an object of $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}})^\circ$, then the following diagram of natural isomorphisms commutes:*

$$\begin{array}{ccccc}
\Lambda' \otimes_\Lambda \mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet) & \xrightarrow[\text{(9.7.1)}]{\sim} & \Lambda' \otimes_\Lambda f_!\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) & \xrightarrow[\text{(8.6)}]{\sim} & f_!(\Lambda' \otimes_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)) \\
{\scriptstyle part\ (i)}\downarrow\wr & & & & {\scriptstyle part\ (i)}\downarrow\wr \\
\mathrm{Sol}_{\acute{e}t}(\Lambda' \otimes_\Lambda f_+\mathcal{M}^\bullet) & \xrightarrow[\text{(3.10)}]{\sim} & \mathrm{Sol}_{\acute{e}t}(f_+(\Lambda' \otimes_\Lambda \mathcal{M}^\bullet)) & \xrightarrow[\text{(9.7.1)}]{\sim} & f_!(\mathrm{Sol}_{\acute{e}t}(\Lambda' \otimes_\Lambda \mathcal{M}^\bullet)).
\end{array}$$

Proof. — Let $\mathcal{M}^\bullet$ be an object of $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^\circ$. Then Proposition 9.8 shows that

$$\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) = \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{O}^\Lambda_{X_{\acute{e}t}})$$

is bounded, and so Lemma 1.13.4 yields a natural transformation

$$\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \to \mathrm{Sol}_{\acute{e}t}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet),$$

which by that lemma is compatible with inseparable base-change. (Here we are taking into account the fact that $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$ is flat over itself, as well as the isomorphism (1.13.1).) This is the natural transformation of part *(i)*, which we must show is an isomorphism. We will not prove this directly; rather, we first establish the commutativity of the diagrams of parts *(ii)* and *(iii)*.

Since it is the natural transformation of Proposition 2.6 which gives rise to the natural transformation of Proposition 9.3, it follows from remark (2.8.1) that the diagram of part *(ii)* commutes. Also, the commutativity of the diagram of part *(iii)* in the case that f is an open immersion is clear. (In this case, all the members of the diagram have vanishing stalks on the complement of Y in X, and so its commutativity can be checked after restricting to Y, where it becomes immediate.) It remains to establish the commutativity of this diagram in the case that f is proper (since any allowable morphism factors as the composite of open immersions and proper maps). This case follows from the compatibility of the adjunction isomorphism of Theorem 4.4.1 with change of coefficient ring.

We now return to proving that the morphism of part *(i)* is an isomorphism. Since it is compatible with inseparable base-change, we may replace k by its perfect closure, and thus assume that k is perfect. We proceed by induction on the dimension of the support Z of $\mathcal{M}^\bullet$. Give Z its reduced induced structure, and let $f : U \to X$ denote the immersion of a dense open affine subset U of Z into X. An excision argument (made permissible by the commutativity of the diagram of part *(iii)* in the case that f is an immersion), together with our induction hypothesis, shows that it is enough to prove the result with $f^!\mathcal{M}^\bullet$ in place of $\mathcal{M}^\bullet$. Thus we may replace $\mathcal{M}^\bullet$ by $f^!\mathcal{M}^\bullet$, and X by U, and so assume that X is a smooth affine k-scheme.

Suppose to begin with that $\mathcal{M}^\bullet$ is a single object $\mathcal{M}$ of $\mu_{lfgu}(X_{\acute{e}t}, \Lambda)$. If $\mathcal{M}$ is a unit (F^r, Λ)-crystal then Lemma 9.4.1 shows that $\mathcal{M}$ is étale locally isomorphic to a tensor product $L \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{\acute{e}t}}$, for some finitely generated projective Λ-module L. Let L^* denote the Λ-dual of L, and let L'^* denote the Λ'-dual of $\Lambda' \otimes_\Lambda L$. Then the morphism of part *(i)* simplifies étale locally to the canonical isomorphism $\Lambda' \otimes_\Lambda L^* \xrightarrow{\sim} L'^*$. This establishes part *(i)* in this case.

If $\mathcal{M}$ is not a unit (F^r, Λ)-crystal then we apply the construction of (5.3.5) to a coherent generator of $\mathcal{M}$ to obtain a resolution of $\mathcal{M}$ by $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-flat locally finitely generated unit $\mathcal{O}^\Lambda_{F^r, X_{\acute{e}t}}$-modules. Since $\mathcal{M}$ is assumed to be of finite Tor-dimension, some finite-length truncation of this resolution again consists of $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-flat modules. Using Proposition 6.9.6 and an excision argument to replace X by a dense open affine subset on which the members of this complex are furthermore coherent as $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-modules, we obtain a bounded resolution of $\mathcal{M}$ by (Λ, F^r)-crystals. A spectral sequence argument now shows that the morphism of part *(i)* is an isomorphism, since it is so for a unit (Λ, F^r)-crystal.

We now proceed by induction on the number of non-vanishing cohomology sheaves of the complex $\mathcal{M}^\bullet$, and given the result of the preceding paragraph we may as well assume that there are at least two such. For ease of notation, apply a shift to $\mathcal{M}^\bullet$ so that its highest non-zero cohomology sheaf appears in degree zero. Then we may replace $\mathcal{M}^\bullet$ by its truncation $\tau_{\leq 0}\mathcal{M}^\bullet$, and so write it as a complex

$$\cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to 0 \to \cdots .$$

By assumption, $H^0(\mathcal{M}^\bullet)$ is a locally finitely generated unit $\mathcal{O}^\Lambda_{F^r, X_{\acute{e}t}}$-module on the affine scheme X, to which we may apply the construction of (5.3.5).

Let $\beta : M \to F_X^{r*}M$ denote a coherent generator of $H^0(\mathcal{M}^\bullet)$, let $P \to M$ be a surjection from a finite rank free $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-module onto M, let $\gamma : P \to F_X^{r*}P$ lift β, and let $\mathcal{P}^\bullet$ denote the object of $\mu_{lfgu}(X_{\acute{e}t}, \Lambda)$ generated by γ. The surjection of P onto M induces a surjection of $\mathcal{P}$ onto $H^0(\mathcal{M}^\bullet)$. Applying the construction of Proposition 5.3.3 to γ yields a short exact sequence

$$0 \to \mathcal{O}^\Lambda_{F^r, X_{\acute{e}t}} \otimes_{\mathcal{O}^\Lambda_{X_{\acute{e}t}}} P \to \mathcal{O}^\Lambda_{F^r, X_{\acute{e}t}} \otimes_{\mathcal{O}^\Lambda_{X_{\acute{e}t}}} P \to \mathcal{P} \to 0.$$

We may lift the surjection of $\mathcal{P}$ onto $H^0(\mathcal{M}^\bullet)$ so as to obtain a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} \otimes P & \longrightarrow & \mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} \otimes P & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{M}^{-2} & \longrightarrow & \mathcal{M}^{-1} & \longrightarrow & \mathcal{M}^{0} & \longrightarrow & 0 & \longrightarrow & \cdots . \end{array}$$

Abbreviate the top row of this diagram by $\mathcal{Q}^\bullet$, and let $\mathcal{C}^\bullet$ denote the cone of the morphism $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$ that it provides. Since both $\mathcal{Q}^\bullet$ and $\mathcal{M}^\bullet$ are of finite Tor-dimension, we see that the same is true of $\mathcal{C}^\bullet$.

A consideration of the long exact sequence of cohomology sheaves provided by this morphism shows that the morphism $H^i(\mathcal{M}^\bullet) \to H^i(\mathcal{C}^\bullet)$ is an isomorphism if $i \leq -2$ or if $i \geq 1$ (in the latter case both cohomology sheaves vanish), and yields the exact sequence

$$0 \to H^{-1}(\mathcal{M}^\bullet) \to H^{-1}(\mathcal{C}^\bullet) \to H^0(\mathcal{Q}^\bullet) \to H^0(\mathcal{M}^\bullet) \to H^0(\mathcal{C}^\bullet) \to 0.$$

By construction the map $H^0(\mathcal{Q}^\bullet) \to H^0(\mathcal{M}^\bullet)$ is surjective, and so we see that $H^0(\mathcal{C}^\bullet) = 0$. Thus $\mathcal{C}^\bullet$ is a complex with one less non-vanishing cohomology sheaf than $\mathcal{M}^\bullet$. Our induction hypothesis implies that the morphism of part *(i)* induces isomorphisms $\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{C}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{C}^\bullet)$ and $\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{Q}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{Q}^\bullet)$. Thus it also induces an isomorphism $\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\Lambda' \overset{\mathbb{L}}{\otimes}_\Lambda \mathcal{M}^\bullet)$. This completes the proof of Proposition 9.10. □

9.11. — Let r' be a multiple of r, $q' = p^{r'}$, $\Lambda' = \Lambda \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$, and suppose that $\mathbb{F}_{q'} \subset k$. The following proposition studies the compatibility of $\mathrm{Sol}_{\acute{e}t}$ with induction and restriction. Since the statement involves both the fields $\mathbb{F}_q$ and $\mathbb{F}_{q'}$, we will denote the corresponding functors $\mathrm{Sol}_{\acute{e}t}$ by $\mathrm{Sol}_{\acute{e}t,q}$ and $\mathrm{Sol}_{\acute{e}tq'}$.

Proposition 9.11.1. — *If X is a smooth k-scheme and Λ a Noetherian $\mathbb{F}_q$-algebra, then there is an isomorphism of functors on $D^b_{lfgu}(\mathcal{O}^{\Lambda'}_{F^r,X_{\acute{e}t}})^\circ$,*

$$\mathrm{Sol}_{\acute{e}t,q} \circ \mathrm{Ind}^q_{q'} \xrightarrow{\sim} \mathrm{Res}^q_{q'} \circ \mathrm{Sol}_{\acute{e}t,q'}, \qquad (9.11.2)$$

which is compatible with inseparable base-change. If furthermore Λ is a finite $\mathbb{F}_q$-algebra, then there is an isomorphisms of functors on $D^b_{lfgu}(\mathcal{O}^{\Lambda'}_{F^r,X_{\acute{e}t}})^\circ$,

$$\mathrm{Ind}^{q'}_q \circ \mathrm{Sol}_{\acute{e}t,q} \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t,q'} \circ \mathrm{Res}^{q'}_q, \qquad (9.11.3)$$

which again is compatible with inseparable base-change.

Proof. — Let $\mathcal{I}^\bullet$ be an resolution of $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$ by injective $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules. Since $\mathcal{O}^{\Lambda'}_{X_{\acute{e}t}}$ is isomorphic to $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$, and since $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules is flat as a right $\mathcal{O}^{\Lambda'}_{F^{r'},X_{\acute{e}t}}$-modules (by Lemma 1.14.1), we see that $\mathcal{I}^\bullet$ is also a resolution of $\mathcal{O}^{\Lambda'}_{X_{\acute{e}t}}$ by injective $\mathcal{O}^{\Lambda'}_{F^{r'},X_{\acute{e}t}}$-modules.

Let $\mathcal{M}^\bullet$ be a complex in $D^-_{lfgu}(\mathcal{O}^{\Lambda'}_{F^{r'},X})$. Then we have the natural isomorphism of complexes of étale sheaves of Λ-modules

$$\begin{aligned}\mathrm{Sol}_{ét,q} \circ \mathrm{Ind}^q_{q'} \mathcal{M}^\bullet = \underline{Hom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{ét}}}(\mathcal{O}^\Lambda_{F^r,X_{ét}} \otimes_{\mathcal{O}^{\Lambda'}_{F^{r'},X_{ét}}} \mathcal{M}^\bullet, \mathcal{I}^\bullet)\\ \xrightarrow{\sim} \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^{r'},X_{ét}}}(\mathcal{M}^\bullet, \mathcal{I}^\bullet) = \mathrm{Res}^q_{q'} \circ \mathrm{Sol}_{ét,q'} \mathcal{M}^\bullet.\end{aligned}$$

This constructs the isomorphism (9.11.2). It is easily seen to be compatible with inseparable base-change.

Now assume that Λ is finite. To construct (9.11.3), let $\mathcal{M}^\bullet$ now denote a complex in $D^-_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$. Then there is a natural morphism of complexes of étale sheaves of Λ'-modules

$$\begin{aligned}\mathrm{Ind}^{q'}_q \circ \mathrm{Sol}_{ét,q} \mathcal{M}^\bullet = \Lambda' \otimes_\Lambda \underline{Hom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{ét}}}(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\\ \longrightarrow \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda'}_{F^{r'},X_{ét}}}(\mathcal{M}^\bullet, \mathcal{I}^\bullet) = \mathrm{Sol}_{ét,q'} \circ \mathrm{Res}^{q'}_q \mathcal{M}^\bullet.\end{aligned}$$

This induces the morphism (9.11.3). Again, it is clearly compatible with inseparable base-change. Thus in order to show this is an isomorphism, we may replace k by its perfect closure, and so assume that k is perfect.

We proceed by induction on the dimension of the support of $\mathcal{M}^\bullet$. A spectral sequence argument allows us to reduce to proving that the morphism under study is an isomorphism when $\mathcal{M}^\bullet$ is a single object $\mathcal{M}$ in $\mu_{lfgu}(X_{ét}, \Lambda)$. An excision argument allows us to restrict to a dense affine open subset U of X. An application of the construction of (5.3.5) to a coherent generator of $\mathcal{M}$ allows us to replace $\mathcal{M}$ by a complex $\mathcal{P}^\bullet$ of locally finitely generated unit $\mathcal{O}^\Lambda_{U,F^r}$-modules which are flat as $\mathcal{O}^\Lambda_{U_{ét}}$-modules, and another spectral sequence argument allows us to reduce to proving the map in question an isomorphism for each member $\mathcal{P}^i$ of $\mathcal{P}^\bullet$. Proposition 6.9.6 together with another excision argument allows us to shrink U so that $\mathcal{P}^\bullet$ becomes $\mathcal{O}^\Lambda_{U_{ét}}$-coherent, as well as flat, and hence a unit (Λ, F^r)-crystal. Thus we are reduced to the case when $\mathcal{M}^\bullet = \mathcal{M}$ is a unit (Λ, F^r)-crystal, and so, by étale localising and applying Lemma 9.4.1, of the form $\mathcal{M} = L \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{ét}}$ for some finitely generated projective Λ-modules L.

Let L^* denote the Λ-dual of L, and L'^* denote the Λ'-dual of $\Lambda' \otimes_\Lambda L$. Then the above map now reduces to the map

$$\begin{aligned}\Lambda' \otimes_\Lambda L^* = \Lambda' \otimes_\Lambda \underline{Hom}_{\mathcal{O}^\Lambda_{F^r,X_{ét}}}(L \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{ét}}, \Lambda \otimes_{\mathbb{F}_q} \mathcal{O}_{X_{ét}})\\ \to \underline{Hom}_{\mathcal{O}_{F^{r'},X_{ét}}}((\Lambda' \otimes_\Lambda L) \otimes_{\mathbb{F}_{q'}} \mathcal{O}_{X_{ét}}, \Lambda' \otimes_{\mathbb{F}_{q'}} \mathcal{O}_{X_{ét}}) \xrightarrow{\sim} L'^*.\end{aligned}$$

Since this is simply the standard isomorphism $\Lambda' \otimes_\Lambda L^* \xrightarrow{\sim} L'^*$, we see that (9.11.3) is an isomorphism, as required. $\square$

10. THE FUNCTOR $\mathrm{M}_{ét}$

10.1. — Suppose that X is a smooth k-scheme. Then the sheaf of rings $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$ is flat over Λ, and so any injective $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-module is also an injective Λ-module. (The forgetful functor "regard an $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-module as a Λ-module" is right adjoint to the exact functor $\mathcal{O}^{\Lambda}_{F^r,X_{ét}} \otimes_{\Lambda} -$.) Thus if $\mathcal{M}^{\bullet}$ is any complex in $D^b(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$, any resolution of $\mathcal{M}^{\bullet}$ by injective $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-modules is also an injective resolution of $\mathcal{M}^{\bullet}$ by injective Λ-sheaves. Thus we regard $\underline{RHom}^{\bullet}_{\Lambda}(-,\mathcal{M}^{\bullet})$ as a contravariant functor

$$\underline{RHom}^{\bullet}_{\Lambda}(-,\mathcal{M}^{\bullet}) : D^-_c(X_{ét},\Lambda) \to D^+(\mathcal{O}^{\Lambda}_{F^r,X_{ét}}).$$

Definition 10.1.1. — If X is a smooth k-scheme, we define the functor

$$\mathrm{M}_{ét} : D^-_c(X_{ét},\Lambda) \to D^+(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$$

as follows:

$$\mathrm{M}_{ét}(\mathcal{F}^{\bullet}) = \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet},\mathcal{O}^{\Lambda}_{X_{ét}})[d_X].$$

Lemma 10.1.2. — *If X is a smooth k-scheme then there is a commutative diagram of natural transformations*

$$\begin{array}{ccccc}
F^{r*}_X \mathrm{M}_{ét}(\mathcal{F}^{\bullet}) & = & F^{r*}_X \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet},\mathcal{O}^{\Lambda}_{X_{ét}})[d_X] & \longrightarrow & \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet},F^{r*}_X\mathcal{O}^{\Lambda}_{X_{ét}}) \\
\downarrow{\scriptstyle \phi_{\mathrm{M}_{ét}(\mathcal{F}^{\bullet})}} & & & \swarrow{\scriptstyle \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet},\phi_{\mathcal{O}^{\Lambda}_{X_{ét}}})} & \\
\mathrm{M}_{ét}(\mathcal{F}^{\bullet}) & = & \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet},\mathcal{O}^{\Lambda}_{X_{ét}})[d_X] & &
\end{array}$$

in which the upper horizontal arrow is an isomorphism. Since $\phi_{\mathcal{O}^{\Lambda}_{X_{ét}}}$ is an isomorphism, we deduce that the same is true of $\phi_{\mathrm{M}_{ét}(\mathcal{F}^{\bullet})}$.

Proof. — Let $\mathcal{I}^{\bullet}$ be a right resolution of $\mathcal{O}^{\Lambda}_{X_{ét}}$ by left $\mathcal{O}^{\Lambda}_{F^r,X_{ét}}$-modules, which are injective as Λ-modules. The sheaf $\mathcal{O}^{\Lambda(r)}_{X_{ét}}$ is locally free as a right $\mathcal{O}^{\Lambda}_{X_{ét}}$-module, hence $F^{r*}_X\mathcal{I}^{\bullet}$ is a direct limit of injective Λ-sheaves, and so (by the usual Noetherian argument, since by [**SGA 4**, IX 2.9] any Λ-sheaf is the direct limit of its constructible subsheaves) we see that $F_X^{r*}\mathcal{I}^\bullet$ is again a complex of injective Λ-modules. Thus the commutative diagram

$$\begin{array}{ccc} F_X^{r*}\,\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,\mathcal{I}^\bullet)[d_X] & \longrightarrow & \underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,F_X^{r*}\mathcal{I}^\bullet) \\ \downarrow & \swarrow{\scriptstyle \underline{Hom}^\bullet(\mathcal{F}^\bullet,\phi_{\mathcal{I}^\bullet})} & \\ \underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,\mathcal{I}^\bullet)[d_X] & & \end{array}$$

is a realisation of the diagram of the lemma on the level of complexes.

To show that the horizontal arrow is an isomorphism, the usual spectral sequence argument reduces us to the case in which the complex $\mathcal{F}^\bullet$ is a single constructible sheaf $\mathcal{F}$. Since $\mathcal{F}$ is Noetherian, we may pass the tensor product with the locally free sheaf $\mathcal{O}_{X_{\acute{e}t}}^{\Lambda(r)}$ through $\underline{Hom}_\Lambda(\mathcal{F},-)$. This completes the proof. □

Remark 10.1.3. — We see from this lemma (via (2.11.1) and the remark following it) that the structural morphisms of the cohomology sheaves of $\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet)$ are isomorphisms. We will see in Proposition 10.4 below that in fact $\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet)$ lies in $D^+_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})$.

10.1.4. — Let X be a smooth k-scheme, let k' be a purely inseparable algebraic extension of k, and set $X' = X\otimes_k k'$. The morphism $X'\to X$ identifies $X'_{\acute{e}t}$ with $X_{\acute{e}t}$, and induces an isomorphism $k'\otimes_k \mathcal{O}^\Lambda_{X'_{\acute{e}t}} \xrightarrow{\sim} \mathcal{O}^\Lambda_{X_{\acute{e}t}}$.

If $\mathcal{I}^\bullet$ is a resolution of $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$ by injective $\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}$-modules, then $k'\otimes_k\mathcal{I}^\bullet$ is a resolution of $\mathcal{O}^\Lambda_{X'_{\acute{e}t}}$ by $\mathcal{O}^\Lambda_{F^r,X'_{\acute{e}t}}$-modules, and so we may find a quasi-isomorphism $k'\otimes_k\mathcal{I}^\bullet\to\mathcal{I}'^\bullet$, where $\mathcal{I}'^\bullet$ is a resolution of $\mathcal{O}^\Lambda_{X'_{\acute{e}t}}$ by injective $\mathcal{O}^\Lambda_{F^r,X'_{\acute{e}t}}$-modules. Thus for any object $\mathcal{F}^\bullet$ of $D^b_c(X_{\acute{e}t})$ there is a natural morphism

$$k'\otimes_k \underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,\mathcal{I}^\bullet)\to\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,k'\otimes_k\mathcal{I}^\bullet)\to\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,\mathcal{I}'^\bullet),$$

and hence a natural transformation $k'\otimes_k\mathrm{M}_{\acute{e}t}\to\mathrm{M}_{\acute{e}t}{}'$ (where $\mathrm{M}_{\acute{e}t}{}'$ denotes the functor $\mathrm{M}_{\acute{e}t}$ computed on X' rather than X).

Lemma 10.1.5. — *The preceding natural transformation is an isomorphism. Thus the functor* $\mathrm{M}_{\acute{e}t}$ *is compatible with inseparable base-change.*

Proof. — Since the objects of $\mathcal{F}^\bullet$ are Noetherian, computing $\underline{Hom}^\bullet(\mathcal{F}^\bullet,-)$ commutes with passage to direct limits. Since k' is a limit of finite dimensional k-vector spaces, we conclude that the morphism $k'\otimes_k\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,\mathcal{I}^\bullet)\to\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,k'\otimes_k\mathcal{I}^\bullet)$ is an isomorphism. Also, we see that $k'\otimes_k\mathcal{I}^\bullet$, being a direct limit of injective Λ-sheaves, is itself an injective Λ-sheaf. Thus $k'\otimes_k\mathcal{I}^\bullet\to\mathcal{I}'^\bullet$ is a quasi-isomorphism of injective Λ-sheaves, and so the morphism $\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,k'\otimes_k\mathcal{I}^\bullet)\to\underline{Hom}_\Lambda^\bullet(\mathcal{F}^\bullet,\mathcal{I}'^\bullet)$ is a quasi-isomorphism. □

10.2. — The following simple lemma will be the basis for all our computations of $\mathrm{M}_{\acute{e}t}$:

Lemma 10.2.1. — *If X is a smooth k-scheme, then* $\mathrm{M}_{\acute{e}t}(\Lambda)=\mathcal{O}^\Lambda_{X_{\acute{e}t}}[d_X]$.

Proof. — This follows immediately from the definition. □

10.3. — In this section we will prove the existence of a natural isomorphism $\mathrm{M}_{ét}f_! = f_+\mathrm{M}_{ét}$ when $f : Y \to X$ is an immersion of smooth k-schemes.

Lemma 10.3.1. — *If $f : Y \to X$ is an open immersion of smooth k-schemes, then there is a natural isomorphism of functors:*

$$\mathrm{M}_{ét}f_! \xrightarrow{\sim} f_+\mathrm{M}_{ét}.$$

Proof. — Let $\mathcal{F}^\bullet$ be an object in $D_c^-(X_{ét}, \Lambda)$, and let $\mathcal{I}^\bullet$ be a resolution of $\mathcal{O}^\Lambda_{X_{ét}}$ by injective $\mathcal{O}^\Lambda_{F^r,X_{ét}}$-modules (which are also Λ-injective, by the above discussion). Then $f^{-1}\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{O}^\Lambda_{Y_{ét}}$ in the category of $\mathcal{O}^\Lambda_{F^r,Y_{ét}}$-modules (since f^{-1} is right adjoint to the exact functor $f_!$).

We have a natural transformation of complexes of $\mathcal{O}^\Lambda_{F^r,X}$

$$\begin{aligned}\mathrm{M}_{ét}(f_!\mathcal{F}^\bullet) = \underline{RHom}^\bullet_\Lambda(f_!\mathcal{F}^\bullet, \mathcal{O}^\Lambda_{X_{ét}})[d_X] \xrightarrow{\sim} \underline{Hom}^\bullet_\Lambda(f_!\mathcal{F}^\bullet, \mathcal{I}^\bullet)[d_X] \\ \xrightarrow{\sim} f_*\underline{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet, f^{-1}\mathcal{I}^\bullet)[d_X] \to Rf_*\underline{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet, f^{-1}\mathcal{I}^\bullet)[d_X] = Rf_*\mathrm{M}_{ét}(\mathcal{F}^\bullet),\end{aligned}$$

where the final equality follows from the fact that X and Y have the same dimension. To check that this is a quasi-isomorphism it suffices to check on the level of complexes of étale Λ-sheaves in which case the composite of the natural maps above becomes the composite of the isomorphisms

$$\mathrm{M}_{ét}(f_!\mathcal{F}^\bullet) = \underline{RHom}^\bullet_\Lambda(f_!\mathcal{F}^\bullet, \mathcal{O}^\Lambda_{X_{ét}}) \xrightarrow{\sim} Rf_*\underline{RHom}^\bullet_\Lambda(\mathcal{F}^\bullet, f^{-1}\mathcal{O}^\Lambda_{X_{ét}}) = Rf_*\mathrm{M}_{ét}(\mathcal{F}^\bullet)$$

induced by the adjointness of $f_!$ and f^{-1}.

Now we saw in Lemma 4.3.1 that

$$f_+\mathrm{M}_{ét}(\mathcal{F}^\bullet) \xrightarrow{\sim} Rf_*\mathrm{M}_{ét}(\mathcal{F}^\bullet).$$

Putting all these natural isomorphisms together, we see that

$$\mathrm{M}_{ét}(f_!\mathcal{F}^\bullet) \xrightarrow{\sim} f_+\mathrm{M}_{ét}(F^\bullet),$$

proving the lemma. □

Proposition 10.3.2. — *If $f : Y \to X$ is a closed immersion of smooth k-schemes, there is a natural isomorphism*

$$f_+\mathrm{M}_{ét} \xrightarrow{\sim} \mathrm{M}_{ét}f_!.$$

Proof. — We begin by describing the natural transformation. Recall by example 5.11.6 that we have canonical morphisms (in the derived category)

$$f_+\mathcal{O}^\Lambda_{Y_{ét}}[d_{Y/X}] \xrightarrow{\sim} f_+f^!\mathcal{O}^\Lambda_{X_{ét}} \xrightarrow{\sim} R\Gamma_Y(\mathcal{O}^\Lambda_{X_{ét}}) \to \mathcal{O}^\Lambda_{X_{ét}},$$

whose composite is the trace map of Proposition 4.4.9. Also, for the closed immersion f_*, no derived functors are required to define the functor f_+; it is given by the simple formula

$$f_+(-) = f_*(\mathcal{O}^\Lambda_{F^r,X_{ét}\leftarrow Y_{ét}} \otimes_{\mathcal{O}^\Lambda_{F^r,Y_{ét}}} -).$$

Thus for any complex $\mathcal{F}^\bullet$ in $D_c^-(Y_{\acute{e}t})$ we have the natural transformation

$$\begin{aligned} f_+\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet) &\xrightarrow{\sim} f_*(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}\leftarrow Y_{\acute{e}t}} \otimes_{\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}} \underline{RHom}^\bullet_\Lambda(\mathcal{F}^\bullet, \mathcal{O}^\Lambda_{Y_{\acute{e}t}})[d_Y]) \\ &\longrightarrow \underline{RHom}^\bullet_\Lambda(f_*\mathcal{F}^\bullet, f_*(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}\leftarrow Y_{\acute{e}t}} \otimes_{\mathcal{O}^\Lambda_{F^r,Y_{\acute{e}t}}} \mathcal{O}^\Lambda_{Y_{\acute{e}t}})[d_Y]) \\ &= \underline{RHom}^\bullet_\Lambda(f_*\mathcal{F}^\bullet, f_+\mathcal{O}^\Lambda_{Y_{\acute{e}t}}[d_{Y/X}])[d_X] \\ &\longrightarrow \underline{RHom}^\bullet_\Lambda(f_*\mathcal{F}^\bullet, \mathcal{O}^\Lambda_{X_{\acute{e}t}})[d_X] \\ &\xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(f_*\mathcal{F}^\bullet), \end{aligned}$$

which we will show is an isomorphism. This can be checked locally on $X_{\acute{e}t}$. Note also that both sides of this homomorphism vanish on the complement of Y in X, so it suffices to verify that we obtain an isomorphism in a neighbourhood in $X_{\acute{e}t}$ of each point of Y. As the last step of our initial reductions, note that the compatibility of $\mathrm{M}_{\acute{e}t}$ with inseparable base-change allows us to replace k by its perfect closure, so that for the duration of the proof we may assume that k is perfect.

By [**De**, Prop. 4.6, p. 93], $\mathcal{F}^\bullet$ is represented by a bounded above complex of constructible, flat Λ-sheaves. Thus by the usual spectral sequence argument, it suffices to treat the case of a single constructible sheaf $\mathcal{F}$ of flat Λ-modules.

Suppose first that the constructible sheaf $\mathcal{F}$ is locally constant. Then being flat over Λ, it is a locally constant sheaf of finitely generated projective modules, and so is étale locally a direct summand of a free Λ-module of finite rank. Thus to show that the natural transformation $f_+\mathrm{M}_{\acute{e}t}(\mathcal{F}) \to \mathrm{M}_{\acute{e}t}(f_!\mathcal{F})$ is an isomorphism, it suffices to treat the case when $\mathcal{F} = \Lambda$, in which case this natural transformation reduces to the composite of the isomorphisms

$$f_+\mathcal{O}^\Lambda_{Y_{\acute{e}t}}[d_Y] \xrightarrow{\sim} R\Gamma_Y(\mathcal{O}^\Lambda_{X_{\acute{e}t}})[d_X] \xrightarrow{\sim} \underline{RHom}^\bullet_\Lambda(f_*\Lambda, \mathcal{O}^\Lambda_{X_{\acute{e}t}})[d_X].$$

We now proceed by induction on the dimension of the support of $\mathcal{F}$. Let Z denote this support. There is a dense open subscheme V of Z (given its reduced induced structure) which is smooth over k, such that over V the sheaf $\mathcal{F}$ restricts to a locally constant sheaf of finitely generated projective Λ-modules $\mathcal{L}$. Let g be the immersion of V into Y. Let $W = Z \setminus V$, a closed subset of X of dimension less than that of Z, and let $h : W \to Y$ be the closed immersion. We have a short exact sequence

$$0 \to g_!\mathcal{L} = g_!g^{-1}\mathcal{F} \to \mathcal{F} \to h_!h^{-1}\mathcal{F} \to 0.$$

By induction, we already know the result for $h_!h^{-1}\mathcal{F}$, so it suffices to prove it for $g_!\mathcal{L}$.

We may factor the immersion g as the composite of an open immersion j with a closed immersion i: $g = ji$. We also factor the immersion fj as the composite of an open immersion l and a closed immersion k: $fj = lk$. We obtain a diagram

$$\begin{array}{ccccccc} f_+\mathrm{M}_{\acute{e}t}(g_!\mathcal{L}) & = & f_+\mathrm{M}_{\acute{e}t}(j_!i_!\mathcal{L}) & \xrightarrow{\sim} & f_+j_+\mathrm{M}_{\acute{e}t}(i_!\mathcal{L}) & = & l_+k_+\mathrm{M}_{\acute{e}t}(i_!\mathcal{L}) \\ \downarrow & & & & & & \downarrow{\scriptstyle\sim} \\ \mathrm{M}_{\acute{e}t}(f_!g_!\mathcal{L}) & = & \mathrm{M}_{\acute{e}t}(l_!k_!i_!\mathcal{L}) & \xrightarrow{\sim} & l_+\mathrm{M}_{\acute{e}t}(k_!i_!\mathcal{L}) & \xleftarrow[\sim]{} & l_+k_+i_+\mathrm{M}_{\acute{e}t}(\mathcal{L}) \end{array}$$

in which the two right pointing horizontal arrows are the natural isomorphisms of Lemma 10.3.1 applied to the open immersions j and l, the left pointing horizontal arrow is induced by the natural transformation $(ki)_+\mathrm{M}_{ét}(\mathcal{L}) \to \mathrm{M}_{ét}((ki)_!\mathcal{L})$, and so is an isomorphism since $\mathcal{L}$ is locally constant, and the right vertical arrow is induced by the inverse of the natural transformation $i_+\mathrm{M}_{ét}(\mathcal{L}) \xrightarrow{\sim} \mathrm{M}_{ét}(i_!\mathcal{L})$ (which again is an isomorphism because $\mathcal{L}$ is locally constant). One easily verifies that this diagram commutes, and thus that the left vertical arrow is also an isomorphism, as required. □

Corollary 10.3.3. — *If $f : Y \to X$ is an immersion of smooth k-schemes, then there is a natural isomorphism*

$$f_+\mathrm{M}_{ét} \xrightarrow{\sim} \mathrm{M}_{ét}f_!.$$

Proof. — Since any immersion decomposes as the composition of an open and closed immersion, this follows from the preceding two results. □

Proposition 10.4. — *If X is a smooth k-scheme, then* $\mathrm{M}_{ét}$ *has image lying in* $D^+_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})$. *Furthermore,* $\mathrm{M}_{ét}$ *restricts to a functor*

$$\mathrm{M}_{ét} : D^b_{ctf}(X_{ét}, \Lambda) \to D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})^{\circ}.$$

Proof. — The compatibility of $\mathrm{M}_{ét}$ with inseparable base-change allows us to replace k by its perfect closure, and so for the duration of the proof we may assume that k is perfect. Let $\mathcal{F}^{\bullet}$ be a complex in $D^-_{ctf}(X_{ét}, \Lambda)$. By [**De**, Prop. 4.6, p. 93], $\mathcal{F}^{\bullet}$ is represented by a bounded above complex of constructible, flat Λ-sheaves, which can be taken to be of finite length in and only if $\mathcal{F}^{\bullet}$ lies in $D^b_{ctf}(\mathcal{O}_{F^r,X})$. A spectral sequence argument then shows that to establish the proposition, it suffices to prove that if $\mathcal{F}$ is a single flat constructible étale Λ-sheaf, then $\mathrm{M}_{ét}(\mathcal{F})$ lies in $D^b_{lfgu}(\mathcal{O}_{F^r,X_{ét}})^{\circ}$.

We proceed by induction on the dimension of the support of $\mathcal{F}$. Let Z denote this support. Then there is a dense open subscheme Y of Z (given its reduced induced structure) which is smooth as a k-scheme, restricted to which $\mathcal{F}$ becomes a locally constant sheaf $\mathcal{L}$ of finitely generated projective Λ-modules. Let $f : Y \to X$ denote the immersion of Y into X. Let $W = Z \setminus Y$ be the complement of Y in Z, a closed subset of X of dimension less than that of Z. Let $i : W \to X$ denote the closed immersion of W into X.

Applying $\mathrm{M}_{ét}$ to the exact sequence $0 \to f_!\mathcal{L} \to \mathcal{F} \to i_!i^{-1}\mathcal{F} \to 0$ of constructible flat Λ-sheaves yields the distinguished triangle

$$\mathrm{M}_{ét}(i_!i^{-1}\mathcal{F}) \to \mathrm{M}_{ét}(\mathcal{F}) \to \mathrm{M}_{ét}(f_!\mathcal{L}) \to \mathrm{M}_{ét}(i_!i^{-1}\mathcal{F})[1].$$

Since $i_!i^{-1}\mathcal{F}$ is supported on W, it follows by induction that $\mathrm{M}_{ét}(i_!i^{-1}\mathcal{F})$ is in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$. Thus to conclude the corresponding result for $\mathrm{M}_{ét}(\mathcal{F})$, it suffices to prove it for $\mathrm{M}_{ét}(f_!\mathcal{L})$.

We have seen that $\mathrm{M}_{ét}(f_!\mathcal{L}) \xrightarrow{\sim} f_+\mathrm{M}_{ét}(\mathcal{L})$. Since f_+ takes $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y_{ét}})^{\circ}$ to $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})^{\circ}$, it is enough to show that $\mathrm{M}_{ét}(\mathcal{L})$ is a complex in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{ét}})^{\circ}$. In fact, we will see that it is a unit (Λ, F^r)-crystal supported in degree $-d_Y$.

Indeed $\mathrm{M}_{ét}(\mathcal{L}) = \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{L}^{\bullet}, \mathcal{O}^{\Lambda}_{Y_{ét}})[d_Y]$. Locally on $Y_{ét}$, $\mathcal{L}$ is isomorphic to a direct summand of Λ^n for some n, and so $\underline{RHom}^{\bullet}_{\Lambda}(\mathcal{L}, \mathcal{O}^{\Lambda}_{Y_{ét}}) = \underline{Hom}_{\Lambda}(\mathcal{L}, \mathcal{O}^{\Lambda}_{Y_{ét}})$ is a sheaf which is étale locally on Y, isomorphic to a direct summand of $(\mathcal{O}^{\Lambda}_{Y_{\acute{e}t}})^n$, and so is a unit (Λ, F^r) crystal, as claimed. This completes the proof of the proposition. □

Proposition 10.5. — *(i) Let X be a smooth k-scheme, and let Λ' be a Noetherian Λ-algebra. If $\mathcal{F}^{\bullet}$ is a complex in $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$, then there is a natural isomorphism*

$$\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathrm{M}_{\acute{e}t}(\mathcal{F}^{\bullet}) \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{F}^{\bullet}),$$

which is compatible with inseparable base-change. (Note that by Proposition 10.4, the $\overset{\mathbb{L}}{\otimes}$ appearing on the left side of the map is well-defined.)

(ii) If $f : Y \to X$ is an immersion of smooth k-schemes, and if $\mathcal{F}^{\bullet}$ is an object of $D^b_{ctf}(Y_{\acute{e}t}, \Lambda)$, then the following diagram of natural isomorphisms commutes:

$$\begin{array}{ccccc}
\Lambda' \otimes_{\Lambda} \mathrm{M}_{\acute{e}t}(f_!\mathcal{F}^{\bullet}) & \underset{(10.3.3)}{\xrightarrow{\sim}} & \Lambda' \otimes_{\Lambda} f_+\mathrm{M}_{\acute{e}t}(\mathcal{F}^{\bullet}) & \underset{(3.10)}{\xrightarrow{\sim}} & f_+(\Lambda' \otimes_{\Lambda} \mathrm{M}_{\acute{e}t}(\mathcal{F}^{\bullet})) \\
{\scriptstyle part\ (i)}\downarrow\wr & & & & {\scriptstyle part\ (i)}\downarrow\wr \\
\mathrm{M}_{\acute{e}t}(\Lambda' \otimes_{\Lambda} f_!\mathcal{F}^{\bullet}) & \underset{(8.6)}{\xrightarrow{\sim}} & \mathrm{M}_{\acute{e}t}(f_!(\Lambda' \otimes_{\Lambda} \mathcal{F}^{\bullet})) & \underset{(10.3.3)}{\xrightarrow{\sim}} & f_+(\mathrm{M}_{\acute{e}t}(\Lambda' \otimes_{\Lambda} \mathcal{F}^{\bullet})).
\end{array}$$

Proof. — We apply (the obvious analogue for $\mathcal{X}_{\acute{e}t}$ of) Proposition B.1.1, taking $\mathcal{A} = \mathcal{B} = \mathbb{F}_q$, $\mathcal{A}' = \Lambda$, $\mathcal{A}'' = \Lambda'$, $\mathcal{M}^{\bullet} = \mathcal{F}^{\bullet}$ and $\mathcal{N}^{\bullet} = \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$. (Note that $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$ is of finite Λ-Tor-dimension, in fact flat, over Λ, and that $\underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}})$ is bounded, by Proposition 10.4.) This yields a morphism

$$\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \underline{RHom}^{\bullet}_{\Lambda}(\mathcal{F}^{\bullet}, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}) \to \underline{RHom}^{\bullet}_{\Lambda'}(\Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{F}^{\bullet}, \Lambda' \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}).$$

Shifting by $[d_X]$ and taking into account the isomorphism (1.13.1) yields the morphism of part *(i)*. Using functoriality of the natural transformation of Proposition B.1.1 in the second variable, it is easy to see that this morphism is compatible with inseparable base-change.

Before proving that the map just constructed is an isomorphism, we will establish the commutativity of the diagram of part *(ii)*. By factoring the immersion f as the composite of an open and closed immersion, we see that it suffices to handle each of these two cases separately. In the case of an open immersion, this commutativity follows from the commutativity of the diagram of section (B.1.4) together with the construction of the natural isomorphism of Proposition 9.5. The case of a closed immersion is more tedious to check, but is nevertheless straightforward, if one takes into account the commutativity of the diagram of section (B.3), as well as the fact that the trace map of Proposition 4.4.9 (which is used in construction of the isomorphism of Proposition 10.3.2) is compatible with change of coefficient ring.

We now turn to proving that the morphism of part *(i)* is an isomorphism. Since it is compatible with inseparable base-change, we may replace k by its perfect closure, and thus assume that k is perfect. Using [**De**, Prop. 4.6, p. 93], we may represent $\mathcal{F}^{\bullet}$ by a bounded complex of constructible, flat étale Λ-sheaves. By induction on the length of this complex, we see that it suffices to prove that the morphism of part *(i)* is an isomorphism in the case that $\mathcal{F}^{\bullet}$ is a single flat constructible Λ-sheaf $\mathcal{F}$.

We now proceed by induction on the dimension of the support of $\mathcal{F}$. Using part *(ii)* of the proposition, an excision argument of the type used in the proof of Proposition 10.4 reduces us to the case when $\mathcal{F}$ is a locally constant sheaf of finitely generated projective Λ-modules. Working étale locally and writing $\mathcal{F}$ as a direct summand of a sheaf of free Λ-modules, we reduce to the case when $\mathcal{F}$ equals Λ. In this case the morphism of part *(i)* reduces to the natural isomorphism $\Lambda' \otimes_\Lambda \mathcal{O}^{\Lambda}_{X_{\acute{e}t}} \xrightarrow{\sim} \mathcal{O}^{\Lambda'}_{X_{\acute{e}t}}$. This completes the proof of the proposition. □

10.6. — Let r' be a multiple of r, $q' = p^{r'}$, $\Lambda' = \Lambda \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$, and suppose that $\mathbb{F}_{q'} \subset k$. The following proposition studies the compatibility of $\mathrm{M}_{\acute{e}t}$ with induction and restriction. Since the statement involves both the fields $\mathbb{F}_q$ and $\mathbb{F}_{q'}$, we will denote the corresponding functors $\mathrm{M}_{\acute{e}t}$ by $\mathrm{M}_{\acute{e}t,q}$ and $\mathrm{M}_{\acute{e}tq'}$.

Proposition 10.6.1. — *Let r' be a multiple of r, $q' = p^{r'}$, $\Lambda' = \Lambda \otimes_{\mathbb{F}_q} \mathbb{F}_{q'}$, and suppose that $\mathbb{F}_{q'} \subset k$. Then we have natural isomorphisms of functors on $D^-_c(X_{\acute{e}t}, \Lambda)$, respectively $D^-_c(X_{\acute{e}t}, \Lambda')$,*

$$G_{,q'} \circ \mathrm{Ind}_q^{q'} \xrightarrow{\sim} \mathrm{Res}_q^{q'} \circ \mathrm{M}_{\acute{e}t,q}, \tag{10.6.2}$$

respectively

$$Ind_{q'}^{q} \circ \mathrm{M}_{\acute{e}t,q'} \xrightarrow{\sim} \mathrm{M}_{\acute{e}t,q} \circ \mathrm{Res}_{q'}^{q}. \tag{10.6.3}$$

Both of these maps are compatible with inseparable base-change.

Proof. — Let $\mathcal{L}^\bullet$ be in $D^-_c(X_{\acute{e}t}, \Lambda)$. Then, keeping in mind the equality $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}} = \mathcal{O}^{\Lambda'}_{X_{\acute{e}t}}$, we have the natural isomorphism

$$\begin{aligned}\mathrm{M}_{\acute{e}t,q'} \circ \mathrm{Ind}_q^{q'} \mathcal{L}^\bullet &= \underline{RHom}^\bullet_{\Lambda'}(\mathcal{L}^\bullet \otimes_\Lambda \Lambda', \mathcal{O}^{\Lambda'}_{X_{\acute{e}t}}) \\ &\xrightarrow{\sim} \underline{RHom}^\bullet_{\Lambda}(\mathcal{L}^\bullet, \mathcal{O}^{\Lambda}_{X_{\acute{e}t}}) = \mathrm{Res}_q^{q'} \circ \mathrm{M}_{\acute{e}t,q}\mathcal{L}^\bullet,\end{aligned}$$

which is clearly compatible with inseparable base-change. This constructs (10.6.2).

Let $\mathcal{I}^\bullet$ be an resolution of $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$ by injective $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules. As observed in the proof of Proposition 9.11.1, $\mathcal{I}^\bullet$ is also a resolution of $\mathcal{O}^{\Lambda'}_{X_{\acute{e}t}}$ by injective $\mathcal{O}^{\Lambda'}_{F^{r'},X_{\acute{e}t}}$-modules.

Let $\mathcal{L}^\bullet$ be in $D^-_{ctf}(X_{\acute{e}t}, \Lambda')$. Then we have a natural map

$$\begin{aligned}\mathrm{Ind}_{q'}^{q} \circ \mathrm{M}_{\acute{e}t,q'}\mathcal{L}^\bullet &= \mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}} \otimes_{\mathcal{O}^{\Lambda'}_{F^{r'},X_{\acute{e}t}}} \underline{Hom}^\bullet_{\Lambda'}(\mathcal{L}^\bullet, \mathcal{I}^\bullet) \\ &\longrightarrow \underline{Hom}^\bullet_{\Lambda}(\mathcal{L}^\bullet, \mathcal{I}^\bullet) = \mathrm{M}_{\acute{e}t,q} \circ \mathrm{Res}_{q'}^{q}\mathcal{L}^\bullet.\end{aligned}$$

This induces the morphism of (10.6.3), and we have to show this map is an isomorphism. It is again clear that this map is compatible with inseparable base-change; in particular in order to prove that it is an isomorphism, we may replace k by its perfect closure, and so assume that k is perfect.

To prove that (10.6.3) is an isomorphism we observe first that the usual spectral sequence argument allows us to reduce to the case when $\mathcal{L}^\bullet$ is a single constructible étale Λ'-sheaf $\mathcal{L}$. We proceed by induction on the dimension of the support of $\mathcal{L}^\bullet$. An excision argument (taking into account the fact that induction and restriction are compatible with $f_!$ and f_+ for any morphism f of smooth k-schemes, and that $\mathrm{M}_{\acute{e}t}$ interchanges $f_!$ and f_+ for any immersion of smooth k-schemes) allows us to assume that $\mathcal{L}$ is locally constant. Working étale locally, we may assume that in fact $\mathcal{L}$ is the constant sheaf corresponding to a finitely generated Λ'-module. Finally, by working with a resolution of $\mathcal{L}$ by free Λ'-sheaves of finite rank, we reduce to the case that $\mathcal{L} = \Lambda'$.

In the case that $\mathcal{L}^\bullet = \mathcal{L} = \Lambda$, the map above reduces to a map

$$\bigoplus_{n=0}^{(r/r)-1} \mathcal{O}_X^\Lambda F^n \xrightarrow{\sim} \mathcal{O}_{F^r, X_{\acute{e}t}}^\Lambda \otimes_{\mathcal{O}_{F^{r'}, X_{\acute{e}t}}^{\Lambda'}} \mathcal{O}_X^\Lambda \longrightarrow \underline{Hom}_\Lambda(\Lambda', \mathcal{O}_X^{\Lambda'})$$
$$\xrightarrow{\sim} \underline{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q'}, \mathcal{O}_X^\Lambda) \xrightarrow{\sim} \mathcal{O}_X^\Lambda \otimes_{\mathbb{F}_q} \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q'}, \mathbb{F}_q).$$

This map in turn is obtained by tensoring the natural map

$$\bigoplus_{n=0}^{(r'/r)-1} \mathbb{F}_q F^n \longrightarrow \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q'}, \mathbb{F}_q)$$

by $\mathcal{O}_{X_{\acute{e}t}}^\Lambda$ over $\mathbb{F}_q$, and the latter map is an isomorphism because the automorphisms $1, F^r, \dots, F^{r'-r}$ of $\mathbb{F}_{q'}/\mathbb{F}_q$ are distinct, hence linearly independent. $\square$

11. THE RIEMANN-HILBERT CORRESPONDENCE FOR UNIT $\mathcal{O}_{F,X}$-MODULES

11.1. — Throughout this section Λ will be assumed to be a finite ring, unless explicitly stated otherwise.

We now have all the machinery necessary to prove our Riemann-Hilbert correspondence between constructible Λ-sheaves on $X_{\acute{e}t}$ and locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules. We begin by noting that there are natural transformations $\eta : \mathrm{id} \to \mathrm{Sol}_{\acute{e}t}\mathrm{M}_{\acute{e}t}$ and $\psi : \mathrm{id} \to \mathrm{M}_{\acute{e}t}\mathrm{Sol}_{\acute{e}t}$. If $\mathcal{I}^\bullet$ is a right resolution of $\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}$ by injective left $\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}$-modules, then for any complex $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^\circ$ we define $\zeta_{\mathcal{M}^\bullet}$ as the composite morphism

$$\begin{aligned}\mathcal{M}^\bullet \to \underline{Hom}^\bullet_\Lambda(\underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\mathcal{M}^\bullet,\mathcal{I}^\bullet),\mathcal{I}^\bullet) \\ \to \underline{Hom}^\bullet_\Lambda(\underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\mathcal{M}^\bullet,\mathcal{I}^\bullet)[d_X],\mathcal{I}^\bullet)[d_X].\end{aligned}$$

Similarly, if $\mathcal{F}^\bullet$ is a complex in $D^b_{ctf}(X_{\acute{e}t},\Lambda)$, we define $\eta_{\mathcal{F}^\bullet}$ as the composite morphism

$$\begin{aligned}\mathcal{F}^\bullet \to \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\underline{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{I}^\bullet),\mathcal{I}^\bullet) \\ \to \underline{Hom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\underline{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{I}^\bullet)[d_X],\mathcal{I}^\bullet)[d_X].\end{aligned}$$

(We refer to [**Con**, 1.3] for a discussion of the sign conventions involved in the definitions of these morphisms.) It follows from the analysis of signs in [**Con**, 1.3] that the composites

$$\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet) \xrightarrow{\zeta_{\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet)}} \mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet))) \xrightarrow{\mathrm{M}_{\acute{e}t}(\eta_{\mathcal{F}^\bullet})} \mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet)$$

and

$$\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \xrightarrow{\eta_{\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)}} \mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet))) \xrightarrow{\mathrm{Sol}_{\acute{e}t}(\zeta_{\mathcal{M}^\bullet})} \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)$$

are the identity morphisms. Also, it is immediate that both ζ and η are compatible with base-change by an inseparable field extension, in an obvious sense.

We will prove that both ζ and η are isomorphisms, and consequently will conclude that $\mathrm{Sol}_{\acute{e}t}$ and $\mathrm{M}_{\acute{e}t}$ induce an equivalence of categories between $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^\circ$ and $D^b_{ctf}(X_{\acute{e}t},\Lambda)$. The proof will be via an excision argument, combining Theorem 9.7.1

and Corollary 10.3.3. Before making this argument, we must verify that these two results are compatible with the natural transformations ζ and η in an appropriate sense.

Proposition 11.2. — *If $f : Y \to X$ is an immersion of smooth k-schemes and if $\mathcal{M}^\bullet$ is an object of $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^{\circ}$, then the diagram of natural isomorphisms*

$$\begin{CD}
f_+\mathcal{M}^\bullet @>{\zeta_{f_+\mathcal{M}^\bullet}}>> \mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(f_+\mathcal{M}^\bullet)) \\
@V{f_+(\zeta_{\mathcal{M}^\bullet})}VV @V{(9.7.1)}V{\sim}V \\
f_+\mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)) @>{\sim}>{(10.3.3)}> \mathrm{M}_{\acute{e}t}(f_!\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet))
\end{CD}$$

(in which each natural isomorphism has been labelled by the result that gives rise to it) commutes.

Similarly, if $\mathcal{F}^\bullet$ is an object of $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$, then the diagram of natural isomorphisms

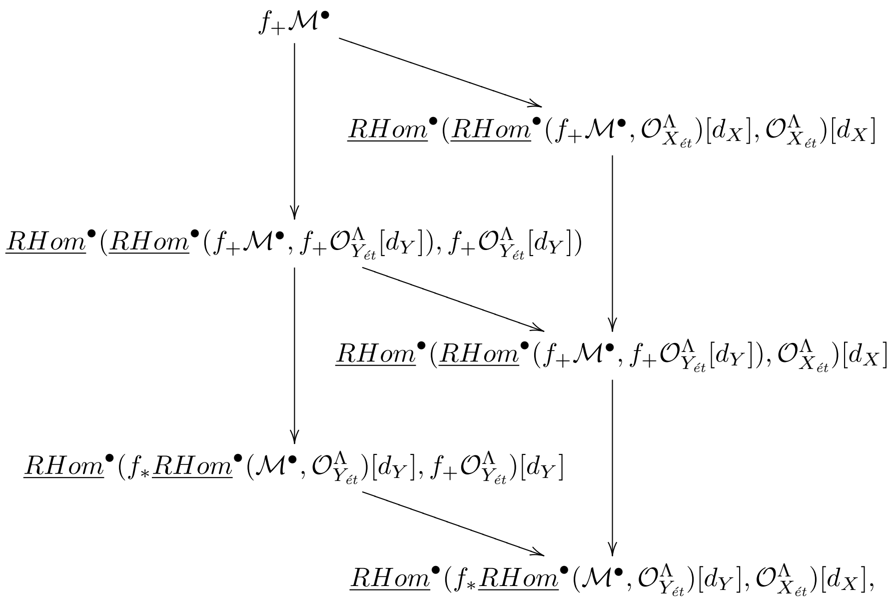
$$\begin{CD}
f_!\mathcal{F}^\bullet @>{\eta_{f_!\mathcal{F}^\bullet}}>> \mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(f_!\mathcal{F}^\bullet)) \\
@V{f_!(\eta_{\mathcal{F}^\bullet})}VV @V{(10.3.3)}V{\sim}V \\
f_!\mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet)) @>{\sim}>{(9.7.1)}> \mathrm{Sol}_{\acute{e}t}(f_+\mathrm{M}_{\acute{e}t}(\mathcal{M}^\bullet))
\end{CD}$$

(in which each natural isomorphism has been labelled by the result that gives rise to it) commutes.

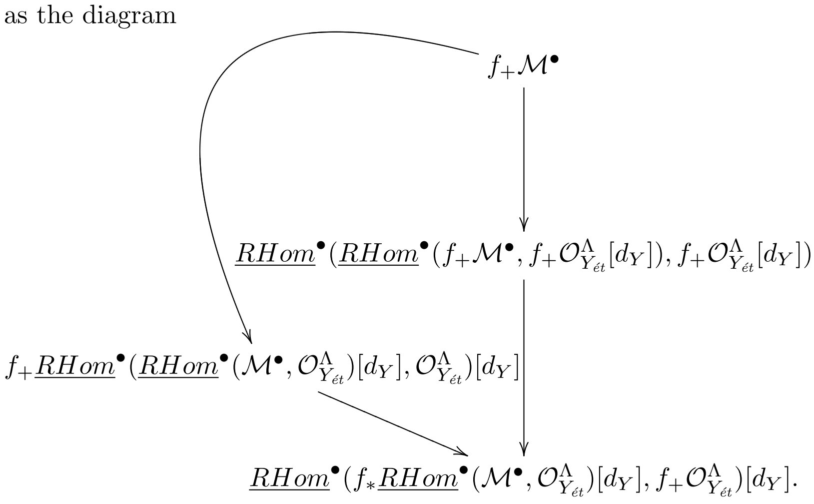
Proof. — The natural transformations of both theorem 9.7.1 and of Corollary 10.3.3 are defined by factoring f as a closed immersion followed by an open immersion. One verifies that the diagrams whose commutativity is to be checked are compatible with compositions in an obvious sense, and thus it suffices to verify that each diagram commutes in the case when f is either an open or a closed immersion.

If f is an open immersion, then the commutativity of both diagrams may be verified after restricting to Y, at which point it is immediate from the constructions.

If f is a closed immersion we have the diagram



as well as the diagram



The first of these obviously commutes, while the second is also seen to commute once one unwinds the "double duality" maps that it involves. If one glues these two diagrams along their common edge one obtains (an expanded version of) the first diagram in the statement of the proposition, and so one sees that this diagram commutes.

One sees that the second diagram in the statement of the lemma arising from the closed immersion f commutes by considering an analogous pair of diagrams, whose construction we leave to the reader. $\square$

Theorem 11.3. — *The natural transformations of functors* $\zeta : \mathrm{id} \to \mathrm{M}_{\acute{e}t}\mathrm{Sol}_{\acute{e}t}$ *and* $\eta : \mathrm{id} \to \mathrm{Sol}_{\acute{e}t}\mathrm{M}_{\acute{e}t}$ *on* $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^{\circ}$ *and* $D^b_{ctf}(X_{\acute{e}t},\Lambda)$ *respectively, are isomorphisms. Thus, the functor* $\mathrm{M}_{\acute{e}t} : D^b_{ctf}(X_{\acute{e}t},\Lambda) \to D^b_{lfgu}(\mathcal{O}_{F^r,X_{\acute{e}t}})^{\circ}$ *is an anti-equivalence of triangulated categories, with* $\mathrm{Sol}_{\acute{e}t} : D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^{\circ} \to D^b_{ctf}(X_{\acute{e}t},\Lambda)$ *providing a quasi-inverse.*

Suppose that $\mathcal{F}^\bullet$ *and* $\mathcal{G}^\bullet$ *are two complexes in* $D^b_{ctf}(X_{\acute{e}t},\Lambda)$, *and that* $\mathcal{M}^\bullet$ *and* $\mathcal{N}^\bullet$ *are two complexes in* $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^{\circ}$. *Then there are natural isomorphisms of complexes of* Λ*-modules,*

$$\mathrm{RHom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\mathrm{M}_{\acute{e}t}(\mathcal{G}^\bullet),\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet)) \tag{11.3.1}$$

and

$$\mathrm{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) \xrightarrow{\sim} \mathrm{RHom}^\bullet_\Lambda(\mathrm{Sol}_{\acute{e}t}(\mathcal{N}^\bullet),\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)), \tag{11.3.2}$$

a natural isomorphism of complexes in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^{\circ}$,

$$\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}} \mathrm{M}_{\acute{e}t}(\mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathcal{G}^\bullet)[d_X], \tag{11.3.3}$$

and a natural isomorphism of complexes in $D^b_{ctf}(X_{\acute{e}t},\Lambda)$,

$$\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\Lambda} \mathrm{Sol}_{\acute{e}t}(\mathcal{N}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}^{\Lambda}_{X_{\acute{e}t}}} \mathcal{N}^\bullet)[d_X]. \tag{11.3.4}$$

If $f : Y \longrightarrow X$ *is any map of smooth* k*-schemes, then there is a natural isomorphism of complexes in* $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,Y_{\acute{e}t}})$,

$$f^!\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(f^{-1}\mathcal{F}^\bullet), \tag{11.3.5}$$

and a natural isomorphism of complexes in $D^b_{ctf}(Y_{\acute{e}t},\Lambda)$,

$$\mathrm{Sol}_{\acute{e}t}f^!(\mathcal{M}^\bullet) \xrightarrow{\sim} f^{-1}\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet). \tag{11.3.6}$$

If furthermore f *is allowable (in the sense of (9.7)) then there is a natural isomorphism of complexes in* $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X_{\acute{e}t}})^{\circ}$,

$$Gf_!(\mathcal{F}^\bullet) \xrightarrow{\sim} f_+\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet), \tag{11.3.7}$$

and a natural isomorphism of complexes in $D^b_{ctf}(X_{\acute{e}t},\Lambda)$,

$$\mathrm{Sol}_{\acute{e}t}f_+(\mathcal{M}^\bullet) \xrightarrow{\sim} f_!\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet). \tag{11.3.8}$$

Finally, this equivalence of categories and each of these natural isomorphisms is compatible in an obvious sense with inseparable base-change and with change of coefficient ring, and interchanges induction and restriction.

Proof. — Assuming for a moment that η and ζ provide the asserted equivalence of categories, we will construct the natural isomorphisms (11.3.1) – (11.3.8). The isomorphisms (11.3.4), (11.3.6) and (11.3.8) were constructed in Propositions 9.9 and 9.3 and Theorem 9.7.1.

We define (11.3.3), (11.3.5) and (11.3.7) to be the corresponding isomorphisms that arise from the equivalence of categories. (If f is an immersion then Corollary 10.3.3 yields an alternative definition of (11.3.7), but this is shown by Proposition 11.2.1 to agree with the more general definition just given.) In each of the results cited, the compatibility with inseparable base-change has been noted.

11.3.9. — To explain the isomorphism (11.3.1) and (11.3.2) requires a slight digression on some sign conventions that are not treated in [**Con**, 1.3].

Suppose that $A^\bullet$, $B^\bullet$ and $C^\bullet$ are three complexes of objects of an additive category. Then there is a natural morphism

$$\psi : \mathrm{Hom}^\bullet(A^\bullet, B^\bullet) \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(B^\bullet, C^\bullet), \mathrm{Hom}^\bullet(A^\bullet, C^\bullet))$$

defined as follows: if $\mathbf{f} = (f^q) \in \prod_q \mathrm{Hom}(A^q, B^{n+q})$ is an element in the degree n member of the left-hand side, then

$$\psi(\mathbf{f}) \in \prod_q \mathrm{Hom}(\prod_p \mathrm{Hom}(B^p, C^{p+q}), \prod_p \mathrm{Hom}(A^p, C^{n+p+q}))$$

(which is the degree n member of the right hand side) is defined via the formula

$$\psi(\mathbf{f})^q : (\phi^p) \to ((-1)^{nq}\phi^{n+p} f^p).$$

The sign is chosen so that (when one uses the sign conventions of [**Con**, 1.3] for the $\mathrm{Hom}^\bullet$ complexes) ψ is a map of complexes. One checks that ψ is compatible in with the "double duality" maps

$$A^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(A^\bullet, C^\bullet), C^\bullet)$$

and

$$B^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(B^\bullet, C^\bullet), C^\bullet),$$

in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Hom}^\bullet(A^\bullet, B^\bullet) & \longrightarrow & \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(B^\bullet, C^\bullet), \mathrm{Hom}^\bullet(A^\bullet, C^\bullet)) \\ \uparrow & & \downarrow \\ \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(A^\bullet, C^\bullet), C^\bullet), B^\bullet) & \longrightarrow & \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(A^\bullet, C^\bullet), C^\bullet), \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(B^\bullet, C^\bullet), C^\bullet)) \end{array}$$

commutes. (Here the upper horizontal arrow is the map ψ, the right hand vertical arrow is the analogous map obtained by replacing the pair $(A^\bullet, B^\bullet)$ with the pair $(\mathrm{Hom}^\bullet(B^\bullet, C^\bullet), \mathrm{Hom}^\bullet(A^\bullet, C^\bullet))$, the left hand vertical arrow is induced by the double duality map for $A^\bullet$, and the lower horizontal arrow is induced by the double duality map for $B^\bullet$.) Also, it is compatible with making a simultaneous translation in $A^\bullet$ and $B^\bullet$. The morphism induced by ψ on the degree n cohomology objects of each side is (up to a sign) the natural morphism

$$\mathrm{Hom}(A^\bullet, B^\bullet[n]) \to \mathrm{Hom}(\mathrm{Hom}^\bullet(B^\bullet[n], C^\bullet), \mathrm{Hom}^\bullet(A^\bullet, C^\bullet)).$$

11.3.10 (Construction of (11.3.1) and (11.3.2)). — Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be two complexes in $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$, and let $\mathcal{I}^\bullet$ be a right resolution of $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$ by injective left $\mathcal{O}^\Lambda_{F^r, X_{\acute{e}t}}$-modules. We may assume that $\mathcal{F}^\bullet$ is a complex of Λ-flat modules, and that $\mathcal{G}^\bullet$ is a complex of injective Λ-modules.

Then we define the morphism (11.3.1) to be the derived category avatar of the composite

$$\begin{aligned}\mathrm{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{G}^\bullet) &\longrightarrow \mathrm{Hom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(\underline{Hom}^\bullet_\Lambda(\mathcal{G}^\bullet,\mathcal{I}^\bullet),\underline{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{I}^\bullet))\\ &\xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(\underline{Hom}^\bullet_\Lambda(\mathcal{G}^\bullet,\mathcal{I}^\bullet)[d_X],\underline{Hom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{I}^\bullet)[d_X]),\end{aligned}$$

where the first morphism is defined via the discussion of (11.3.9). Note that since the members of $\mathcal{F}^\bullet$ are Λ-flat, $\underline{Hom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(\mathcal{F}^\bullet,\mathcal{I}^\bullet)$ is a complex of injective left $\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}$-modules. Thus this does realise on the level of complexes a morphism

$$\mathrm{RHom}^\bullet_\Lambda(\mathcal{F}^\bullet,\mathcal{G}^\bullet)\to\mathrm{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}}(\mathrm{M}_{\acute{e}t}(\mathcal{G}^\bullet),\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet))$$

between objects of $D^+(X_{\acute{e}t},\Lambda)$. To see that this is an isomorphism, it suffices to consider the resulting map on cohomology modules; since $\mathrm{M}_{\acute{e}t}$ is an equivalence of categories, these maps are indeed isomorphisms.

The morphism (11.3.2) can be defined in a similar way, or simply by applying the equivalence of categories to (11.3.1). Both definitions yield the same morphism, since the construction of (11.3.9) is compatible with double duality and translations. Finally, we observed in remark (9.2.1) and Lemma 10.1.5 that $\mathrm{M}_{\acute{e}t}$ and $\mathrm{Sol}_{\acute{e}t}$ are compatible with inseparable base-change, while if Λ' is a finite Λ-algebra then Propositions 9.10 and 10.5.8 show that they are compatible with the change of coefficient ring functor $\overset{\mathbb{L}}{\otimes}_\Lambda \Lambda'$. That they interchange induction and restriction follows from Propositions 9.11.1 and 10.6.1. (It is easy to check that the natural isomorphisms of these two Propositions are compatible with the natural isomorphisms ζ and η.) It is also straightforward to check that the natural transformations constructed above are compatible with these various operations.

11.3.11. — It remains to prove that ζ and η are natural isomorphisms. It has already been observed that these morphisms are compatible with inseparable base-change; in particular, we may replace k by its perfect closure, and so assume that k is perfect.

Let us suppose first that $\mathcal{F}^\bullet$ is a complex in $D^b_{ctf}(X_{\acute{e}t},\Lambda)$, and show that $\eta:\mathcal{F}^\bullet\to\mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\mathcal{F}^\bullet))$ is an isomorphism.

We may assume that $\mathcal{F}^\bullet$ is a finite length complex of flat constructible Λ-modules. We will argue by induction on the dimension of the support Z of $\mathcal{F}^\bullet$. Let Y be a dense open subset of Z which is a smooth k-scheme and over which $\mathcal{F}^\bullet$ restricts to a complex $\mathcal{L}^\bullet$ of locally constant sheaves of finitely generated projective Λ-modules. Let $W=Z\setminus Y$ be the complement of Y in Z; then W is a closed subset of X of dimension less than that of Z. Let $i:W\to X$ be the closed immersion of W into X, and $f:Y\to X$ the immersion of Y into X.

There is a distinguished triangle

$$f_!f^{-1}\mathcal{F}^\bullet\to\mathcal{F}^\bullet\to i_!i^{-1}\mathcal{F}^\bullet\to f_!f^{-1}\mathcal{F}^\bullet[1]$$

By induction, we already know the result for $i_! i^{-1}\mathcal{F}^\bullet$, so it suffices to prove it for $f_! f^{-1}\mathcal{F}^\bullet = f_!\mathcal{L}^\bullet$. But by 9.7 and 10.3.3 we have

$$f_!\mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\mathcal{L}^\bullet)) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(f_+\mathrm{M}_{\acute{e}t}(\mathcal{L}^\bullet)) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(f_!\mathcal{L}^\bullet))$$

and Proposition 11.2 shows that the morphism $\eta_{f_!\mathcal{L}^\bullet} : f_!\mathcal{L}^\bullet \to \mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(f_!\mathcal{L}^\bullet))$ is obtained by composing this natural isomorphism with the morphism obtained by applying $f_!$ to the morphism $\eta_{\mathcal{L}^\bullet} : \mathcal{L}^\bullet \to \mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\mathcal{L}^\bullet))$. Hence it suffices to show that this latter morphism is an isomorphism. It is enough to check this for each one of the members $\mathcal{L} = \mathcal{L}^i$ of the complex $\mathcal{L}^\bullet$. Working étale locally, we may assume that $\mathcal{L}$ is a constant sheaf of finite projective Λ-modules. Writing $\mathcal{L}$ as a direct summand of a constant sheaf of finite free Λ-modules, we reduce to showing that $\ngeq \to \mathrm{Sol}_{\acute{e}t}(\mathrm{M}_{\acute{e}t}(\Lambda))$ is an isomorphism. Now $\mathrm{M}_{\acute{e}t}(\Lambda) = \mathcal{O}^\Lambda_{X_{\acute{e}t}}[d_X]$, by Lemma 10.2.1, and the claim follows by example 9.3.1. We have thus proved that η is an isomorphism.

11.3.12. — We now turn to proving that ζ is an isomorphism. The proof is quite analogous to the preceding proof that η is an isomorphism. There is a slight complication, however, caused by the fact that we have not proved that any complex in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})$ can be represented by a complex of unit (Λ, F^r)-crystals on a dense open subset of its support. (This would be the analogue of the replacement of $\mathcal{F}^\bullet$ by $\mathcal{L}^\bullet$ in the preceding argument.) That such a representation exists is a consequence of the theorem we are trying to prove (consider the proof of Proposition 10.4), but we have not been able to find a proof of this fact that does not depend on Theorem 11.3. In the absence of this result, we proceed by an argument similar to that used in the proof of Proposition 9.10.

Let $\mathcal{M}^\bullet$ be a complex in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})$; our goal is to prove that $\zeta : \mathcal{M}^\bullet \to \mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet))$ is an isomorphism. We begin by noting that if $\mathcal{M}^\bullet$ is a single unit (Λ, F^r)-crystal, then the result follows by example 9.3.1 and Lemma 10.2.1, as above.

In general, we proceed by induction on the dimension of the support Z of $\mathcal{M}^\bullet$. Give Z its reduced induced structure, and let $f : U \to X$ denote the immersion of a dense open affine subset U of Z into X. An excision argument, together with our induction hypothesis, shows that it is enough to prove the result with $f_+f^!\mathcal{M}^\bullet$ in place of $\mathcal{M}^\bullet$, and Proposition 11.2.1 shows that $\zeta_{f_+f^!\mathcal{M}^\bullet}$ is an isomorphism provided $\zeta_{f^!\mathcal{M}^\bullet}$ is. Thus we may replace $\mathcal{M}^\bullet$ by $f^!\mathcal{M}^\bullet$, and X by U, and so assume that X is a smooth affine k-scheme.

Suppose to begin with that $\mathcal{M}^\bullet$ is a single object $\mathcal{M}$ of $\mu_{lfgu}(X_{\acute{e}t}, \Lambda)$. We may apply the construction of (5.3.5) to a coherent generator of $\mathcal{M}$ to obtain a resolution of $\mathcal{M}$ by $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-flat locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}}$-modules. Since $\mathcal{M}$ is assumed to be of finite Tor-dimension, some finite-length truncation of this resolution again consists of $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-flat modules. Using Proposition 6.9.6 and an excision argument to replace X by a dense open affine subset on which the members of this complex are furthermore coherent as $\mathcal{O}^\Lambda_{X_{\acute{e}t}}$-modules, we obtain a bounded resolution of $\mathcal{M}$ by (Λ, F^r)-crystals. A spectral sequence argument now shows that ζ is an isomorphism, since it is so for a unit (Λ, F^r)-crystal.

We now proceed by induction on the number of non-vanishing cohomology sheaves of the complex $\mathcal{M}^\bullet$, and in light of the result of the preceding paragraph we may assume that there are at least two such. As in the proof of Proposition 9.10 we may include $\mathcal{M}^\bullet$ in a distinguished triangle $\cdots \to \mathcal{Q}^\bullet \to \mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \cdots$ of objects in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X_{\acute{e}t}})^\circ$ such that $\mathcal{Q}^\bullet$ has a single non-vanishing cohomology sheaf, which maps surjectively onto the highest degree non-vanishing cohomology sheaf of $\mathcal{M}^\bullet$. As noted in the proof of that proposition, this implies that $\mathcal{C}^\bullet$ is a complex with one less non-vanishing cohomology sheaf than $\mathcal{M}^\bullet$, and our induction hypothesis then implies that ζ induces isomorphisms $\mathcal{C}^\bullet \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathcal{C}^\bullet))$ and $\mathcal{Q}^\bullet \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathcal{Q}^\bullet))$. Thus ζ also induces an isomorphism $\mathcal{M}^\bullet \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet))$. This completes the proof of Theorem 11.3.

□

11.4. — We are now ready to prove the Riemann-Hilbert correspondence between $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$ and $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$.

Definition 11.4.1. — Define the functor

$$\mathrm{M} : D^b_{ctf}(X_{\acute{e}t}, \Lambda) \to D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$$

to be the composition of the functors $\mathrm{M}_{\acute{e}t}$ and $R\pi_{X*}$.

Define the functor

$$\mathrm{Sol} : D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ \to D^b_{ctf}(X_{\acute{e}t}, \Lambda)$$

to be composition of the functors π_X^* and $\mathrm{Sol}_{\acute{e}t}$.

Theorem 11.4.2. — *The functors* M *and* Sol *are quasi-inverse to one another and hence induce an anti-equivalence of triangulated categories between* $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$ *and* $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$*, which respects* $\mathrm{RHom}^\bullet$ *and* $\overset{\mathbb{L}}{\otimes}$ *(up to a shift of* $d_{Y/X}$*), exchanges* f^{-1} *and* $f^!$*,* $f_!$ *and* f_+ *(for allowable morphisms* f*), and induction and restriction, and is compatible with inseparable base-change and descent, as well as with change of ring.*

Proof. — This follows immediately from theorem 11.3, which provide the analogous result in the étale setting, together with the étale descent results of section 7. □

11.5. — Suppose that Λ is a product of finite fields, so that $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X}) = D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$. Theorem 11.4.2 shows that the triangulated category $D^b_c(X_{\acute{e}t}, \Lambda)$ admits a t-structure whose heart is equivalent to the category of locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,X}$-modules. The purpose of this section is to give an explicit description of this t-structure.

11.5.1. — In the analogous situation of $\mathscr{D}$-modules, one has the notion of a perverse complex of sheaves with constructible cohomology sheaves $\mathcal{F}^\bullet$. It is defined using the following condition (which is dual to condition *(p)* of [**Bo**, IV, §22]):

(*) for any immersion $i : Y \to X$ *there exists a dense open subscheme* $Y_0 \subset Y$ *such that the cohomology sheaves of* $i^{-1}\mathcal{F}^\bullet|_{Y_0}$ *are concentrated in degrees* $\leq -\dim Y$.

Then $\mathcal{F}^\bullet$ satisfies (*) if and only if the corresponding complex $\mathcal{M}^\bullet$ of $\mathscr{D}$-modules has its cohomology sheaves supported in non-negative degrees. (Here $\mathcal{F}^\bullet$ and $\mathcal{M}^\bullet$ are assumed to be related via $\mathcal{F}^\bullet \xrightarrow{\sim} \underline{RHom}^\bullet_{\mathscr{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[d_X]$, which is the the contravariant form of the Riemann-Hilbert correspondence.) In particular (since the Riemann-Hilbert correspondence is compatible with taking duals) $\mathcal{F}^\bullet$ is perverse (that is, the complex $\mathcal{M}^\bullet$ reduces to a single $\mathscr{D}$-module placed in degree zero) if and only if both $\mathcal{F}^\bullet$ and its Verdier dual $D\mathcal{F}^\bullet$ satisfy (*).

One can eliminate the reference to duality by introducing the condition

(**) for any immersion $i : Y \to X$ there exists a dense open subscheme $Y_0 \subset Y$ such that the cohomology sheaves of $\mathcal{F}^\bullet$ with support on Y_0, that is, writing i_0 for the immersion $Y_0 \to X$, the cohomology sheaves of $i_0^!\mathcal{F}^\bullet$, are concentrated in degrees $\geq -\dim Y$.

Since $i_0^!$ is naturally isomorphic to $D \circ i_0^{-1} \circ D$, we see that the complex of $\mathscr{D}$-modules $\mathcal{M}^\bullet$ attached to $\mathcal{F}^\bullet$ is supported in non-positive degrees if and only if $\mathcal{F}^\bullet$ satisfies condition (**). In particular, perverse sheaves are precisely the complexes $\mathcal{F}^\bullet$ satisfying both conditions (*) and (**).

Returning to the situation under consideration in this paper, Gabber has noted that although there is no duality functor, and although the functor $i_0^!$ does not preserve the property of having constructible cohomology sheaves for a general immersion i_0, nevertheless the conditions (*) and (**) make sense. Furthermore, he has shown that they define a t-structure on the category of sheaves of Λ-modules on any k-scheme X.

11.5.2. — Before making a precise statement of Gabber's result, we introduce some notation and terminology. Let X be a k-scheme. For $x \in X$, denote by $i_x : x \to X$ the natural inclusion. Recall that a (bounded) perversity function on X is a function $\mathbf{p} : X \to \mathbb{Z}$ such that for all $x \in X$ and $y \in \overline{\{x\}}$ we have $\mathbf{p}(y) \geq \mathbf{p}(x)$. Given such a function, define full sub-categories ${}^{\mathbf{p}}D^{\leq 0}$ and ${}^{\mathbf{p}}D^{\geq 0}$ of $D^b_c(X_{ét}, \Lambda)$ by the conditions:

$\mathcal{F}^\bullet$ is in ${}^{\mathbf{p}}D^{\leq 0}$ if and only if for all $x \in X$ $H^i(i_x^{-1}\mathcal{F}^\bullet) = 0$ for $i > \mathbf{p}(x)$.

$\mathcal{F}^\bullet$ is in ${}^{\mathbf{p}}D^{\geq 0}$ if and only if for all $x \in X$ $H^i(i_x^!\mathcal{F}^\bullet) = 0$ for $i < \mathbf{p}(x)$.

Theorem 11.5.3. — *(Gabber) The subcategories ${}^{\mathbf{p}}D^{\leq 0}$ and ${}^{\mathbf{p}}D^{\geq 0}$ underly a (necessarily unique) t-structure on $D^b_c(X_{ét}, \Lambda)$.*

Proof. — This is essentially [**Ga**, Thm. 10.3]. More precisely, he proves the theorem in the case $\Lambda = \mathbb{Z}/p$. The general case follows immediately from this one, by restriction of scalars.

Gabber's argument depends on first considering the analogous subcategories ${}^{\mathbf{p}}D^{\leq 0}$ and ${}^{\mathbf{p}}D^{\geq 0}$ of $D^b(X_{ét}, R)$ (using the same conditions as above), for any sheaf of rings R, and checking that these underly a t-structure. He then proves that if $R = \mathcal{O}_{X_{ét}}$, the t-structure on $D^b(X_{ét}, \mathcal{O}_{X_{ét}})$ induces a t-structure on the full subcategory consisting of complexes with coherent cohomology sheaves. Finally, he uses Artin-Schreier theory to pass from this case to the case of $D^b_c(X_{ét}, \mathbb{Z}/p)$. □

The middle perversity is the perversity $p : X \to \mathbb{Z}$ defined by

$$\mathbf{p}(x) = -\dim \overline{\{x\}}.$$

The t-structure defined on $D^b_c(X_{ét}, \Lambda)$ by the middle perversity is a precise analogue of the perverse t-structure defined by conditions (*) and (**) above in the case of the usual Riemann-Hilbert correspondence.

We will recover Theorem 11.5.3 in the special case that X is smooth over k and p is the middle perversity as a consequence of the following result.

Theorem 11.5.4. — *Let $\mathbf{p} : X \to \mathbb{Z}$ be the middle perversity. Then under the equivalence of categories of Theorem 11.4.2, the essential image of the full subcategory $D^{\geq 0}_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ is equal to the full subcategory ${}^{\mathbf{p}}D^{\leq 0}$ of $D^b_c(X_{\acute{e}t},\Lambda)$, while the essential image of the full subcategory $D^{\leq 0}_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ is equal to the full subcategory ${}^{\mathbf{p}}D^{\geq 0}$ of $D^b_c(X_{\acute{e}t},\Lambda)$. In particular, the full subcategories ${}^{\mathbf{p}}D^{\leq 0}$ and ${}^{\mathbf{p}}D^{\geq 0}$ define a t-structure on $D^b_c(X_{\acute{e}t},\Lambda)$, and a complex $\mathcal{F}^\bullet$ in $D^b_c(X_{\acute{e}t},\Lambda)$ is of the form $\mathrm{Sol}(\mathcal{M})$ for a locally finitely generated unit $\mathcal{O}^{\Lambda}_{F^r,X}$-module $\mathcal{M}$ if and only if $\mathcal{F}^\bullet$ lies in ${}^{\mathbf{p}}D^{\leq 0}\cap{}^{\mathbf{p}}D^{\geq 0}$.*

Proof. — By the usual arguments, we may replace k by a purely inseparable extension, and so assume that k is perfect. We do this from now on.

We begin by proving the statement of the theorem relating $D^{\geq 0}(\mathcal{O}^{\Lambda}_{F^r,X})$ and ${}^{\mathbf{p}}D^{\leq 0}$. Let $\mathcal{M}^\bullet$ be a complex in D^b_{lfgu}, and suppose first that the cohomology sheaves of $\mathcal{M}^\bullet$ are concentrated in non-negative degrees.

Fix $x \in X$. Then $H^i(i_x^{-1}\mathcal{F}^\bullet)$ is equal to zero if and only if there is a non-empty open subset of $\overline{\{x\}}$ such that (letting $i_U : U \to X$ denote the inclusion) $H^i(i_U^{-1}\mathcal{F}^\bullet) = 0$. Let U be such an open subset, and factor the immersion i_U as a composite $U \xrightarrow{i} X' \xrightarrow{j} X$, where i is a closed immersion and j an open immersion.

Since the cohomology sheaves of $\mathcal{M}^\bullet$ are concentrated in non-negative degrees, the same is true of the cohomology sheaves of $j^!\mathcal{M}^\bullet = j^{-1}\mathcal{M}^\bullet$, and so also of those of $i^!j^!\mathcal{M}^\bullet = i_U^!\mathcal{M}^\bullet$, by Proposition 2.10.4. Proposition 6.9.6 shows that we may shrink U so that the cohomology sheaves of $i_U^!\mathcal{M}^\bullet$ are unit (Λ,F^r)-crystals, and so in particular acyclic for $\mathrm{Hom}_{\mathcal{O}^{\Lambda}_{F^r,U_{\acute{e}t}}}(-,\mathcal{O}^{\Lambda}_{U_{\acute{e}t}})$, by Lemma 9.3.2. Thus shrinking U, we compute that

$$\begin{aligned} i_U^{-1}\mathrm{Sol}(\mathcal{M}^\bullet) &= \mathrm{Sol}(i_U^!\mathcal{M}^\bullet) = \\ &\mathrm{RHom}^\bullet_{\mathcal{O}^{\Lambda}_{F^r,U_{\acute{e}t}}}(\pi_U^* i_U^!\mathcal{M}^\bullet,\mathcal{O}^{\Lambda}_{U_{\acute{e}t}})[d_U] = \mathrm{Hom}_{\mathcal{O}^{\Lambda}_{F^r,U_{\acute{e}t}}}(\pi_U^* i_U^!\mathcal{M}^\bullet,\mathcal{O}^{\Lambda}_{U_{\acute{e}t}})[d_U] \end{aligned}$$

is supported in degrees $\leq -d_U$, and so conclude that $H^i(i_x^{-1}\mathrm{Sol}(\mathcal{M}^\bullet)) = 0$ for degrees $i > -\mathbf{p}(x)$. Since x was an arbitrary point of X, we conclude that Sol takes $D^{\geq 0}_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ into ${}^{\mathbf{p}}D^{\leq 0}$, as required.

We now turn to proving the converse. To do this, it will be enough to prove that for any complex $\mathcal{M}^\bullet$ in $D^{\geq 0}_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ for which $H^0(\mathcal{M}^\bullet) \neq 0$, there is a point x in X such that $H^{-\mathbf{p}(x)}(i_x^{-1}\mathrm{Sol}(\mathcal{M}^\bullet)) \neq 0$. To this end, observe that by Proposition 6.9.6, we may find an immersion $i_U : U \to X$ with irreducible domain and such that $i_U^! H^0(\mathcal{M}^\bullet)$ is a non-zero unit (Λ,F^r)-crystal. We may furthermore shrink U so that each cohomology sheaf of $i_U^!\mathcal{M}^\bullet$ is a unit (Λ,F^r)-crystal.

Applying $i_U^!$ to the distinguished triangle

$$H^0(\mathcal{M}^\bullet) \to \mathcal{M}^\bullet \to \tau_{>0}\mathcal{M}^\bullet$$

yields the distinguished triangle

$$i_U^! H^0(\mathcal{M}^\bullet) \to i_U^!\mathcal{M}^\bullet \to i_U^!\tau_{>0}\mathcal{M}^\bullet.$$

Using the fact (observed in the preceding paragraph) that $i_U^!$ has non-negative cohomological amplitude, we see that this is equal to the distinguished triangle

$$H^0(i_U^!\mathcal{M}^\bullet) \to i_U^!\mathcal{M}^\bullet \to \tau_{>0} i_U^!\mathcal{M}^\bullet.$$

Applying $\mathrm{Sol}(\pi_U^* -)$ as in the preceding paragraph, and using the fact that unit (Λ, F^r)-modules are acyclic for $\mathrm{Hom}_{\mathcal{O}^\Lambda_{F^r,U_{ét}}}(-, \mathcal{O}^\Lambda_{U_{ét}})$, yields the distinguished triangle

$$\tau_{<-d_U}\mathrm{Sol}(i_U^!\mathcal{M}^\bullet) \to \mathrm{Sol}(i_U^!\mathcal{M}^\bullet) \to \mathrm{Sol}(i_U^! H^0(\mathcal{M}^\bullet)).$$

Thus $H^{-d_U}(\mathrm{Sol}(i_U^!\mathcal{M}^\bullet)) \xrightarrow{\sim} \mathrm{Sol}(i_U^! H^0(\mathcal{M}^\bullet))$ is a non-zero étale Λ-local system, since $i_U^! H^0(\mathcal{M}^\bullet)$ is a non-zero unit (Λ, F^r)-crystal. Letting x denote the generic point of U, we find that $H^{-\mathbf{p}(x)}(i_x^{-1}\mathrm{Sol}(\mathcal{M}^\bullet)) \neq 0$, as required.

We now prove the statement of the theorem relating $D^{\leq 0}(\mathcal{O}^\Lambda_{F^r,X})$ and ${}^{\mathbf{p}}D^{\geq 0}$. Let $\mathcal{M}^\bullet$ be a complex in D^b_{lfgu}, and suppose first that the cohomology sheaves of $\mathcal{M}^\bullet$ are concentrated in non-positive degrees.

Fix $x \in X$. Then $H^i(i_x^!\mathcal{F}^\bullet)$ is equal to zero if and only if there is a non-empty open subset of $\overline{\{x\}}$ such that (letting $i_U : U \to X$ denote the inclusion) $H^i(i_U^!\mathcal{F}^\bullet) = 0$. Let U be such an open subset, and factor the immersion i_U as a composite $U \xrightarrow{i} X' \xrightarrow{j} X$, where i is a closed immersion and j an open immersion.

Since the cohomology sheaves of $\mathcal{M}^\bullet$ are concentrated in non-positive degrees, the same is true of the cohomology sheaves of $j^!\mathcal{M}^\bullet = j^{-1}\mathcal{M}^\bullet$. We now compute that

$$\begin{aligned} i_U^!\mathrm{Sol}(\mathcal{M}^\bullet) &= i^! j^! \mathrm{Sol}(\mathcal{M}^\bullet) = i^!\mathrm{Sol}(j^!\mathcal{M}^\bullet) \\ &= \underline{RHom}^\bullet_\Lambda(i_*\Lambda, \mathrm{Sol}(j^!\mathcal{M}^\bullet)) = \underline{RHom}^\bullet_\Lambda(i_*\mathrm{Sol}(\mathcal{O}^\Lambda_U[d_U]), \mathrm{Sol}(j^!\mathcal{M}^\bullet)) \\ &= \underline{RHom}^\bullet_\Lambda(\mathrm{Sol}(i_+\mathcal{O}^\Lambda_U[d_U]), \mathrm{Sol}(j^!\mathcal{M}^\bullet)) = \underline{RHom}^\bullet_{\mathcal{O}^\Lambda_{F^r,X'_{ét}}}(j^!\pi_X^*\mathcal{M}^\bullet, i_+\mathcal{O}^\Lambda_{U_{ét}}[d_U]). \end{aligned}$$

Since the cohomology sheaves of $j^!\mathcal{M}^\bullet$ are concentrated in non-positive degrees, and since $i_+\mathcal{O}^\Lambda_{U_{ét}}$ sits in degree zero (remark 3.4.1), we see that the cohomology sheaves of $i_U^!\mathrm{Sol}(\mathcal{M}^\bullet)$ are concentrated in degrees $\geq -d_U$. This shows that Sol takes $D^{\leq 0}_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$ into ${}^{\mathbf{p}}D^{\geq 0}$, as required.

We now turn to proving the converse. To do this, it will be enough to prove that for any complex $\mathcal{M}^\bullet$ in $D^{\leq 0}_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$ for which $H^0(\mathcal{M}^\bullet) \neq 0$, there is a point x in X such that $H^{-\mathbf{p}(x)}(i_x^!\mathrm{Sol}(\mathcal{M}^\bullet)) \neq 0$. To this end, observe that by Propositions 5.12.1 and 6.9.6, we may find an immersion $i_U : U \to X$ with irreducible domain, factored as above into a product of open and closed immersions $U \xrightarrow{i} X' \xrightarrow{j} X$, such that $j^! H^0(\mathcal{M}^\bullet)$ is equal to $i_+\mathcal{E}$, where $\mathcal{E}$ is a non-zero unit (F^r, Λ)-crystal on U. The calculation of the preceding paragraph, applied to the members of the distinguished triangle

$$\tau_{<0}\mathcal{M}^\bullet \to \mathcal{M}^\bullet \to H^0(\mathcal{M}^\bullet)$$

then shows that

$$H^{-d_U}(i_U^!\mathrm{Sol}(\mathcal{M}^\bullet)) = \underline{Hom}_{\mathcal{O}^\Lambda_{F^r,X'_{ét}}}(i_+\pi_U^*\mathcal{E}, i_+\mathcal{O}^\Lambda_{U_{ét}}) = \underline{Hom}_{\mathcal{O}^\Lambda_{F^r,U_{ét}}}(\pi_U^*\mathcal{E}, \mathcal{O}^\Lambda_{U_{ét}}),$$

which is non-zero, since $\mathcal{E}$ is a unit (Λ, F^r)-crystal. Letting x denote the generic point of U, we find that $H^{-\mathbf{p}(x)}(i_x^! \mathrm{Sol}(\mathcal{M}^\bullet))$ is non-zero, as required. This completes the proof of the theorem. $\square$

Remark 11.5.5. — It seems reasonable to call complexes in $D_c^b(X_{ét}, \Lambda)$ which lie in the heart of the equivalent t-structures in the statement of Theorem 11.5.4 "perverse sheaves" (or "perverse Λ-sheaves", if we wish to emphasise the ring of coefficients Λ).

11.5.6. — Let us take $r = 1$ and $\Lambda = \mathbb{Z}/p$. It follows immediately from [**Lyu**, Thm. 3.2] that the category $\mu_{lfgu}(X, \mathbb{Z}/p)$ is Artinian. Thus the same is true of the category of perverse $\mathbb{Z}/p$-sheaves on X. (Gabber has also given a direct proof that this latter category is Artinian in [**Ga**].) Using this fact, it is easy to define, for any immersion $j : Y \to X$, an intermediate extension functor

$$j_{!+} : \mu_{lfgu}(Y, \mathbb{Z}/p) \to \mu_{lfgu}(X, \mathbb{Z}/p);$$

for any object $\mathcal{M}$ of the source, the lfgu $\mathcal{O}_{F,X}$-module $j_{!+}\mathcal{M}$ is the minimal subobject of $H^0(j_+\mathcal{M})$ in $\mu_{lfgu}(X, \mathbb{Z}/p)$ whose restriction to Y equals $\mathcal{M}$. (See [**EK 2**, Cor. 4.2.2] for a detailed proof of the existence of this minimal subobject.) In [**EK 2**, Lem. 4.3.1], we show that under the Riemann-Hilbert correspondence, this functor corresponds to the functor $j_{!*}$ on the corresponding categories of perverse sheaves, defined in the following way:

$$j_{!*}\mathcal{L}^\bullet = \mathrm{Im}({}^{\mathbf{p}}H^0(j_!\mathcal{L}^\bullet) \to {}^{\mathbf{p}}H^0(Rj_*\mathcal{L}^\bullet)),$$

for any perverse $\mathbb{Z}/p$-sheaf $\mathcal{L}^\bullet$ on U. (Here ${}^{\mathbf{p}}H^0(-)$ denotes the 0th cohomology sheaf computed with respect to the perverse t-structure on $D^b(X_{ét}, \mathbb{Z}/p)$ given by the pervesity $\mathbf{p}$, and the image is computed in the abelian category of perverse sheaves.)

We also prove that any simple object of $\mu_{lfgu}(X, \mathbb{Z}/p)$ is (up to isomorphism) of the form $j_{!+}\mathcal{N}$, for some immersion $j : Y \to X$ and some simple F-crystal $\mathcal{N}$ on Y [**EK 2**, Cor. 4.2.3]. Correspondingly, one finds that any simple perverse $\mathbb{Z}/p$-sheaf on X is (up to isomorphism) of the form $j_{!*}\mathcal{L}[d_Y]$, where $\mathcal{L}$ is an irreducible local system of $\mathbb{Z}/p$-sheaves on Y [**EK 2**, Cor. 4.3.3]. (Recall that the analogue of this latter statement is a fundamental result in the theory of ℓ-adic perverse sheaves of [**BBD**].)

11.6. — Suppose that X is a smooth k-scheme. A natural question is whether the functor $D^b(\mu_{lfgu}(X, \Lambda)) \to D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$ is an equivalence of categories. Note that in the context of holonomic $\mathscr{D}$-modules the corresponding result has been proved by Beilinson [**Be**]: a bounded complex of $\mathscr{D}$-modules with regular holonomic cohomology sheaves is represented by a complex of regular holonomic $\mathscr{D}$-modules. In the case that $r = 1$ and $\Lambda = \mathbb{F}_p$ this result will be proved in §17. The proof uses in an essential way the Riemann-Hilbert correspondence proved here, but also the theory of $\mathscr{D}_{F,X}$-modules to be developed in §§13–17.

12. L-FUNCTIONS FOR UNIT F^r-MODULES

12.1. — In this section we will define (in certain cases) L-functions for complexes in $D_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$ and $D_{ctf}(X_{\acute{e}t}, \Lambda)$ when X is a smooth scheme over a finite field k. Thus for the remainder of our discussion we assume that k is a finite field containing $\mathbb{F}_q$. In fact we give two definitions of L-functions, one when Λ is any reduced Noetherian ring (which is applicable to complexes in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})$) and one when Λ is a finite ring (which is applicable to complexes in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$); however, these agree for reduced finite rings (i.e. for a product of finite fields). Unfortunately we do not know of a more general definition which encompasses both of these.

12.1.1. — Suppose first that Λ is a product of finitely many fields. We begin by defining L-functions in the case that X is a point. Let $k' = \mathbb{F}_{p^s}$ be a finite extension of k and write $x = \operatorname{Spec} k'$. Note that $\mathbb{F}_q \subset k \subset \mathbb{F}_{p^s}$, by assumption. If $\mathcal{M}^{\bullet}$ is a complex in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,x})$, then the restriction $\operatorname{Res}^{p^s}_q \mathcal{M}^{\bullet}$ is a complex in $D^b_{lfgu}(\mathcal{O}^{\Lambda'}_{F^s,x})$, where $\Lambda' = \Lambda \otimes_{\mathbb{F}_q} k'$. For each integer i, we consider the inverse of the structural isomorphism

$$\phi^{-1}_{\operatorname{Res}^{p^s}_q H^i(\mathcal{M}^{\bullet})} : H^i(\mathcal{M}^{\bullet}) \xrightarrow{\sim} F^{s*}_x H^i(\mathcal{M}^{\bullet}) = H^i(\mathcal{M}^{\bullet}),$$

where the last equality holds because F^s_x is the identity on x.

We define

$$L_u(x, \mathcal{M}^{\bullet}) = \prod_i \det{}_{\Lambda'}(1 - \phi^{-1}_{\operatorname{Res}^{p^s}_q H^i(\mathcal{M}^{\bullet})} T^s | \operatorname{Res}^{p^s}_q H^i(\mathcal{M}^{\bullet}))^{(-1)^{i+1}}.$$

One sees that this lies in $\Lambda'[\![T]\!]$, and in fact it lies in $\Lambda[\![T]\!]$, as will follow from Lemma 12.1.2 below. Also note that if $\mathbb{F}_q \subset \mathbb{F}_{q'} \subset k$ then

$$L_u(x, \operatorname{Res}^{q'}_q \mathcal{M}^{\bullet}) = L_u(x, \mathcal{M}^{\bullet}),$$

as follows directly from the definition together with transitivity of restriction.

Lemma 12.1.2. — *With the notation of (12.1.1) the factor*

$$\det{}_{\Lambda'}(1 - \phi^{-1}_{\operatorname{Res}^{p^s}_q H^i(\mathcal{M}^{\bullet})} T^s | \operatorname{Res}^{p^s}_q \mathcal{M}^i)$$

of $L_u(x, \mathcal{M}^{\bullet})$ lies in $\Lambda[\![T]\!]$.

Proof. — In fact consider any finitely generated Λ'-module $\mathcal{M}$ and any map $\phi_{\mathcal{M}}^{-1} : \mathcal{M} \to F^{r*}\mathcal{M}$ which we do not assume is an isomorphism. Then we claim that the characteristic polynomial of the Λ'-linear map

$$\phi^{-1}_{\mathrm{Res}_q^{p^s} \mathcal{M}} =: F^{s-r*}\phi_{\mathcal{M}}^{-1} \circ \cdots \circ F^{r*}\phi_{\mathcal{M}}^{-1} \circ \phi_{\mathcal{M}}^{-1}$$

has coefficients in Λ. Choose a finite projective $\Lambda \otimes_{\mathbb{F}_q} k'$-module $\mathcal{N}$ such that $\mathcal{M} \oplus \mathcal{N}$ is finite free, and extend $\phi_{\mathcal{M}}^{-1}$ to $\mathcal{M} \oplus \mathcal{N}$ by setting it equal to 0 on $\mathcal{N}$. Clearly it is enough to prove the claim with $\mathcal{M} \oplus \mathcal{N}$ in place of $\mathcal{M}$, so we may assume that $\mathcal{M}$ is a finite free Λ'-module. In this case the result follows by choosing a basis of $\mathcal{M}$ (which also gives a basis of $F^{ri*}\mathcal{M}$ for each i) and noting that the matrix describing $\phi^{-1}_{\mathrm{Res}_q^{p^s} \mathcal{M}}$ is then invariant under the action of $\mathrm{Gal}(k'/\mathbb{F}_q)$, since this group is generated by F^r. Hence this matrix must have coefficients in Λ. □

12.1.3. — Now let X be any smooth k-scheme. Denote by $M(X)$ the set of closed points of X, and for each $x \in M(X)$, write $i_x : x \to X$ for the natural closed immersion. Let $\mathcal{M}^\bullet$ be a complex in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,x})$. We define the L-function $L_u(X, \mathcal{M}^\bullet) \in \Lambda[\![T]\!]$ by the formula

$$L_u(X, \mathcal{M}^\bullet) = \prod_{x \in M(X)} L_u(x, i_x^! \mathcal{M}^\bullet).$$

That this L-function has coefficients in Λ follows from Lemma 12.1.2. If $\mathbb{F}_q \subset \mathbb{F}_{q'} \subset k$ then

$$L_u(X, \mathrm{Res}_q^{q'} \mathcal{M}^\bullet) = L_u(X, \mathcal{M}^\bullet),$$

as follows from the corresponding property of L-functions of points. Note also that this L-function depends only on the scheme X itself (and of course the complex $\mathcal{M}^\bullet$), and not on the particular finite field k over which it is a scheme. (Note that usually factors in the L-function of an F-crystal are defined using the characteristic polynomial of the structural morphism, not its inverse. Our definition has been chosen so as to be compatible with the Riemann-Hilbert correspondence that we have constructed, as Proposition 12.3.1 below will show.)

12.1.4. — We are going to define L-functions for complexes in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$ for arbitrary reduced, Noetherian Λ. Denote by $Q(\Lambda)$ the total ring of fractions of Λ. Since Λ is reduced and Noetherian, $Q(\Lambda)$ is a finite product of fields. If X is a smooth k scheme, and $\mathcal{M}^\bullet$ is a complex in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$ we set

$$L_u(X, \mathcal{M}^\bullet) = L_u(X, \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda Q(\Lambda)),$$

where the expression on the right is defined using the construction of (12.1.2).

12.1.5. — Suppose now that Λ is a finite ring. We will give a definition of $L_u(X, \mathcal{M}^\bullet)$ for $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$, which agrees with our previous one if Λ happens to be reduced.

For general X the L-functions will be defined by the same formula as in (12.1.3), so we only have to define them for X a point $x = \mathrm{Spec}\, k'$, where the notation is the same as that in (12.1.1).

If $\mathcal{M}^\bullet$ is a complex in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,x})^\circ$, then it is represented by a finite length complex $\mathcal{P}^\bullet$ of locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,x}$-modules whose members are finitely generated and projective as $k'\otimes_{\mathbb{F}_q}\Lambda$-modules. To see this, note that $\mathcal{M}^\bullet = \mathrm{M}_{\acute{e}t}(\mathcal{L}^\bullet)$ for some $\mathcal{L}^\bullet$ in $D^b_{ctf}(x_{\acute{e}t},\Lambda)$, and $\mathcal{L}^\bullet$ can be represented by a finite length complex of finitely generated and flat, and hence projective, Λ-modules. Replacing $\mathcal{L}^\bullet$ by such a complex, we have

$$\mathcal{M}^\bullet = \underline{Hom}^\bullet_\Lambda(\mathcal{P}^\bullet, k'\otimes_{\mathbb{F}_q}\Lambda),$$

and the result follows. Just as in (12.1.1) we have a morphism

$$\phi^{-1}_{\mathrm{Res}^{p^s}_q\mathcal{P}^\bullet} : \mathcal{P}^\bullet \xrightarrow{\sim} F^{s*}_x\mathcal{P}^\bullet = \mathcal{P}^\bullet,$$

and we define (with the notation of (12.1.1))

$$L_u(x,\mathcal{P}^\bullet) = \prod_i \det{}_{\Lambda'}(1-\phi^{-1}_{\mathrm{Res}^{p^s}_q\mathcal{P}^i}T^s|\mathrm{Res}^{p^s}_q\mathcal{P}^i)^{(-1)^{i+1}}.$$

Lemma 12.1.6. — *$L_u(x,\mathcal{P}^\bullet)$ depends only on $\mathcal{M}^\bullet$ and not on the choice of $\mathcal{P}^\bullet$.*

Proof. — Suppose that $\mathcal{P}^\bullet_1$ and $\mathcal{P}^\bullet_2$ are two different choices for the complex $\mathcal{P}^\bullet$ above. Using Lemma 9.3.2, we have $\mathrm{Sol}_{\acute{e}t}(\mathcal{P}^\bullet_i) = \underline{Hom}^\bullet_{k'\otimes_{\mathbb{F}_q}\Lambda}(\mathcal{P}^\bullet_i,\mathcal{O}^\Lambda_x)$. Denote the right hand side by $\mathcal{L}^\bullet_i$. It is a finite length complex of finitely generated projective étale Λ-sheaves. Such complexes may be regarded as finite length complexes of discrete $(G_{k'},\Lambda)$-modules, where $G_{k'}$ is the absolute Galois group of k'. Furthermore, $G_{k'}$ acts on all the terms of $\mathcal{L}^\bullet_i$ $(i=1,2)$ through some finite quotient G, and hence there exists a finite length complex $\mathcal{L}^\bullet_3$ of $\Lambda[G]$-modules mapping quasi-isomorphically to $\mathcal{L}^\bullet_1$ and $\mathcal{L}^\bullet_2$, such that the terms of $\mathcal{L}^\bullet_3$ are finitely generated and projective as Λ-modules. (To see this, take a left resolution of $\mathcal{L}^\bullet_1$ or $\mathcal{L}^\bullet_2$ by free $\Lambda[G]$-modules, and truncate.) Passing back to the sheaf theoretic point of view, we see that $\mathcal{P}^\bullet_3 = \underline{Hom}_\Lambda(\mathcal{L}^\bullet_3,\mathcal{O}^\Lambda_x)$ also represents $\mathcal{M}^\bullet$ and admits a quasi-isomorphism from each of $\mathcal{P}^\bullet_1$ and $\mathcal{P}^\bullet_2$. Thus it is enough to prove the lemma under the additional assumption that there is a quasi-isomorphism $\psi:\mathcal{P}^\bullet_1\to\mathcal{P}^\bullet_2$.

In this case we have an exact sequence of complexes of (Λ,F^r)-crystals

$$0\to\mathcal{P}^\bullet_1\to\mathrm{cyl}(\psi)\to\mathrm{cone}(\psi)\to 0,$$

and $\mathrm{cone}(\psi)$ is an acyclic complex of unit (Λ,F^r)- crystals. It is straightforward to check that $L_u(x,\mathrm{cone}(\psi))=1$. On the other hand, the definition of $\mathrm{cyl}(\psi)$ implies that $L_u(x,\mathrm{cyl}(\psi)) = L_u(x,\mathcal{P}^\bullet_2)$. Thus we obtain

$$L_u(x,\mathcal{P}^\bullet_2) = L_u(x,\mathrm{cyl}(\psi)) = L_u(x,\mathcal{P}^\bullet_1)\cdot L_u(x,\mathrm{cone}(\psi)) = L_u(x,\mathcal{P}^\bullet_1).$$

□

Remarks 12.1.7. — (*i*) Both types of L-functions that we have defined are multiplicative with respect to distinguished triangles. (In fact this was used to conclude the preceding argument.)

(*ii*) The definitions of (12.1.4) and (12.1.5) are compatible for Λ a finite, reduced ring. This follows easily from the fact that the Euler characteristic of a finite length complex is equal to that of its cohomology.

(*iii*) In the definition of (12.1.5) we have used a trivial case of the result mentioned in (11.6), which says that a complex in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$ can be represented by a finite length complex of flat locally finitely generated unit Λ-modules.

(*iv*) It is not *a priori* obvious that the L-function defined in (12.1.4) is compatible with a change of coefficients $\Lambda \to \Lambda'$ between reduced rings. We will prove that this is the case in section 12.4. In general the proof makes use of de Jong's results on resolution of singularities in characteristic p [**deJ**, 4.1].

12.2. — We keep the notation of the previous paragraph. We will define L-functions for complexes in $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$, following [**De**, p. 116]. (In particular, the L-functions that we define will depend only on X, and not on the particular finite field k over which we view it as a scheme.)

12.2.1. — If $x = \operatorname{Spec} k'$ with $k' = \mathbb{F}_{p^s}$ a finite extension of k and $\mathcal{L}^{\bullet}$ in $D^b_{ctf}(x_{\acute{e}t}, \Lambda)$ denotes a finite length complex of flat constructible Λ-modules, then we define $L_{\acute{e}t}(x, \mathcal{L}^{\bullet}) \in \Lambda[\![T]\!]$ by the formula

$$L_{\acute{e}t}(x, \mathcal{L}^{\bullet}) = \prod_i \det\nolimits_{\Lambda}(1 - (F^s)^{-1}T^s | \mathcal{L}^i_{\bar{x}})^{(-1)^{i+1}},$$

where $\bar{x} = \operatorname{Spec} \bar{k}$ for some algebraic closure $\bar{k}$ of k containing k', and $F^s : a \mapsto a^{p^s}$ is regarded as an automorphism of $\bar{k}$ over k (with inverse $(F^s)^{-1}$) which acts on the fibre $\mathcal{L}^i_{\bar{x}}$ by functoriality. Note that if $\mathbb{F}_q \subset \mathbb{F}_{q'} \subset k$ then

$$L_{\acute{e}t}(X, \operatorname{Ind}_q^{q'} \mathcal{L}^{\bullet}) = L_{\acute{e}t}(X, \mathcal{L}^{\bullet}),$$

since determinants are invariant under base-change.

12.2.2. — If $\mathcal{L}^{\bullet}$ is in $D^b_{ctf}(X_{\acute{e}t}, \Lambda)$ we define the L-function $L_{\acute{e}t}(X, \mathcal{L}) \in \Lambda[\![T]\!]$ by

$$L_{\acute{e}t}(X, \mathcal{L}^{\bullet}) = \prod_{x \in M(X)} L_{\acute{e}t}(x, i_x^{-1}\mathcal{L}^{\bullet}).$$

If $\mathbb{F}_q \subset \mathbb{F}_{q'} \subset k$ then

$$L_{\acute{e}t}(X, \operatorname{Ind}_q^{q'} \mathcal{L}^{\bullet}) = L_{\acute{e}t}(X, \mathcal{L}^{\bullet}),$$

as follows from the corresponding property for the L-functions of points.

12.3. — For Λ a finite reduced ring, we can show that L-functions respect the functor f_+ by using the Riemann-Hilbert correspondence, and the analogous fact for constructible étale sheaves.

Proposition 12.3.1. — *Suppose that Λ is a finite ring. If $\mathcal{M}^{\bullet}$ is in $D^b_{lfgu}(\mathcal{O}^{\Lambda}_{F^r,X})^{\circ}$ and $\mathcal{L}^{\bullet} = \operatorname{Sol}(\mathcal{M}^{\bullet})$ then we have*

$$L_u(X, \mathcal{M}^{\bullet}) = L_{\acute{e}t}(X, \mathcal{L}^{\bullet}).$$

Proof. — As Sol is compatible with pull-backs, we reduce immediately to the case $X = x = \operatorname{Spec} k'$, with k' a finite extension of k. Considering k' rather than k to be the ground field does not change the L-function, so we may also assume that $k' = k$. Furthermore, since Sol interchanges induction and restriction, and L_u is invariant under restriction and $L_{ét}$ is invariant under induction, we may replace $\mathbb{F}_q$ by k, and assume that $k = \mathbb{F}_q$.

As in (12.1.5) we may assume that $\mathcal{M}^\bullet$ is a finite length complex of unit (Λ, F^r)-crystals, and then by dévissage that $\mathcal{M}^\bullet = M$ is a single unit (Λ, F^r) crystal. Thus $\mathcal{M}$ is simply a projective Λ-module equipped with an automorphism $\phi_{\mathcal{M}}$.

The complex $\mathcal{L}^\bullet = \mathrm{Sol}(\mathcal{M})$ is then a single locally constant sheaf $\mathcal{L}$, which when regarded as a $\mathrm{Gal}(\bar{k}/k)$-module by taking its fibre over $\mathrm{Spec}\,\bar{k}$ is isomorphic to the kernel of the morphism of $\mathrm{Gal}(\bar{k}/k)$-modules

$$\mathrm{Hom}_\Lambda(\mathcal{M}, \Lambda \otimes_k \bar{k}) \xrightarrow{1-F^r\circ(\phi_{\mathcal{M}}^{-1})^*} \mathrm{Hom}_\Lambda(\mathcal{M}, \Lambda \otimes_k \bar{k})$$

(where $(\phi_{\mathcal{M}}^{-1})^*$ denotes the morphism of Hom-groups induced by the automorphism $\phi_{\mathcal{M}}^{-1}$ of $\mathcal{M}$, and F^r is the automorphism of $\bar{k}$ defined by $a \mapsto a^q$). Thus we see that $(F^r)^{-1}$ and $(\phi_{\mathcal{M}}^{-1})^*$ have the same action on $\mathcal{L}$.

A linear transformation and its adjoint have the same characteristic polynomial, and this characteristic polynomial is invariant under extension of scalars. Also, the natural morphism $\bar{k} \otimes_k \mathcal{L} \to \mathrm{Hom}_\Lambda(\mathcal{M}, \Lambda \otimes_k \bar{k})$ is an isomorphism. Putting these observations together we find that

$$\begin{aligned}
L_u(\mathrm{Spec}\,k, \mathcal{M}) &= \det{}_\Lambda(1 - \phi_{\mathcal{M}}^{-1} T^r | \mathcal{M}) \\
&= \det{}_\Lambda(1 - (\phi_{\mathcal{M}}^{-1})^* T^r | \mathrm{Hom}_\Lambda(\mathcal{M}, \Lambda)) \\
&= \det{}_{\Lambda\otimes_k\bar{k}}(1 - (\phi_{\mathcal{M}}^{-1})^* T^r | \mathrm{Hom}_\Lambda(\mathcal{M}, \Lambda \otimes_k \bar{k})) \\
&= \det{}_{\Lambda\otimes_k\bar{k}}(1 - (\phi_{\mathcal{M}}^{-1})^* T^r | \bar{k} \otimes_k \mathcal{L}_{\bar{k}}) \\
&= \det{}_k(1 - (\phi_{\mathcal{M}}^{-1})^* T^r | \mathcal{L}_{\bar{k}}) \\
&= \det{}_k(1 - (F^r)^{-1} T^r | \mathcal{L}_{\bar{k}}) = L_{ét}(\mathrm{Spec}\,k, \mathcal{L}).
\end{aligned}$$

□

Theorem 12.3.2. — *Let $f : Y \to X$ be a morphism of smooth k-schemes. Suppose that Λ is finite and reduced (i.e. a product of finite fields). If $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,Y})$ and $\mathcal{L}^\bullet = \mathrm{Sol}(\mathcal{M}^\bullet)$ then*

$$L_u(Y, \mathcal{M}^\bullet) = L_{ét}(Y, \mathcal{L}^\bullet) = L_{ét}(X, f_!\mathcal{L}^\bullet) = L_u(X, f_+\mathcal{M}^\bullet).$$

Proof. — The first equality follow from the previous proposition, while the middle one is a result of Deligne [**De**, 22, p. 116]. (An alternative proof is presented in the article [**Cr**, 5.1] of Crew. Yet another proof of the theorem (and in fact of a more general result which was conjectured by Katz) is given in [**EK 1**] as an application of the techniques developed in this paper. See (12.5) below for further remarks.) The right-most equality follows from the previous proposition together with the Riemann-Hilbert correspondence, in the case that f is allowable. To see this equality in the general case, note that we may write Y as the finite disjoint union of locally closed affine sub-schemes, and that since L-functions are multiplicative with respect to distinguished triangles and are computed point by point, it suffices to verify this equality for the restriction of $\mathcal{M}^\bullet$ to each of these affine subschemes. Since any map whose source is an affine scheme is allowable, this proves the stated equality. □

12.4. — Given the definitions of the previous section, it is natural to ask whether all the L-functions $L_u(X, \mathcal{M}^\bullet)$ have coefficients in Λ, and whether one can prove a trace formula for them as in Theorem 12.3.2. In this section we show that the answer to the second question is affirmative if Λ is reduced, and that the answer to the first question is affirmative if Λ is normal. Moreover, we will establish that our L-functions are compatible with change of scalars $\Lambda \to \Lambda'$ between reduced rings.

The main technique is a specialisation argument, which allows us to reduce to the case of Λ a finite field already proved above.

Theorem 12.4.1. — *Suppose that Λ is a regular ring, Λ' a reduced ring, and $\lambda : \Lambda \to \Lambda'$ a map of $\mathbb{F}_q$-algebras. For any smooth k-scheme X and and any complex $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$ we have that $L_u(X, \mathcal{M}^\bullet)$ is in $\Lambda[\![T]\!]$ and*

$$L_u(X, \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda') = \lambda(L_u(X, \mathcal{M}^\bullet)). \tag{12.4.2}$$

Proof. — First note that since Λ is regular any module over it has finite projective dimension, so that $\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda'$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$ and the left hand side of (12.4.2) makes sense.

¿From the definition, it is enough to consider the case $X = x = \operatorname{Spec} k'$, and as usual we may replace k by k' and so assume that $k' = k$. Since the L-functions only depend on $\operatorname{Res}^{p^s}_q \mathcal{M}^\bullet$, where $p^s = |k|$, we may also replace q, Λ and Λ' by p^s, $\Lambda \otimes_{\mathbb{F}_q} k$ and $\Lambda' \otimes_{\mathbb{F}_q} k$ respectively, and so assume that $s = r$.

By dévissage and the multiplicative properties of L-functions, we may assume that $\mathcal{M}^\bullet$ is a single locally finitely generated unit $\mathcal{O}^\Lambda_{F^r,x}$-module $\mathcal{M}$ concentrated in degree 0. Let $\beta : M \to F^{r*}M$ be a generating morphism for $\mathcal{M}$, with M a finite Λ-module. Since Λ is regular we may take a finite left resolution $N^\bullet$ of M by projective Λ-modules, and lift β to a map $\beta' : N^\bullet \to F^{r*}N^\bullet$. Let $\mathcal{N}^\bullet$ be the complex of $\mathcal{O}^\Lambda_{F^r,x}$-modules generated by β'. By dévissage, it is enough to prove the proposition with one of the terms of $\mathcal{N}^\bullet$ in place of $\mathcal{M}$, and so we may assume that the generator M of $\mathcal{M}$ is a projective Λ-module.

Since $X = \operatorname{Spec} \mathbb{F}_q$, we have that β induces a map $\beta : M \to F^{r*}M = M$. Now it is easy to see that we have

$$L_u(x, \mathcal{M} \overset{\mathbb{L}}{\otimes}_\Lambda Q(\Lambda)) = \det_\Lambda(1 - \beta T^r | M),$$

which already shows that the left hand side is in $\Lambda[\![T]\!]$. On the other hand, $\mathcal{M} \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda' = \mathcal{M} \otimes_\Lambda \Lambda'$, since $\mathcal{M}$ is flat over Λ, and $\mathcal{M} \otimes_\Lambda \Lambda'$ has generator

$$\beta' : M \otimes_\Lambda \Lambda' \overset{\beta\otimes 1}{\to} M \otimes_\Lambda \Lambda'.$$

Thus we have

$$L_u(x, \mathcal{M} \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda') = \det_{\Lambda'}(1 - \beta' T^r | M \otimes_\Lambda \Lambda') = \lambda(\det_\Lambda(1 - \beta T^r | M)) = \lambda(L_u(x, \mathcal{M})).$$

□

Theorem 12.4.3. — *Let $f : Y \to X$ be a map of smooth k-schemes, Λ a reduced Noetherian $\mathbb{F}_q$-algebra, and $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,Y})$. Then we have*

$$L_u(X, f_+\mathcal{M}^\bullet) = L_u(Y, \mathcal{M}^\bullet).$$

Proof. — By dévissage we reduce to the case of $\mathcal{M}^\bullet$ a single finitely generated unit $\mathcal{O}^\Lambda_{F^r,Y}$-module $\mathcal{M}$ concentrated in degree 0.

Now we are going to modify the ring Λ. To begin with, just from the definition of the L-functions, we can replace Λ by $Q(\Lambda)$, and so assume that Λ is a finite product of fields. It is then clear that we may further reduce to the case in which Λ is a single field. Next, since $\mathcal{M}$ has a generating morphism $\beta : M \to F^{r*}M$ with M a finite $\mathcal{O}^\Lambda_Y$-module, if we write the field Λ as a limit of finite type $\mathbb{F}_q$-subalgebras $\Lambda = \varinjlim_i \Lambda_i$ then we may descend our situation to one of the domains Λ_i. In other words there exists a finite $\mathcal{O}^{\Lambda_i}_Y$-module M' and a morphism $\beta' : M' \to F^{r*}M'$ such that $M \xrightarrow{\sim} M' \otimes_{\Lambda_i} \Lambda$ and $\beta' \otimes 1$ become β under this identification. Then we can replace $\mathcal{M}$ by the $\mathcal{O}^{\Lambda_i}_{F^r,Y}$-module generated by β' and Λ by Λ_i. Note here that we have used the fact that the field $\Lambda = Q(\Lambda)$ is flat over Λ_i, since it is a $Q(\Lambda_i)$ algebra, so that $M' \otimes_{\Lambda_i} \Lambda = M' \overset{\mathbb{L}}{\otimes}_{\Lambda_i} \Lambda$.

We have reduced to the case where Λ is a domain of finite type over $\mathbb{F}_q$. Choose an element u of Λ such that $\Lambda[1/u]$ is a regular ring. Then we may replace Λ by $\Lambda[1/u]$ (since these both have $Q(\Lambda)$ as their quotient field) and so assume that Λ is regular.

Now let $\mathfrak{m} \subset \Lambda$ be a maximal ideal, and write $\lambda_\mathfrak{m} : \Lambda \to \Lambda/\mathfrak{m}\Lambda$ for the projection. Using Lemma 12.4.1, Proposition 3.10, and Theorem 12.3.2 we compute

$$\begin{aligned}\lambda_\mathfrak{m}(L_u(X, f_+\mathcal{M})) = L_u(X, f_+\mathcal{M} \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{m}) &= L_u(X, f_+(\mathcal{M} \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{m})) \\ &= L_u(Y, \mathcal{M} \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{m}) = \lambda_\mathfrak{m}(L_u(Y, \mathcal{M}))\end{aligned}$$

Since this holds for every maximal ideal $\mathfrak{m}$, and the maximal points are dense in $\operatorname{Spec}\Lambda$ we must actually have $L_u(X, f_+\mathcal{M}) = L_u(Y, \mathcal{M})$, as required □

Corollary 12.4.4. — *If X is a smooth k-scheme, Λ is a reduced Noetherian ring, and $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})$, then $L_u(X, \mathcal{M}^\bullet)$ is rational.*

Proof. — If X is a point, the result is obvious. The general case follows by applying Theorem 12.4.3. □

Theorem 12.4.5. — *With the above notation, if Λ is a normal ring, then $L_u(X, \mathcal{M}^\bullet)$ is in $\Lambda[\![T]\!]$.*

Proof. — Since Λ is normal, it decomposes as a product of normal domains, and it clearly suffices to prove the theorem for each of these normal domains individually. Thus we assume for the remainder of the proof that Λ is a normal domain.

If we let $\mathfrak{p} \subset \Lambda$ be a height one prime ideal then $\Lambda_\mathfrak{p}$ is regular, and so Lemma 12.4.1 implies that $L_u(X, \mathcal{M}^\bullet)$ lies in $\Lambda_\mathfrak{p}[\![T]\!]$. Since Λ is a normal domain we also have $\Lambda = \bigcap_\mathfrak{p} \Lambda_\mathfrak{p}$. The result follows. □

Theorem 12.4.6. — *Suppose that X is a smooth k-scheme, that Λ is a reduced finite type $\mathbb{F}_q$-algebra, that $\mathcal{M}^\bullet$ is a complex in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F^r,X})^\circ$, and that $L_u(X,\mathcal{M}^\bullet)$ lies in $\Lambda[\![T]\!]$. (The preceding result shows that this last condition is automatic if Λ is normal.) Let $\lambda : \Lambda \to \Lambda'$ be a map of reduced Noetherian rings. Then we have*

$$L_u(X, \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda') = \lambda(L_u(X,\mathcal{M}^\bullet)).$$

Proof. — ¿From the very construction of L-functions, we see that we may replace Λ' by its flat extension $Q(\Lambda')$, which is a product of fields. It suffices furthermore to prove the result with $Q(\Lambda')$ replaced by one of its field factors. Thus we may assume that Λ' is a field, which we do for the remainder of this paragraph. Then if λ is an injection, it is an injection of domains, and so induces an injection of fields $Q(\Lambda) \to \Lambda'$, for which the result is obvious. In general λ factors as $\Lambda \to \Lambda/\mathfrak{p} \to \Lambda'$, where $\mathfrak{p}$ is a prime ideal of Λ, and the second arrow is an injection. Thus it suffices to prove the result for the first of these morphisms.

For the remainder of the proof we let $\mathfrak{p}$ denote a prime ideal of Λ, and assume that λ is the surjection $\Lambda \to \Lambda/\mathfrak{p}$. We apply the desingularisation results of [**deJ**, 4.1] to each of the irreducible components of $\operatorname{Spec}\Lambda$. (It is at this point that we assume that Λ is of finite type over $\mathbb{F}_q$.) Thus we obtain a smooth variety Z over $\mathbb{F}_q$ and a surjective proper map $Z \to \operatorname{Spec}\Lambda$, with the additional property that each irreducible component of Z dominates an irreducible component of $\operatorname{Spec}\Lambda$. Choose a point $\tilde{\mathfrak{p}}$ of Z mapping to $\mathfrak{p}$, and a dense open affine subset $\operatorname{Spec}\tilde{\Lambda} \subset Z$ containing $\tilde{\mathfrak{p}}$. Then $\operatorname{Spec}\tilde{\Lambda}$ dominates $\operatorname{Spec}\Lambda$, and the corresponding injection of rings $\eta : \Lambda \to \tilde{\Lambda}$ extends to an injection of total quotient rings $Q(\eta) : Q(\Lambda) \to Q(\tilde{\Lambda})$, as well as inducing an injection of domains $\bar{\eta} : \Lambda/\mathfrak{p} \to \tilde{\Lambda}/\tilde{\mathfrak{p}}$. Thus we have a commutative diagram

$$\begin{array}{ccccc} Q(\Lambda) & \longleftarrow & \Lambda & \xrightarrow{\lambda} & \Lambda/\mathfrak{p} \\ \Big\downarrow{\scriptstyle Q(\eta)} & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \bar{\eta}} \\ Q(\tilde{\Lambda}) & \longleftarrow & \tilde{\Lambda} & \xrightarrow{\tilde{\lambda}} & \tilde{\Lambda}/\tilde{\mathfrak{p}} \end{array}$$

in which all three vertical maps are injective, and we deduce, using the known case of injections between domains (or of injections between products of fields, which immediately reduces to the case of injections of fields) that $\eta(L_u(X,\mathcal{M}^\bullet)) = L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \tilde{\Lambda})$ and that $\bar{\eta}(L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{p})) = L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \tilde{\Lambda}/\tilde{\mathfrak{p}})$. We now apply Lemma 12.4.1 to the morphism $\tilde{\lambda}$ (remembering that $\tilde{\Lambda}$ is regular) to compute that

$$\begin{aligned}(\bar{\eta}\circ\lambda)(L_u(X,\mathcal{M}^\bullet)) = (\tilde{\lambda}\circ\eta)(L_u(X,\mathcal{M}^\bullet)) &= \tilde{\lambda}(L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \tilde{\Lambda})) \\ &= L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \tilde{\Lambda}/\tilde{\mathfrak{p}}) = \bar{\eta}(L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{p})).\end{aligned}$$

Since $\bar{\eta}$ is injective, we conclude that $\lambda(L_u(X,\mathcal{M}^\bullet)) = L_u(X,\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{p})$. □

Remarks 12.4.7. — (*i*) We do not know if Theorem 12.4.5 holds without assuming Λ normal.

(*ii*) Corollary 12.4.4 generalises some of the results of [**TW**]. In particular it implies Goss's conjecture (proved in [**TW**]) that the local L-function attached to a Drinfeld module is rational - see [**TW**, §7].

(*iii*) Theorem 12.4.6 can be proved more generally. The assumption that Λ is of finite type over $\mathbb{F}_q$ was used only to know that there exists a regular ring $\tilde{\Lambda}$ containing Λ. However this is true in many cases when Λ is not of finite type over $\mathbb{F}_q$.

12.5. — Theorem 12.4.3 can also be generalised to Λ where $p \neq 0$. In [**EK 1**] we prove the following result.

Theorem 12.5.1. — *Let k be a finite field of characteristic p containing $\mathbb{F}_q$, and let $f : Y \to X$ be a morphism of (not necessarily smooth) k-schemes. If $\mathcal{L}^\bullet$ is a bounded complex of $W_n(\mathbb{F}_q)$-sheaves of finite Tor-dimension and having constructible cohomology sheaves on $Y_{ét}$ (where $W_n(\mathbb{F}_q)$ denotes the truncated ring of Witt vectors of $\mathbb{F}_q$ of length n, for some given positive integer n) then the quotient $L_{ét}(Y, \mathcal{L}^\bullet)/L_{ét}(X, f_!\mathcal{L}^\bullet)$ (where the L-functions are defined in the usual manner, generalising the definition of (12.2) in the case when $n = 1$), a priori an element of $1 + TW_n(\mathbb{F}_q)[\![T]\!]$, is in fact an element of $1 + pTW_n(\mathbb{F}_q)[T]$.*

This theorem has the following corollary:

Corollary 12.5.2. — *Let k be a finite field containing $\mathbb{F}_q$ and X be a smooth k-scheme. Let $\mathcal{L}$ be a lisse $W(\mathbb{F}_q)$-sheaf on $X_{ét}$. Then the ratio of L-functions $L_{ét}(X, \mathcal{L})/L_{ét}(\operatorname{Spec} k, f_!\mathcal{L})$, which is an element of $W(\mathbb{F}_q)[\![T]\!]$, defines an invertible rigid analytic function on the closed unit p-adic disc.*

Proof. — For each positive integer n we see that

$$L_{ét}(X, \mathcal{L})/L_{ét}(\operatorname{Spec} k, f_!\mathcal{L}) \equiv L_{ét}(X, \mathcal{L}/p^n)/L_{ét}(\operatorname{Spec} k, f_!\mathcal{L}/p^n) \pmod{p^n},$$

which by the theorem lies in $1 + pTW_n(\mathbb{F}_q)[T]$. Taking the limit as n goes to infinity proves the corollary. □

12.5.3. — This result proves part of a conjecture of Katz [**Ka 2**, Conj. 6.1]. In *loc. cit.*, in addition to conjecturing our corollary 12.5.2, Katz conjectures that the L-function $L_{ét}(X, \mathcal{L})$ extends to a meromorphic function of T on the entire affine line. (Since $L_{ét}(\operatorname{Spec} k, f_!\mathcal{L})$ is manifestly a rational function, this is equivalent to the quotient $L_{ét}(X, \mathcal{L})/L_{ét}(\operatorname{Spec} k, f_!\mathcal{L})$ so extending, where $f : X \to \operatorname{Spec} k$ is the structural morphism of the k-scheme X.) However, this result has been disproved for general p-adic lisse sheaves by Wan [**Wa**]. Note that from Corollary 12.5.2, one sees that $L_{ét}(X, \mathcal{L})$ is a meromorphic function on the closed unit disc (since as already observed, it is the product of the quotient $L_{ét}(X, \mathcal{L})/L_{ét}(\operatorname{Spec} k, f_!\mathcal{L})$, which is defined on the closed unit disc by Corollary 12.5.2, and the rational function $L_{ét}(\operatorname{Spec} k, f_!\mathcal{L})$).

INTRODUCTION TO §§13–17: $\mathscr{D}_{F,X}$-MODULES

In §§1–12, we worked over a field k of characteristic p, and (in particular) constructed for a smooth k-scheme X an anti-equivalence of categories between the bounded derived category of constructible étale sheaves of $\mathbb{F}_p$-vector spaces on X, and a certain triangulated category of $\mathcal{O}_X[F]$-modules, where F denotes the absolute Frobenius. We showed that this anti-equivalence respects three of Grothendieck's six operations (that is, those three that are actually defined in this situation).

In §§13–17 we extend these results to a situation where p is nilpotent, but $p \neq 0$. Namely, we consider a perfect field k of characteristic p, and a smooth $W_n(k)$-scheme X. Denote by $D^b(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ the bounded derived category of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves, and by $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ the full triangulated sub-category of $D^b(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ consisting of complexes with constructible cohomology sheaves that have finite Tor dimension over $\mathbb{Z}/p^n\mathbb{Z}$. We construct an anti-equivalence between $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ and a certain triangulated category of "arithmetic $\mathscr{D}$-modules equipped with an action of Frobenius."

We now describe the contents of §§13–17 of the paper in more detail. In §13, we introduce a certain sheaf of rings $\mathscr{D}_{F,X}$. This sheaf of rings is obtained by adding a local lift of Frobenius to the ring $\mathscr{D}_X$ of arithmetic differential operators introduced by Berthelot [**Ber 2**]. A key point is that the resulting ring is independent of the chosen lifting, up to canonical isomorphism. Hence we get a sheaf of rings on X. Formally, the construction of $\mathscr{D}_{F,X}$ is based on the observation of Berthelot [**Ber 3**] that even when one does not have a lift of Frobenius, one nevertheless has a functor F^* on the category of $\mathscr{D}_X$-modules. We then define $\mathscr{D}_{F,X} = \bigoplus_{r \geq 0} (F^*)^r \mathscr{D}_X$, and show that it has a ring structure.

We collect in §13 various results about $\mathscr{D}_{F,X}$-modules. The most important of these are characterisations of left and right $\mathscr{D}_{F,X}$-modules in terms of the functors F^* and $F^!$ introduced by Berthelot. Namely, giving a left $\mathscr{D}_{F,X}$-module is equivalent to giving a left $\mathscr{D}_X$-module $\mathcal{E}$ together with a map of $\mathscr{D}_X$-modules $F^*\mathcal{E} \to \mathcal{E}$. Giving a right $\mathscr{D}_{F,X}$-module is equivalent to giving a right $\mathscr{D}_X$-module together with a map $\mathcal{M} \to F^!\mathcal{M}$.

One may also consider a sheaf of rings $\mathscr{D}_{F^!,X}$, defined with reference to the $F^!$ functor on $\mathscr{D}_X$-modules, rather than the F^* functor. Left $\mathscr{D}_{F^!,X}$-modules are characterised as being left $\mathscr{D}_X$-modules $\mathcal{E}$ that are equipped with a map of left $\mathscr{D}_X$-modules $\mathcal{E} \to F^*\mathcal{E}$, while right $\mathscr{D}_{F^!,X}$-modules are characterised as being right $\mathscr{D}_X$-modules $\mathcal{M}$ that are equipped with a map of right $\mathscr{D}_X$-modules $F^!\mathcal{M} \to \mathcal{M}$.

Thus if one considers both $\mathscr{D}_{F,X}$-modules and $\mathscr{D}_{F^!,X}$-modules, a certain symmetry is maintained, which is lost if one considers $\mathscr{D}_{F,X}$-modules alone. Also, modules over these two rings interact with one another when one considers tensor products or Hom functors. (See section (13.8) below.) Nevertheless, we focus primarily on $\mathscr{D}_{F,X}$-modules in this note, since such modules suffice for the construction of our Riemann-Hilbert correspondence.

In §14 we introduce direct and inverse images f_+ and $f^!$ for $\mathscr{D}_{F,X}$-modules, and prove some of their properties. As usual, the definitions depend on the construction of suitable bimodules. The most delicate result here is that f_+ is left adjoint to $f^!$ when f is a proper morphism. It is this result that underlies the fact that the anti-equivalence of categories, constructed later on, respects direct images. To prove this result we rely heavily on ideas of Virrion [**Vi 1**], [**Vi 2**], and in particular on her interpretation of certain complexes in terms of residual complexes on nilpotent neighbourhoods of the diagonal $X \hookrightarrow X^{k+1}$. However, our method differs somewhat from hers, since we replace her use of the dual Čech-Alexander complex [**Vi 2**, I, §2] by an application of the unnormalised bar complex construction (see (14.4.2)). Our approach has the advantage that several compatibilities that our construction is required to satisfy can be deduced from known compatibilities in the theory of the trace map for residual complexes. This allows us to avoid some of the more painful calculations in [**Vi 2**]. Although we work with $\mathscr{D}_{F,X}$-modules, the reader will check that our method also works for $\mathscr{D}_X$-modules, and hence gives a new construction for the trace map of [**Vi 2**].

Since the theory of arithmetic $\mathscr{D}$-modules does have a duality functor (see [**Ber 1**]), the reader may wonder if the theory of $\mathscr{D}_{F,X}$-modules does also. This would allow one to extend the three operations $f_+, f^!$ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ to the full six operations. Unfortunately this seems not be the case, because a good duality theory for $\mathscr{D}_{F,X}$ would involve Tate twists, which one cannot make when p is nilpotent.

In §15, we study unit and locally finitely generated unit (lfgu) $\mathscr{D}_{F,X}$-modules. A unit $\mathscr{D}_{F,X}$-module is a quasi-coherent $\mathscr{D}_{F,X}$-module $\mathcal{M}$ such that the map $F^*\mathcal{M} \to \mathcal{M}$ is an isomorphism. An lfgu module is a unit $\mathscr{D}_{F,X}$-module that is finitely generated locally on X. A key point is that when $n = 1$, there is an equivalence between the category of unit $\mathscr{D}_{F,X}$-modules and the category of unit $\mathcal{O}_{F,X}$-modules, which was studied in §5. This equivalence preserves the property of being locally finitely generated. Using this, we are able, by dévissage, to reduce problems about unit and lfgu $\mathscr{D}_{F,X}$- modules to analogous questions about $\mathcal{O}_{F,X\otimes k}$-modules, which have already been handled above.

In §16, we construct the Riemann-Hilbert correspondence alluded to earlier. To explain it, denote by $D^b(\mathscr{D}_{F,X})$ the bounded derived category of $\mathscr{D}_{F,X}$-modules, and by $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ the full triangulated sub-category of $D^b(\mathscr{D}_{F,X})$ consisting of complexes that have lfgu cohomology sheaves, and finite Tor dimension over $\mathcal{O}_X$. We construct an anti-equivalence between this category and $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$, and we show that it interchanges $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ and $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}}$, $f^!$ and f^{-1}, and f_+ and $f_!$. The proofs proceed mostly by reduction to the results of §11, using the dévissage technique explained above.

Let μ_{lfgu} denote the abelian category of lfgu $\mathscr{D}_{F,X}$-modules. In §17, we show that the natural functor $D^b(\mu_{lfgu})^\circ \to D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ is an equivalence of categories. This is an analogue in our context of a theorem of Beilinson concerning the derived category of regular holonomic $\mathscr{D}$-modules [**Be**]. Its proof depends on the Riemann-Hilbert correspondence.

Let us note that in the first two sections, we work in more generality than indicated here: namely, we study $\mathscr{D}^{(v)}_{F,X}$-modules for any level $v \in \mathbb{N}^{\geq 0} \bigcup \{\infty\}$, and establish the results indicated above at any such level. Since unit modules always have a natural module structure over the full ring of differential operators, beginning with the third section we restrict our attention to $\mathscr{D}_{F,X}$-modules.

Finally we mention some extensions of the results of this paper that we have not included here:

(1) The most important extension would be to pass to the limit in all our constructions (in a sense which will not be made precise here) and so develop a theory of $\mathscr{D}^\dagger_{F,X}$-modules. Although many of the constructions are formal, and almost completely analogous to those performed in the work of Berthelot (see for example [**Ber 1**, §§3–4]), they nevertheless take some space to write out, and we decided not to include them here.

(2) There is a connection with F-crystals. If X is a smooth k-scheme, then it is easy to define a category of F-crystals on the infinitesimal site of X/W_n (i.e. no divided powers), that play the role of lfgu modules: namely these should be locally provided by lfgu $\mathscr{D}_{F,X}$-modules on liftings of X to a smooth W_n-scheme. Using the results of §16, one can then easily define an anti-equivalence between $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$, and a suitable triangulated category of F-crystals. This has the advantage that one needs only a smooth k-scheme to get a Riemann-Hilbert type correspondence. The draw back is that this correspondence is not as interesting, because, for example, we do not know of a definition of direct image functors on the categories of such crystals.

(3) Assume that $\mathbb{F}_q \subset k$, where $q = p^r$. As in §§1–12, one could consider $\mathscr{D}_{F^r,X}$-modules, and even $\mathscr{D}_{F^r,X} \otimes_{W_n(\mathbb{F}_q)} \Lambda$-modules for some noetherian $W_n(\mathbb{F}_q)$ algebra Λ. Since the changes necessary for these variants are all explained in §§1–12, it seems safe to leave their implementation as an exercise for the reader.

13. $\mathscr{D}^{(v)}_{F,X}$-MODULES.

13.1. — We fix a prime p, an integer n, and a perfect field k of characteristic p. We let $W_n(k)$, or just W_n if k is understood, denote the ring of Witt vectors of length n with coefficients in k.

We let σ denote the canonical Frobenius automorphism of W_n. If X is a k-scheme, we let F_X denote the absolute Frobenius endomorphism of X.

13.1.1. — Let X be a smooth W_n-scheme. Liftings of the absolute Frobenius on $X \otimes_{W_n} k$ will play a crucial role in this paper. If X is affine, then such a lifting always exists: it exists locally by smoothness, and the obstruction to a global lifting lives in the cohomology of a coherent sheaf, hence vanishes.

If $f : Y \to X$ is a smooth map of smooth W_n-schemes, and F_X is a lifting of the absolute Frobenius to X, then there is a lifting F_Y of the absolute Frobenius to Y that is compatible with f and F_X, using the smoothness of f, and the fact that Y is affine, as above.

If f is étale then there is a unique lift F_Y with the above properties (and so in this case Y need not be affine).

13.1.2. — If X is a smooth W_n-scheme, write X' for the pull-back of X by $\sigma : W_n \to W_n$, and $F_{W_n} : X' \to X$ for the canonical projection.

Suppose one is given a lifting F to X of the absolute Frobenius of $X \otimes_{W_n} k$. By definition, the map $F : X \to X$ factors through X'. We write $F_{X/W_n} : X \to X'$ for this factorisation, and call this map the relative Frobenius of X/W_n. It is a lifting of the relative Frobenius of $X \otimes_{W_n} k$.

13.1.3. — Let X be a smooth W_n-scheme. Assume that there exists a lifting F of the Frobenius on X. We denote by $\mathcal{O}_{F,X}$ the twisted polynomial algebra $\mathcal{O}_X[F]$ defined by the relation $F(a)F = Fa$ for $a \in \mathcal{O}_X$. As an $\mathcal{O}_X$-bimodule we have

$$\mathcal{O}_{F,X} = \bigoplus_{r\geq 0} F^{r*}\mathcal{O}_X \xrightarrow{\sim} \bigoplus_{r\geq 0} \mathcal{O}_X^{(r)}$$

where $\mathcal{O}_X^{(r)}$ denotes $\mathcal{O}_X$ viewed as a left $\mathcal{O}_X$ module in the usual way, and as a right $\mathcal{O}_X$-module via the map $F^r : \mathcal{O}_X \to \mathcal{O}_X$.

Proposition 13.1.4. — *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then giving a left $\mathcal{O}_{F,X}$-module structure on $\mathcal{M}$ is equivalent to giving a map of $\mathcal{O}_X$-modules $\psi : F^*\mathcal{M} \to \mathcal{M}$.*

Proof. — Given a morphism $\psi : F^*\mathcal{M} \to \mathcal{M}$ we define

$$\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\sim} \bigoplus_{r\geq 0} F^{r*}\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\oplus F^{r*}\psi} \mathcal{M}.$$

One verifies immediately that this gives $\mathcal{M}$ the structure of a left $\mathcal{O}_{F,X}$-module. Conversely, given an $\mathcal{O}_{F,X}$-module $\mathcal{M}$, we have in particular a morphism

$$\psi : F^*\mathcal{M} = \mathcal{O}_X F \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}.$$

□

13.1.5. — If $\mathcal{M}$ is a left $\mathcal{O}_{F,X}$-module, we call the map $F^*\mathcal{M} \to \mathcal{M}$ defined by Proposition 13.1.4 its *structural morphism*, and denote it by $\psi_{\mathcal{M}}$. We denote by $\alpha_{\mathcal{M}} : \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ the map describing the left action of $\mathcal{O}_{F,X}$ on $\mathcal{M}$. Note that for any non-negative integer r, $F^{r*}\mathcal{M}$ has a natural structure of left $\mathcal{O}_{F,X}$-module, defined by pulling back the structural morphism of $\mathcal{M}$ by F^{r*}.

Consider an $\mathcal{O}_X$-module M. Then $\mathcal{M} = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M$ is naturally a left $\mathcal{O}_{F,X}$-module. Left $\mathcal{O}_{F,X}$-modules of this type are called *induced.* The following proposition shows that any left $\mathcal{O}_{F,X}$ module has a canonical two term resolution by induced $\mathcal{O}_{F,X}$-modules

Proposition 13.1.6. — *Let $\mathcal{M}$ be an $\mathcal{O}_{F,X}$ module. Then there is an exact sequence of $\mathcal{O}_{F,X}$ modules*

$$0 \longrightarrow \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} F^*\mathcal{M} \longrightarrow \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\alpha_{\mathcal{M}}} \mathcal{M} \longrightarrow 0.$$

Proof. — Consider the map

$$F^*\mathcal{M} \xrightarrow{a\otimes m \mapsto a\otimes m - 1\otimes\psi_{\mathcal{M}}(a\otimes m)} F^*\mathcal{M} \oplus \mathcal{M} \subset \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

One sees immediately that this is an $\mathcal{O}_X$-linear map, when $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M}$ is regarded as an $\mathcal{O}_X$-module via the left $\mathcal{O}_X$-module structure on $\mathcal{O}_{F,X}$. It follows that there is an induced map of induced $\mathcal{O}_{F,X}$-modules

$$\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} F^*\mathcal{M} \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

given by

$$\sum_{i\geq 0} a_i F^i \otimes m_i \mapsto \sum_{i\geq 0} (a_i F^{i+1} \otimes m_i - a_i F^i \psi(1 \otimes m_i))$$

(where the m_i are sections of $\mathcal{M}$). This map is immediately seen to be injective, because if r is chosen such that $a_r F^r \otimes m_r \neq 0$ and $a_i = 0$ for $i > r$, then there is nothing in the sum above to cancel the term $a_r F^{r+1} \otimes m_r$. The image of our map is clearly in the kernel of the natural map $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$. We claim that this image is equal to the kernel. To see this consider an element $h = \sum_{i=0}^r a_i F^i \otimes m_i$ such that $\sum_{i=0}^r a_i F^i(m_i) = 0 \in \mathcal{M}$. We will show that h is in the image of our map by induction on r. If $r = 0$, we have $h = 0$, so there is nothing to prove. In general we have

$$h = \sum_{i=0}^{r}(a_i F^i \otimes m_i - a_i F^i(m_i)) = \sum_{i=1}^{r}(a_i F^i \otimes m_i - a_i F^i(m_i)).$$

By induction on r, it suffices to show that $F^r \otimes m_r - F^r(m_r)$ is in the image of our map. We have

$$F^r \otimes m_r - F^r(m_r) = (F^r \otimes m_r - F^{r-1} \otimes \psi(1 \otimes m_r)) + (F^{r-1} \otimes \psi(1 \otimes m_r) - F^r(m_r))$$

Now the first term on the right hand side above is the image of $F^{r-1} \otimes m_r$, while the second is in the required image by induction. □

13.2. — Let $v \in \mathbb{N}^{\geq 0} \cup \{\infty\}$; furthermore, assume that $v \geq 1$ if $p = 2$. Keeping the notation above, we denote by $\mathscr{D}^{(v)}_{X/W_n}$, or simply $\mathscr{D}^{(v)}_X$ (when W_n-scheme structure on X is understood), the sheaf of rings of differential operators of order v on $\mathcal{O}_X$ over W_n – see [**Ber 2**, §2]. When $v = \infty$, we sometimes denote this sheaf simply by $\mathscr{D}_X$. More generally, omission of the index v in some piece of notation means that we are working with $v = \infty$.

For any left $\mathscr{D}^{(v)}_X$-module $\mathcal{M}$ we denote by $\beta_{\mathcal{M}} : \mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ the map that gives $\mathcal{M}$ its $\mathscr{D}^{(v)}_X$-module structure.

Proposition 13.2.1. — *Let X be a smooth W_n-scheme, and $\mathcal{M}$ a left $\mathscr{D}^{(v)}_X$-module. Then for any Frobenius lifting F on X, $F^*\mathcal{M}$ has the canonical structure of a left $\mathscr{D}^{(v+1)}_X$-module, that is independent of F up to canonical isomorphism.*

Proof. — Base-change by σ equips $F^*_{W_n}\mathcal{M}$ with a structure of left $\mathscr{D}^{(v)}_{X'}$-modules Ber 3 2.1.3. Since F_{W_n} is independent of the choice of F, this structure on $F^*_{W_n}$ is evidently also independent of the choice of F. (Note that we are using the notation of (13.1.2).)

Thus Ber 3 2.2.3 shows that $F^*\mathcal{M} = F^*_{X/W_n}F^*_{W_n}\mathcal{M}$ has the structure of a left $\mathscr{D}^{(v+1)}_X$-module. If F_1, F_2 are any two liftings of Frobenius on X then by [**Ber 3**, 2.2.5]. there is a canonical isomorphism of left $\mathscr{D}^{(v+1)}_X$-modules

$$\tau_{F_1,F_2} : F_1^*\mathcal{M} = F^*_{1,X/W_n}F^*_{W_n}\mathcal{M} \xrightarrow{\sim} F^*_{2,X/W_n}F^*_{W_n}\mathcal{M} = F_2^*\mathcal{M}$$

that for any three lifts F_1, F_2, F_3 satisfies the transitivity property $\tau_{F_1,F_3} = \tau_{F_1,F_2} \circ \tau_{F_2,F_3}$. □

13.2.2. — Let X be any smooth W_n-scheme, and $\mathcal{M}$ a left $\mathscr{D}^{(v)}_X$-module. Then 13.1.1 shows that locally on X there exists a Frobenius lift F, so we can define $F^*\mathcal{M}$ by the above proposition. The above proposition shows that this left $\mathscr{D}^{(v+1)}_X$-module is independent of the choice of F up to canonical isomorphism. Thus, even when no global lifting F exists on X, we can nevertheless define a functor F^* on the category of left $\mathscr{D}^{(v)}_X$-modules on X. (As always, we are assuming that $v \geq 1$ if $p = 2$.) This crucial observation is a mild variant of [**Ber 3**, 2.2.6], which is the analogous result for lifts of the relative Frobenius.

The functor F^* commutes with the left $\mathscr{D}^{(v)}_X$-module pull-back functor f^* for f an arbitrary map of smooth W_n-schemes $f : X \to Y$; more precisely, there is a canonical isomorphism of functors $F^*f^* \xrightarrow{\sim} f^*F^*$ (as follows immediately from the arguments of [**Ber 3**]).

13.3. — Let X be a smooth W_n-scheme. We write

$$\mathscr{D}^{(v)}_{F,X} = \bigoplus_{r\geq 0} (F^*)^r \mathscr{D}^{(v)}_X, \tag{13.3.1}$$

where F^* is the functor defined in (13.2.2). By Proposition 13.2.1 this is naturally a left $\mathscr{D}^{(v)}_X$-module on X. This subsection will be concerned with showing that $\mathscr{D}^{(v)}_{F,X}$ has a natural structure of a sheaf of rings, extending the ring structure on $\mathscr{D}^{(v)}_X$.

We begin by defining a map of left $\mathscr{D}^{(v)}_X$-modules

$$\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} \to \mathscr{D}^{(v)}_{F,X},$$

where the term on the left is a $\mathscr{D}^{(v)}_X$-module via the first factor. First note that, using the left $\mathscr{D}^{(v)}_X$-module structure on $(F^*)^m \mathscr{D}^{(v)}_X$, we have, for any non-negative integer m, a natural map

$$\mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m \mathscr{D}^{(v)}_X \xrightarrow{\sim} (F^*)^m \mathscr{D}^{(v)}_X.$$

Using the functor $(F^*)^r$, for any non-negative integer r we then obtain an isomorphism

$$(F^*)^r \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m \mathscr{D}^{(v)}_X \xrightarrow{\sim} (F^*)^{m+r} \mathscr{D}^{(v)}_X.$$

Combining these for all $m, r \geq 0$ gives the required pairing.

13.3.2. — To verify that the above pairing makes $\mathscr{D}^{(v)}_{F,X}$ a sheaf of associative rings we will give a more explicit local description of $\mathscr{D}^{(v)}_{F,X}$ and of the above pairing on it.

We work locally on X, so that we may and do fix a lifting of Frobenius F on X. In this situation we have

$$\mathscr{D}^{(v)}_{F,X} = \bigoplus_{r\geq 0} (F^*)^r \mathscr{D}^{(v)}_X \xrightarrow{\sim} \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X.$$

Consider a left $\mathscr{D}^{(v)}_X$-module $\mathcal{M}$, together with a map of left $\mathscr{D}^{(v)}_X$-modules $\psi_{\mathcal{M}} : F^*\mathcal{M} \to \mathcal{M}$. We call $\mathcal{M}$ equipped with such a datum, a left $\mathscr{D}^{(v)}_{F,X}$-*pre-module*. (The term "module" is not justified before we know that $\mathscr{D}^{(v)}_{F,X}$ is a ring !).

A morphism of left pre-modules is defined to be a morphism of $\mathscr{D}^{(v)}_X$-modules $h : \mathcal{M} \to \mathcal{N}$ that is compatible with the maps $\psi_{\mathcal{M}}, \psi_N$ in the sense that we have $\psi_{\mathcal{N}} \circ F^*h = h \circ \psi_{\mathcal{M}}$.

If $\mathcal{M}$ is a left $\mathscr{D}^{(v)}_{F,X}$-pre-module, then it is, in particular, a left $\mathcal{O}_{F,X}$-module, by Proposition 13.1.4. Using the left $\mathscr{D}^{(v)}_X$ and $\mathcal{O}_{F,X}$-module structures on $\mathcal{M}$, we obtain a map

$$\gamma_{\mathcal{M}} : \mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} \xrightarrow{\sim} \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}.$$

In particular, $\mathscr{D}^{(v)}_{F,X}$ is a left $\mathscr{D}^{(v)}_{F,X}$-pre-module via the natural inclusion $F^*\mathscr{D}^{(v)}_{F,X} \subset \mathscr{D}^{(v)}_{F,X}$. It is easy to see that if we apply the above construction to $\mathscr{D}^{(v)}_{F,X}$ itself, then the resulting map $\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} \to \mathscr{D}^{(v)}_{F,X}$ is equal to the pairing of (13.3).

More generally if $\mathcal{M}$ is any left $\mathscr{D}^{(v)}_X$-module, then $\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M}$ is a $\mathscr{D}^{(v)}_{F,X}$ pre-module, via the first factor.

Lemma 13.3.3. — *Let $\mathcal{M}$ be a left $\mathscr{D}^{(v)}_{F,X}$-pre-module. Then the map $\gamma_{\mathcal{M}}$: $\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} \to \mathcal{M}$ is a map of left $\mathscr{D}^{(v)}_{F,X}$-pre-modules, if the left hand side is regarded as a left $\mathscr{D}^{(v)}_{F,X}$-pre-module via the first factor.*

Proof. — Since $\mathcal{M}$ is a module over the ring $\mathscr{D}^{(v)}_X$, the morphism $\beta_{\mathcal{M}} : \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} \to \mathcal{M}$ is a map of left $\mathscr{D}^{(v)}_X$-modules, when the term on the left is regarded as a left $\mathscr{D}^{(v)}_X$-module via the first factor. By functoriality it follows that $(F^*)^r\beta_{\mathcal{M}}$ is also a map of left $\mathscr{D}^{(v)}_X$-modules. Composing this with the map of left $\mathscr{D}^{(v)}_X$-modules $(F^*)^r\mathcal{M} \to \mathcal{M}$ induced by $\psi_{\mathcal{M}}$ yields a map of left $\mathscr{D}^{(v)}_X$-modules

$$(F^*)^r\mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} \to (F^*)^r\mathcal{M} \to \mathcal{M}.$$

The direct sum of these maps is the map $\gamma_{\mathcal{M}}$, which is therefore a map of left $\mathscr{D}^{(v)}_X$-modules.

Since $\gamma_{\mathcal{M}}$ is visibly a map of $\mathcal{O}_{F,X}$-modules, we see that it must be a map of left $\mathscr{D}^{(v)}_{F,X}$-pre-modules. □

Lemma 13.3.4. — *Let $h : \mathcal{M} \to \mathcal{N}$ be a map of left $\mathscr{D}^{(v)}_{F,X}$-pre-modules. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} & \xrightarrow{\mathrm{id}_{\mathscr{D}^{(v)}_{F,X}} \otimes h} & \mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{N} \\ \downarrow{\scriptstyle \gamma_{\mathcal{M}}} & & \downarrow{\scriptstyle \gamma_{\mathcal{N}}} \\ \mathcal{M} & \xrightarrow{h} & \mathcal{N} \end{array}$$

Proof. — This is immediate from the functoriality of the construction of $\gamma_{\mathcal{M}}$ and γ_N. □

Corollary 13.3.5. — *$\mathscr{D}^{(v)}_{F,X}$ is a sheaf of associative W_n-algebras. The natural embedding $\mathscr{D}^{(v)}_X \to \mathscr{D}^{(v)}_{F,X}$ is a W_n-algebra homomorphism.*

Proof. — The multiplication on $\mathscr{D}^{(v)}_{F,X}$ was described in (13.3) and (13.3.2). Lemma 13.3.3 implies that the multiplication $\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} \to \mathscr{D}^{(v)}_{F,X}$ is a map of left $\mathscr{D}^{(v)}_{F,X}$-pre-modules, when the left hand side is regarded as a left $\mathscr{D}^{(v)}_{F,X}$-pre-module via the first factor. The associativity of the multiplication now follows by applying Lemma

13.3.4 with h equal to this multiplication map. That the map $\mathscr{D}_X^{(v)} \to \mathscr{D}_{F,X}^{(v)}$ is an algebra homomorphism follows directly from the construction of the multiplication on $\mathscr{D}_{F,X}^{(v)}$. □

13.3.6. — We conclude this subsection by providing a characterisation of left $\mathscr{D}_{F,X}^{(v)}$-modules. Let X be a smooth W_n-scheme. We do not assume that there exists a global lift of Frobenius on X, but we nevertheless have the functor F^* on $\mathscr{D}_X^{(v)}$-modules, explained in (13.2.2). We generalise the definition of (13.3.2), and call a left $\mathscr{D}_{F,X}^{(v)}$-pre-module a $\mathscr{D}_X^{(v)}$-module $\mathcal{M}$ together with a morphism $\psi_{\mathcal{M}} : F^*\mathcal{M} \to \mathcal{M}$. Morphisms of left $\mathscr{D}_{F,X}^{(v)}$-pre-modules are defined as in (13.3.2). We then have:

Proposition 13.3.7. — *There is an equivalence of categories between the category of left $\mathscr{D}_{F,X}^{(v)}$-pre-modules, and the category of left $\mathscr{D}_{F,X}^{(v)}$-modules.*

Proof. — If $\mathcal{M}$ is a left $\mathscr{D}_{F,X}^{(v)}$-module then we have in particular a map

$$F^*\mathcal{M} \xrightarrow{\sim} F^*\mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \subset \mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \to \mathcal{M}.$$

Since $\mathcal{M}$ is assumed to be a $\mathscr{D}_{F,X}^{(v)}$-module, this is a map of left $\mathscr{D}_X^{(v)}$-modules.

Conversely, let $\mathcal{M}$ be a left $\mathscr{D}_X^{(v)}$-module, and suppose that we are given a map of left $\mathscr{D}_X^{(v)}$-modules $\psi_{\mathcal{M}} : F^*\mathcal{M} \to \mathcal{M}$. Pulling back $\psi_{\mathcal{M}}$ by F repeatedly we obtain maps of left $\mathscr{D}_X^{(v)}$-modules $\psi_{\mathcal{M},r} : (F^*)^r\mathcal{M} \to \mathcal{M}$, which combine to yield a map of left $\mathscr{D}_X^{(v)}$-modules

$$\mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_X} \mathcal{M} = \bigoplus_{r\geq 0}(F^*)^r\mathcal{M} \xrightarrow{\oplus \psi_{\mathcal{M},r}} \mathcal{M}.$$

Lemma 13.3.3 shows that this is a map of left $\mathscr{D}_{F,X}^{(v)}$-pre-modules, and Lemma 13.3.4 then implies that it is a map of left $\mathscr{D}_{F,X}^{(v)}$-modules. One easily checks that the two constructions we have given are quasi-inverse to one another. □

Corollary 13.3.8. — *There is a functor F^* from the category of left $\mathscr{D}_{F,X}^{(v)}$-modules to the category of left $\mathscr{D}_{F,X}^{(v+1)}$-modules, that agrees with F^* on underlying $\mathscr{D}_X^{(v)}$-modules, and which is an equivalence of categories.*

Proof. — If $\mathcal{M}$ is a left $\mathscr{D}_{F,X}^{(v)}$-module, then pulling back the map $\psi_{\mathcal{M}}$ by F^* gives a map of left $\mathscr{D}_X^{(v+1)}$-modules

$$F^*F^*\mathcal{M} \to F^*\mathcal{M}.$$

This endows $F^*\mathcal{M}$ with the structure of a left $\mathscr{D}_{F,X}^{(v+1)}$-module. The fact that F^* is an equivalence of categories follows from the corresponding result for $\mathscr{D}_X^{(v)}$-modules [**Ber 3**, 2.3.6], and Proposition 13.3.7. □

13.4. — We want to give a description of right $\mathscr{D}_{F,X}^{(v)}$-modules, analogous to Proposition 13.3.7. The functor F^* that plays a key role in theory of left $\mathscr{D}_{F,X}^{(v)}$-modules will be replaced by the functor $F^!$ of coherent duality theory.

More precisely, suppose first that X is a smooth W_n-scheme equipped with a lifting F of the absolute Frobenius. If $\mathcal{M}$ is a right $\mathscr{D}_X^{(v)}$-module, then we define

$$F^!\mathcal{M} = F^{-1}Hom_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{M}) \tag{13.4.1}$$

In fact this definition makes sense for any $\mathcal{O}_X$-module $\mathcal{M}$, but if $\mathcal{M}$ is a right $\mathscr{D}_X^{(v)}$-module then we have the following proposition, which is the analogue of Proposition 13.2.1 for right $\mathscr{D}_X^{(v)}$-modules.

Proposition 13.4.2. — *Let X be a smooth W_n-scheme, equipped with a lifting F of the absolute Frobenius. If $\mathcal{M}$ is a right $\mathscr{D}_X^{(v)}$-module, then $F^!\mathcal{M}$ has a natural structure of right $\mathscr{D}_X^{(v+1)}$-module. The right $\mathscr{D}_X^{(v+1)}$-module $F^!\mathcal{M}$ is independent of the choice of F up to canonical isomorphism.*

Proof. — [**Ber 3**, 2.4.1, 2.4.5] proves the above proposition, but with the relative Frobenius F_{X/W_n} in place of F. The result of the proposition follows as in Proposition 13.2.1, by applying [**Ber 3**, 2.4.1] to $F^!_{W_n}\mathcal{M}$. (Note that as always we are assuming that $v \geq 1$ if $p = 2$.) □

13.4.3. — Let X be a smooth W_n-scheme, and $\mathcal{M}$ a right $\mathscr{D}_X^{(v)}$-module. By (13.1.1) there exists a lift F of the absolute Frobenius locally on X, so we can define $F^!\mathcal{M}$. Proposition 1.5.2 implies that $F^!\mathcal{M}$ is independent of the choice of the local lift F, up to canonical isomorphism. Thus, even when no global lift exists we can nevertheless define a functor $F^!$ on the category of right $\mathscr{D}_X^{(v)}$-modules. (Again, we require $v \geq 1$ if $p = 2$.)

This observation is a mild variant of [**Ber 3**, 2.4.5], where the analogous functor is defined in the relative situation. Namely, let X' denote the pull-back of X by the Frobenius on W_n. If X admits a lifting of the absolute Frobenius, then there is an induced map $F_{X/W_n} : X \to X'$, and one can define the functor $F^!_{X/W_n}$ from right $\mathscr{D}_{X'}^{(v)}$ to right $\mathscr{D}_X^{(v+1)}$- modules. If $\mathcal{M}'$ is a right $\mathscr{D}_{X'}^{(v)}$-module, then $F^!_{X/W_n}$ turns out to be independent of the choice of lifting up to canonical isomorphism, so that we can define a functor $F^!_{X/W_n}$ from right $\mathscr{D}_{X'}^{(v)}$-modules to right $\mathscr{D}_X^{(v+1)}$-modules, without assuming the existence of a global lifting of Frobenius.

13.5. — Let X be a smooth W_n-scheme. We call a right $\mathscr{D}_X^{(v)}$-module $\mathcal{M}$ equipped with a map of right $\mathscr{D}_X^{(v)}$-modules $\psi_\mathcal{M} : \mathcal{M} \to F^!\mathcal{M}$ a right $\mathscr{D}_{F,X}^{(v)}$-pre-module. A morphism of right $\mathscr{D}_{F,X}^{(v)}$-pre-modules $(\mathcal{M}, \psi_\mathcal{M}) \to (\mathcal{N}, \psi_N)$ is a morphism $h : \mathcal{M} \to \mathcal{N}$ of right $\mathscr{D}_X^{(v)}$-modules such that $\psi_\mathcal{N} \circ h = F^!h \circ \psi_\mathcal{M}$.

The aim of this subsection is to prove an analogue of Proposition 13.3.7. We will need two lemmas.

Lemma 13.5.1. — *There are canonical isomorphisms of $(\mathscr{D}_X^{(v+1)}, \mathscr{D}_X^{(v+1)})$-bimodules*

$$\mathscr{D}_X^{(v+1)} \xrightarrow{\sim} F^*F^!\mathscr{D}_X^{(v)} \xrightarrow{\sim} F^!F^*\mathscr{D}_X^{(v)} \tag{13.5.2}$$

Proof. — The second isomorphism follows as in [**Ber 3**, 2.5.1], from the formal properties of a $\mathscr{D}^{(v)}_X$-bimodule: that the two $\mathscr{D}^{(v)}_X$ actions commute.

We adopt the notation of (13.4.3). By [**Ber 3**, 2.5.2], we have an isomorphism of $\mathscr{D}^{(v+1)}_X$-bimodules $\mathscr{D}^{(v+1)}_X \xrightarrow{\sim} F^*_{X/W_n} F^!_{X/W_n} \mathscr{D}^{(v)}_{X'}$, and it is easy to see that one has an isomorphism $\mathscr{D}^{(v)}_{X'} \xrightarrow{\sim} F^*_{W_n} F^!_{W_n} \mathscr{D}^{(v)}_X$. Thus we obtain

$$\mathscr{D}^{(v+1)}_X \xrightarrow{\sim} F^*_{X/W_n} F^!_{X/W_n} F^*_{W_n} F^!_{W_n} \mathscr{D}^{(v)}_X \xrightarrow{\sim} F^*_{X/W_n} F^*_{W_n} F^!_{X/W_n} F^!_{W_n} \mathscr{D}^{(v)}_X \xrightarrow{\sim} F^* F^! \mathscr{D}^{(v)}_X.$$

Here the second isomorphism follows as before, as a formal consequence of the definition of a bimodule.

In fact the result of [**Ber 3**, 2.5.2], is proved assuming that there is a fixed lifting F of the Frobenius to X. Thus, one has to check that our isomorphism (or equivalently the one of *loc. cit.*) is actually independent of the choice of F. It is enough to check this after a base change, so we may assume that k is an algebraically closed field.

Suppose that $F_{1,X}, F_{2,X}$ are two liftings of the absolute Frobenius to X. The construction above yields two isomorphisms $\mathscr{D}^{(v+1)}_X \xrightarrow{\sim} F^* F^! \mathscr{D}^{(v)}_X$ corresponding to the choice of $F_{1,X}$ and $F_{2,X}$ respectively. Composing the first with the inverse of the second gives an automorphism of the $(\mathscr{D}^{(v+1)}_X, \mathscr{D}^{(v+1)}_X)$ bimodule $\mathscr{D}^{(v+1)}_X$. Such an automorphism is given by multiplication by a unit $a(F_{1,X}, F_{2,X})$ in the centre of $\mathscr{D}_X$.

This centre is equal to W_n. Indeed, locally on X, we can embed X into a smooth $W(k)$-scheme $\mathfrak{X}$ and lift $F_{1,X}$ and $F_{2,X}$ to automorphisms $F_{1,\mathfrak{X}}$ and $F_{2,\mathfrak{X}}$ that cover the absolute Frobenius on $W(k)$. Now the reductions of $F_{1,\mathfrak{X}}$ and $F_{2,\mathfrak{X}}$ modulo p^i give rise to a unit in the centre of $\mathscr{D}^{(v+1)}_{\mathfrak{X}\otimes W_i(k)}$. These units are compatible, and hence give a unit in the centre of $\mathscr{D}^{(v+1)}_{\mathfrak{X}} =: \varprojlim D^{(v+1)}\mathfrak{X} \otimes W_i(k)$. Let $x \in \mathfrak{X}$ be a closed point, with residue field $k(x) = k$. Then the completion of the local ring $\mathcal{O}_{\mathfrak{X},x}$ is isomorphic to a power series ring over $W(k)$. It follows that our unit must be contained in $W(k)$ for every closed point $x \in \mathfrak{X}$. Reducing modulo p^n proves our claim.

Now suppose that $f : X \to Y$ is a smooth map of smooth W_n-schemes, and that we have liftings $F_{1,Y}$ and $F_{2,Y}$ of the absolute Frobenius to Y such that $F_{i,X}$ and $F_{i,Y}$ are compatible via f, for $i = 1, 2$. It is not hard to see from the construction of [**Ber 3**] that we must have $a(F_{1,X}, F_{2,X}) = a(F_{1,Y}, F_{2,Y})$. Applying this in the case where $Y = \operatorname{Spec}(W_n)$ and $F_1 = F_2 = \sigma$ we see that we must have $a(F_{1,X}, F_{2,X}) = 1$, as required. □

Lemma 13.5.3. — *If $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_X$-module and $\mathcal{E}$ is a left $\mathscr{D}^{(v)}_X$-module, then there is a canonical W_n-linear isomorphism*

$$F^!\mathcal{M} \otimes_{\mathscr{D}^{(v+1)}_X} F^*\mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E}$$

that is functorial in $\mathcal{M}$ and $\mathcal{E}$.

Proof. — Let X' be the pull-back of X by the absolute Frobenius on W_n. By [**Ber 3**, 2.5.7], there is an isomorphism

$$F^!\mathcal{M}\otimes_{\mathscr{D}_X^{(v+1)}} F^*\mathcal{E} \xrightarrow{\sim} F^!_{X/W_n}F^!_{W_n}\mathcal{M}\otimes_{\mathscr{D}_X^{(v+1)}} F^*_{X/W_n}F^*_{W_n}\mathcal{E} \xrightarrow{\sim} F^!_{W_n}\mathcal{M}\otimes_{\mathscr{D}_{X'}^{(v)}} F^*_{W_n}\mathcal{E}.$$

In fact Berthelot's result is proved in the context of of a fixed Frobenius lift, so one has to check that the isomorphism of [**Ber 3**, 2.5.7] is independent of the choice of F, and hence glues over all of X. However, examining the constructions of [**Ber 3**], it is not hard to see that this follows once one knows that the isomorphism [**Ber 3**, 2.5.2] is independent of the choice of F, because [**Ber 3**, 2.5.7] is constructed using formal manipulations of [**Ber 3**, 2.5.2], which makes sense without assuming a local lift of Frobenius. However we already saw in the proof of Lemma 13.5.1 that this isomorphism is indeed independent of the choice of Frobenius lift.

It remains to show that there is a canonical isomorphism

$$F^!_{W_n}\mathcal{M}\otimes_{\mathscr{D}_{X'}^{(v)}} F^*_{W_n}\mathcal{E} \xrightarrow{\sim} \mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}\mathcal{E}.$$

We have

$$F^!_{W_n}\mathcal{M} = \mathit{Hom}_{\mathcal{O}_X}(F_{W_n*}\mathcal{O}_{X'},\mathcal{M}) = \mathrm{Hom}_{W_n}(F_{W_n*}W_n,\mathcal{M}).$$

Under these identifications the required isomorphism is given by $h\otimes F^*e\mapsto h(1)\otimes e$.

Alternatively, we could have deduced the lemma from Lemma 13.5.1, using the same formal manipulations that one uses to deduce 2.5.7 from 2.5.2 in [**Ber 3**]]. □

Proposition 13.5.4. — *There is an equivalence of categories between right $\mathscr{D}^{(v)}_{F,X}$-pre-modules and right $\mathscr{D}^{(v)}_{F,X}$-modules.*

Proof. — We will construct functors between the two categories, but leave it to the reader to check that these really are inverses.

Suppose that $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,X}$ module. We have a map

$$\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X \subset \mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}\mathscr{D}_{F,X}\to\mathcal{M},$$

where the last map is given by multiplication on the right. Since $\mathcal{M}$ is assumed to be a right $\mathscr{D}^{(v)}_{F,X}$-module, this is a map of right $\mathscr{D}^{(v)}_X$-modules, where the left hand side is viewed as a right $\mathscr{D}^{(v)}_X$ module via the second factor $F^*\mathscr{D}^{(v)}_X$. Applying the functor $F^!$ and using Lemma 13.5.1, we obtain a morphism of right $\mathscr{D}^{(v)}_X$-modules

$$\mathcal{M}\to\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}\mathscr{D}_X^{(v+1)}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}F^!F^*\mathscr{D}_X^{(v)}\xrightarrow{\sim}F^!(\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}F^*\mathscr{D}_X^{(v)})\to F^!\mathcal{M}.$$

Conversely, suppose we are given a right $\mathscr{D}^{(v)}_{F,X}$-pre-module $(\mathcal{M},\psi_{\mathcal{M}})$. Let r be a non-negative integer. Repeatedly applying $F^!$ to $\psi_{\mathcal{M}}$ and composing the resulting maps gives a morphism of right $\mathscr{D}^{(v)}_X$-modules $\psi_{\mathcal{M},r}:\mathcal{M}\to(F^!)^r\mathcal{M}$. We therefore obtain a map of right $\mathscr{D}^{(v)}_X$-modules

$$\begin{aligned}\gamma_{\mathcal{M},r}:\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}(F^*)^r\mathscr{D}_X^{(v)}&\xrightarrow{\psi_{\mathcal{M},r}\otimes 1}(F^!)^r\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}(F^*)^r\mathscr{D}_X^{(v)}\\ &\longrightarrow(F^!)^r\mathcal{M}\otimes_{\mathscr{D}_X^{(v+r)}}(F^*)^r\mathscr{D}_X^{(v)}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}}\mathscr{D}_X^{(v)}=\mathcal{M}.\end{aligned}$$

Here the third map is obtained by applying Lemma 13.5.3 n times. Below, it will be convenient to denote by $\beta_{\mathcal{M},r}$ the composite of the last three maps above. Taking the direct sum of the maps $\gamma_{\mathcal{M},r}$ we get a map

$$\mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} = \bigoplus_{r\geq 0} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}^{(v)}_X \xrightarrow{\oplus \gamma_{\mathcal{M},r}} \mathcal{M}.$$

We have to check that this map makes $\mathcal{M}$ into a right $\mathscr{D}^{(v)}_{F,X}$ module. In fact this turns out to be formal: what we have to check is that for non-negative integers m, r the following diagram commutes

$$\begin{array}{ccc}
(F^!)^{m+r}\mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}^{(v)}_X & \longrightarrow & (F^!)^{m+r}\mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^{m+r} \mathscr{D}^{(v)}_X \\
\downarrow{\scriptstyle \beta_{(F^!)^r\mathcal{M},m}\otimes 1} & & \\
(F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}_X & & \downarrow{\scriptstyle \beta_{\mathcal{M},m+r}} \\
\downarrow{\scriptstyle \beta_{\mathcal{M},r}} & & \\
\mathcal{M} & = & \mathcal{M}
\end{array}$$

where the top map is given by the multiplication in $\mathscr{D}^{(v)}_{F,X}$. Using the definition of the maps $\beta_{\mathcal{M},r}$ one reduces to the checking the commutativity of the following diagram

$$\begin{array}{ccc}
(F^!)^m (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}^{(v)}_X & \longrightarrow & (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}^{(v)}_X \\
\downarrow & & \| \\
(F^!)^m (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m (F^*)^r \mathscr{D}^{(v)}_X & \longrightarrow & (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}_X
\end{array}$$

where the map on the left is given by multiplication in $\mathscr{D}^{(v)}_{F,X}$, the bottom map is induced by the map in Lemma 13.5.3, and the top map is $\beta_{(F^!)^r\mathcal{M},m} \otimes 1$. In fact this follows from a more general situation: Suppose that $\mathcal{E}$ is a quasi-coherent left $\mathscr{D}^{(v)}_X$-module. The multiplication map $\mu_{\mathcal{E}} : \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E} \to \mathcal{E}$ is a map of left $\mathscr{D}^{(v)}_X$ modules, hence we also have a map of left $\mathscr{D}^{(v)}_X$-modules $(F^*)^m \mu_{\mathcal{E}} : (F^*)^m \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E} \to (F^*)^m \mathcal{E}$. We claim the following diagram commutes

$$\begin{array}{ccc}
(F^!)^m (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m \mathscr{D}^{(v)}_X \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E} & \longrightarrow & (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E} \\
\downarrow{\scriptstyle 1\otimes (F^*)^m \mu_{\mathcal{E}}} & & \| \\
(F^!)^m (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (F^*)^m \mathcal{E} & \longrightarrow & (F^!)^r \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E}
\end{array}$$

where the map on the bottom is induced by the map of Lemma 13.5.3, and the top map is $\beta_{(F^!)^r\mathcal{M},m} \otimes 1$. When $\mathcal{E} = (F^*)^r \mathscr{D}^{(v)}_X$ this diagram becomes the previous one. The question of whether the diagram commutes is local on X. Hence we may assume that we have a surjection of left $\mathscr{D}^{(v)}_X$- modules $\mathcal{E}' \to \mathcal{E}$ with $\mathcal{E}'$ a free $\mathscr{D}^{(v)}_X$-module. By the functoriality of all the maps in the diagram above, we are reduced to checking that the diagram commutes in the case $\mathcal{E} = \mathscr{D}^{(v)}_X$, when this commutativity is obvious. □

Corollary 13.5.5. — *There is a functor $F^!$ from the category of right $\mathscr{D}^{(v)}_{F,X}$-modules to the category of right $\mathscr{D}^{(v+1)}_{F,X}$, that agrees with $F^!$ on underlying $\mathscr{D}^{(v)}_X$-modules, and which is an equivalence of categories.*

Proof. — If $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,X}$-module, then we may pull back the map $\psi_{\mathcal{M}}$ by $F^!$ to obtain a map of right $\mathscr{D}^{(v+1)}_X$-modules $F^!\mathcal{M} \to F^!F^!\mathcal{M}$. By Proposition 13.5.4 this implies that $F^!\mathcal{M}$ has a structure of a right $\mathscr{D}^{(v+1)}_{F,X}$-module. That $F^!$ is an equivalence of categories follows from the corresponding result for right $\mathscr{D}_X$-modules [**Ber 3**, 2.4.6] and Proposition 13.5.4. □

Corollary 13.5.6. — *The right $\mathscr{D}_X$-action on the canonical bundle ω_X extends canonically to a right $\mathscr{D}_{F,X}$-action*

Proof. — Using [**Ber 3**, 2.4.2], one deduces that there is a canonical isomorphism of right $\mathscr{D}_X$-modules $\omega_X \xrightarrow{\sim} F^!\omega_X$. The result now follows from Proposition 13.5.4. □

Lemma 13.5.7. — *Let $\mathcal{M}$ be a $(\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_X)$-bimodule. Then there is a canonical isomorphism of $(\mathscr{D}^{(v+1)}_X, \mathscr{D}^{(v+1)}_X)$-bimodules*

$$F^*F^!\mathcal{M} \xrightarrow{\sim} F^!F^*\mathcal{M}$$

Proof. — Suppose first that X admits a lift of Frobenius F. Then the required isomorphism is easy to construct, as in [**Ber 3**, 2.5.1]. We have to check that this isomorphism does not depend on the choice of F. If $\mathcal{M}' \to \mathcal{M}$ is a surjection of $(\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_X)$-bimodules, then by the functoriality of our isomorphism, it is enough to prove this independence for $\mathcal{M}'$.

Write $\mathcal{M}$ as a quotient of a locally free $\mathcal{O}_X$-module E. There is a surjection of $(\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_X)$-bimodules

$$\mathscr{D}^{(v)}_X \otimes_{W_n} E \otimes_{W_n} \mathscr{D}^{(v)}_X \to \mathcal{M}.$$

For the bimodule on the left, the fact that F^* and $F^!$ are additive functors implies that we have obvious (and in particular intrinsically defined) isomorphisms

$$F^!F^*(\mathscr{D}^{(v)}_X \otimes E \otimes \mathscr{D}^{(v)}_X) \xrightarrow{\sim} F^*\mathscr{D}^{(v)}_X \otimes E \otimes F^!\mathscr{D}^{(v)}_X \xrightarrow{\sim} F^*F^!(\mathscr{D}^{(v)}_X \otimes E \otimes \mathscr{D}^{(v)}_X),$$

where all the tensor products are taken over W_n. □

Proposition 13.5.8. — *There is an isomorphism of $(\mathscr{D}^{(v+1)}_{F,X}, \mathscr{D}^{(v+1)}_{F,X})$-bimodules,*

$$\mathscr{D}^{(v+1)}_{F,X} \xrightarrow{\sim} F^*F^!\mathscr{D}^{(v)}_{F,X}$$

Proof. — The bimodule structure on $F^*F^!\mathscr{D}^{(v)}_{F,X}$ is deduced from Corollaries 13.3.8 and 13.5.5.

By Lemmas 13.5.1 and 13.5.7 we have for each non-negative integer r, the following sequence of isomorphisms of $(\mathscr{D}_X^{(v+1)}, \mathscr{D}_X^{(v+1)})$-bimodules

$$(F^*)^r \mathscr{D}_X^{(v+1)} \xrightarrow{\sim} (F^*)^r F^* F^! \mathscr{D}_X^{(v)} \xrightarrow{\sim} F^* F^! (F^*)^r \mathscr{D}_X^{(v)}$$

Taking the direct sum of these over $n \geq 0$, gives an isomorphism of $(\mathscr{D}_X^{(v+1)}, \mathscr{D}_X^{(v+1)})$-bimodules $\mathscr{D}_{F,X}^{(v+1)} \xrightarrow{\sim} F^* F^! \mathscr{D}_{F,X}^{(v)}$

To check that this is a map of $(\mathscr{D}_{F,X}^{(v+1)}, \mathscr{D}_{F,X}^{(v+1)})$-bimodules is an easy exercise, and left to the reader. (It may be done by "pure thought", or by choosing a local lift of Frobenius, and using the explicit local definition of the isomorphism in 13.5.7). □

Corollary 13.5.9. — *Let $\mathcal{M}$ be a right $\mathscr{D}_{F,X}^{(v)}$-module, and $\mathcal{E}$ a left $\mathscr{D}_{F,X}^{(v)}$-module. There is a canonical W_n-linear isomorphism*

$$F^! \mathcal{M} \otimes_{\mathscr{D}_{F,X}^{(v+1)}} F^* \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}_{F,X}^{(v)}} \mathcal{E}.$$

Proof. — The proof of this is formally identical to the proof of [**Ber 3**, 2.5.7] using Proposition 13.5.8 in place of [**Ber 3**, 2.5.2]. □

Corollary 13.5.10. — *The equivalence of categories F^* of Corollary 13.3.8 has an explicit quasi-inverse, given by the formula*

$$\mathcal{E} \mapsto F^! \mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_{F,X}^{(v+1)}} \mathcal{E}.$$

Proof. — This follows from Proposition 13.5.8, and Corollary 13.5.9. □

13.6. — Let $\mathcal{M}$ be a (left) $\mathscr{D}_X^{(v)}$-module. Then $\mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M}$ is naturally a (left) $\mathscr{D}_{F,X}^{(v)}$-module, which we call the left $\mathscr{D}_{F,X}^{(v)}$-module induced from $\mathcal{M}$. We want to explain the analogue of Proposition 13.1.6.

Proposition 13.6.1. — *Let $\mathcal{M}$ be a $\mathscr{D}_{F,X}^{(v)}$-module. There is a canonical exact sequence of $\mathscr{D}_{F,X}^{(v)}$-modules*

$$0 \to \mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} F^* \mathcal{M} \to \mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \to \mathcal{M} \to 0,$$

where the first two terms are regarded as left $\mathscr{D}_{F,X}^{(v)}$ modules via the first factor $\mathscr{D}_{F,X}^{(v)}$.

Proof. — By Proposition 13.3.7 there is associated to $\mathcal{M}$ a map of $\mathscr{D}_X^{(v)}$-modules $\psi : F^* \mathcal{M} \to \mathcal{M}$. We define a map as in Proposition 13.1.6

$$F^* \mathcal{M} \overset{h \mapsto h \oplus -\psi(h)}{\longrightarrow} F^* \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \oplus \mathcal{M} \subset \mathscr{D}_{F,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M}.$$

This is a map of $\mathscr{D}_X^{(v)}$-modules, as ψ is. Here we have identified $F^* \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M}$ and $F^* \mathcal{M}$ via the isomorphism obtained by applying F^* to the natural isomorphism $\mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}$. Note that if we fix a lifting F of the absolute Frobenius, then this map is given by $a \otimes m \mapsto aF \otimes m - \psi(a \otimes m)$, for $a \otimes m \in F^* \mathcal{M} = \mathcal{O}_X F \otimes_{\mathcal{O}_X} \mathcal{M}$.

The above map of left $\mathscr{D}^{(v)}_X$-modules, induces a map of induced $\mathscr{D}^{(v)}_{F,X}$-modules

$$\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} F^*\mathcal{M} \to \mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M}. \tag{13.6.2}$$

We claim that (13.6.2) is injective, and that its image is equal to the kernel of the multiplication $\mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{M} \to \mathcal{M}$. This claim is local on X, so we may assume that there exists a Frobenius lifting F on X, and write $\mathscr{D}^{(v)}_{F,X} = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X$. In this case (13.6.2), when viewed as a map of $\mathcal{O}_{F,X}$-modules, reduces to the map of Proposition 13.1.6. The proposition follows. □

Corollary 13.6.3. — *If $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,X}$-module, then $\mathcal{M}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_{F,X}$-module if and only if it has finite Tor dimension as a right $\mathscr{D}^{(v)}_X$-module.*

Proof. — Suppose that $\mathcal{M}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_{F,X}$-module. Let $\mathcal{N}$ be any left $\mathscr{D}^{(v)}_X$-module. Then we have

$$\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_X} \mathcal{N} \xrightarrow{\sim} \mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} (\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_X} \mathcal{N})$$

and the right hand side is a finite length complex, since $\mathscr{D}^{(v)}_{F,X}$ is locally free as a right $\mathscr{D}^{(v)}_X$-module.

Conversely, suppose that $\mathcal{M}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_X$-module. We have to show that there is an integer $i > 0$ such that for any left $\mathscr{D}^{(v)}_{F,X}$-module $\mathcal{N}$, $\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} \mathcal{N}$ is acyclic in degree $< -i$. By Proposition 13.6.1, it is enough to show this for $\mathcal{N}$ of the form $\mathcal{N} = \mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_X} N$ for some left $\mathscr{D}^{(v)}_X$-module N. However, in this case we have

$$\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} \mathcal{N} = \mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_X} N,$$

so the existence of the integer i follows from the factor that $\mathcal{M}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_X$-module. □

13.6.4. — There is a similar notion of induced right $\mathscr{D}^{(v)}_{F,X}$-module: if M is a right $\mathscr{D}^{(v)}_X$- module, then $\mathcal{M} = M \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X}$ is a right $\mathscr{D}^{(v)}_{F,X}$-module.

The following result shows that if $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,X}$-module, then $F^!\mathcal{M}$ has a canonical two term resolution by induced modules.

Proposition 13.6.5. — *Let $\mathcal{M}$ be a right $\mathscr{D}^{(v)}_{F,X}$-module, and denote by $\mathcal{M}'$ the right $\mathscr{D}^{(v)}_X$-module $\mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} F^*\mathscr{D}^{(v)}_X$. There is a canonical exact sequence of right $\mathscr{D}^{(v)}_{F,X}$-modules*

$$0 \to \mathcal{M}' \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} \to \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} \to \mathcal{M} \to 0$$

where the first two terms of the exact sequence are right $\mathscr{D}^{(v)}_{F,X}$-modules, via the second factor in the tensor product

Proof. — Let $\mathcal{M}''$ be the right $\mathscr{D}_X^{(v-1)}$-module $\mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)}$. Then we have isomorphisms of right $\mathscr{D}_{F,X}^{(v)}$-modules

$$\mathcal{M}' \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_{F,X}^{(v)}$$
$$\xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)} \otimes_{\mathscr{D}_X^{(v-1)}} \mathscr{D}_{F,X}^{(v)} \xrightarrow{\sim} \mathcal{M}'' \otimes_{\mathscr{D}_X^{(v-1)}} \mathscr{D}_{F,X}^{(v)}$$

Proposition 13.5.8 implies that we have $\mathcal{M} \xrightarrow{\sim} F^!\mathcal{M}''$. Proposition 13.5.4 implies that $\mathcal{M}''$ has a canonical right $D_{F,X}^{(v-1)}$-module structure which induces the right $\mathscr{D}_{F,X}^{(v)}$-module structure on $\mathcal{M}$ by applying $F^!$. In particular we have map of right $\mathscr{D}_X^{(v-1)}$-modules $\psi_{\mathcal{M}''} : \mathcal{M}'' \to F^!\mathcal{M}'' = \mathcal{M}$.

Denote by β the composite the isomorphism of right $\mathscr{D}_X^{(v-1)}$-modules

$$\mathcal{M}'' \xrightarrow{\sim} F^!\mathcal{M}'' \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)}$$

given by Lemma 13.5.3. We define a map of right $\mathscr{D}_X^{(v-1)}$-modules

$$\mathcal{M}'' \overset{m \mapsto \psi_{\mathcal{M}''}(m) \oplus -\beta(m)}{\longrightarrow} F^!\mathcal{M}'' \oplus F^!\mathcal{M}'' \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)}$$
$$\xrightarrow{\sim} \mathcal{M} \oplus \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v)} \subset \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)}$$

We get an induced map of right $\mathscr{D}_{F,X}^{(v)}$-modules

$$\mathcal{M} \otimes_{\mathscr{D}_X^{(v-1)}} \mathscr{D}_{F,X}^{(v)} \to \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)} \tag{13.6.6}$$

To prove the proposition, it will suffice to show that (13.6.6) is an injection, and that its image, is exactly equal to the kernel of the natural map of right $\mathscr{D}_{F,X}^{(v)}$-modules

$$\mu : \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)} \to \mathcal{M}.$$

Consider an element $m = \sum_{i=0}^{r} m_i \in \mathcal{M}'' \otimes_{\mathscr{D}_X^{(v-1)}} \mathscr{D}_{F,X}^{(v)}$ with $m_i \in \mathcal{M}'' \otimes_{\mathscr{D}_X^{(v-1)}} (F^*)^i\mathscr{D}_X^{(v)}$ and $m_r \neq 0$. The image of m under the composite of (13.6.6) with the projection

$$\mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)} \to \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} (F^*)^{r+1}\mathscr{D}_X^{(v)}$$

is equal to the image of m_r under composite of the isomorphisms

$$\mathcal{M}'' \otimes_{\mathscr{D}_X^{(v-1)}} (F^*)^r\mathscr{D}_X^{(v)} \overset{-\beta\otimes 1}{\xrightarrow{\sim}} F^!\mathcal{M}'' \otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)} \otimes_{\mathscr{D}_X^{(v-1)}} (F^*)^r\mathscr{D}_X \tag{13.6.7}$$
$$\xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} (F^*)^{r+1}\mathscr{D}_X^{(v)}$$

Since $m_r \neq 0$, we see, in particular, that the image of m under (13.6.6) is non-zero, whence (13.6.6) is injective.

To check that the image of (13.6.6) is in the kernel of μ, it is enough to check that μ kills the image of $\mathcal{M}''$ itself. For this we consider the following diagram

$$\begin{array}{ccccc} \mathcal{M}'' & \xrightarrow{\psi_{\mathcal{M}''}} & \mathcal{M} & \xleftarrow{\sim} & \\ \downarrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \searrow \\ F^!\mathcal{M}''\otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)} & \xrightarrow[F^!\psi_{\mathcal{M}''}\otimes 1]{} & F^!\mathcal{M}\otimes_{\mathscr{D}_X^{(v)}} F^*\mathscr{D}_X^{(v-1)} & \longrightarrow & F^!\mathcal{M}\otimes_{\mathscr{D}_X^{(v+1)}} F^*\mathscr{D}_X^{(v)} \end{array}$$

(the diagonal arrow, marked $\sim$, goes from $F^!\mathcal{M}\otimes_{\mathscr{D}_X^{(v+1)}} F^*\mathscr{D}_X^{(v)}$ to $\mathcal{M}$)

Note that the left rectangle in the above diagram commutes by the functoriality of the isomorphism β. On the other hand the maps obtained by composing $F^!\psi_{\mathcal{M}''}\otimes 1 = \psi_{\mathcal{M}}\otimes 1$ with the two maps going around the triangle on the left are equal, since the right $\mathscr{D}_{F,X}^{(v-1)}$-module structure on $\mathcal{M}$ is obtained from by restriction of scalars from its right $\mathscr{D}_{F,X}^{(v)}$-module structure. This implies that the outer boundary of the above diagram commutes. Since $\mu\circ\beta$ is the map obtained by going around the diagram counterclockwise, one sees that μ kills the image of $\mathcal{M}''$ under the map (13.6.6).

Finally we check that (13.6.6) is a surjection onto the kernel of μ. Suppose that $m = \sum_{i=0}^{r} m_i$ with $m_i \in \mathcal{M}\otimes_{\mathscr{D}_X^{(v)}} (F^*)^i\mathscr{D}_X^{(v)}$, and $\mu(m) = 0$. To show that m is in the image of (13.6.6), we proceed by induction on r. If $r = 0$ then we must have $m = 0$. If $r > 0$, choose $h \in \mathcal{M}''\otimes_{\mathscr{D}_X^{(v-1)}} (F^*)^{r-1}\mathscr{D}_X^{(v)}$ such that (13.6.7) maps h to m_r. Let m' denote the image of h under (13.6.6). Then we have $m = m - m' + m'$. Now $m - m'$ is in the image of (13.6.6) by induction, while m' is in the image by definition. □

13.7. — Proposition 13.3.7 (resp. 13.5.4) shows that if $\mathcal{M}$ is a left (resp. right) $\mathscr{D}_X^{(v)}$-module, then extending this structure on $\mathcal{M}$ to a left (resp. right) $\mathscr{D}_{F,X}^{(v)}$-module structure is equivalent to equipping $\mathcal{M}$ with a pre-module structure – that is, a map of left (resp. right) $\mathscr{D}_X^{(v)}$-modules $F^*\mathcal{M}\to\mathcal{M}$ (resp. $\mathcal{M}\to F^!\mathcal{M}$).

It is natural to also consider left (resp. right) $\mathscr{D}_X^{(v)}$-modules together with a map of $\mathscr{D}_X^{(v)}$-modules $\mathcal{M}\to F^*\mathcal{M}$ (resp. $F^!\mathcal{M}\to\mathcal{M}$). (Objects of the first type appear, for example, in Berthelot's treatment of F-crystals from the $\mathscr{D}$-module point of view [**Ber 1**, 5.1.1]. This section is devoted to giving an analogous module-theoretic interpretation of such objects.

13.7.1. — Let X be a smooth W_n-scheme, and let v be an element of $\mathbb{N}^{\geq 0}\bigcup\{\infty\}$; as always, we assume that $v\geq 1$ if $p = 2$. We define

$$\mathscr{D}_{F^!,X}^{(v)} = \bigoplus_{r\geq 0}(F^!)^r\mathscr{D}_X^{(v)}.$$

(Of course, we are using the functor $F^!$ on right $\mathscr{D}_X^{(v)}$-modules described in (13.4.3).) Each summand in this direct sum is naturally a right $\mathscr{D}_X^{(v)}$-module; hence for any integer $r\geq 0$ we have an isomorphism of right $\mathscr{D}_X^{(v)}$-modules

$$(F^!)^r\mathscr{D}_X^{(v)}\otimes_{\mathscr{D}_X^{(v)}}\mathscr{D}_X^{(v)} \xrightarrow{\sim} (F^!)^r\mathscr{D}_X^{(v)}.$$

Applying $(F^!)^m$ for any integer $m \geq r$ gives an isomorphism

$$(F^!)^r \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} (F^!)^m \mathscr{D}_X^{(v)} \xrightarrow{\sim} (F^!)^{r+m} \mathscr{D}_X^{(v)}.$$

Combining these for all $m, r \geq 0$ gives a map of right $\mathscr{D}_X^{(v)}$-modules

$$\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}^{(v)}_{F^!,X} \to \mathscr{D}^{(v)}_{F^!,X}. \tag{13.7.2}$$

13.7.3. — Let X be a smooth W_n-scheme. We call a left $\mathscr{D}_X^{(v)}$-module $\mathcal{M}$ equipped with a map of left $\mathscr{D}_X^{(v)}$-modules $\psi_{\mathcal{M}} : \mathcal{M} \to F^*\mathcal{M}$ a left $\mathscr{D}^{(v)}_{F^!,X}$-pre-module. A morphism of left $\mathscr{D}^{(v)}_{F^!,X}$-pre-modules $(\mathcal{M}, \psi_{\mathcal{M}}) \to (\mathcal{N}, \psi_{\mathcal{N}})$ is defined to be a morphism $h : \mathcal{M} \to \mathcal{N}$ of left $\mathscr{D}_X^{(v)}$-modules for which $\psi_{\mathcal{N}} \circ h = F^*h \circ \psi_{\mathcal{N}}$. (Here we are using the functor F^* on left $\mathscr{D}_X^{(v)}$-modules described in (13.2.2).)

Fix a left $\mathscr{D}^{(v)}_{F^!,X}$-pre-module $(\mathcal{M}, \psi_{\mathcal{M}})$, and let r be a non-negative integer. Repeatedly applying F^* to $\psi_{\mathcal{M}}$ and composing the resulting maps yields a morphism $\psi_{\mathcal{M},r} : \mathcal{M} \to (F^*)^r\mathcal{M}$ of left $\mathscr{D}_X^{(v)}$-modules. We thus obtain a map of left $\mathscr{D}_X^{(v)}$-modules

$$\begin{aligned}\gamma_{\mathcal{M},r} : (F^!)^r \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} &\xrightarrow{\psi_{\mathcal{M},r}\otimes 1} (F^!)^r \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} (F^*)^r\mathcal{M} \\ &\longrightarrow (F^!)^r \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v+r)}} (F^*)^r\mathcal{M} \xrightarrow{\sim} \mathscr{D}_X^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}.\end{aligned}$$

(The third arrow is provided by Lemma 13.5.3.) Taking the direct sum over all $r \geq 0$ yields a map

$$\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{M} \to \mathcal{M}. \tag{13.7.4}$$

13.7.5. — We call a right $\mathscr{D}_X^{(v)}$-module equipped with a map of right $\mathscr{D}_X^{(v)}$-modules $F^!\mathcal{M} \to \mathcal{M}$ a right $\mathscr{D}^{(v)}_{F^!,X}$-pre-module. A morphism of right $\mathscr{D}^{(v)}_{F^!,X}$-pre-modules $(\mathcal{M}, \psi_{\mathcal{M}}) \to (\mathcal{N}, \psi_{\mathcal{N}})$ is defined to be a morphism $h : \mathcal{M} \to \mathcal{N}$ of right $\mathscr{D}_X^{(v)}$-modules for which $\psi_{\mathcal{N}} \circ h = F^!h \circ \psi_{\mathcal{N}}$. (Here we are using the functor $F^!$ on right $\mathscr{D}_X^{(v)}$-modules described in (13.4.3).)

Fix a right $\mathscr{D}^{(v)}_{F^!,X}$-pre-module $(\mathcal{M}, \psi_{\mathcal{M}})$. Repeatedly applying $F^!$ to $\psi_{\mathcal{M}}$ and composing the resulting maps yields a morphism $\psi_{\mathcal{M},r} : (F^!)^r\mathcal{M} \to \mathcal{M}$ of right $\mathscr{D}_X^{(v)}$-modules. We thus obtain a morphism of right $\mathscr{D}_X^{(v)}$-modules

$$\mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} (F^!)^r \mathscr{D}_X^{(v)} \xrightarrow{\sim} (F^!)^r\mathcal{M} \to \mathcal{M}.$$

Taking the direct sum over all $r \geq 0$ yields a map

$$\mathcal{M} \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}^{(v)}_{F^!,X} \to \mathcal{M}. \tag{13.7.6}$$

Proposition 13.7.7. — *Let X be a smooth W_n-scheme.*

(i) The map (13.7.2) makes $\mathscr{D}^{(v)}_{F^!,X}$ a sheaf of associative W_n-algebras, such that the natural embedding $\mathscr{D}_X^{(v)} \to \mathscr{D}^{(v)}_{F^!,X}$ is a W_n-algebra homomorphism.

(*ii*) *If $\mathcal{M}$ is a left $\mathscr{D}^{(v)}_{F^!,X}$-pre-module, then the map (13.7.4) gives $\mathcal{M}$ the structure of a left $\mathscr{D}^{(v)}_{F^!,X}$-module.*

(*iii*) *The association of a left $\mathscr{D}^{(v)}_{F^!,X}$-module to a left $\mathscr{D}^{(v)}_{F^!,X}$-pre-module given by (ii) establishes an equivalence of categories between the category of left $\mathscr{D}^{(v)}_{F^!,X}$-pre-modules and the category of left $\mathscr{D}^{(v)}_{F^!,X}$-modules.*

(*iv*) *If $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F^!,X}$-pre-module, then the map (13.7.6) gives $\mathcal{M}$ the structure of a right $\mathscr{D}^{(v)}_{F^!,X}$-module.*

(*v*) *The association of a right $\mathscr{D}^{(v)}_{F^!,X}$-module to a right $\mathscr{D}^{(v)}_{F^!,X}$-pre-module given by (iv) establishes an equivalence of categories between the category of right $\mathscr{D}^{(v)}_{F^!,X}$-pre-modules and the category of right $\mathscr{D}^{(v)}_{F^!,X}$-modules.*

Proof. — The proofs depend on formal considerations similar to those used in the proofs of Corollary 13.3.5 and Propositions 13.3.7 and 13.5.4. □

13.7.8. — Recall that (for any value of v) there is a natural isomorphism of left $\mathscr{D}^{(v)}_X$-modules $F^*\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X$. Thus not only is $\mathcal{O}_X$ naturally a left $\mathscr{D}^{(v)}_{F,X}$-module, it is also naturally a left $\mathscr{D}^{(v)}_{F^!,X}$-module (as one sees by applying part (ii) of Proposition 13.7.7 to the inverse of this isomorphism). Similarly, the isomorphism $\omega_X \xrightarrow{\sim} F^!\omega_X$ recalled in the proof of Corollary 13.5.6 allows one to define a right $\mathscr{D}^{(v)}_{F^!,X}$-module structure on ω_X.

13.8. — Recall from [**Ber 1**, 1.3.6] that if $\mathcal{E}$ and $\mathcal{F}$ are two left $\mathscr{D}^{(v)}_X$-modules (resp. a right $\mathscr{D}^{(v)}_X$-module and a left $\mathscr{D}^{(v)}_X$-module), then the tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is naturally a left $\mathscr{D}^{(v)}_X$-module (resp. a right $\mathscr{D}^{(v)}_X$-module). We will consider the various refinements of this result that occur when $\mathcal{E}$ and $\mathcal{F}$ are equipped with Frobenius structures.

Lemma 13.8.1. — *If $\mathcal{E}$ and $\mathcal{F}$ are two left $\mathscr{D}^{(v)}_X$-modules, then the natural isomorphism of $\mathcal{O}_X$-modules*

$$\omega_X \otimes_{\mathcal{O}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\sim} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism of right $\mathscr{D}^{(v)}_X$-modules.

Proof. — This "associativity" property can be seen directly from the local formulas for the $\mathscr{D}^{(v)}_X$-module structures on the tensor products in [**Ber 1**, 1.3.6] and [**Ber 3**, 1.1.7]. □

Lemma 13.8.2. — *Let $\mathcal{E}$ be a left $\mathscr{D}^{(v)}_X$-module, and $\mathcal{M}$ a right $\mathscr{D}^{(v)}_X$-module. Then there is a canonical isomorphism of right $\mathscr{D}^{(v)}_X$-modules*

$$\omega_X \otimes_{\mathcal{O}_X} F^*\mathcal{E} \xrightarrow{\sim} F^!(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E})$$

and a canonical isomorphism of left $\mathscr{D}_X^{(v)}$-modules

$$F^*(\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \xrightarrow{\sim} F^!\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

Proof. — The isomorphisms are a mild variant of [**Ber 3**, 2.4.5.1, 2.4.5.2], where they are proved for the relative Frobenius. We adopt the notation of (13.4.3). To prove the first isomorphism, we apply the analogous result in the relative situation to $\mathcal{E}' = F_{W_n}^*\mathcal{E}$. We obtain

$$\begin{aligned}\omega_X \otimes_{\mathcal{O}_X} F^*\mathcal{E} = \omega_X \otimes_{\mathcal{O}_X} F_{X/W_n}^*\mathcal{E}' &\xrightarrow{\sim} F_{X/W_n}^!(\omega_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{E}') \\ &\xrightarrow{\sim} F_{X/W_n}^!(\omega_{X'} \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} F_{X/W_n}^!(F_{W_n}^!\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \\ &\xrightarrow{\sim} F_{X/W_n}^! F_{W_n}^!(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} F^!(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E})\end{aligned}$$

The second isomorphism is proved in a similar way. □

Proposition 13.8.3. — *(i) Let $\mathcal{E}$ and $\mathcal{F}$ be two left $\mathscr{D}_{F,X}^{(v)}$-modules. Then $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}$ carries a natural structure of left $\mathscr{D}_{F,X}^{(v)}$-module.*

(ii) Let $\mathcal{E}$ and $\mathcal{F}$ be left $\mathscr{D}_{F^!,X}^{(v)}$-modules. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ carries a natural structure of left $\mathscr{D}_{F^!,X}^{(v)}$-module.

(iii) Let $\mathcal{M}$ be a right $\mathscr{D}_{F,X}^{(v)}$-module and $\mathcal{E}$ be a left $\mathscr{D}_{F^!,X}^{(v)}$-module. Then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ carries a natural structure of right $\mathscr{D}_{F,X}^{(v)}$-module.

(iv) Let $\mathcal{M}$ be a right $\mathscr{D}_{F^!,X}^{(v)}$-module and $\mathcal{E}$ be a left $\mathscr{D}_{F,X}^{(v)}$-module. Then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ carries a natural structure of right $\mathscr{D}_{F^!,X}^{(v)}$-module.

Proof. — In each case it suffices to construct the corresponding pre-module structure on the tensor product.

(i) [**Ber 3**, 3.3.1], and its proof, show that there is a natural isomorphism of $\mathscr{D}_X^{(v)}$-modules

$$F^*\mathcal{E} \otimes_{\mathcal{O}_X} F^*\mathcal{F} \xrightarrow{\sim} F^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

More precisely the discussion of *loc. cit* proves the analogue of the above isomorphism for the relative Frobenius, and it is easy to deduce the above isomorphism from the relative case, as in (for example) Lemma 13.5.1 and 13.5.3. Hence we obtain a map of $\mathscr{D}_X^{(v)}$-modules

$$F^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\sim} F^*\mathcal{E} \otimes_{\mathcal{O}_X} F^*\mathcal{F} \xrightarrow{\psi_{\mathcal{E}}\otimes\psi_{\mathcal{F}}} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F},$$

which endows $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ with the requisite pre-module structure.

(ii) One uses the same isomorphism as in the proof of (i) to construct a pre-module structure

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\psi_{\mathcal{E}}\otimes\psi_{\mathcal{F}}} F^*\mathcal{E} \otimes_{\mathcal{O}_X} F^*\mathcal{F} \xrightarrow{\sim} F^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

(iii) There is a natural isomorphism of right $\mathscr{D}_X^{(v)}$-modules

$$F^!\mathcal{M} \otimes_{\mathcal{O}_X} F^*\mathcal{E} \xrightarrow{\sim} F^!(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}). \tag{13.8.4}$$

To see this, set $\mathcal{M}' = \mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}$. Lemma 13.8.1 yields a natural isomorphism of right $\mathscr{D}^{(v)}_X$-modules

$$\omega_X \otimes_{\mathcal{O}_X} (\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes_{\mathcal{O}_X} \mathcal{E} = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}.$$

Applying $F^!$ to the above isomorphism, and using the isomorphisms of Lemma 13.8.2, we obtain the required isomorphism. Thus we obtain a map of right $\mathscr{D}^{(v)}_X$-modules

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\psi_{\mathcal{M}} \otimes \psi_{\mathcal{E}}} F^!\mathcal{M} \otimes_{\mathcal{O}_X} F^*\mathcal{E} \xrightarrow{\sim} F^!(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}),$$

endowing $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ with a right $\mathscr{D}^{(v)}_{F^!,X}$-pre-module structure.

(iv) One uses the isomorphism (13.8.4) to construct a pre-module structure

$$F^!(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} F^!\mathcal{M} \otimes_{\mathcal{O}_X} F^*\mathcal{E} \xrightarrow{\psi_{\mathcal{M}} \otimes \psi_{\mathcal{E}}} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}.$$

□

13.8.5. — Recall (again from [**Ber 1**, 1.3.6]) that if $\mathcal{E}$ and $\mathcal{F}$ are two left $\mathscr{D}^{(v)}_X$-modules (resp. a left and a right $\mathscr{D}^{(v)}_X$-module, resp. two right $\mathscr{D}^{(v)}_X$-modules) then $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is naturally a left $\mathscr{D}^{(v)}_X$-module (resp. a right $\mathscr{D}^{(v)}_X$-module, resp. a left $\mathscr{D}^{(v)}_X$-module).

Proposition 13.8.6. — *(i) If $\mathcal{E}$ and $\mathcal{F}$ are left modules over the algebras $\mathscr{D}^{(v)}_{F,X}$ and $\mathscr{D}^{(v)}_{F^!,X}$, respectively, then $S\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is naturally a left $\mathscr{D}^{(v)}_{F^!,X}$-module.*

(ii) If $\mathcal{E}$ and $\mathcal{F}$ are left modules over the algebras $\mathscr{D}^{(v)}_{F^!,X}$ and $\mathscr{D}^{(v)}_{F,X}$, respectively, then $\underline{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is naturally a left $\mathscr{D}^{(v)}_{F,X}$-module.

(iii) If $\mathcal{E}$ and $\mathcal{M}$ are left and right modules, respectively, over the algebra $\mathscr{D}^{(v)}_{F,X}$, then $\underline{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$ is naturally a right $\mathscr{D}^{(v)}_{F,X}$-module.

(iv) If $\mathcal{E}$ and $\mathcal{M}$ are left and right modules, respectively, over the algebra $\mathscr{D}^{(v)}_{F^!,X}$, then $\underline{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$ is naturally a right $\mathscr{D}^{(v)}_{F^!,X}$-module.

(v) If $\mathcal{M}$ and $\mathcal{N}$ are right modules over the algebras $\mathscr{D}^{(v)}_{F,X}$ and $\mathscr{D}^{(v)}_{F^!,X}$, respectively, then $\underline{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is naturally a left $\mathscr{D}^{(v)}_{F,X}$-module.

(vi) If $\mathcal{M}$ and $\mathcal{N}$ are right modules over the algebras $\mathscr{D}^{(v)}_{F^!,X}$ and $\mathscr{D}^{(v)}_{F,X}$, respectively, then $\underline{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is naturally a left $\mathscr{D}^{(v)}_{F^!,X}$-module.

Proof. — This is proved in a similar fashion to Proposition 13.8.3. The required natural isomorphisms are: an isomorphism of left $\mathscr{D}^{(v)}_X$-modules

$$F^*\underline{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}_X}(F^*\mathcal{E}, F^*\mathcal{F}),$$

if $\mathcal{E}$ and $\mathcal{F}$ are left $\mathscr{D}^{(v)}_X$-modules; an isomorphism of right $\mathscr{D}^{(v)}_X$-modules

$$F^!\underline{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M}) \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}_X}(F^*\mathcal{E}, F^!\mathcal{M}),$$

if $\mathcal{E}$ is a left $\mathscr{D}^{(v)}_X$-module and $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_X$-module; an isomorphism of left $\mathscr{D}^{(v)}_X$-modules

$$F^*\underline{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}_X}(F^!\mathcal{M},F^!\mathcal{N}),$$

if $\mathcal{M}$ and $\mathcal{N}$ are right $\mathscr{D}^{(v)}_X$-modules. The first isomorphism is easily constructed using the functoriality of F^*, and the other two can be deduced from the first one using Lemma 13.8.2. We leave the (straightforward) details to the reader. □

13.8.7. — As observed in (13.7.8), the canonical bundle ω_X is naturally both a right $\mathscr{D}^{(v)}_{F,X}$-module and a right $\mathscr{D}^{(v)}_{F^!,X}$-module. It follows from Proposition 13.8.3 that if $\mathcal{E}$ is a left $\mathscr{D}^{(v)}_{F,X}$-module (resp. a left $\mathscr{D}^{(v)}_{F^!,X}$-module) then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is naturally a right $\mathscr{D}^{(v)}_{F^!,X}$-module (resp. a right $\mathscr{D}^{(v)}_{F,X}$-module). Similarly, it follows from Proposition 13.8.6 that if $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,X}$-module (resp. a right $\mathscr{D}^{(v)}_{F^!,X}$-module) then $\mathcal{M}\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is naturally a left $\mathscr{D}^{(v)}_{F^!,X}$-module (resp. a left $\mathscr{D}^{(v)}_{F,X}$-module). (Here we have also used the natural isomorphism $\mathcal{M}\otimes_{\mathcal{O}_X}\omega_X^{-1} \xrightarrow{\sim} \underline{Hom}_{\mathcal{O}_X}(\omega_X,\mathcal{M})$.)

13.9. — The natural $(\mathscr{D}^{(v)}_X,\mathscr{D}^{(v)}_X)$-bimodule structure on $\mathscr{D}^{(v)}_X$ induces a pair of commuting right $\mathscr{D}^{(v)}_X$-module structures on the tensor product $\omega_X\otimes_{\mathcal{O}_X}\mathscr{D}^{(v)}_X$, and a pair of commuting left $\mathscr{D}^{(v)}_X$-module structures on the tensor product $\mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$. There are canonical isomorphisms (in fact an involutions)

(13.9.1) $\quad \omega_X\otimes_{\mathcal{O}_X}\mathscr{D}^{(v)}_X \xrightarrow{\sim} \omega_X\otimes_{\mathcal{O}_X}\mathscr{D}^{(v)}_X$ and $\mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1} \xrightarrow{\sim} \mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$,

known as the transposition isomorphisms, that interchanges these two module structures [**Ber 3**, 1.3.4.1, 1.3.4.3]. We will present some extensions of these isomorphisms to the context of $\mathscr{D}^{(v)}$-modules with Frobenius structure.

Fix an integer $r \geq 0$. First note that, since $(F^!)^r\mathscr{D}^{(v)}_X$ is a $\mathscr{D}^{(v)}_X$-bimodule, the tensor product $\omega_X\otimes_{\mathcal{O}_X}(F^!)^r\mathscr{D}^{(v)}_X$ is equipped with a pair of commuting right $\mathscr{D}^{(v)}_X$-module structures: the first arising from the right $\mathscr{D}^{(v)}_X$-module structure on $F^!\mathscr{D}^{(v)}_X$, and the second arising from the twisting by ω_X of the left $\mathscr{D}^{(v)}_X$-module structure on $(F^!)^r\mathscr{D}^{(v)}_X$. Similarly, $\omega_X\otimes_{\mathcal{O}_X}(F^*)^r\mathscr{D}^{(v)}_X$ is equipped with a pair of commuting right $\mathscr{D}^{(v)}_X$-module structures. On the other hand, $(F^*)^r\mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is equipped with a pair of left $\mathscr{D}^{(v)}_X$-module structures, as is $(F^!)^r\mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$. In each case, we will refer to the obvious module structure as the first module structure, and the module structure arising via the twist as the second module structure.

Lemma 13.9.2. — (*i*) *There is a natural isomorphism*

$$\omega_X\otimes_{\mathcal{O}_X}(F^*)^r\mathscr{D}^{(v)}_X \xrightarrow{\sim} \omega_X\otimes_{\mathcal{O}_X}(F^!)^r\mathscr{D}^{(v)}_X$$

that interchanges the first and second right $\mathscr{D}^{(v)}_X$-module structures on its source and target.

(*ii*) *There is a natural isomorphism*

$$(F^*)^r\mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1} \xrightarrow{\sim} (F^!)^r\mathscr{D}^{(v)}_X\otimes_{\mathcal{O}_X}\omega_X^{-1},$$

that interchanges the first and second left $\mathscr{D}_X^{(v)}$*-module structures on its source and target.*

Proof. — Consider the transposition isomorphism $\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(v)} \xrightarrow{\sim} \omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(v)}$. This interchanges the first and second right $\mathscr{D}_X^{(v)}$-module structures on its source and target. If we apply $F^!$ to this isomorphism, computed with respect to the second right $\mathscr{D}_X^{(v)}$-module structure on the source, and with respect to the first right $\mathscr{D}_X^{(v)}$-module structure on the target, we obtain an isomorphism

$$(F^!)^r(\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(v)}) \xrightarrow{\sim} \omega_X \otimes_{\mathcal{O}_X} (F^!)^r \mathscr{D}_X^{(v)}.$$

Now Lemma 13.8.2 yields an isomorphism

$$\omega_X \otimes_{\mathcal{O}_X} (F^*)^r \mathscr{D}_X^{(v)} \xrightarrow{\sim} (F^!)^r(\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(v)}).$$

Composing this with the preceding isomorphism yields the isomorphism of part (i).

The isomorphism of part (ii) is constructed in a similar fashion. (Alternatively, it can be deduced from the first as [**Ber 3**, 1.3.4.3] is deduced from [**Ber 3**, 1.3.4.1].) □

13.9.3. — The discussion of (13.8.7) shows that each of $\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_{F,X}^{(v)}$ and $\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_{F^!,X}^{(v)}$ is equipped with commuting right $\mathscr{D}_{F,X}^{(v)}$-module and $\mathscr{D}_{F^!,X}^{(v)}$-module structures, while each of $\mathscr{D}_{F,X}^{(v)} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ and $\mathscr{D}_{F^!,X}^{(v)} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ is equipped with commuting left $\mathscr{D}_{F,X}^{(v)}$ and $\mathscr{D}_{F^!,X}^{(v)}$-module structures.

Proposition 13.9.4. — (*i*) *There is a natural isomorphism*

$$\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_{F,X}^{(v)} \xrightarrow{\sim} \omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_{F^!,X}^{(v)}$$

that respects both the right $\mathscr{D}_{F,X}^{(v)}$*-module and the right* $\mathscr{D}_{F^!,X}^{(v)}$*-modules structure on each of its source and target.*

(*ii*) *There is a natural isomorphism*

$$\mathscr{D}_{F,X}^{(v)} \otimes \omega_X^{-1} \xrightarrow{\sim} \mathscr{D}_{F^!,X}^{(v)} \otimes_{\mathcal{O}_X} \omega_X^{-1}$$

that respects both the left $\mathscr{D}_{F,X}^{(v)}$*-module and the left* $\mathscr{D}_{F^!,X}^{(v)}$*-modules structure on each of its source and target.*

Proof. — Taking the direct sum over all $r \geq 0$ of the isomorphisms of Lemma 13.9.2 yields the required isomorphisms. (We leave it to the reader to check that each is compatible with the $\mathscr{D}_{F,X}^{(v)}$ and $\mathscr{D}_{F^!,X}^{(v)}$-module structures on its source and target.) □

13.9.5. — If $\mathcal{E}$ is a left $\mathscr{D}_X^{(v)}$-module, then $\mathscr{D}_{F^!,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{E}$ is a left $\mathscr{D}_{F^!,X}^{(v)}$-module, and so by the discussion of (13.8.7) the tensor product $\omega_X \otimes_{\mathcal{O}_X} (\mathscr{D}_{F^!,X}^{(v)} \otimes_{\mathscr{D}_X^{(v)}} \mathcal{E})$ is a right $\mathscr{D}_{F,X}^{(v)}$-module. On the other hand, the tensor product $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is a right $\mathscr{D}_X^{(v)}$-module, and hence $(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}_X^{(v)}} \mathscr{D}_{F,X}^{(v)}$ is also a right $\mathscr{D}_{F,X}^{(v)}$-module.

Proposition 13.9.6. — *In the situation of (13.9.5) there is a natural isomorphism of right $\mathscr{D}^{(v)}_{F,X}$-modules*

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{F,X} \xrightarrow{\sim} \omega_X \otimes_{\mathcal{O}_X} (\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E}).$$

Proof. — Fix an integer $r \geq 0$. We have isomorphisms of right $\mathscr{D}^{(v)}_X$-modules

$$\begin{aligned}(13.9.7)\quad &(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} (F^*)^r \mathscr{D}^{(v)}_X \xrightarrow{\sim} \omega_X \otimes_{\mathscr{D}^{(v)}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} (F^*)^r \mathscr{D}^{(v)}_X)\\ &\xrightarrow{\sim} \omega_X \otimes_{\mathscr{D}^{(v)}_X} ((F^*)^r \mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} (\omega_X \otimes_{\mathcal{O}_X} (F^*)^r \mathscr{D}^{(v)}_X) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E}\\ &\xrightarrow{\sim} (\omega_X \otimes_{\mathcal{O}_X} (F^!)^r \mathscr{D}^{(v)}_X) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E} \xrightarrow{\sim} \omega_X \otimes_{\mathscr{D}^{(v)}_X} ((F^!)^r \mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E}),\end{aligned}$$

the first, third, and fifth being the natural associativity given by Lemma 13.9.8 below, the second being induced by transposition of the second and third factors, and the fourth by the isomorphism of Lemma 13.9.2 (i).

Taking the direct sum over all values of r of the maps obtained by taking the composite in (13.9.7), yields the isomorphism of the proposition. (We leave it for the reader to check that this isomorphism respects the $\mathscr{D}^{(v)}_{F,X}$-module structures on its source and target.) □

Lemma 13.9.8. — *If $\mathcal{E}$ and $\mathcal{F}$ are two left $\mathscr{D}^{(v)}_X$-modules, and $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_X$-module, then the natural associativity isomorphism*

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

induces an isomorphism

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

In particular, taking $\mathcal{F} = \mathcal{O}_X$, we obtain a natural isomorphism

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{O}_X \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E}.$$

Proof. — For any smooth W_n-scheme X, and left $\mathscr{D}^{(v)}_X$-module $\mathcal{E}$, recall from [**Ber 3**, 1.3.3] the isomorphism of $(\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_X)$-bimodules $\gamma_{\mathcal{E}} : \mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X$. Thus, we have natural isomorphisms

$$\begin{aligned}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{F} &\xrightarrow{\sim} (\mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{F}\\ &\xrightarrow[\sim]{\mathrm{id}\otimes\gamma_{\mathcal{E}}\otimes\mathrm{id}} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}).\end{aligned}$$

Since $\gamma_{\mathcal{E}}$ restricts to the identity on $\mathcal{E} \subset \mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E}$, we see that this isomorphism is covered by the associativity isomorphism. □

13.9.9. — If $\mathcal{E}$ and $\mathcal{F}$ are two left $\mathscr{D}^{(v)}_X$-modules, then there is a natural isomorphism

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{F} \xrightarrow{\sim} \omega_X \otimes_{\mathscr{D}^{(v)}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\sim} \omega_X \otimes_{\mathscr{D}^{(v)}_X} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_{\mathscr{D}^{(v)}_X} \mathcal{E}.$$

(Here the first and third isomorphisms are provided by Lemma 13.9.8, while the second isomorphism is given by transposing the factors.) This isomorphism is covered by the isomorphism $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \omega_X \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$ given by transposing the second and third factors. The following Lemma gives a version of this for $\mathscr{D}^{(v)}_X$-modules with Frobenius structures.

Lemma 13.9.10. — *If $\mathcal{E}$ is a left $\mathscr{D}^{(v)}_{F,X}$-module and $\mathcal{F}$ is a left $\mathscr{D}^{(v)}_{F^!,X}$-module, then the natural isomorphism $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \omega_Y \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$ obtained by interchanging the second and third factors induces an isomorphism*

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F^!,X}} \mathcal{F} \xrightarrow{\sim} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_{\mathscr{D}^{(v)}_{F,X}} \mathcal{E}.$$

Proof. — Using (13.8.7) and Proposition 13.9.4, we obtain isomorphisms of right $\mathscr{D}^{(v)}_{F^!,X}$-modules

$$\omega_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \omega \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_{F,X} \otimes_{\mathscr{D}^{(v)}_{F,X}} \mathcal{E} \xrightarrow{\sim} (\omega \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_{F^!,X}) \otimes_{\mathscr{D}^{(v)}_{F,X}} \mathcal{E} \tag{13.9.11}$$

Applying $\otimes_{\mathscr{D}^{(v)}_{F^!,X}} \mathcal{F}$ to both sides of (13.9.11) yields

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F^!,X}} \mathcal{F} \xrightarrow{\sim} (\omega \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathscr{D}^{(v)}_{F^!,X}} \mathcal{F}) \otimes_{\mathscr{D}^{(v)}_{F,X}} \mathcal{E} \xrightarrow{\sim} (\omega \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_{\mathscr{D}^{(v)}_{F,X}} \mathcal{E}.$$

□

14. DIRECT AND INVERSE IMAGES FOR $\mathscr{D}_{F,X}^{(v)}$-MODULES

14.1. — In this section we will define direct and inverse image functors for $\mathscr{D}_{F,X}^{(v)}$-modules, and establish their basic properties. Since these functors are only defined on the level of derived categories, we begin with the definition of the relevant triangulated categories.

If X is a smooth W_n-scheme we denote by $\mu(\mathscr{D}_{F,X}^{(v)})$ the full subcategory of the abelian category of left $\mathscr{D}_{F,X}^{(v)}$-modules consisting of those modules which are quasi-coherent as $\mathcal{O}_X$-modules.

We denote by $D(\mathscr{D}_{F,X}^{(v)})$ the derived category of the category of left $\mathscr{D}_{F,X}^{(v)}$-modules, and by $D_{qc}(\mathscr{D}_{F,X}^{(v)})$ the full triangulated sub-category of $D(\mathscr{D}_{F,X}^{(v)})$ consisting of complexes whose cohomology sheaves are in $\mu(\mathscr{D}_{F,X}^{(v)})$

If $\bullet$ is one of $\phi, -, +, b$ then we denote by $D^{\bullet}(\mathscr{D}_{F,X}^{(v)})$ and $D^{\bullet}_{qc}(\mathscr{D}_{F,X}^{(v)})$ the full triangulated subcategories of $D(\mathscr{D}_{F,X}^{(v)})$ and $D_{qc}(\mathscr{D}_{F,X}^{(v)})$ respectively, consisting of complexes that satisfy the appropriate boundedness condition.

We denote by $D^b(\mathscr{D}_{F,X}^{(v)})^{\circ}$ (resp. $D^b_{qc}(\mathscr{D}_{F,X}^{(v)})^{\circ}$) the full triangulated subcategory of $D^b(\mathscr{D}_{F,X}^{(v)})$ (resp. $D^b_{qc}(\mathscr{D}_{F,X}^{(v)})$) consisting of complexes that have finite Tor dimension when considered as complexes of $\mathcal{O}_X$-modules. A theorem of Bernstein implies that the natural map $D^b(\mu(\mathscr{D}_{F,X}^{(v)})) \to D^b_{qc}(\mathscr{D}_{F,X}^{(v)})$ is an equivalence of categories (see [**Bo**, VI 2.10]). In particular, any complex in $D^b_{qc}(\mathscr{D}_{F,X}^{(v)})$ is represented by a complex of quasi-coherent $\mathscr{D}_{F,X}^{(v)}$-modules.

14.2. — In this sub-section we are going to define inverse image functors for $\mathscr{D}_{F,X}^{(v)}$-modules.

14.2.1. — Let $f : Y \to X$ be a map of smooth W_n-schemes. By [**Ber 3**, 2.1.2], if $\mathcal{M}$ is a $\mathscr{D}_X^{(v)}$-module, then $f^*\mathcal{M}$ has a structure of a left $\mathscr{D}_Y^{(v)}$-module. If $g : Z \to Y$ is a map of smooth W_n-schemes, then there is a canonical isomorphism $(f \circ g)^* \xrightarrow{\sim} g^* \circ f^*$. There are analogous results for $\mathscr{D}_{F,X}^{(v)}$-modules.

Proposition 14.2.2. — *With the notation above, suppose that $\mathcal{M}$ is a $\mathscr{D}^{(v)}_{F,X}$-module. Then $f^*\mathcal{M}$ has a natural structure of a $\mathscr{D}^{(v)}_{F,Y}$-module. Moreover, there is a natural isomorphism of $\mathscr{D}^{(v)}_{F,Y}$-modules $(f\circ g)^*\mathcal{M} \xrightarrow{\sim} g^*f^*\mathcal{M}$.*

Proof. — Consider the map of $\mathscr{D}^{(v)}_{X}$-modules $F^*\mathcal{M}\to\mathcal{M}$ given by Proposition 13.3.7. Applying f^* we get a map of $\mathscr{D}^{(v)}_{Y}$-modules

$$F^*f^*\mathcal{M} \xrightarrow{\sim} f^*F^*\mathcal{M}\to\mathcal{M},$$

that endows $f^*\mathcal{M}$ with a structure of $\mathscr{D}^{(v)}_{F,Y}$-module, by Proposition 13.3.7.

The verification that the natural isomorphism of $\mathcal{O}_Z$-modules $(f\circ g)^*\mathcal{M} \xrightarrow{\sim} g^*f^*\mathcal{M}$ is in fact a map of $\mathscr{D}^{(v)}_{F,Z}$ modules is straightforward. □

14.2.3. — We keep the notation above. Proposition 14.2.2 implies, in particular, that $f^*\mathscr{D}^{(v)}_{F,X}$ has a canonical structure of left $\mathscr{D}^{(v)}_{F,Y}$-module. It has an obvious structure of right $f^{-1}\mathscr{D}^{(v)}_{F,X}$-module, and these two actions commute, by functoriality of f^*. Thus $f^*\mathscr{D}^{(v)}_{F,X}$ is a $(\mathscr{D}^{(v)}_{F,Y}, f^{-1}\mathscr{D}^{(v)}_{F,X})$-bimodule, that we denote by $\mathscr{D}^{(v)}_{F,Y\to X}$.

We define a functor

$$f^! : D(\mathscr{D}^{(v)}_{F,X})\to D(\mathscr{D}^{(v)}_{F,Y})$$

by the formula

$$f^!\mathcal{M}^\bullet = \mathscr{D}^{(v)}_{F,Y\to X}\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}^{(v)}_{F,X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}],$$

where $\mathcal{M}^\bullet$ is a complex in $D^-(\mathscr{D}^{(v)}_{F,X})$, and $d_{Y/X}$ denotes the relative dimension of Y over X.

The derived tensor product may be computed either by taking a left resolution of $\mathscr{D}^{(v)}_{F,Y\to X}$ by $(\mathscr{D}^{(v)}_{F,Y}, f^{-1}\mathscr{D}^{(v)}_{F,X})$ bimodules that are flat as right $f^{-1}\mathscr{D}^{(v)}_{F,X}$-modules, or by replacing $\mathcal{M}^\bullet$ by a quasi-isomorphic complex of flat left $\mathscr{D}^{(v)}_{F,X}$-modules, if such a complex exists. It is defined on the entire derived category (rather than the subcategory of complexes bounded above) because $\mathscr{D}^{(v)}_{F,Y\to X}$ has finite Tor-dimension as a right $f^{-1}\mathscr{D}^{(v)}_{F,X}$-module (using the fact that $\mathcal{O}_Y$ has finite Tor-dimension as an $f^{-1}\mathcal{O}_X$-module – see Lemma 14.2.7 below).

Lemma 14.2.4. — *There are canonical isomorphisms of $(\mathscr{D}^{(v)}_{F,Z}, (f\circ g)^{-1}\mathscr{D}^{(v)}_{F,X})$ bimodules*

$$\mathscr{D}^{(v)}_{F,Z\to X} \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Z\to Y}\overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X} \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Z\to Y}\otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X}.$$

Proof. — We have isomorphisms of left $\mathscr{D}^{(v)}_{F,Z}$-modules

$$\begin{aligned}\mathscr{D}^{(v)}_{F,Z\to X} \xrightarrow{\sim} g^*\mathscr{D}^{(v)}_{F,Y\to X} \xrightarrow{\sim} g^*\mathscr{D}^{(v)}_{F,Y}\otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X}&\\ \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Z\to Y}\otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X}.&\end{aligned}$$

Moreover, it is easy to see that this is also a map of right $(f \circ g)^{-1}\mathscr{D}^{(v)}_{F,X}$-modules. Thus, to prove the lemma, it remains to show that the complex $\mathscr{D}^{(v)}_{F,Z\to Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X}$ is concentrated in degree 0. On underlying $\mathcal{O}_Z$-modules, this complex is equal to

$$g^*\mathscr{D}^{(v)}_{F,Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X} \xrightarrow{\sim} \mathcal{O}_Z \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}_Y} g^{-1}f^*\mathscr{D}^{(v)}_{F,X}.$$

The complex on the right is concentrated in degree 0, since $f^*\mathscr{D}^{(v)}_{F,X}$ is flat as a left $\mathcal{O}_Y$-module, because $\mathscr{D}^{(v)}_{F,X}$ is flat as a left $\mathcal{O}_X$-module. ☐

Corollary 14.2.5. — *There is a canonical isomorphism of functors* $(f \circ g)^! \xrightarrow{\sim} g^! \circ f^!$.

Proof. — If $\mathcal{M}^\bullet$ is any complex in $D(\mathscr{D}^{(v)}_{F,X})$ then using Lemma 14.2.4, we obtain isomorphisms

$$\begin{aligned}(f\circ g)^!\mathcal{M}^\bullet[-d_{Z/X}] &= \mathscr{D}^{(v)}_{F,Z\to X} \overset{\mathbb{L}}{\otimes}_{g^{-1}f^{-1}\mathscr{D}^{(v)}_{F,X}} g^{-1}f^{-1}\mathcal{M}^\bullet \\ &\xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Z\to Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}\mathscr{D}^{(v)}_{F,Y\to X} \overset{\mathbb{L}}{\otimes}_{g^{-1}f^{-1}\mathscr{D}^{(v)}_{F,X}} g^{-1}f^{-1}\mathcal{M}^\bullet \\ &\xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Z\to Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}(\mathscr{D}^{(v)}_{F,Y\to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}^{(v)}_{F,X}} f^{-1}\mathcal{M}^\bullet) \xrightarrow{\sim} g^!f^!\mathcal{M}^\bullet[-d_{Z/X}].\end{aligned}$$

☐

Proposition 14.2.6. — *The functor $f^!$ restricts to functors*

$$f^! : D_{qc}(\mathscr{D}^{(v)}_{F,X}) \to D_{qc}(\mathscr{D}^{(v)}_{F,Y})$$

$$f^! : D^b(\mathscr{D}^{(v)}_{F,X}) \to D^b(\mathscr{D}^{(v)}_{F,Y})$$

and

$$f^! : D^b(\mathscr{D}^{(v)}_{F,X})^\circ \to D^b(\mathscr{D}^{(v)}_{F,Y})^\circ$$

Proof. — The first claim follow immediately from the fact that on the level of $\mathcal{O}_X$-modules, the functor $f^!$ is simply the derived functor of usual $\mathcal{O}_X$-module pull-back f^*, composed with a shift.

We prove the second and third claims. Let $\mathcal{M}^\bullet$ be in $D^-(\mathscr{D}^{(v)}_{F,X})$. On underlying $\mathcal{O}_Y$-modules we have

$$f^!\mathcal{M}^\bullet[-d_{Y/X}] = f^*\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}^{(v)}_{F,X}} \mathcal{M}^\bullet = \mathcal{O}_Y \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_X} \mathcal{M}^\bullet.$$

If $\mathcal{M}^\bullet$ is a bounded complex, then the Lemma 14.2.7 below implies that the complex on the right is bounded. This proves the second claim. If $\mathcal{M}^\bullet$ has finite Tor dimension as a complex of $\mathcal{O}_X$ -modules, then the complex on the right has finite Tor dimension as a complex of $\mathcal{O}_Y$-modules, which proves the third claim. ☐

Lemma 14.2.7. — *With the notation above, $\mathcal{O}_Y$ has finite Tor dimension as an $f^{-1}\mathcal{O}_X$-module.*

Proof. — Factoring f as its graph followed by a projection $Y \to X \times_{W_n} Y \to X$, one sees that it is enough to show the lemma in the two cases when f is a closed immersion, and when it is a smooth map. In the former case, the result follows from the fact that f is a local complete intersection. In the second, f is smooth, hence, in particular, flat. □

14.2.8. — Forgetting the Frobenius structure yields forgetful functors $D(\mathscr{D}^{(v)}_{F,X}) \to D(\mathscr{D}^{(v)}_{X})$ and $D(\mathscr{D}^{(v)}_{F,Y}) \to D(\mathscr{D}^{(v)}_{Y})$, and the resulting diagram

$$\begin{array}{ccc} D(\mathscr{D}^{(v)}_{F,X}) & \longrightarrow & D(\mathscr{D}^{(v)}_{X}) \\ \downarrow f^! & & \downarrow (f^!)^{\mathscr{D}} \\ D(\mathscr{D}^{(v)}_{F,Y}) & \longrightarrow & D(\mathscr{D}^{(v)}_{Y}) \end{array}$$

commutes. (Here we have written $(f^!)^{\mathscr{D}}$ to denote the inverse image functor on complexes of $\mathscr{D}^{(v)}_{X}$-modules. Recall from [**Ber 1**, 2.2.3] that this functor is defined analogously to the functor $f^!$, using the $(\mathscr{D}^{(v)}_{Y}, f^{-1}\mathscr{D}^{(v)}_{X})$-bimodule $f^*\mathscr{D}^{(v)}_{X}$. In light of this, the claimed commutativity is obvious.)

14.2.9. — There is an obvious analogue of (14.2.2) for left $\mathscr{D}^{(v)}_{F^!,X}$-modules. Thus $f^*\mathscr{D}^{(v)}_{F^!,X}$ is naturally a $(\mathscr{D}^{(v)}_{F^!,Y}, f^{-1}\mathscr{D}^{(v)}_{F^!,X})$-bimodule, which we denote by $\mathscr{D}^{(v)}_{F^!,Y\to X}$. We can introduce appropriate derived categories of $\mathscr{D}^{(v)}_{F^!,}$-modules, and use the bimodule $\mathscr{D}^{(v)}_{F^!,Y\to X}$ to define a derived inverse image functor $f^!$. The analogues of the preceding results clearly hold.

14.3. — In this subsection, we will define direct image functors for $\mathscr{D}^{(v)}_{F}$-modules and $\mathscr{D}^{(v)}_{F^!,}$-modules under a map $f : Y \to X$ of smooth W_n-schemes. The bimodules $\mathscr{D}^{(v)}_{F,Y\to X}$ and $\mathscr{D}^{(v)}_{F^!,Y\to X}$ defined above allow one to define the direct image functor on right modules. We use the usual trick of tensoring with the canonical bundle, together with (13.8.7), to define the direct image functor on left modules.

14.3.1. — Let $\mathcal{E}^\bullet$ be a complex of left $\mathscr{D}^{(v)}_{F,Y}$-modules. The tensor product $\omega_Y \otimes_{\mathcal{O}_X} \mathcal{E}^\bullet$ is then a complex of right $\mathscr{D}^{(v)}_{F^!,Y}$-modules. We may form the left derived tensor product with the $(\mathscr{D}^{(v)}_{F^!,Y}, f^{-1}\mathscr{D}^{(v)}_{F^!,X})$-bimodule $\mathscr{D}^{(v)}_{F^!,Y\to X}$ of (14.2.10), to obtain a complex $(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F^!,Y}} \mathscr{D}^{(v)}_{F^!,Y\to X}$ of right $f^{-1}\mathscr{D}^{(v)}_{F^!,X}$-modules. Finally, the tensor product $((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F^!,Y}} \mathscr{D}^{(v)}_{F^!,Y\to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}$ is a complex of left $f^{-1}\mathscr{D}^{(v)}_{F,X}$-modules. Applying Rf_* yields a complex of left $\mathscr{D}^{(v)}_{F,X}$-modules. Thus we obtain a functor $D(\mathscr{D}^{(v)}_{F,Y}) \to D(\mathscr{D}^{(v)}_{F,X})$.

14.3.2. — As usual, it is helpful to give a description of the functor defined above in terms of an $(f^{-1}\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,Y})$-bimodule. Lemma 13.9.10 (applied to a resolution of $\mathscr{D}^{(v)}_{F,Y\to X}$ by $(\mathscr{D}^{(v)}_{F^!,Y}, f^{-1}\mathscr{D}^{(v)}_{F^!,X})$-bimodules that are locally free as $\mathscr{D}^{(v)}_{F^!,Y}$-modules) yields an isomorphism of right $f^{-1}\mathscr{D}^{(v)}_{F^!,X}$-modules

$$(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F^!,Y}} \mathscr{D}^{(v)}_{F^!,Y\to X} \xrightarrow{\ \sim\ } (\omega_Y \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_{F^!,Y\to X}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathcal{E}^\bullet.$$

(Here the right $f^{-1}\mathscr{D}^{(v)}_{F^!,X}$-module structure is defined on both source and target via the right $f^{-1}\mathscr{D}^{(v)}_{F^!,X}$-module structure on $\mathscr{D}^{(v)}_{F^!,Y\to X}$. That this isomorphism is compatible with these module structures follows from its naturality.) Thus tensoring with $f^{-1}\omega_X^{-1}$ over $f^{-1}\mathcal{O}_X$, and so converting these right $f^{-1}\mathscr{D}^{(v)}_{F^!,X}$-modules to left $f^{-1}\mathscr{D}^{(v)}_{F,X}$-modules, we obtain an isomorphism

$$\begin{aligned}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F^!,Y}} \mathscr{D}^{(v)}_{F^!,Y\to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}& \\ \xrightarrow{\ \sim\ } ((\omega_Y \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_{F^!,Y\to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}) &\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathcal{E}^\bullet.\end{aligned}$$

The tensor product $(\omega_Y \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_{F^!,Y\to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}$ appearing in the target of this isomorphism is the sought-after $(f^{-1}\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,Y})$-bimodule.

14.3.3. — We will give another description of the bimodule of (14.3.2), which eliminates the reference to $\mathscr{D}^{(v)}_{F^!,Y\to X}$. Recall that Proposition 13.9.4 yields an isomorphism

$$\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathcal{O}_X} \omega_X^{-1} \xrightarrow{\ \sim\ } \mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1},$$

which respects the commuting left $\mathscr{D}^{(v)}_{F^!,X}$-module and left $\mathscr{D}^{(v)}_{F,X}$-module structures on each of its source and target. Applying f^* (with respect to the left $\mathscr{D}^{(v)}_{F^!,X}$-module structures) to each of the source and target of this map, we obtain objects equipped with commuting left $\mathscr{D}^{(v)}_{F^!,Y}$-module and left $f^{-1}\mathscr{D}^{(v)}_{F,X}$-module structures, and an isomorphism

$$f^*(\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \xrightarrow{\ \sim\ } f^*(\mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1})$$

that respects these left module structures. Tensoring both the source and the target of this isomorphism with ω_Y over $\mathcal{O}_Y$, each of them becomes a $(f^{-1}\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,Y})$-bimodule, and we obtain an isomorphism of bimodules

$$\omega_Y \otimes_{\mathcal{O}_Y} f^*(\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \xrightarrow{\ \sim\ } \omega_Y \otimes_{\mathcal{O}_Y} f^*(\mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}).$$

Clearly the source of this map is isomorphic, as a $(f^{-1}\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,Y})$-bimodule, to $(\omega_Y \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_{F^!,Y\to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}$. We denote its target by $\mathscr{D}^{(v)}_{F,X\leftarrow Y}$.

14.3.4. — Let $\mathscr{D}^{(v)}_{X\leftarrow Y}$ be the $(f^{-1}\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_Y)$-bimodule defined in [**Ber 3**, 3.4.1]. It is clear from the construction that there is a morphism of $(f^{-1}\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_Y)$-bimodules $\mathscr{D}^{(v)}_{X\leftarrow Y} \to \mathscr{D}^{(v)}_{F,X\leftarrow Y}$, which induces an isomorphism of $(f^{-1}\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_Y)$-bimodules

$$f^{-1}\mathscr{D}^{(v)}_{F,X} \otimes_{f^{-1}\mathscr{D}^{(v)}_X} \mathscr{D}^{(v)}_{X\leftarrow Y} \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,X\leftarrow Y}.$$

This observation will be important in the proof of the following proposition.

Proposition 14.3.5. — *$\mathscr{D}^{(v)}_{F,X\leftarrow Y}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_{F,Y}$-module.*

Proof. — By Corollary 13.6.3 it is enough to show that $\mathscr{D}^{(v)}_{F,X\leftarrow Y}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_Y$-module. Since $\mathscr{D}^{(v)}_{F,X}$ is locally free as a right $\mathscr{D}^{(v)}_X$-module, the isomorphism of (14.3.4) shows that it is enough to show that $\mathscr{D}^{(v)}_{X\leftarrow Y}$ has finite Tor dimension as a right $\mathscr{D}^{(v)}_Y$-module. This is proved in [**Ber 4**]. (It is announced in [**Ber 3**, 3.4.3]). □

14.3.6. — We now define a functor $f_+ : D(\mathscr{D}^{(v)}_{F,Y}) \to D(\mathscr{D}^{(v)}_{F,X})$. For any complex $\mathcal{M}^\bullet$ in $D(\mathscr{D}^{(v)}_{F,Y})$, we define

$$f_+\mathcal{M}^\bullet = Rf_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathcal{M}^\bullet).$$

That this functor is well-defined follows from Proposition 14.3.5 and Grothendieck's vanishing theorem (which shows that Rf_* is of finite cohomological amplitude). It follows from the discussions of (14.3.2) and (14.3.3) that it is naturally isomorphic to the functor defined in (14.3.1).

Proposition 14.3.7. — *Let $g : Z \to Y$ be a map of smooth W_n-schemes. Then there are canonical isomorphisms of $(g^{-1}f^{-1}\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,Z})$-bimodules*

$$\mathscr{D}^{(v)}_{F,X\leftarrow Z} \xrightarrow{\sim} g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\leftarrow Z} \xrightarrow{\sim} g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\leftarrow Z}.$$

Proof. — We prove the second isomorphism first. In order to show that the complex $g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\leftarrow Z}$ is concentrated in degree 0 it suffices to check on the underlying complex of $\mathcal{O}_Z$-modules.

Note that $\mathscr{D}^{(v)}_{F,X}$ is locally free as a right $\mathcal{O}_X$-module. Indeed, to see this, it suffices to see that $\mathscr{D}^{(v)}_X$ is locally free as a right $\mathcal{O}_X$-module, and this follows for example from [**Ber 3**, 1.3.1], using the fact that $\mathscr{D}^{(v)}_X$ is locally free as a left $\mathcal{O}_X$-module. This immediately implies that $\mathscr{D}^{(v)}_{F,X\leftarrow Y}$ is locally free as a right $\mathcal{O}_Y$-module.

We compute

$$
\begin{aligned}
&g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\leftarrow Z}\\
&\qquad\xrightarrow{\sim} g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^*(\mathscr{D}^{(v)}_{F,Y}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_Z}\omega_Z\\
&\qquad\xrightarrow{\sim} g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^{-1}(\mathscr{D}^{(v)}_{F,Y}\otimes_{\mathcal{O}_Y}\omega_Y^{-1}) \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}_Y}\omega_Z\\
&\qquad\qquad\xrightarrow{\sim} g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}_Y}\omega_{Z/Y}.
\end{aligned}
$$

Now the final expression is a complex concentrated in degree 0, since we saw above, that $\mathscr{D}^{(v)}_{F,X\leftarrow Y}$ is a flat right $\mathcal{O}_Y$-module. In the second isomorphism of the calculation above we have used the fact that $\mathscr{D}^{(v)}_{F,Y}$ is locally free as a right $\mathcal{O}_Y$-module, and that the pull-back g^* appearing in the second line is taken with respect to the left $\mathscr{D}^{(v)}_Y$-module structure on $\mathscr{D}^{(v)}_{F,Y}\otimes_{\mathcal{O}_Y}\omega_Y^{-1}$ induced by the right $\mathscr{D}^{(v)}_Y$-module structure on $\mathscr{D}^{(v)}_{F,Y}$ (this will also be important below).

Before making the calculation that will establish the first isomorphism, we make some observations in a more general setting. Suppose that $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,Y}$-module, and that $\mathcal{E}$ is a $(\mathscr{D}^{(v)}_{F,Y},\mathscr{D}^{(v)}_Y)$-bimodule. Then $\mathcal{M}\otimes_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{E}$ is a right $\mathscr{D}^{(v)}_Y$-module, and $\mathcal{E}\otimes_{\mathcal{O}_Y}\omega_Y^{-1}$ has the structure of a left $\mathscr{D}^{(v)}_{F,Y}$-module, and a left $\mathscr{D}^{(v)}_Y$ module, with these two actions commuting. There is an obvious W_n-linear isomorphism

$$(\mathcal{M}\otimes_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{E})\otimes_{\mathcal{O}_Y}\omega_Y^{-1}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathscr{D}^{(v)}_{F,Y}}(\mathcal{E}\otimes_{\mathcal{O}_Y}\omega_Y^{-1}).$$

Both sides of this isomorphism are left $\mathscr{D}^{(v)}_Y$-modules, and one checks immediately that this is an isomorphism of $\mathscr{D}^{(v)}_Y$- modules. Applying the pull-back functor g^* for $\mathscr{D}^{(v)}_Y$-modules, we obtain an isomorphism

$$g^*((\mathcal{M}\otimes_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{E})\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\xrightarrow{\sim}g^{-1}\mathcal{M}\otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}}g^*(\mathcal{E}\otimes_{\mathcal{O}_Y}\omega_Y^{-1}).$$

Applying this isomorphism with $\mathcal{E}=\mathscr{D}^{(v)}_{F,Y}$ and $\mathcal{M}=\mathscr{D}^{(v)}_{F,X\leftarrow Y}$ we obtain isomorphisms of $(g^{-1}f^{-1}\mathscr{D}^{(v)}_{F,X},\mathscr{D}^{(v)}_Z)$-bimodules

$$
\begin{aligned}
&g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y}\otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}}\mathscr{D}^{(v)}_{F,Y\leftarrow Z}\\
&\qquad= g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y}\otimes_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} g^*(\mathscr{D}^{(v)}_{F,Y}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_Z}\omega_Z\\
&\qquad\xrightarrow{\sim} g^*((\mathscr{D}^{(v)}_{F,X\leftarrow Y}\otimes_{\mathscr{D}^{(v)}_{F,Y}}\mathscr{D}^{(v)}_{F,Y})\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_Z}\omega_Z\\
&\qquad\xrightarrow{\sim} g^*(f^*(\mathscr{D}^{(v)}_{F,X}\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_Y}\omega_Y\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_Z}\omega_Z\\
&\qquad\qquad\xrightarrow{\sim} g^*f^*(\mathscr{D}^{(v)}_{F,X}\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_Z}\omega_Z\xrightarrow{\sim}\mathscr{D}^{(v)}_{F,X\leftarrow Z}.
\end{aligned}
$$

It is straightforward, but somewhat tedious to check that the composite of the isomorphisms above is in fact a map of $(g^{-1}f^{-1}\mathscr{D}^{(v)}_{F,X},\mathscr{D}^{(v)}_{F,Z})$-bimodules □

Corollary 14.3.8. — *With f and g as above, there is a canonical isomorphism of functors $(f\circ g)_+ \xrightarrow{\sim} f_+g_+$.*

Proof. — Let $\mathcal{M}^\bullet$ be in $D(\mathscr{D}^{(v)}_{F,Z})$. Using the previous proposition we compute

$$\begin{aligned}(fg)_+\mathcal{M}^\bullet &= R(f\circ g)_*(\mathscr{D}^{(v)}_{F,X\leftarrow Z} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Z}} \mathcal{M}^\bullet)\\ &\xrightarrow{\sim} Rf_*Rg_*(g^{-1}\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\leftarrow Z} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Z}} \mathcal{M}^\bullet)\\ &\xrightarrow{\sim} Rf_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} Rg_*(\mathscr{D}^{(v)}_{F,Y\leftarrow Z} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Z}} \mathcal{M}^\bullet)) = f_+g_+\mathcal{M}^\bullet.\end{aligned}$$

□

Proposition 14.3.9. — *The functor f_+ restricts to functors*

$$f_+ : D^b(\mathscr{D}^{(v)}_{F,Y}) \to D^b(\mathscr{D}^{(v)}_{F,X}),$$

$$f_+ : D_{qc}(\mathscr{D}^{(v)}_{F,Y}) \to D_{qc}(\mathscr{D}^{(v)}_{F,X}),$$

and

$$f_+ : D^-_{qc}(\mathscr{D}^{(v)}_{F,Y})^\circ \to D^-_{qc}(\mathscr{D}^{(v)}_{F,X})^\circ.$$

Proof. — The first claim follows immediately from Proposition 14.3.5. To prove the second (resp. third) claim, note that by Proposition 13.6.1, it suffices to prove it for a complex $\mathcal{M}^\bullet$ in $D(\mathscr{D}^{(v)}_{F,Y})$ that has the form

$$\mathcal{M}^\bullet = \mathscr{D}^{(v)}_{F,Y} \otimes_{\mathscr{D}^{(v)}_Y} M^\bullet = \mathscr{D}^{(v)}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_Y} M^\bullet$$

where $M^\bullet$ is a complex of quasi-coherent $\mathscr{D}^{(v)}_Y$-modules (resp. complex of quasi-coherent $\mathscr{D}^{(v)}_Y$-modules, of finite Tor dimension as a complex of $\mathcal{O}_Y$-modules). We compute

$$f_+\mathcal{M}^\bullet = Rf_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_X} M^\bullet) = Rf_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_Y} M^\bullet).$$

Now the isomorphism of (14.3.4) shows that the expression on the right is equal to $\bigoplus_{r\geq 0}(F^*)^r f_+M^\bullet$, where $f_+M^\bullet$ denotes the $\mathscr{D}^{(v)}$-module direct image. The claims are local on X, so we may assume that X admits a lifting F of the absolute Frobenius. Since F is a finite flat map, we see that to show the second (resp. third) claim it suffices to show that $f_+M^\bullet$ has quasi-coherent cohomology (resp. has quasi-coherent cohomology and is of finite Tor dimension as a complex of $\mathcal{O}_X$-modules). These results are proved in [**Ber 4**] (see [**Ber 1**, 2.4.2, 2.4.5] for the announcements). □

14.3.10. — The functor f_+ that we have defined is *not* compatible with the analogous functor for $\mathscr{D}^{(v)}_Y$-modules defined in [**Ber 3**, 3.4.3], for general v. Let $f^{\mathscr{D}}_+$ denote this latter functor. The inclusion of $(\mathscr{D}^{(v)}_X, \mathscr{D}^{(v)}_X)$-bimodules $\mathscr{D}^{(v)}_X \to \mathscr{D}^{(v)}_{F,X}$ induces an inclusion of $(f^{-1}\mathscr{D}_X^{(v)}, \mathscr{D}_Y^{(v)})$-bimodules $\iota : \mathscr{D}_{X\leftarrow Y}^{(v)} \to \mathscr{D}_{F,X\leftarrow Y}^{(v)}$. Thus, we have maps in the derived category of $(f^{-1}\mathscr{D}_X^{(v)}, \mathscr{D}_{F,Y}^{(v)})$-bimodules

$$\mathscr{D}_{X\leftarrow Y}^{(v)} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_Y^{(v)}} \mathscr{D}_{F,Y}^{(v)} \to \mathscr{D}_{X\leftarrow Y}^{(v)} \otimes_{\mathscr{D}_Y^{(v)}} \mathscr{D}_{F,Y}^{(v)} \overset{\partial_1\otimes\partial_2\mapsto\iota(\partial_1)\partial_2}{\longrightarrow} \mathscr{D}_{F,Y\leftarrow X}^{(v)}.$$

Applying the functor $Rf_*(- \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}^{(v)}} \mathcal{M}^\bullet)$ to the composite of the above maps yields a map of $\mathscr{D}_X^{(v)}$-modules

$$f_+^{\mathscr{D}}\mathcal{M}^\bullet \to f_+\mathcal{M}^\bullet. \tag{14.3.11}$$

This map is not an isomorphism in general, but is an isomorphism when $v = \infty$. Let us explain why.

It will be simpler to consider the corresponding functors on right $\mathscr{D}_{F^!,Y}^{(v)}$-modules. Let $\mathcal{M}^\bullet$ be a complex of right $\mathscr{D}_{F^!,Y}^{(v)}$-modules. We have to consider whether or not the composite

$$\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F^!,Y}^{(v)}} f^*\mathscr{D}_{F^!,X}^{(v)},$$

induced by the sequence of natural maps

$$\mathscr{D}_{F^!,Y}^{(v)} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to \mathscr{D}_{F^!,Y}^{(v)} \otimes_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to f^*\mathscr{D}_{F^!,X}^{(v)},$$

is an isomorphism. Thus we have to consider whether or not the composite of these latter maps is an isomorphism. These maps in turn may be written as the direct sum of maps

$$(F^!)^r\mathscr{D}_Y^{(v)} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to (F^!)^r\mathscr{D}_Y^{(v)} \otimes_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to f^*(F^!)^r\mathscr{D}_X^{(v)},$$

as r ranges over the non-negative integers. To see whether or not the composite of these maps is an isomorphism, it suffices to consider the corresponding sequence of maps obtained by applying the functor $(F^r)^*$ (since this functor induces an equivalence between the category of left $\mathscr{D}_Y^{(v)}$-modules and the category of left $\mathscr{D}_Y^{v+r}$-modules). If we apply this functor (and use the isomorphisms $(F^r)^*(F^!)^r\mathscr{D}_Y^{(v)} \overset{\sim}{\longrightarrow} \mathscr{D}_Y^{(v+r)}$ and $(F^*)^r f^*(F^!)^r\mathscr{D}_X^{(v)} \overset{\sim}{\longrightarrow} f^*(F^*)^r(F^!)^r\mathscr{D}_X^{(v)} \overset{\sim}{\longrightarrow} f^*\mathscr{D}_X^{(v+r)}$) we obtain the sequence of maps

$$\mathscr{D}_Y^{(v+r)} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to \mathscr{D}_Y^{(v+r)} \otimes_{\mathscr{D}_Y^{(v)}} f^*\mathscr{D}_X^{(v)} \to f^*\mathscr{D}_X^{(v+r)}. \tag{14.3.12}$$

For a finite level $v \in \mathbb{N}^{\geq 0}$, it is typically the case that neither of the maps appearing in (14.3.12), nor their composite, are isomorphisms. On the other hand, if $v = \infty$, then $v+r = v$, and so both maps are obviously isomorphisms. This proves our claims concerning the natural transformation (14.3.11).

14.3.13. — If $f : Y \to X$ is a morphism of smooth W_n-schemes, then we can define a direct image functor for complexes of $\mathscr{D}_{F^!,Y}^{(v)}$-modules. Again, it admits two descriptions. In the first description, we convert from left modules to right modules.

More precisely, if $\mathcal{E}^\bullet$ is a complex of left $\mathscr{D}^{(v)}_{F^!,Y}$-modules, the direct image functor is defined as

$$\mathcal{E}^\bullet \mapsto Rf_*(((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}).$$

In the second description, we introduce the $(f^{-1}\mathscr{D}^{(v)}_{F^!,X}, \mathscr{D}^{(v)}_{F^!,Y})$-bimodule

$$\mathscr{D}^{(v)}_{F^!,X\leftarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} f^*(\mathscr{D}^{(v)}_{F^!,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \xrightarrow{\sim} \omega_Y \otimes_{\mathcal{O}_Y} f^*(\mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1})$$

(the isomorphism being induced by the isomorphism of Proposition 13.9.4), and consider the functor

$$\mathcal{E}^\bullet \mapsto Rf_*(\mathscr{D}^{(v)}_{F^!,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F^!,Y}} \mathcal{E}^\bullet).$$

Considerations analogous to those of (14.3.2) shows that these two functors are naturally isomorphic, and the obvious analogues of the preceding results hold for these functors.

14.3.14. — There is one respect in which the direct image functor for $\mathscr{D}^{(v)}_{F^!,Y}$-modules is better behaved than that for $\mathscr{D}^{(v)}_{F,Y}$-modules. Namely, it is compatible with the direct image functor $f_+^{\mathscr{D}}$ on $\mathscr{D}^{(v)}_Y$-modules. To see this, it is easier to consider the direct image of right $\mathscr{D}^{(v)}_{F,Y}$-modules. Considerations analogous to those of (14.3.10) shows that we have to consider whether or not the composite

$$\mathscr{D}^{(v)}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X \to \mathscr{D}^{(v)}_{F,Y} \otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X \to f^*\mathscr{D}^{(v)}_{F,X} \tag{14.3.15}$$

is an isomorphism. If we work locally on X and Y, we may assume that both X and Y are equipped with a local lift of Frobenius, compatibly with the map f. We may then rewrite the sequence of maps as

$$\begin{aligned}(14.3.16)\quad & \mathcal{O}_{F,Y} \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_Y \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X \\ & \to \mathcal{O}_{F,Y} \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_Y \otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_Y \to f^*(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X).\end{aligned}$$

Since $\mathcal{O}_{F,Y}$ is locally free as a right $\mathcal{O}_Y$-module, and since there is a natural isomorphism $\mathcal{O}_{F,Y} \xrightarrow{\sim} f^*\mathcal{O}_{F,X}$, we that both maps in (14.3.16) are isomorphisms, and thus that this is also true of both maps in (14.3.15). This establishes our claim.

14.3.17. — Let us note one final compatibility. If $\mathcal{E}^\bullet$ is a complex of left $\mathscr{D}^{(v)}_{F^!,Y}$-modules, then there is a structural morphism of left $\mathscr{D}^{(v)}_Y$-modules $\mathcal{E}^\bullet \to F^*\mathcal{E}^\bullet$. If we forget the left $\mathscr{D}^{(v)}_{F^!,Y}$-module structure on the members of $\mathcal{E}^\bullet$, and merely remember this map as a map in the derived category $D(\mathscr{D}^{(v)}_Y)$, then we may regard $\mathcal{E}^\bullet$ as an $F - \mathscr{D}^{(v)}_Y$-complex, in (an obvious modification of) the sense of [**Ber 1**, 5.1.1]. It is easily checked if $\mathcal{E}^\bullet$ is an $F - \mathscr{D}^{(v)}_Y$-complex, then the direct image $f_+^{\mathscr{D}}\mathcal{E}$ is again an $F - \mathscr{D}^{(v)}_Y$-complex in a natural fashion. (The corresponding fact for $F - \mathscr{D}^\dagger_{\mathbb{Q}}$-complexes is observed in [**Ber 1**, 5.1.2]) We leave it to the reader to check that the direct image functor on complexes of $\mathscr{D}^{(v)}_{F^!,Y}$-modules that we have defined is compatible with the forgetful functor to $F-\mathscr{D}^{(v)}_Y$-complexes, and the direct image functor $f^{\mathscr{D}}_+$ acting on such complexes.

14.4. — In this subsection we will construct a trace map, which is the key to the adjointness of f_+ and $f^!$ for proper f, proved in (14.5). This adjointness plays a key roles in showing that the Riemann-Hilbert correspondence constructed later respects direct images.

As usual, the construction of the trace map is one of the most delicate points in the theory. We begin with the following result, about $\mathscr{D}_{F,X}$-modules

Proposition 14.4.1. — *Let $f: Y \to X$ be a proper map of smooth W_n-schemes. There is a map in the derived category of right $\mathscr{D}_{F,X}$-modules*

$$Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}) \to \omega_X[d_X].$$

Proof. — The proof will proceed in several steps, and is partly inspired by the techniques of Virrion [**Vi 2**]. We begin by recalling some results about the unnormalised bar resolution. □

14.4.2. — Let A be a commutative ring, and B an associative, unital A-algebra. We denote by $\mu: B\otimes_A B \to B$ the multiplication map.

Recall that the unnormalised bar resolution of B, $C_\bullet(B)$, is given in degree $-n$ by $C_n(B) = B^{\otimes(n+2)}$ for $n \geq 0$ with the differential $d: B^{\otimes(n+2)} \to B^{\otimes(n+1)}$ given by $d = \sum_{i=0}^{n-1}(-1)^i d_i$ where $d_i = \mathrm{id}^{\otimes n-i}\otimes\mu\otimes\mathrm{id}^{\otimes i}$. Here all the tensor products are taken over A and $B^{\otimes n} =: B\otimes_A B\otimes_A \cdots \otimes_A B$.

The augmentation map $\mu: B^{\otimes 2} \to B$ makes $C_\bullet(B)$ into a left resolution of B by (B,B)-bimodules. In fact, the map $B^{\otimes n} \to B^{\otimes n+1}$ given by $b_1\otimes\cdots\otimes b_n \mapsto b_1\otimes\cdots\otimes b_n\otimes 1$ is a contracting homotopy for the augmented complex $C_\bullet(B) \to B$ viewed as a complex of left B-modules. Thus, if M is a right B-module, then $M\otimes_B C_\bullet(B)$ is a resolution of M by right B-modules. We also have the analogous statement for left B-modules.

Finally, suppose that B is flat over A, and that N is a left B-module that is flat over A. Then we have isomorphisms in the derived category of left B-modules

$$B^{\otimes(n+2)} \overset{\mathbb{L}}{\otimes}_B N \xrightarrow{\sim} B^{\otimes(n+1)} \overset{\mathbb{L}}{\otimes}_A N \xrightarrow{\sim} B^{\otimes(n+1)}\otimes_A N \xrightarrow{\sim} B^{\otimes(n+2)}\otimes_B N.$$

It follows that $M\otimes_B C_\bullet(B)\otimes_B N$ computes $M \overset{\mathbb{L}}{\otimes}_B N$.

We will use the above considerations, in a sheafified situation. Namely, we will usually take $A = \mathcal{O}_X$, and B either $\mathscr{D}_X$ or $\mathscr{D}_{F,X}$.

The reader who wishes to compare our calculations to those of Virrion [**Vi 2**] should note that the complex $C_\bullet(\mathscr{D}^{(v)}_X)\otimes_{\mathscr{D}^{(v)}_X}\mathcal{O}_X$, which is a resolution of $\mathcal{O}_X$ by left $\mathscr{D}^{(v)}_X$-modules, is closely related to the "dual Čech-Alexander" resolution of Virrion.

14.4.3. — Recall that for any map $g: X_1 \to X_2$ of finite type W_n-schemes, there exists a functor $g^\triangle$ taking residual complexes on X_2 to residual complexes on X_1, and, if g is proper, a morphism of functors $g_*g^\triangle \to \mathrm{id}$ [**Ha 1**, VI]. The functor $g^\triangle$ realises, on the level of complexes, the functor $g^!$ on the derived category of quasi-coherent sheaves (when applied to residual complexes) [**Ha 1**, VI].

The $\vartriangle$-pull-back construction is compatible with composition of morphisms. More precisely, for (composable) maps h_1, h_2 between W_n-schemes, there is an isomorphism $c_{h_1,h_2} : (h_1 \circ h_2)^{\vartriangle} \xrightarrow{\sim} h_2^{\vartriangle} h_1^{\vartriangle}$ [**Ha 1**, VI 3.1].

For any (not necessarily smooth) W_n-scheme X, we set $K_X = p_X^{\vartriangle} W_n$, where $p_X : X \to \operatorname{Spec} W_n$ is the natural projection. The compatibility of $\vartriangle$-pull-back with compositions implies that K_X is equipped with a costratification. (We are using here, and will use frequently below, the fact that $f^!$ and $f^{\vartriangle}$ coincide as functors on residual complexes when f is a finite map.) This implies that if g is as above, then $g^{\vartriangle} K_{X_2}$ depends only on the reduction of g modulo p up to canonical isomorphism. Thus, $g^{\vartriangle} K_{X_2}$ makes sense, even if g is only defined modulo p, provided that g has a lifting locally on X_1 and X_2. Moreover, in such a situation $g^{\vartriangle} K_{X_2}$ is still equipped with a costratification, because the isomorphisms c_{h_1,h_2} satisfy an "associativity" property for the composition of three maps [**Ha 1**, VI 3.1, Var 1)]. The same property implies that c_{h_1,h_2} exists, even if h_1 and h_2 are only defined modulo p, as long as h_1 and h_2 are locally liftable.

In particular, if X admits a local lifting of the absolute Frobenius, then $F^{\vartriangle} K_X$ is a well defined complex on X, equipped with a costratification. We also have a map

$$\psi_{K_X} : K_X = p_X^{\vartriangle} W_n \xrightarrow{\sim} p_X^{\vartriangle} F^{\vartriangle} W_n \xrightarrow{\sim} F^{\vartriangle} p_X^{\vartriangle} W_n = F^{\vartriangle} K_X.$$

That this map respects costratifications follows from [**Ha 1**, VI 3.1, Var 1)]. Similarly, if X_1 and X_2 admit local liftings of Frobenius, then *loc. cit.* implies that the maps

$$\psi_{g^{\vartriangle} K_{X_2}} : g^{\vartriangle} K_{X_2} \xrightarrow{g^{\vartriangle}\psi_{K_{X_2}}} g^{\vartriangle} F^{\vartriangle} K_{X_2} \xrightarrow[\sim]{c_{p_X,F} \circ c_{F,p_X}^{-1}} F^{\vartriangle} g^{\vartriangle} K_{X_2},$$

and the isomorphisms

$$K_{X_1} \xrightarrow{\sim} p_{X_1}^{\vartriangle} W_n \xrightarrow{\sim} g^{\vartriangle} p_{X_2}^{\vartriangle} W_n \xrightarrow{\sim} g^{\vartriangle} K_{X_2}$$

are compatible with costratifications, and with the maps $\psi_{K_{X_1}}$ and $\psi_{g^{\vartriangle} K_{X_2}}$.

If X_1 is smooth these observations translate, via [**Ber 3**, 1.1] and Proposition 13.5.4, to say that K_{X_1} and $g^{\vartriangle} K_{X_2}$ are right $\mathscr{D}_{F,X_1}$-modules, and that there is an isomorphism of right $\mathscr{D}_{F,X_1}$-modules $K_{X_1} \xrightarrow{\sim} g^{\vartriangle} K_{X_2}$.

If X is smooth, then K_X is the unique residual complex quasi-isomorphic to $\omega_X[d_X]$ [**Ha 1**, VI, §1]. It is a resolution of $\omega_X[d_X]$ by injective $\mathcal{O}_X$-modules. The definition of the right $\mathscr{D}_{F,X}$-module structure on $\omega_X = p_X^! W_n[-d_X]$ in [**Ber 3**, 2.4.2] and Corollary 13.5.6, using costratifications, shows that K_X is actually a resolution of $\omega_X[d_X]$ by right $\mathscr{D}_{F,X}$-modules.

14.4.4. — Let $f : Y \to X$ be a map of smooth W_n-schemes. By (14.4.2), the (total complex of the) complex of right $f^{-1}\mathscr{D}_{F,Y}$-modules $K_Y \otimes_{\mathscr{D}_{F,Y}} C_\bullet(\mathscr{D}_{F,Y}) \otimes_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}$ represents $K_Y \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}$. Following [**Vi 2**], we will interpret this complex in terms of residual complexes.

Fix an integer $k \geq 0$. For each positive integer r we denote by $P_Y^r(k)$ (resp. $P_{Y/X}^r(k)$) the nilpotent neighbourhood of order r of the image of Y in Y^{k+1} (resp. $Y^k \times$

X.) (cf. [**Vi 2**, II 2.1]). Since the above are regular closed immersions, the structure sheaves $\mathcal{P}^r_Y(k)$ and $\mathcal{P}^r_{Y/X}(k)$ of $P^r_Y(k)$ and $P^r_{Y/X}(k)$ respectively, are locally free of finite rank when considered as $\mathcal{O}_Y$-modules via the first projection to Y [**Ber 2**, 1.5].

By definition [**Ber 2**, 2.2], we have

$$\mathscr{D}_Y = \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(1), \mathcal{O}_Y),$$

where the $\underline{Hom}_{\mathcal{O}_Y}$ is taken with respect to the left $\mathcal{O}_Y$-module structure on $\mathcal{P}^r_Y(1)$. Similarly, we have

$$\begin{aligned} f^*\mathscr{D}_X &= \varinjlim_r f^*\underline{Hom}_{\mathcal{O}_X}(\mathcal{P}^r_X(1), \mathcal{O}_X) \\ &\xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(f^*\mathcal{P}^r_X(1), \mathcal{O}_Y) \xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_{Y/X}(1), \mathcal{O}_Y) \end{aligned}$$

Here we have used $f^*\mathcal{P}^r_X(1) \xrightarrow{\sim} \mathcal{P}^r_{Y/X}(1)$, which follows easily from the results of [**Ber 2**, 1.5].

Next we have

$$(\mathscr{D}_Y)^{\otimes_{\mathcal{O}_Y} k} \xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(1), \mathcal{O}_Y)^{\otimes k} \xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(1)^{\otimes k}, \mathcal{O}_Y),$$

where the second isomorphism is given by

$$f_1 \otimes \cdots \otimes f_k \mapsto f_1 \circ (1 \otimes f_2) \cdots \circ (1^{\otimes k-1} \otimes f_k).$$

Finally for $r, r', k, k' \geq 0$ there is a natural map $\mathcal{P}^{r+r'}_Y(k+k') \to \mathcal{P}^r_Y(k) \otimes_{\mathcal{O}_Y} \mathcal{P}^{r'}_Y(k')$, which becomes an isomorphism after applying $\underline{Hom}_{\mathcal{O}_Y}(-, \mathcal{O}_Y)$ and passing to the limit over r and r' (cf. [**Ber 2**, 2.1.4] and [**Vi 2**, II 2.1.1]). Thus we obtain isomorphisms

$$(\mathscr{D}_Y)^{\otimes k} \xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(1)^{\otimes k}, \mathcal{O}_Y) \xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(k), \mathcal{O}_Y).$$

Similarly, we have

$$\begin{aligned} \mathscr{D}_Y^{\otimes k-1} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_X &\xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(1)^{\otimes k-1} \otimes_{\mathcal{O}_Y} P^r_{Y/X}(1), \mathcal{O}_Y) \\ &\xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_{Y/X}(k), \mathcal{O}_Y). \end{aligned}$$

For $i = 0, \dots k-1$ denote by $p^{r,k}_i$ (resp. $p^{r,k\prime}_i$) the map from $P^r_Y(k)$ (resp. $P^r_{Y/X}(k)$) to Y induced by the projection from Y^{k+1} (resp. $Y^k \times X$) to the $(i+1)^{\text{st}}$ factor. We have

$$\begin{aligned} (14.4.5) \quad K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k} &\xrightarrow{\sim} \varinjlim_r K_Y \otimes_{\mathcal{O}_Y} \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(k), \mathcal{O}_Y) \\ &\xrightarrow{\sim} \varinjlim_r \underline{Hom}_{\mathcal{O}_Y}(\mathcal{P}^r_Y(k), K_Y) = \varinjlim_r p^{r,k}_{0*} p^{r,k!}_0 K_Y. \end{aligned}$$

and similarly

$$(14.4.6) \qquad K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k-1} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_X \xrightarrow{\sim} \varinjlim_r p^{r,k\prime}_{0*} p^{r,k\prime!}_0 K_Y.$$

Next, we introduce pull-backs by Frobenius. For this it will be convenient to remark that $P^r_Y(k)$ is also the nilpotent neighbourhood of order r of the diagonal $Y \hookrightarrow Y^{k+1}/\operatorname{Spec}\mathbb{Z}$, when the product is taken over $\operatorname{Spec}\mathbb{Z}$. This follows because the residue field of W_n is perfect, and p is nilpotent on W_n, which implies that the ideal sheaf of the inclusion $Y^{k+1} \hookrightarrow Y^{k+1}/\operatorname{Spec}\mathbb{Z}$ is equal to its own square. A similar remark applies to $P^r_{Y/X}(k)$,

Let $\underline{s} = (s_0, s_1, \ldots, s_{k-1})$ be a k-tuple of non-negative integers. If F is a local lift of Frobenius on Y, then we have maps

$$Y^{k+1}/\operatorname{Spec}\mathbb{Z} \overset{F^{s_0}\times F^{s_1}\times\cdots\times F^{s_{k-1}}\times \mathrm{id}}{\longrightarrow} Y^{k+1}/\operatorname{Spec}\mathbb{Z}.$$

and

$$(Y^k \times X)/\operatorname{Spec}\mathbb{Z} \overset{F^{s_0}\times F^{s_1}\times\cdots\times F^{s_{k-1}}\times \mathrm{id}}{\longrightarrow} Y^k \times X/\operatorname{Spec}\mathbb{Z}.$$

The remarks of the previous paragraph imply that these induce maps $P^r_Y(k) \to P^r_Y(k)$ and $P^r_{Y/X}(k) \to P^r_{Y/X}(k)$, which we also denote by $F^{s_0} \times F^{s_1} \times \cdots \times F^{s_{k-1}} \times \mathrm{id}$.

As usual, if $\mathcal{M}$ is a quasi-coherent sheaf on $P^r_Y(k)$, which is equipped with a costratification, then $(F^{\underline{s}} \times \mathrm{id})^!\mathcal{M}$ is independent of the choice of F up to canonical isomorphism, so that there is a well defined sheaf on $(F^{\underline{s}} \times \mathrm{id})^!\mathcal{M}$ on $P^r_Y(k)$ even if no lift F exists. Similarly, if $\mathcal{M}$ is a quasi-coherent sheaf on $P^r_{Y/X}(k)$, equipped with a costratification, then $(F^{\underline{s}} \times \mathrm{id})^!\mathcal{M}$ is a well defined sheaf on $P^r_{Y/X}(k)$.

We write $F^{\underline{s}*}\mathscr{D}_Y^{\otimes k}$ for the pull-back of $\mathscr{D}_Y^{\otimes k}$ (resp. $\mathscr{D}_Y^{\otimes k-1} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_X$) by $(F^{s_0} \otimes \cdots \otimes F^{s_{k-1}})^*$. The k-fold product $K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k}$ has k commuting structures of right $\mathscr{D}_Y$-module corresponding to the k ways of writing it as the product of a left $\mathscr{D}_Y$-module and a right $\mathscr{D}_Y$-module. (More precisely,for each i between 1 and k we write

$$K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k} = (K_Y \otimes_{\mathcal{O}_Y} \otimes \cdots \otimes_{\mathcal{O}_Y} \mathscr{D}_Y) \otimes_{\mathcal{O}_Y} (\mathscr{D}_Y \otimes_{\mathcal{O}_Y} \cdots \otimes_{\mathcal{O}_Y} \mathscr{D}_Y),$$

where there are $i-1$ copies of $\mathscr{D}_Y$ in the first bracketed factor, and $k-i+1$ copies of $\mathscr{D}_Y$ in the second bracketed factor. We regard the first factor as a right $\mathscr{D}_Y$-module via the $\mathscr{D}_Y$-action on its right-most factor, and the second factor as a left $\mathscr{D}_Y$-module via the $\mathscr{D}_Y$-action on its left-most factor. The indicated factorisation then equips $\mathcal{K}_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k}$ with a right $\mathscr{D}_Y$-module structure. We ignore for the moment the right $\mathscr{D}_Y$-module structure induced by the last factor in the product). We denote by $F^{\underline{s}!}(K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k})$ the pull-back of $K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k}$ by $F^{s_0 !} \otimes \cdots \otimes F^{s_{k-1}!}$. In other words, we pull back $\mathscr{D}_Y^{\otimes k}$ by $F^{s_i !}$ with respect to its $i+1^{\text{st}}$ right $\mathscr{D}_Y$-module structure. The result is independent of the order in which one applies these pull-backs.

Now for $i = 0, 1 \ldots k-1$ set $t_i = \sum_{j=i}^{k-1} s_j$. Using (14.4.3), (13.8.4), (13.5.2) and (14.4.5) we compute

$$\begin{aligned}(14.4.7)\quad K_Y \otimes_{\mathcal{O}_Y} F^{\underline{s}*}\mathscr{D}_Y^{\otimes k} &\overset{\sim}{\longrightarrow} (F^!)^{t_0} K_Y \otimes_{\mathcal{O}_Y} F^{\underline{s}*}\mathscr{D}_Y^{\otimes k} \\ &\overset{\sim}{\longrightarrow} (F^!)^{\underline{t}}(K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_Y^{\otimes k}) \overset{\sim}{\longrightarrow} \varinjlim_r p_{0*}^{r,k}(F^{\underline{t}} \times \mathrm{id})^! p_0^{r,k!} K_Y\end{aligned}$$

Similarly, we obtain an isomorphism

$$(14.4.8)\qquad K_Y \otimes_{\mathcal{O}_Y} F^{\underline{s}*}(\mathscr{D}_Y^{\otimes k-1} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_X) \overset{\sim}{\longrightarrow} \varinjlim_r p_{0*}^{r,k\prime}(F^{\underline{t}} \times \mathrm{id})^! p_0^{r,k\prime !} K_Y.$$

14.4.9. — We are going to construct a map of complexes of right $\mathscr{D}_{F,X}$-modules

$$f_*(K_Y \otimes_{\mathscr{D}_{F,Y}} C_\bullet(\mathscr{D}_{F,Y}) \otimes_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}) \to K_X \otimes_{\mathscr{D}_{F,X}} C_\bullet(\mathscr{D}_{F,X}). \tag{14.4.10}$$

The calculations of (14.4.8) show, in particular, that the argument of the functor f_* above is a double complex whose terms are acyclic for f_*, since they are direct limits of push-forwards by finite maps of terms of residual complexes on certain thickenings of Y. (Here we are again using the fact that $(\cdot)^!$ and $(\cdot)^\triangle$ coincide on residual complexes when $(\cdot)$ is a finite map.) Thus, according to (14.4.2) such a map induces a map in the derived category of right $\mathscr{D}_{F,X}$-modules

$$Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}) \to \omega_X[d_X] \tag{14.4.11}$$

as required by Proposition 14.4.1. We construct the required map by specifying it term by term. Let k be a non-negative integer. In degree $-k$ we have

$$\begin{aligned} K_Y \otimes_{\mathscr{D}_{F,Y}} C_{-k}(\mathscr{D}_{F,Y}) \otimes_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X} &\xrightarrow{\sim} K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_{F,Y}^{\otimes k} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_{F,X} \\ &\xrightarrow{\sim} \bigoplus_{\underline{s}} K_Y \otimes_{\mathcal{O}_Y} F^{\underline{s}*}(\mathscr{D}_Y^{\otimes k} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_X) \end{aligned} \tag{14.4.12}$$

where $\underline{s}$ runs over $k+1$-tuples of non-negative integers. Similarly we have

$$K_X \otimes_{\mathscr{D}_X} C_{-k}(\mathscr{D}_{F,X}) \xrightarrow{\sim} \bigoplus_{\underline{s}} K_X \otimes_{\mathcal{O}_X} F^{\underline{s}*}(\mathscr{D}_X^{\otimes k+1}) \tag{14.4.13}$$

Applying (14.4.8), and (14.4.7) with X in place of Y, we see that it suffices to construct for each $\underline{t}$ as above, a map

$$f_*(p_{0*}^{r,k+1\prime}(F^{\underline{t}} \times \mathrm{id})^\triangle p_0^{r,k+1\prime\triangle} K_Y) \to p_{0*}^{r,k+1\prime\prime}(F^{\underline{t}} \times \mathrm{id})^\triangle p_0^{r,k+1\prime\prime\triangle} K_X \tag{14.4.14}$$

where $p_i^{r,k\prime\prime}$ denotes the analogue of the map $p_i^{r,k}$ but with X in place of Y. At this we have replaced ! by $\triangle$ in the notation to emphasise that we are dealing with residual complexes. (Recall again that these two functors are, by definition, equal for finite maps.)

Now let $f^{r,k} : P^r_{Y/X}(k) \to P^r_X(k)$ denote the map induced by the projection $Y^k \times X \overset{f^k\times\mathrm{id}}{\to} X^{k+1}$ The isomorphism $p_0^{r,k+1\prime\prime} \circ f^{r,k+1} = f \circ p_0^{r,k+1\prime}$ yields an isomorphism of functors $p_{0*}^{r,k+1\prime\prime} \circ f_*^{r,k+1} = f_* \circ p_{0*}^{r,k+1\prime}$. Thus, to construct the map of (14.4.14), it is enough to construct a map

$$f_*^{r,k+1}(F^{\underline{t}} \times \mathrm{id})^\triangle p_0^{r,k+1\prime\triangle} K_Y \to (F^{\underline{t}} \times \mathrm{id})^\triangle p_0^{r,k+1\prime\prime\triangle} K_X \tag{14.4.15}$$

By the adjointness between $f_*^{r,k+1}$ and $f^{r,k+1\triangle}$, and using the isomorphism $K_Y \xrightarrow{\sim} f^\triangle K_X$ discussed in (14.4.3) this means we have to construct a map (which will turn out to be an isomorphism)

$$(F^{\underline{t}} \times \mathrm{id})^\triangle p_0^{r,k+1\prime\triangle} f^\triangle K_X \to f^{r,k+1\triangle}(F^{\underline{t}} \times \mathrm{id})^\triangle p_0^{r,k+1\prime\prime\triangle} K_X \tag{14.4.16}$$

Now after reducing our whole situation modulo p we have

$$f \circ p_0^{r,k+1\prime} \circ (F^{\underline{t}} \times \mathrm{id}) = p_0^{r,k+1\prime\prime} \circ F^{\underline{t}} \times \mathrm{id} \circ f^{r,k+1},$$

(of course $F^{\underline{t}} \times \mathrm{id}$ only exists globally modulo p) so the discussion in (14.4.3) and [**Ha 1**, VI 3.1, Var 1)] shows that there is indeed an isomorphism between the two sides of (14.4.16).

14.4.17. — So far, we have constructed the map (14.4.10) as a map of graded sheaves. We have to check that this is a map of *complexes of right $\mathscr{D}_{F,X}$-modules.* We first verify that (14.4.10) is compatible with differentials. For the differentials coming from K_X and K_Y this is immediate from the functorial properties of our construction. The differentials coming from the complexes $C_\bullet(\mathscr{D}_{F,Y})$ and $C_\bullet(\mathscr{D}_{F,X})$ can be written as $d = \sum_{i=0}^{k}(-1)^i d_i$ corresponding to the analogous expression in (14.4.2). We will show that each of the d_i commute with the map of (14.4.10). Our technique for doing this involves giving a geometric interpretation to the differential in the complexes appearing in (14.4.10) (here we always mean the differential coming from the complex $C_\bullet(\mathscr{D}_{F,Y})$, and not from K_Y). The required compatibility will then follow from functorial properties of the trace map for residual complexes.

Before doing this, it will be convenient to slightly reinterpret the isomorphism (14.4.14). Note that by [**Ha 1**, VI 3.1, Var 1)] the isomorphism (14.4.16) fits into a commutative diagram of isomorphisms

$$\text{(14.4.18)}\qquad \begin{array}{ccc} (F^{\underline{t}} \times \mathrm{id})^{\vartriangle} p_0^{r,k+1\prime\vartriangle} f^{\vartriangle} K_X & \xrightarrow{\sim} & f^{r,k+1\vartriangle}(F^{\underline{t}} \times \mathrm{id})^{\vartriangle} p_0^{r,k+1\prime\prime\vartriangle} K_X \\ \downarrow\wr & & \downarrow\wr \\ p_0^{r,k+1\prime\vartriangle} f^{\vartriangle} F^{t_0\vartriangle} K_X & \xrightarrow{\sim} & f^{r,k+1\vartriangle} p_0^{r,k+1\prime\prime\vartriangle} F^{t_0\vartriangle} K_X \end{array}$$

So via the isomorphism given by the left hand vertical map (14.4.15) corresponds to the map

$$f_*^{r,k+1} p_0^{r,k+1\prime\vartriangle} f^{\vartriangle} F^{t_0\vartriangle} K_X \xrightarrow{\sim} f_*^{r,k+1} f^{r,k+1\vartriangle} p_0^{r,k+1''\vartriangle} F^{t_0\vartriangle} K_X \xrightarrow{tr_{f^{r,k+1}}} p_0^{r,k+1''\vartriangle} F^{t_0\vartriangle} K_X,$$

where $tr_{f^{r,k+1}}$ denotes the trace map for residual complexes [**Ha 1**, VI 4.2].

For $i = 0, \dots k$, the map d_i is obtained by contracting the $i+1^{\text{th}}$ and $i+2^{\text{th}}$ term in the product $K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_{F,Y}^{\otimes k} \otimes_{\mathcal{O}_Y} \mathscr{D}_{F,X}$ (resp. $K_X \otimes_{\mathcal{O}_X} \mathscr{D}_{F,X}^{\otimes k+1}$). We begin by showing that (14.4.10) is compatible with d_i for $i = 1, \dots k$. Let $\mu_i : P^r_{Y/X}(k+1) \to P^r_{Y/X}(k)$ (resp. $P^r_X(k+1) \to P^r_X(k)$) denote the map induced by the map $Y^{k+1} \times X \to Y^k \times X$ (resp. $X^{k+2} \to X$) obtained by omitting the $i+1^{\text{th}}$ factor.

We begin by showing that (14.4.10) is compatible with d_i for $i = 1, \dots k$. For such i, we have a map

$$\text{(14.4.19)}\quad p_{0*}^{r,k+1'} p_0^{r,k+1'\vartriangle} f^{\vartriangle} F^{t_0\vartriangle} K_X \xrightarrow{\sim} p_{0*}^{r,k'} \mu_{i*} \mu_i^{\vartriangle} p_0^{r,k'\vartriangle} f^{\vartriangle} F^{t_0\vartriangle} K_X \xrightarrow{tr_{\mu_i}} p_{0*}^{r,k} p_0^{r,k\vartriangle} f^{\vartriangle} F^{t_0\vartriangle} K_X.$$

An unwinding of definitions shows that, via the isomorphisms (14.4.7), (14.4.8), and the isomorphism induced by the left vertical map of (14.4.18), d_i is obtained by summing (14.4.19) over $\underline{t}$ and passing to the limit over r. The compatibility of d_i with

(14.4.10) now follows from the commutativity of the following diagram of functors applied to the residual complex $F^{t_0\Delta}K_X$:

$$
\begin{array}{ccc}
f_* p_{0*}^{r,k+1'} p_0^{r,k+1'\Delta} f^\Delta & \xrightarrow{\sim} & p_{0*}^{r,k+1''} f_*^{r,k+1} f^{r,k+1\Delta} p_0^{r,k+1''\Delta} \\
\downarrow \sim & & \downarrow tr_{f^{r,k+1}} \\
f_* p_{0*}^{r,k'} \mu_{i*}\mu_i^\Delta p_0^{r,k\Delta} f^\Delta & & p_{0*}^{r,k+1''} p_0^{r,k+1''\Delta} \\
\downarrow tr_{\mu_i} & & \downarrow \sim \\
f_* p_{0*}^{r,k'} p_0^{r,k\Delta} f^\Delta & & p_{0*}^{r,k''} \mu_{i*}\mu_i^\Delta p_0^{r,k''\Delta} \\
\downarrow \sim & \swarrow tr_{\mu_i} & \\
p_{0*}^{r,k''} f_*^{r,k} f^{r,k\Delta} p_0^{r,k''\Delta} & & \\
\downarrow tr_{f^{r,k}} & & \\
p_{0*}^{r,k''} p_0^{r,k''\Delta} & &
\end{array}
$$

The commutativity of this diagram follows from the compatibility of the trace map for residual complexes with composition of maps [**Ha 1**, VI 4.2, TRA 1)], applied to $\mu_i \circ f^{r,k} = f^{r,k+1} \circ \mu_i$.

It remains to show that (14.4.10) is compatible with d_0. We have maps

$$
\begin{aligned}
(14.4.20)\quad & p_{0*}^{r,k+1'} p_0^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X \underset{p_{0*}^{r,k+1'}\varepsilon_Y}{\xrightarrow{\sim}} p_{0*}^{r,k+1'} p_1^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X \\
& \xrightarrow{\sim} p_{1*}^{r,k+1'} p_1^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X \xrightarrow{\sim} p_{0*}^{r,k'} \mu_{0*}\mu_0^\Delta p_0^{r,k'\Delta} f^\Delta F^{t_0\Delta} K_X \\
& \overset{tr_{\mu_0}}{\rightarrow} p_{0*}^{r,k} p_0^{r,k\Delta} f^\Delta F^{t_0\Delta} K_X .
\end{aligned}
$$

Here ε_Y is the map $p_0^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X \xrightarrow{\sim} p_0^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X$ obtained using the costratification on $f^\Delta F^{t_0\Delta} K_X$, while the second isomorphism is simply the fact that all the projections $p_i^{r,k+1'}$ are homeomorphisms. Unwinding definitions shows that d_0 is obtained via (14.4.7), (14.4.8), and (14.4.18) by summing (14.4.20) over $\underline{t}$, and passing to the limit over r. The compatibility of d_0 with (14.4.10) now follows as before from the compatibility of the trace with composition of maps applied to $\mu_0 \circ f^{r,k} = f^{r,k+1} \circ \mu_0$, together with commutativity of the following diagram

$$
\begin{array}{ccc}
p_0^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X & \xrightarrow{\sim} & f^{r,k+1\Delta} p_0^{r,k+1''\Delta} F^{t_0\Delta} K_X \\
\downarrow \varepsilon_Y & & \downarrow \varepsilon_X \\
p_1^{r,k+1'\Delta} f^\Delta F^{t_0\Delta} K_X & \longrightarrow & f^{r,k+1\Delta} p_1^{r,k+1''\Delta} F^{t_0\Delta} K_X
\end{array}
$$

The commutativity of this diagram follows in turn from the definition of right $\mathscr{D}$-modules structures on $F^{t_0\triangle}K_X$ and $f^{\triangle}F^{t_0\triangle}K_X$, discussed in (14.4.3), together with [**Ha 1**, VI 3.1, Var 1)].

Finally, we have to show that (14.4.10) is a map of right $\mathscr{D}_{F,X}$-modules. Since the method for this is largely the same as the proof that (14.4.10) is compatible with d_i for $i = 1, \dots, k$ we will only sketch it. First, just as we constructed (14.4.10), one can construct a map

$$(14.4.21)\quad f_*(K_Y \otimes_{\mathscr{D}_{F,Y}} C_\bullet(\mathscr{D}_{F,Y}) \otimes_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathscr{D}_{F,X}) \\ \to K_X \otimes_{\mathscr{D}_{F,X}} C_\bullet(\mathscr{D}_{F,X}) \otimes_{\mathcal{O}_X} \mathscr{D}_{F,X}$$

This is done by interpreting the left (resp. right) hand side of (14.4.21) in terms of residual complexes on nilpotent neighbourhoods of the image of Y in $Y^k \times X \times X$ (resp. X^{k+2}), and applying the trace for residual complexes.

The same argument that showed the compatibility of d_i with (14.4.10) for $i = 1, \dots, k$ shows that (14.4.21) and (14.4.10) are compatible via the maps μ_{k+1} which contract the last two factors $\mathscr{D}_{F,X}$ in the left and right hand side of (14.4.21). However, the compatibility of the trace map with base change shows that (14.4.21), is nothing but the map (14.4.10) tensored by $\otimes_{\mathcal{O}_X}\mathscr{D}_{F,X}$. Thus, the compatibility of μ_{k+1} with (14.4.10) and (14.4.21) becomes the compatibility of (14.4.10) with the right $\mathscr{D}_{F,X}$-module structures.

This completes the proof that (14.4.11) is a map of complexes of right $\mathscr{D}_{F,X}$-modules, and hence the proof of Proposition 14.4.1.

Corollary 14.4.22. — *Let $f : Y \to X$ be as in Proposition 14.4.1, and $v \in \mathbb{N}\cup\{\infty\}$. There is a map in the derived category of right $\mathscr{D}^{(v)}_{F,X}$-modules*

$$Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}) \to \omega_X[d_X].$$

Proof. — We have the following maps in the derived category of $(\mathscr{D}_{F,Y}, f^{-1}\mathscr{D}^{(v)}_{F,X})$-bimodules

$$\mathscr{D}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} \to \mathscr{D}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}_{F,X} \to \mathscr{D}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}.$$

Here the second map is induced by the derived category analogue of the morphism $\otimes_{\mathscr{D}^{(v)}_{F,Y}} \to \otimes_{\mathscr{D}_{F,Y}}$ of bifunctors on the category of $\mathscr{D}_{F,Y}$-module. We leave it as an exercise to the reader to check that this map really induces a map of functors on derived categories (Hint: The fact that any quasi-coherent $\mathcal{O}_Y$-module is a quotient of a locally free one implies the analogous fact for $\mathscr{D}_{F,Y}$ and $\mathscr{D}^{(v)}_{F,Y}$-modules.) Applying the functor $Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} -)$ to the composite of the above maps yields a map of right $f^{-1}\mathscr{D}_{F,X}$-modules

$$Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}) \to Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}).$$

Composing this map with the map of Proposition 14.4.1 yields the map of the corollary. □

14.4.23. — We end this subsection by explaining a compatibility between the morphism of Corollary 14.4.22 and the usual trace map. Namely, we have a commutative diagram in the derived category of $\mathcal{O}_X$-modules

$$\begin{array}{ccc} Rf_*(\omega_Y[d_Y]) & \xrightarrow{\;tr_f\;} & \omega_X[d_X] \\ \big\downarrow\wr & & \Big\| \\ Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y}) & & \Big\| \\ \big\downarrow & & \Big\| \\ Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}) & \longrightarrow & \omega_X[d_X] \end{array}$$

Here the top map is the usual trace map of coherent duality theory, while the bottom one is given by forgetting the right $\mathscr{D}^{(v)}_{F,X}$-module structures in the map Corollary 14.4.22.

The construction of Corollary 14.4.22 shows that to check the diagram commutes it is enough to handle the case of $\mathscr{D}_{F,Y}$-modules. In this case, we have maps

$$\begin{aligned}(14.4.24)\qquad f_*(K_Y \otimes_{\mathscr{D}_{F,Y}} C_\bullet(\mathscr{D}_{F,Y})) &\to f_*(K_Y \otimes_{\mathscr{D}_{F,Y}} C_\bullet(\mathscr{D}_{F,Y}) \otimes_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}) \\ &\to K_X \otimes_{\mathscr{D}_{F,X}} C_\bullet(\mathscr{D}_{F,X})\end{aligned}$$

with the second map given by (14.4.10) Now the discussion of (14.4.2) shows that the first term in (14.4.24) is a resolution of $f_*(K_Y)$, while the third is a resolution of K_X (note that the terms in all our complexes are acyclic for f_*). We have to check that the diagram

$$\begin{array}{ccc} f_*(K_Y \otimes_{\mathscr{D}_{F,Y}} C_\bullet(\mathscr{D}_{F,Y})) & \longrightarrow & f_*(K_Y) \\ \big\downarrow{\scriptstyle (14.4.24)} & & \big\downarrow{\scriptstyle tr_f} \\ K_X \otimes_{\mathscr{D}_{F,X}} C_\bullet(\mathscr{D}_{F,X}) & \longrightarrow & K_X \end{array}$$

commutes. For this we only have to examine the effect of (14.4.24) on the term $f_*(K_Y \otimes_{\mathscr{D}_{F,Y}} C_0(\mathscr{D}_{F,Y}))$. So we have to check the commutativity of the following diagram

$$\begin{array}{ccc} f_*(K_Y \otimes_{\mathcal{O}_Y} \mathscr{D}_{F,Y}) & \longrightarrow & f_*(K_Y) \\ \big\downarrow & & \big\downarrow \\ f_*(K_Y \otimes_{\mathcal{O}_Y} f^*\mathscr{D}_{F,X}) & & \big\downarrow \\ \big\downarrow & & \big\downarrow \\ K_X \otimes_{\mathcal{O}_X} \mathscr{D}_{F,X} & \longrightarrow & K_X \end{array}$$

The commutativity of this diagram follows by an argument similar to the one we used in (14.4.17) to show that (14.4.10) was compatible with d_0. In particular, it is based on the compatibility of the trace for residual complexes with composition of maps, and the commutativity of the diagram

$$\begin{array}{ccc} P^r_Y(1) & \xrightarrow{p_1^{r,1}} & Y \\ \downarrow & & \Big\downarrow \\ P^r_{Y/X}(1) & & \Big\downarrow \\ \Big\downarrow{\scriptstyle f^{r,1}} & & \Big\downarrow \\ P^r_X(1) & \xrightarrow{p_1^{r,1\prime\prime}} & X \end{array}$$

14.5. — In this subsection, we will use the trace map constructed in (14.4) to show that f_+ is right adjoint to $f^!$, for f a proper map of smooth W_n-schemes.

Proposition 14.5.1. — *Let $f : Y \to X$ be a proper map of smooth W_n-schemes. There is a canonical map in the derived category of $(\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,X})$-bimodules, called the "trace map"*

$$f_+ f^! \mathscr{D}^{(v)}_{F,X} \to \mathscr{D}^{(v)}_{F,X}.$$

Proof. — We begin with a lemma.

Lemma 14.5.2. — *Let $f : Y \to X$ be a map of smooth W_n-schemes. If $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_Y$-module, and $\mathcal{E}$ is a left $\mathscr{D}^{(v)}_X$-module then there is a canonical isomorphism of right $f^{-1}\mathscr{D}^{(v)}_X$-modules*

$$(14.5.3) \qquad (\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E}.$$

If $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,Y}$-module, and $\mathcal{E}$ is a left $\mathscr{D}^{(v)}_{F^!,X}$-module, then there is a canonical isomorphism of right $f^{-1}\mathscr{D}^{(v)}_{F,X}$-modules

$$(14.5.4) \qquad (\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E}.$$

Proof. — Let $\mathcal{M}$ be a right $\mathscr{D}^{(v)}_Y$-module, and $\mathcal{E}$ a left $\mathscr{D}^{(v)}_X$-module. We define (14.5.3) to be the composite of the natural isomorphisms

$$\begin{aligned} (\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X &\xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_Y} (f^*\mathcal{E} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}^{(v)}_X) \\ &\xrightarrow{\sim} \mathcal{M} \otimes_{\mathscr{D}^{(v)}_Y} f^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X) \overset{\mathrm{id}\otimes f^*\gamma_{\mathcal{E}}^{-1}}{\xrightarrow{\sim}} \mathcal{M} \otimes_{\mathcal{O}_Y} f^*(\mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E}) \\ &\xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}^{(v)}_X \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E}, \end{aligned}$$

with the first isomorphism being provided by Lemma 13.9.8, and where $\gamma_{\mathcal{E}}^{-1}$ is the inverse of the transposition isomorphism of [**Ber 3**, 1.3.3]. By virtue of the naturality of the isomorphisms of Lemma 13.9.8, this isomorphism respects the right $f^{-1}\mathscr{D}^{(v)}_X$-module structures on its source and target.

To construct (14.5.4) note that we have an isomorphism of $(\mathscr{D}^{(v)}_{F,Y}, f^{-1}\mathscr{D}^{(v)}_X)$-bimodules $f^*\mathscr{D}^{(v)}_{F,X} \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Y} \otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X$. Thus as a right $f^{-1}\mathscr{D}^{(v)}_X$-module the left (resp. right) side of (14.5.4) is isomorphic to the left (resp. right) side of (14.5.3). We define (14.5.4) by requiring that on underlying right $f^{-1}\mathscr{D}^{(v)}_X$-modules (14.5.4) reduces to (14.5.3) via the above isomorphisms.

We claim that the resulting map, which we will denote by χ, respects the right $f^{-1}\mathscr{D}_{F,X}$-module structures on the two sides of (14.5.4). To see this, note that, by the functoriality of our construction, if $\mathcal{M}' \to \mathcal{M}$ is a surjection of right $\mathscr{D}^{(v)}_{F,Y}$-modules, then it is enough to check the claim with $\mathcal{M}'$ in place of $\mathcal{M}$. Thus, we may assume that $\mathcal{M}$ is a locally free $\mathscr{D}^{(v)}_{F,Y}$-module. Moreover, we may work locally on X and Y, so we are reduced to the case $\mathcal{M} = \mathscr{D}^{(v)}_{F,Y}$. Again, working locally on X and Y we may assume that there exists lifts F of the absolute Frobenius to X and Y respectively, and that these lifts are compatible with f : $f \circ F = F \circ f$.

Let $1 \otimes e \otimes 1$ be a section of the left hand side of (14.5.4), with e a local section of $\mathcal{E}$, and let $\phi : \mathcal{E} \to F^*\mathcal{E}$ denote the map defining the $\mathscr{D}^{(v)}_{F^!,X}$-pre-module structure on $\mathcal{E}$. Write $\phi(e) = aFe'$, with a a section of $\mathcal{O}_X$, and e' a section of $\mathcal{E}$. We again denote by e the image of e in $f^*\mathcal{E}$, and similarly for a and e'. A simple calculation shows that in $(\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}$, we have $(1 \otimes e \otimes 1)F = aF \otimes e \otimes 1$. Thus, we compute

$$\chi(1\otimes e\otimes 1)F = (1\otimes 1\otimes e)F = 1\otimes aF\otimes e = aF\otimes 1\otimes e = \chi(aF\otimes e\otimes 1) = \chi((1\otimes e\otimes 1)F),$$

where in the fourth equality we have used that, by functoriality, χ is $\mathscr{D}^{(v)}_{F,Y}$-linear (when $\mathcal{M} = \mathscr{D}^{(v)}_{F,Y}$). Using this linearity again, and the fact that we already know that χ is a map of right $f^{-1}\mathscr{D}^{(v)}_X$-modules, we find that for any local sections m and ∂ of M and $\mathscr{D}^{(v)}_{F,X}$ respectively, we have

$$\chi(m \otimes e \otimes 1)\partial = \chi(m \otimes e \otimes \partial).$$

Thus for any sections ∂_1, ∂_2 of $\mathscr{D}_{F,X}$ we have

$$\chi(m \otimes e \otimes \partial_1)\partial_2 = \chi(m \otimes e \otimes 1)\partial_1\partial_2 = \chi(m \otimes e \otimes \partial_1\partial_2).$$

Since any local section of $(\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}$, can be written as a sum of sections of the form $m \otimes e \otimes \partial_1$, this shows that χ in indeed a map of right $f^{-1}\mathscr{D}_{F,X}$-modules. □

14.5.5. — To prove Proposition 14.5.1, we will apply the above lemma with $\mathcal{E} = \mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}$, where the pull-back $f^*\mathcal{E}$ is to be computed with respect to the left $\mathscr{D}^{(v)}_X$-module structure on $\mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ obtained from the right $\mathscr{D}^{(v)}_X$-module structure on $\mathscr{D}^{(v)}_{F,X}$.

Note that if $\mathcal{M}$ is a right $\mathscr{D}^{(v)}_{F,Y}$-module, that is locally free as a right $\mathscr{D}^{(v)}_Y$-module, then $\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}$ is locally free as a right $\mathscr{D}^{(v)}_Y$-module, for example by using the isomorphism $\gamma_{f^*\mathcal{E}}$. It follows that $\mathcal{M}\otimes_{\mathcal{O}_Y} f^*\mathcal{E}$ is acyclic for the functor $\otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}$, because on underlying $\mathscr{D}^{(v)}_Y$-modules this functor is simply $\otimes_{\mathscr{D}^{(v)}_Y} f^*\mathscr{D}^{(v)}_X$. Similarly, $\mathcal{M}$ itself is acyclic for the functor $\otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}$. Thus, if we take a resolution of ω_Y by locally free right $\mathscr{D}^{(v)}_{F,Y}$-modules (which are, in particular, locally free as right $\mathscr{D}^{(v)}_Y$-modules), and apply (14.5.4), we obtain, using the functoriality of (14.5.4), an isomorphism in the derived category of $(\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,X})$-bimodules

$$(14.5.6)\quad \begin{aligned} f_+f^!\mathscr{D}^{(v)}_{F,X} &= Rf_*((\omega_Y\otimes_{\mathcal{O}_Y} f^*\mathcal{E})\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}[d_{Y/X}]) \\ &\xrightarrow{\sim} Rf_*(\omega_Y \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E}[d_{Y/X}]) \\ &\xrightarrow{\sim} Rf_*(\omega_Y[d_Y] \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X}) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{E}[-d_X] \end{aligned}$$

Composing with the trace map of Proposition 14.4.1, we obtain maps in the derived category of $(\mathscr{D}^{(v)}_{F,X}, \mathscr{D}^{(v)}_{F,X})$-bimodules

$$f_+f^!\mathscr{D}^{(v)}_{F,X} \to \omega_X[d_X]\otimes_{\mathcal{O}_X}(\mathscr{D}_{F,X}\otimes_{\mathcal{O}_X}\omega_X^{-1})[-d_X] \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,X}.$$

The composite of these maps yields the map of Proposition 14.5.1.

□

14.5.7. — The map of Proposition 14.5.1 satisfies a certain compatibility with the usual trace map for quasi-coherent sheaves. To explain it, note that on the level of $(\mathcal{O}_X, \mathcal{O}_X)$-bimodules, we have

$$\begin{aligned} f_+\mathscr{D}^{(v)}_{F,Y}[d_{Y/X}] &\xrightarrow{\sim} Rf_*(f^*(\mathscr{D}^{(v)}_{F,X}\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_Y}\omega_Y)[d_{Y/X}] \\ &\xrightarrow{\sim} \mathscr{D}^{(v)}_{F,X}\otimes_{\mathcal{O}_X} Rf_*(\omega_{Y/X}[d_{Y/X}]) \xrightarrow{\sim} \mathscr{D}^{(v)}_{F,X}\otimes_{\mathcal{O}_X} Rf_*f^!_{qc}\mathcal{O}_X, \end{aligned}$$

where we have set $\omega_{Y/X} = \omega_Y\otimes_{\mathcal{O}_Y} f^*\omega_X^{-1}$, and written $f^!_{qc}$ to denote the functor that is normally denoted $f^!$ in the duality theory of quasi-coherent sheaves. Then we have a commutative diagram in the derived category of $(\mathcal{O}_X, \mathcal{O}_X)$-bimodules

$$(14.5.8)\quad \begin{array}{ccccc} f_+\mathscr{D}^{(v)}_{F,Y}[d_{Y/X}] & \xrightarrow{\sim} & \mathscr{D}^{(v)}_{F,X}\otimes_{\mathcal{O}_X} Rf_*f^!_{qc}\mathcal{O}_X & \longrightarrow & \mathscr{D}^{(v)}_{F,X} \\ \downarrow & & & & \| \\ f_+f^!\mathscr{D}^{(v)}_{F,X} & & \longrightarrow & & \mathscr{D}^{(v)}_{F,X} \end{array}$$

where the second map in the top line is given by tensoring the trace map of quasi-coherent duality theory by $\mathscr{D}^{(v)}_{F,X}$, the vertical map on the left is given by shifting the projection $\mathscr{D}^{(v)}_{F,Y} \to f^*\mathscr{D}^{(v)}_{F,X}$ by $[d_{Y/X}]$, while the map on the bottom is that of Proposition 14.5.1.

To verify that the diagram commutes we begin by noting that for any $\mathscr{D}^{(v)}_X$-module $\mathcal{E}$ we have a diagram

$$
\begin{array}{ccccc}
f^*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathscr{D}^{(v)}_Y & \xrightarrow[\sim]{\gamma^{-1}_{f^*\mathcal{E}}} & \mathscr{D}^{(v)}_Y \otimes_{\mathcal{O}_Y} f^*\mathcal{E} & \xrightarrow{\sim} & \mathscr{D}^{(v)}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E} \\
\downarrow & & & & \\
f^*\mathcal{E} \otimes_{\mathcal{O}_Y} f^*\mathscr{D}^{(v)}_X & & & & \downarrow \\
\downarrow \wr & & & & \\
f^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathscr{D}^{(v)}_X) & \xrightarrow[\sim]{f^*\gamma_{\mathcal{E}}} & f^*(\mathscr{D}^{(v)}_X \otimes_{\mathcal{O}_X} \mathcal{E}) & \xrightarrow{\sim} & f^*\mathscr{D}_X \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E}
\end{array}
$$

The commutativity of this diagram can be seen either via an explicit local calculation, using the local formulas for the maps $\gamma_{\mathcal{E}}$ and $\gamma_{f^*\mathcal{E}}$, or, more easily, by using stratifications, and the definitions of the maps $\gamma_{\mathcal{E}}$ and $\gamma_{f^*\mathcal{E}}$ in [**Ber 3**, 1.3.1]. Using the diagram above, we easily deduce the commutativity of the following diagram, where the notation is that of Lemma 14.5.2:

(14.5.9)
$$
\begin{array}{ccc}
\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E} & = & \mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E} \\
\downarrow \wr & & \downarrow \wr \\
(\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y} & & \mathcal{M} \otimes_{\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y} \otimes_{\mathcal{O}_Y} f^*\mathcal{E} \\
\downarrow & & \downarrow \\
(\mathcal{M} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} & \xrightarrow{(14.5.4)} & \mathcal{M} \otimes_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E}
\end{array}
$$

Now consider the following diagram, where we write $\mathcal{E} = \mathscr{D}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}$:

$$\begin{array}{ccc}
f_+\mathscr{D}^{(v)}_{F,Y}[d_{Y/X}] & \longrightarrow & f_+f^!\mathscr{D}^{(v)}_{F,X} \\
\| & & \| \\
Rf_*(\omega_Y \otimes_{\mathcal{O}_Y} f^*\mathcal{E}[d_{Y/X}]) & \longrightarrow & Rf_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_{F,Y}} f^*\mathscr{D}_{F,X}[d_{Y/X}]) \\
\| & & \big\downarrow {\scriptstyle (14.5.4)}\ \wr \\
Rf_*(\omega_Y \otimes_{\mathcal{O}_Y} f^*\mathcal{E}[d_{Y/X}]) & \longrightarrow & Rf_*(\omega_Y \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^*\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E})[d_{Y/X}] \\
\big\downarrow \wr & & \\
Rf_*(\omega_Y[d_{Y/X}]) \otimes_{\mathcal{O}_X} \mathcal{E} & & \big\downarrow {\scriptstyle (14.4.1)} \\
\big\downarrow {\scriptstyle tr_f \otimes 1} & & \\
\omega_X \otimes_{\mathcal{O}_X} \mathcal{E} & = & \omega_X \otimes_{\mathcal{O}_X} \mathcal{E}
\end{array}$$

(where the labels indicate the result we have used to construct the relevant map). If we identify $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ with $\mathscr{D}^{(v)}_{F,X}$, then the boundary of the above diagram can be identified with the diagram (14.5.8). The above diagram commutes, because the top rectangle commutes by definition, the middle rectangle commutes because (14.5.9) does, while the bottom rectangle commutes by the discussion in (14.4.23).

Lemma 14.5.10. — *Let $f : Y \to X$ be a map of smooth W_n-scheme, let $\mathcal{M}^\bullet$ be in $D^b_{qc}(\mathscr{D}^{(v)}_{F,Y})$, and let $\mathcal{N}^\bullet$ be in $D^b_{qc}(\mathscr{D}^{(v)}_{F,Y})$. Then there is a natural map*

$$Rf_*\underline{RHom}_{\mathscr{D}^{(v)}_{F,Y}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}_{\mathscr{D}^{(v)}_{F,X}}(f_+\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet).$$

Proof. — This is the analogue of Proposition 4.4.2, and the proof is formally identical, the main point being the construction of enough objects that are acyclic for the operations of tensoring by certain bimodules. □

Lemma 14.5.11. — *Let $\mathcal{N}^\bullet$ be in $D^-_{qc}(\mathscr{D}^{(v)}_{F,X})$, and $f : Y \to X$ a map of smooth W_n-schemes. There is a canonical isomorphism*

$$f_+f^!\mathcal{N}^\bullet \xrightarrow{\sim} f_+f^!\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} \mathcal{N}^\bullet.$$

Proof. — Consider first a bounded above complex of $(f^{-1}\mathscr{D}^{(v)}_{F,X}, f^{-1}\mathscr{D}^{(v)}_{F,X})$-bimodules $\mathcal{M}^\bullet$. We may assume that $\mathcal{N}^\bullet$ is a complex of locally free $\mathscr{D}^{(v)}_{F,X}$-modules. Then using the projection formula we compute

$$\begin{aligned} Rf_*(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}^{(v)}_{F,X}} f^{-1}\mathcal{N}^\bullet) &= Rf_*(\mathcal{M}^\bullet \otimes_{f^{-1}\mathscr{D}^{(v)}_{F,X}} f^{-1}\mathcal{N}^\bullet) \\ &\xrightarrow{\sim} Rf_*(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} \mathcal{N}^\bullet. \end{aligned}$$

The lemma now follows from the above isomorphism, if we take

$$\mathcal{M}^\bullet = \mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} f^! \mathscr{D}^{(v)}_{F,X} = \mathscr{D}^{(v)}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}} \mathscr{D}^{(v)}_{F,Y\to X}.$$

□

Theorem 14.5.12. — *Let $f : Y \to X$ be a proper map. If $\mathcal{M}^\bullet$ is in $D^b_{qc}(\mathscr{D}^{(v)}_{F,Y})$ and $\mathcal{N}^\bullet$ is in $D^b_{qc}(\mathscr{D}^{(v)}_{F,X})$ then there is a canonical isomorphism*

$$Rf_* \underline{RHom}_{\mathscr{D}^{(v)}_{F,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet) \xrightarrow{\sim} \underline{RHom}_{\mathscr{D}^{(v)}_{F,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet).$$

Proof. — We begin by constructing the required morphism. Combining Lemmas 14.5.10, 14.5.11, and Proposition 14.5.1, we obtain natural maps

$$(14.5.13)\quad Rf_*(\underline{RHom}_{\mathscr{D}^{(v)}_{F,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet)) \to \underline{RHom}_{\mathscr{D}^{(v)}_{F,X}}(f_+\mathcal{M}^\bullet, f_+f^!\mathcal{N}^\bullet)$$
$$\to \underline{RHom}_{\mathscr{D}^{(v)}_{F,X}}(f_+\mathcal{M}^\bullet, f_+f^!\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} \mathcal{N}^\bullet) \to \underline{RHom}_{\mathscr{D}^{(v)}_{F,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet).$$

We have to show that this map is an isomorphism.

For this, we may assume that $\mathcal{M}$ is a complex whose term in degree $i \in \mathbb{Z}$ are of the form $\mathscr{D}^{(v)}_{F,Y} \otimes_{\mathcal{O}_Y} M^i$ with M^i a locally free $\mathcal{O}_Y$-module, and $M^i = 0$ for i sufficiently large. Indeed this follows from the fact that $\mathcal{M}$ is represented by a complex of quasi-coherent $\mathscr{D}^{(v)}_{F,Y}$-modules, by Bernstein's theorem, and the fact that any such module is a quotient of one that is of the form $\mathscr{D}^{(v)}_{F,Y} \otimes_{\mathcal{O}_Y} M$ with M locally free.

Let $\sigma \in \mathbb{Z}$. We have the "brutal truncations" $\mathcal{M}^{<\sigma}$ and $\mathcal{M}^{\geq\sigma}$, and an exact triangle

$$\mathcal{M}^{\geq\sigma} \to \mathcal{M}^\bullet \to \mathcal{M}^{<\sigma} \to \mathcal{M}^{\geq\sigma}[1].$$

Fix an integer i. Since the functor f_+ has finite cohomological amplitude, we can choose σ small enough that if we replace $\mathcal{M}^\bullet$ by $\mathcal{M}^{<\sigma}$ in (14.5.13), then the cohomology of both sides of the resulting morphism vanish in degree $\geq i$. Thus, to show that (14.5.13) induces an isomorphism on cohomology in degree i, it suffices to show it with $\mathcal{M}^{\geq\sigma}$ in place of $\mathcal{M}^\bullet$. Finally, by dévissage we are reduced to the case where $\mathcal{M}^\bullet$ is a single $\mathscr{D}^{(v)}_{F,Y}$-module in degree 0, of the form $\mathcal{M} = \mathscr{D}^{(v)}_{F,Y} \otimes_{\mathcal{O}_Y} M$ with M a locally free $\mathcal{O}_Y$-module. In this case we have

$$(14.5.14)\quad f_+\mathcal{M} \xrightarrow{\sim} Rf_*(f^*(\mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes_{\mathcal{O}_Y} \omega_Y \otimes_{\mathcal{O}_Y} M)$$
$$\xrightarrow{\sim} \mathscr{D}^{(v)}_{F,X} \otimes_{\mathcal{O}_X} Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M),$$

and hence a commutative diagram

$$\begin{array}{ccc} Rf_*\underline{RHom}_{\mathcal{O}_Y}(M, f^*\mathcal{N}^\bullet[d_{Y/X}]) & \longrightarrow & \underline{RHom}_{\mathcal{O}_X}(Rf_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} M), \mathcal{N}^\bullet) \\ \downarrow\wr & & \downarrow\wr \\ Rf_*\underline{RHom}_{\mathscr{D}^{(v)}_{F,Y}}(\mathcal{M}, f^!\mathcal{N}^\bullet) & \xrightarrow{(14.5.13)} & \underline{RHom}_{\mathscr{D}^{(v)}_{F,X}}(f_+\mathcal{M}, \mathcal{N}^\bullet) \end{array}$$

Here the right vertical map is deduced from the isomorphism (14.5.14). The compatibility explained in (14.5.7) shows that the top map is simply the map induced by duality for quasi-coherent sheaves:

$$\begin{aligned}&Rf_*\underline{RHom}_{\mathcal{O}_Y}(M, f^*\mathcal{N}^\bullet[d_{Y/X}])\\ &\qquad\xrightarrow{\sim} Rf_*\underline{RHom}_{\mathcal{O}_Y}(\omega_{Y/X}\otimes_{\mathcal{O}_Y} M, \omega_{Y/X}\otimes_{\mathcal{O}_Y} f^*\mathcal{N}^\bullet[d_{Y/X}])\\ &= Rf_*\underline{RHom}_{\mathcal{O}_Y}(\omega_{Y/X}\otimes_{\mathcal{O}_Y} M, f^!_{qc}\mathcal{N}^\bullet) \xrightarrow{\sim} \underline{RHom}_{\mathcal{O}_X}(Rf_*(\omega_{Y/X}\otimes_{\mathcal{O}_Y} M), \mathcal{N}^\bullet).\end{aligned}$$

In particular, this map is an isomorphism, whence so is (14.5.13). □

Corollary 14.5.15. — *Let $f: Y \to X$ be a proper map. Suppose that $\mathcal{M}^\bullet$ is in $D^b_{qc}(\mathscr{D}^{(v)}_{F,Y})$ and $\mathcal{N}^\bullet$ is in $D^-_{qc}(\mathscr{D}^{(v)}_{F,X})$. Then there are canonical adjunction morphisms*

$$f_+f^!\mathcal{N}^\bullet \to \mathcal{N}^\bullet \text{ and } \mathcal{M}^\bullet \to f^!f_+\mathcal{M}^\bullet.$$

If f is a closed immersion, then the second map exists even if we only assume that $\mathcal{M}^\bullet$ is in $\mathscr{D}^-(\mathscr{D}^{(v)}_{F,Y})$.

Proof. — For $\mathcal{M}^\bullet$ in $D^b_{qc}(\mathscr{D}^{(v)}_{F,Y})$ and $\mathcal{N}^\bullet$ in $D^b_{qc}(\mathscr{D}^{(v)}_{F,X})$ this follows immediately from the adjointness of (14.5.12). In general, the first morphism is given by the composite

$$f_+f^!\mathcal{N}^\bullet \xrightarrow{\sim} f_+f^!\mathscr{D}^{(v)}_{F,X}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}}\mathcal{N}^\bullet \to \mathcal{N}^\bullet$$

where the first map is given by (14.5.11) and the second by the adjunction morphism already constructed for bounded complexes, applied to $\mathscr{D}^{(v)}_{F,X}$.

Similarly if f is a closed immersion we have

$$\begin{aligned}\mathcal{M}^\bullet &\xrightarrow{\sim} \mathscr{D}^{(v)}_{F,Y}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{M}^\bullet \to f^!f_+\mathscr{D}^{(v)}_{F,Y}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{M}^\bullet\\ &\xrightarrow{\sim} f^{-1}(\mathscr{D}^{(v)}_{F,Y\to X}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} f_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y})[d_{Y/X}])\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{M}^\bullet\\ &\xrightarrow{\sim} f^{-1}(\mathscr{D}^{(v)}_{F,Y\to X}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,X}} f_*(\mathscr{D}^{(v)}_{F,X\leftarrow Y}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{(v)}_{F,Y}}\mathcal{M}^\bullet[d_{Y/X}])) = f^!f_+\mathcal{M}^\bullet\end{aligned}$$

□

14.6. — To complete the formalism of our "three operations" we note that Proposition 13.8.3 (i) implies that we have a bi-functor

$$D^-(\mathscr{D}^{(v)}_{F,X})\times D^-(\mathscr{D}^{(v)}_{F,X}) \xrightarrow{(\mathcal{M}^\bullet,\mathcal{N}^\bullet)\mapsto\mathcal{M}^\bullet\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{N}^\bullet} D^-(\mathscr{D}^{(v)}_{F,X}).$$

This restricts to a bi-functor

$$D^b(\mathscr{D}^{(v)}_{F,X})\times D^b(\mathscr{D}^{(v)}_{F,X})^\circ \to D^b(\mathscr{D}^{(v)}_{F,X}).$$

14.7. — In this section we will compare the functors defined above with those defined in §§2,3. We will use the notation of that paper, in addition to the notation introduced above. Suppose that X is a smooth k-scheme (i.e. $n = 1$). Note that we have functors

$$D^\bullet(\mathscr{D}^{(v)}_{F,X}) \to D^\bullet(\mathcal{O}_{F,X}),$$
$$D^\bullet(\mathcal{O}_{F,X}) \to D^\bullet(\mathscr{D}^{(v)}_{F,X}),$$

where the first is given by regarding a complex of $\mathscr{D}^{(v)}_{F,X}$-modules as a complex of $\mathcal{O}_{F,X}$ modules, and the second is given by $\mathcal{M}^\bullet \mapsto \mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}^\bullet$.

Proposition 14.7.1. — *Let $f : Y \to X$ be a map of smooth k- schemes. We have diagrams*

$$\begin{array}{ccc} D^\bullet(\mathscr{D}^{(v)}_{F,X}) & \longrightarrow & D^\bullet(\mathcal{O}_{F,X}) \\ \downarrow{\scriptstyle f^!} & & \downarrow{\scriptstyle f^!} \\ D^\bullet(\mathscr{D}^{(v)}_{F,Y}) & \longrightarrow & D^\bullet(\mathcal{O}_{F,Y}) \end{array}$$

and

$$\begin{array}{ccc} D^\bullet(\mathcal{O}_{F,Y}) & \longrightarrow & D^\bullet(\mathscr{D}^{(v)}_{F,Y}) \\ \downarrow{\scriptstyle f_+} & & \downarrow{\scriptstyle f_+} \\ D^\bullet(\mathcal{O}_{F,X}) & \longrightarrow & D^\bullet(\mathscr{D}^{(v)}_{F,X}) \end{array}$$

which commute up to natural isomorphism.

Proof. — The existence of the first commutative diagram is clear from the construction of $f^!$ in each of the two cases. (On the level of the underlying $\mathcal{O}_X$-modules it is just $\mathcal{O}_Y \overset{\mathbb{L}}{\otimes} -[d_{Y/X}]$.)

The second diagram is more delicate, since the construction of f_+ in the case of complexes of $\mathcal{O}_{F,X}$-modules given in above is not obviously compatible with the corresponding construction in the case of complexes of $\mathscr{D}^{(v)}_{F,X}$-modules given above. The required compatibility may be verified using the fact that the Cartier operator is compatible with composition of morphisms of smooth k-schemes - see A.2.3 below. applied to the composite $Y \to X \to k$.

In a large number of cases (which will be sufficient for applications) there is an easier way to construct the required diagram: Suppose that f is an *allowable* morphism in the sense of §9.7. This means that f may be factored as a composite of an immersion and a smooth proper map between *smooth* k-schemes. For such a morphism the construction of the second diagram is reduced to the special cases where f is either an open immersion, or a proper map. If f is an open immersion, then the existence of such a diagram is clear, as f_+ both for $\mathcal{O}_{F,X}$ and $\mathscr{D}^{(v)}_{F,X}$-modules is simply Rf_*. For a proper map, note that Theorem 14.5.12 implies in particular that f_+ is left adjoint to $f^!$ for $\mathscr{D}^{(v)}_{F,X}$-modules, and Theorem 4.4.1 implies the analogous fact for $\mathscr{D}^{(v)}_{F,X}$-modules. Suppose $\mathcal{M}^\bullet$ is in $D^b(\mathcal{O}_{F,Y})$ and $\mathcal{N}^\bullet$ is in $D^b(\mathscr{D}^{(v)}_{F,X})$. Using the first commutative diagram, we obtain

$$\begin{aligned} Hom_{\mathscr{D}^{(v)}_{F,X}}(\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) &\xrightarrow{\sim} Hom_{\mathcal{O}_{F,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) \\ &\xrightarrow{\sim} Hom_{\mathcal{O}_{F,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet) \xrightarrow{\sim} Hom_{\mathscr{D}^{(v)}_{F,Y}}(\mathscr{D}^{(v)}_{F,Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,Y}} \mathcal{M}^\bullet, f^!\mathcal{N}^\bullet) \\ &\xrightarrow{\sim} Hom_{\mathscr{D}^{(v)}_{F,X}}(f_+(\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}^\bullet), \mathcal{N}^\bullet). \end{aligned}$$

This implies that

$$\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} f_+\mathcal{M}^\bullet \xrightarrow{\sim} f_+(\mathscr{D}^{(v)}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}^\bullet)$$

as required □

14.7.2. — As with $f^!$, the functor $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ is not compatible with the extension of scalars functors $D^\bullet(\mathcal{O}_{F,X}) \to D^\bullet(\mathscr{D}^{(v)}_{F,X})$, but with restriction of scalars: we have a commutative diagram.

$$\begin{array}{ccc} D^-(\mathscr{D}^{(v)}_{F,X}) \times D^b_{qc}(\mathscr{D}^{(v)}_{F,X}) & \longrightarrow & D^-(\mathcal{O}_{F,X}) \times D^b_{qc}(\mathcal{O}_{F,X}) \\ \downarrow \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} & & \downarrow \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \\ D^-(\mathscr{D}^{(v)}_{F,X}) & \longrightarrow & D^-(\mathcal{O}_{F,X}) \end{array}$$

14.8. — Suppose that X is a smooth W_n-scheme. For any positive integer $j \leq n$ there is a functor

$$D^-(\mathscr{D}^{(v)}_{F,X}) \xrightarrow{\mathcal{M}^\bullet \mapsto \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}} D^-(\mathscr{D}^{(v)}_{F,X\otimes_{W_n} W_j}).$$

Proposition 14.8.1. — *Let $f : Y \to X$ be a map of smooth W_n-schemes. If $\mathcal{M}^\bullet$ is in $D^-(\mathscr{D}^{(v)}_{F,X})$ then there are canonical isomorphisms*

$$f^!\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z} \xrightarrow{\sim} f^!(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}).$$

If $\mathcal{M}^\bullet$ is in $D^-(\mathscr{D}^{(v)}_{F,Y})$ then there are canonical isomorphisms

$$f_+\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z} \xrightarrow{\sim} f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}).$$

If f is proper, and $\mathcal{M}^\bullet$ is in $D^b_{qc}(\mathscr{D}^{(v)}_{F,Y})$, then these isomorphisms are compatible with the isomorphism of Theorem 14.5.12.

Proof. — The first two claims follow immediately from the definitions. The final claim follows using the compatibility of the trace map with base change [**Con**]. □

15. UNIT $\mathscr{D}_{F,X}$-MODULES

15.1. — We call a left $\mathscr{D}_{F,X}^{(v)}$-module $\mathcal{M}$ a *pseudo-unit* $\mathscr{D}_{F,X}^{(v)}$-module if the structural morphism $F^*\mathcal{M} \to \mathcal{M}$ is an isomorphism. By Corollary 13.3.8, a pseudo-unit $\mathscr{D}_{F,X}^{(v)}$-module is automatically equipped with the structure of a $\mathscr{D}_{F,X}^{(\infty)} = \mathscr{D}_{F,X}$-module. Thus for the remainder of the paper we will deal with pseudo-unit $\mathscr{D}_{F,X}$-modules.

A *unit* $\mathscr{D}_{F,X}$-module is a *pseudo-unit* $\mathscr{D}_{F,X}$-module, which is quasi-coherent as an $\mathcal{O}_X$-module.

Proposition 15.1.1. — *Pseudo-unit (resp. unit) $\mathscr{D}_{F,X}$-modules form a thick subcategory of the category of $\mathscr{D}_{F,X}$-modules.*

Proof. — We must show that if

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4 \to \mathcal{M}_5$$

is an exact sequence of $\mathscr{D}_{F,X}$ modules, such that $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$ and $\mathcal{M}_5$ are all pseudo-unit (resp. unit) $\mathscr{D}_{F,X}$-modules, then $\mathcal{M}_3$ is also a pseudo-unit (resp. unit) $\mathscr{D}_{F,X}$-module.

The fact that $\phi_{\mathcal{M}_3}$ is an isomorphism follows from the exactness of F_X^*, together with the 5-lemma. (Compare Lemma 5.2.) Since quasi-coherent $\mathscr{D}_{F,X}$-modules form a thick subcategory of all modules, we see that if $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$ and $\mathcal{M}_5$ are unit, then $\mathcal{M}_3$ is quasi-coherent. □

15.1.2. — We have already seen in Corollary 13.5.10 that the functor F^* admits a quasi-inverse given by $\mathcal{E} \mapsto F^!\mathscr{D}_{F,X} \otimes_{\mathscr{D}_{F,X}} \mathcal{E}$. For r a positive integer, we denote the r-fold iterate of this inverse functor by $(F^*)^{-r}$. For any $\mathscr{D}_{F,X}$-module $\mathcal{M}$, we can apply $(F^*)^{-1}$ to the morphism $F^*\mathcal{M} \to \mathcal{M}$ and obtain a morphism $\mathcal{M} \to (F^*)^{-1}\mathcal{M}$.

We write

$$\mathscr{D}_{F,F^!,X} = \varinjlim_r (F^*)^{-r}\mathscr{D}_{F,X} \tag{15.1.3}$$

Proposition 15.1.4. — *The $\mathscr{D}_{F,X}$-module $\mathscr{D}_{F,F^!,X}$ has a structure of a sheaf of associative algebras, which naturally contains $\mathscr{D}_{F,X}$ and $\mathscr{D}_{F^!,X}$ as subrings.*

There is an equivalence of categories between the category of $\mathscr{D}_{F,F^!,X}$-modules, and the category of pseudo-unit $\mathscr{D}_{F,X}$-modules.

Proof. — We have $\mathscr{D}_{F,F^!,X} = \oplus_{r\in\mathbb{Z}}(F^*)^r\mathscr{D}_X$ The multiplication on $\mathscr{D}_{F,F^!,X}$ is defined by maps

$$(F^*)^r\mathscr{D}_X \otimes_{\mathscr{D}_X} (F^*)^s\mathscr{D}_X \to (F^*)^{r+s}\mathscr{D}_X,$$

one for each value of r and s obtained by applying $(F^*)^r$ to the isomorphism $\mathscr{D}_X \otimes_{\mathscr{D}_X} (F^*)^s\mathscr{D}_X \to (F^*)^s\mathscr{D}_X$, and keeping in mind Lemmas 13.5.1 and 13.5.7 if r and s have opposite signs.

One checks as in the proof of Corollary 13.3.5 that this makes $\mathscr{D}_{F,F^!,X}$ into a sheaf of associative rings, and it is clear from the definition that it contains $\mathscr{D}_{F,X}$ and $\mathscr{D}_{F^!,X}$ as subrings.

To prove the last claim of the proposition, consider a $\mathscr{D}_{F,F^!,X}$-module $\mathcal{M}$. Then $\mathcal{M}$ is, in particular, both a $\mathscr{D}_{F,X}$-module and a $D_{F^!,X}$-module, and thus is equipped with maps of $\mathscr{D}_X$-modules $\phi_{\mathcal{M}} : F^*\mathcal{M} \to \mathcal{M}$ and $\phi'_{\mathcal{M}} : F^!\mathscr{D}_X \otimes_{\mathscr{D}_X} \mathcal{M} \to \mathcal{M}$. Since $\mathcal{M}$ is a $\mathscr{D}_{F,F^!,X}$-module, the composites

$$\mathcal{M} \xrightarrow{\sim} F^*\mathscr{D}_X \otimes_{\mathscr{D}_X} F^!\mathscr{D}_X \otimes_{\mathscr{D}_X} \mathcal{M} \overset{F^*\phi'_{\mathcal{M}}}{\to} F^*\mathscr{D}_X \otimes_{\mathscr{D}_X} \mathcal{M} \overset{\phi_{\mathcal{M}}}{\to} \mathcal{M}$$

and

$$\mathcal{M} \xrightarrow{\sim} F^!\mathscr{D}_X \otimes_{\mathscr{D}_X} F^*\mathscr{D}_X \otimes_{\mathscr{D}_X} \mathcal{M} \overset{1\otimes\phi_{\mathcal{M}}}{\to} F^!\mathscr{D}_X \otimes_{\mathscr{D}_X} \mathcal{M} \overset{\phi'_{\mathcal{M}}}{\to} \mathcal{M}$$

are both the identity map. But this condition means that $\phi_{\mathcal{M}}$ is an isomorphism with inverse $F^*\phi'_{\mathcal{M}}$.

Conversely, given a $\mathscr{D}_X$-module $\mathcal{M}$ and an isomorphism $\phi_{\mathcal{M}} : F^*\mathcal{M} \to \mathcal{M}$ we can define $\phi'_{\mathcal{M}}$ by

$$\phi'_{\mathcal{M}} : F^!\mathscr{D}_X \otimes_{\mathscr{D}_X} \mathcal{M} \overset{1}{\otimes} \phi^{-1}_{\mathcal{M}} \to F^!\mathscr{D}_X \otimes_{\mathscr{D}_X} F^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$$

where the last isomorphism is that of Lemma 13.5.3. Then $\phi_{\mathcal{M}}$ and $\phi'_{\mathcal{M}}$ give $\mathcal{M}$ the structures of $\mathscr{D}_{F,X}$-module and $\mathscr{D}_{F^!,X}$ respectively, and one checks easily that there is a unique $\mathscr{D}_{F,F^!,X}$-module structure on $\mathcal{M}$ compatible with these. □

Corollary 15.1.5. — (*i*) *The forgetful functor from the category of pseudo-unit (resp. unit) $\mathscr{D}_{F,X}$-modules to the category of all $\mathscr{D}_{F,X}$-modules (resp. quasi-coherent $\mathscr{D}_{F,X}$-modules) admits a left adjoint U and a right adjoint V (resp. $\tilde{V}$).*

(*ii*) *U is exact while V (resp. $\tilde{V}$) is left exact*

(*iii*) *For $\mathcal{M}$ a pseudo-unit (resp. unit) $\mathscr{D}_{F,X}$-module, the adjunction morphism $\mathcal{M} \to V(\mathcal{M})$ (resp. $\mathcal{M} \to \tilde{V}(\mathcal{M})$) is an isomorphism*

Proof. — Let $\mathcal{M}$ be a $\mathscr{D}_{F,X}$-module. Set

$$U(\mathcal{M}) = \mathscr{D}_{F,F^!,X} \otimes_{\mathscr{D}_{F,X}} \mathcal{M}$$

It is evidently exact and left adjoint to the forgetful functor by Proposition 15.1.4. Since

$$(F^*)^{-r}\mathscr{D}_{F,X} \otimes_{\mathscr{D}_{F,X}} \mathcal{M} \xrightarrow{\sim} (F^!)^r\mathscr{D}_{F,X} \otimes_{\mathscr{D}_{F,X}} \mathcal{M} = (F^*)^{-r}(\mathcal{M})$$

we have $U(\mathcal{M}) = \varinjlim_{r} (F^*)^{-r}\mathcal{M}$. In particular one sees that if $\mathcal{M}$ is quasi-coherent, then $U(\mathcal{M})$ is quasi-coherent. This proves the claims concerning U.

Next we set

$$V(\mathcal{M}) = \underline{Hom}_{\mathscr{D}_{F,X}}(\mathscr{D}_{F,F^!,X}, \mathcal{M}).$$

The functor V is evidently left exact, and right adjoint to the forgetful functor. To construct $\tilde{V}$, we recall that the forgetful functor from the category of sheaves of $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathcal{O}_X$-modules has a right adjoint $\mathcal{N} \rightsquigarrow \tilde{\mathcal{N}}$ by [**Ha 1**, Appendix, Cor. 1], and we set $\tilde{V}(\mathcal{M}) = \widetilde{V(\mathcal{M})}$. Then $\tilde{V}$ clearly has the required adjointness property.

The final claim follows from the adjunction property of V (resp. $\tilde{V}$) which implies that $\mathcal{M}$ and $V(\mathcal{M})$ (resp. $\tilde{V}(\mathcal{M})$) represent the same functor on the category of pseudo-unit (resp. unit) $\mathscr{D}_{F,X}$-modules. □

Corollary 15.1.6. — *The category of unit $\mathscr{D}_{F,X}$-modules has enough injectives. Furthermore, a unit $\mathscr{D}_{F,X}$-module is injective in the category of unit $\mathscr{D}_{F,X}$-modules if and only if it is injective in the category of quasi-coherent $\mathscr{D}_{F,X}$-modules.*

Proof. — The category of quasi-coherent $\mathscr{D}_{F,F^!,X}$-modules has enough injectives, so the first claim follows from Proposition 15.1.4. This claim can also be deduced from the fact that the category of quasi-coherent $\mathscr{D}_{F,X}$-modules has enough injectives, as the functor $\tilde{V}$ takes injectives to injectives, being right adjoint to an exact functor.

Now for any unit $\mathscr{D}_{F,X}$-module $\mathcal{J}$ there is a natural isomorphism $\mathcal{J} \xrightarrow{\sim} \tilde{V}(\mathcal{J})$. We have already observed that $\tilde{V}$ preserves injectives, and since the forgetful functor from unit $\mathscr{D}_{F,X}$-modules to quasi-coherent $\mathscr{D}_{F,X}$-modules is right adjoint to the exact functor U, it also preserves injectives. Thus we see that $\mathcal{J}$ is injective as a unit $\mathscr{D}_{F,X}$-module if and only if it is injective as a quasi-coherent $\mathscr{D}_{F,X}$-module. This completes the proof of the corollary. □

15.2. — In this sub-section we show that when $n = 1$, the theory of unit $\mathscr{D}_{F,X}$-modules can be reduced to that of unit $\mathcal{O}_{F,X}$-modules, which was studied in §5. Together with a dévissage technique to be introduced later, this often allows us to reduce question about unit $\mathscr{D}_{F,X}$-modules on a smooth W_n-scheme X to questions about unit $\mathcal{O}_{F,X\otimes k}$ modules.

Suppose that X is a smooth k-scheme. It is well-known that if $\mathcal{M}$ is a unit $\mathcal{O}_{F,X}$-module, then there is a unique way to endow $\mathcal{M}$ with a connection compatible with its structural morphism $\phi_{\mathcal{M}}$, and that this connection necessarily extends to a stratification. In other words, the left $\mathcal{O}_{F,X}$-module structure on $\mathcal{M}$ extends in a unique way to a left $\mathscr{D}_{F,X}$-module structure. (This is discussed in [**Ka 2**, §6] in the case that $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module, and in the general case in [**Lyu**, §5]. It is also a consequence of the Frobenius descent results of [**Ber 3**].) In proposition 15.2.3 below we will prove a strengthened form of this result.

15.2.1. — For any integer $v \geq 0$, we let $\overline{\mathscr{D}}_X^{(v)}$ denote the image of $\mathscr{D}_X^{(v)}$ in $\mathscr{D}_X$; then $\overline{\mathscr{D}}_X^{(v)}$ can be described as the subring of $\mathscr{D}_X$ generated by the differential operators of order at most p^m. Similarly, we denote by $\overline{\mathscr{D}}^{(v)}_{F,X}$ the image of $\mathscr{D}^{(v)}_{F,X}$ in $\mathscr{D}_{F,X}$. We also write $\overline{\mathscr{D}}^{(-1)}_{F,X} = \mathcal{O}_{F,X}$.

Lemma 15.2.2. — *The $\overline{\mathscr{D}}^{(v)}_{F,X}$ form an increasing chain of sheaves of subrings of $\mathscr{D}_{F,X}$, with $\mathscr{D}_{F,X}$ as their direct limit. They have the property that for any $v \geq 0$, $\overline{\mathscr{D}}^{(v)}_{F,X} F \subset \mathcal{O}_X F \overline{\mathscr{D}}^{(v-1)}_{F,X}$.*

Proof. — The first statement of the lemma is immediate. To see the second, note that $\overline{\mathscr{D}}^{(v-1)}_X$ is a $\mathscr{D}^{(v-1)}_X$-submodule of $\mathscr{D}^{(v)}_X$, and so $F^*\overline{\mathscr{D}}^{(v-1)}_X$ is a $\mathscr{D}^{(v)}_X$-submodule of $F^*\mathscr{D}^{(v)}_X$. In other words,

$$\mathscr{D}^{(v)}_X \cdot F^*\overline{\mathscr{D}}^{(v-1)}_X \subset F^*\overline{\mathscr{D}}^{(v-1)}_X,$$

and so for any integer r,

$$(F^r)^*\mathscr{D}^{(v)}_X \cdot F^*\overline{\mathscr{D}}^{(v-1)}_X \subset (F^{(r+1)})^*\overline{D}^{(v-1)}_X.$$

In particular,

$$(F^r)^*\mathscr{D}^{(v)}_X \cdot F^*\mathcal{O}_X \subset (F^{(r+1)})^*\overline{\mathscr{D}}^{(v-1)}_X$$

for all r, and thus we have $\overline{\mathscr{D}}^{(v)}_{F,X} \cdot F \subset \mathcal{O}_X F \overline{\mathscr{D}}^{(v-1)}_{F,X}$, where both sides are now regarded as subsets of $\mathscr{D}_{F,X}$. □

Proposition 15.2.3. — *If X is a smooth k-scheme and $\mathcal{M}$ is a unit $\mathcal{O}_{F,X}$-module then the natural morphism $\mathcal{M} \xrightarrow{1\otimes \mathrm{id}} \mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}$ in $D^-(\mathscr{D}_{F,X})$ is an isomorphism. In particular, $\mathcal{M}$ is canonically endowed with a left $\mathscr{D}_{F,X}$-module structure.*

Proof. — The claims of the proposition may be checked locally, so we assume that X is affine. Let $\beta : M \to F_X^* M$ be a generator of $\mathcal{M}$, let $P^\bullet$ be a free resolution of M, and lift β to

$$\beta^\bullet : P^\bullet \to F_X^* P^\bullet.$$

Let $\mathcal{P}^i$ be the unit $\mathcal{O}_{F,X}$-module generated by β^i. Then the complex $\mathcal{P}^\bullet$ is a resolution of $\mathcal{M}$ (compare 5.3.5), and so it suffices to prove that the morphism

$$\mathcal{P}^i \to \mathscr{D}_{F,X} \otimes_{\mathcal{O}_{F,X}} \mathcal{P}^i$$

is an isomorphism for each i.

Let us fix some value of i, and write $P^i = \mathcal{O}_X^I$ for some set I. Then also $F_X^* P^i = \mathcal{O}_X^I$, and so β^i is simply an $I \times I$ matrix μ whose entries are global sections of $\mathcal{O}_X$, each of whose columns has only finitely many non-zero entries. Thus $\mathcal{P}^i$ has the free presentation

$$0 \to \mathcal{O}^I_{F,X} \xrightarrow{1-\mu F} \mathcal{O}^I_{F,X} \to \mathcal{P}^i \to 0,$$

and so $\mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{P}^i$ is represented by the complex $\mathscr{D}^I_{F,X} \xrightarrow{1-\mu F} \mathscr{D}^I_{F,X}$. We must show that the morphism of complexes

$$(\mathcal{O}^I_{F,X} \xrightarrow{1-\mu F} \mathcal{O}^I_{F,X}) \to (\mathscr{D}^I_{F,X} \xrightarrow{1-\mu F} \mathscr{D}^I_{F,X})$$

is a quasi-isomorphism.

The argument in the proof of Proposition 5.3.3 that proves that $\mathcal{O}^I_{F,X} \stackrel{1-\mu F}{\longrightarrow} \mathcal{O}^I_{F,X}$ is injective also proves that $\mathscr{D}^I_{F,X} \stackrel{1-\mu F}{\longrightarrow} \mathscr{D}^I_{F,X}$ is injective. Thus it suffices to show that the morphism

$$coker(\mathcal{O}^I_{F,X} \stackrel{1-\mu F}{\longrightarrow} \mathcal{O}^I_{F,X}) \to coker(\mathscr{D}^I_{F,X} \stackrel{1-\mu F}{\longrightarrow} \mathscr{D}^I_{F,X})$$

is an isomorphism.

For any integer m, we see by Lemma 15.2.2 that $(\overline{\mathscr{D}}^{(v)}_{F,X})^I \mu F \subset (\overline{\mathscr{D}}^{(v-1)}_{F,X})^I$, and thus

$$(\overline{\mathscr{D}}^{(v)}_{F,X})^I \subset (\overline{\mathscr{D}}^{(v-1)}_{F,X})^I + \mathscr{D}^I_{F,X}(1-\mu F).$$

By descending induction on m, we conclude that

$$(\overline{\mathscr{D}}^{(v)}_{F,X})^I \subset (D^{(-1)}_{F,X})^I + \mathscr{D}^I_{F,X}(1-\mu F) = \mathcal{O}^I_{F,X} + \mathscr{D}^I_{F,X}(1-\mu F).$$

Since v was arbitrary we conclude that

$$\mathscr{D}^I_{F,X} \subset \mathcal{O}^I_{F,X} + \mathscr{D}^I_{F,X}(1-\mu F).$$

Thus the morphism of cokernels is surjective.

To show that this morphism is injective, we must show that

$$\mathcal{O}^I_{F,X} \bigcap \mathscr{D}^I_{F,X}(1-\mu F) = \mathcal{O}^I_{F,X}(1-\mu F).$$

Let e_i ($i \in I$) denote the standard basis elements of both $\mathcal{O}^I_{F,X}$ and $\mathscr{D}^I_{F,X}$. Let $P_{i_1}, \ldots, P_{i_n}$ be an n-tuple of elements of $\mathscr{D}_{F,X}$ giving rise to a section $P_{i_1}e_{i_1} + \cdots + P_{i_n}e_{i_n}$ of $\mathscr{D}^I_{F,X}$ such that $(P_{i_1}e_{i_1} + \cdots + P_{i_n}e_{i_n})(1-\mu F)$ lies in $\mathcal{O}^I_{F,X}$. We may find v such that $P_{i_1}, \ldots, P_{i_n}$ are sections of $\mathscr{D}^{(v)}_{F,X}$. Then

$$(P_{i_1}e_{i_1} + \cdots + P_{i_n}e_{i_n})(1-\mu F) + (P_{i_1}e_{i_1} + \cdots + P_{i_n}e_{i_n})\mu F$$

is the sum of sections of $\mathcal{O}^I_{F,X}$ and $(\overline{\mathscr{D}}^{(v-1)}_{F,X})^I$. Thus $P_{i_1}, \ldots, P_{i_n}$ is in fact an n-tuple of sections of $\overline{D}^{(v-1)}_{F,X}$. By descending induction on v, we conclude that $P_{i_1}, \ldots, P_{i_n}$ is indeed an n-tuple of sections of $\mathcal{O}_{F,X}$, and thus that the morphism of cokernels is injective. This completes the proof of the proposition. □

Corollary 15.2.4. — *Suppose that $\mathcal{M}$ is a unit $\mathscr{D}_{F,X}$-module. Then the natural morphism $\mathscr{D}_{F,X} \otimes_{\mathcal{O}_{F,X}} \mathcal{M} \to \mathcal{M}$ (given by left multiplication of $\mathscr{D}_{F,X}$ on $\mathcal{M}$) is an isomorphism.*

Proof. — The composite $\mathcal{M} \stackrel{1\otimes \mathrm{id}}{\longrightarrow} \mathscr{D}_{F,X} \otimes_{\mathcal{O}_{F,X}} \mathcal{M} \to \mathcal{M}$ is simply the identity morphism of $\mathcal{M}$, and the first arrow is an isomorphism, by the proposition. Thus the second arrow is also an isomorphism. □

15.3. — In this sub-section we begin to investigate left $\mathscr{D}_{F,X}$-modules $\mathcal{M}$ on a smooth W_n-scheme X that are locally finitely generated unit modules. This means that $\mathcal{M}$ is a unit $\mathscr{D}_{F,X}$-module, and that locally on X, there is an $\mathcal{O}_X$-coherent submodule $M \subset \mathcal{M}$ such that the natural map $\mathscr{D}_{F,X} \otimes_{\mathcal{O}_X} M \to \mathcal{M}$ is a surjection. We will usually refer to such a module as an lfgu module.

Lemma 15.3.1. — *If X is a smooth k-scheme, and if $\mathcal{M}$ is a sheaf of quasi-coherent left $\mathscr{D}_{F,X}$-modules whose structural morphism is surjective, then $\mathcal{M}$ is locally finitely generated as a sheaf of left $\mathscr{D}_{F,X}$-modules if and only if $\mathcal{M}$ is locally finitely generated as a sheaf of left $\mathcal{O}_{F,X}$-modules.*

Proof. — Since $\mathcal{O}_{F,X}$ is a sheaf of subrings of $\mathscr{D}_{F,X}$, the "if" direction is trivial. Now suppose that $\mathcal{M}$ is locally finitely generated as a $\mathscr{D}_{F,X}$-module. Then we may find a coherent subsheaf M of $\mathcal{M}$ such that $\mathcal{M} = \mathscr{D}_{F,X}M$. We will begin by showing that in fact $\mathcal{M} = \overline{\mathscr{D}}^{(v)}_{F,X}M$ for some non-negative v.

By assumption $\phi_{\mathcal{M}}$ is surjective, which is to say that

$$\mathcal{M} = \mathcal{O}_X F\mathcal{M} = \mathcal{O}_X F\mathscr{D}_{F,X}M = \bigcup_{v=0}^{\infty} \mathcal{O}_X F\overline{\mathscr{D}}^{(v)}_{F,X}M.$$

Since M is coherent, we find that $M \subset \mathcal{O}_X F\overline{\mathscr{D}}^{(v)}_{F,X}M$ for some finite v. Using Lemma 15.2.2 we have

$$M \subset \mathcal{O}_X F\overline{\mathscr{D}}^{(v)}_{F,X}M \subset \mathcal{O}_X F\overline{\mathscr{D}}^{(v)}_{F,X}\mathcal{O}_X F\overline{\mathscr{D}}^{(v)}_{F,X}M \subset \mathcal{O}_X F^2\overline{\mathscr{D}}^{(v)}_{F,X}M \subset \cdots \subset \mathcal{O}_X F^s\overline{\mathscr{D}}^{(v)}_{F,X}M$$

for every positive integer s. Now if v' is any integer, choose $s \geq v' - v$. Then

$$\overline{D}^{(v')}_{F,X}F^s \subset \mathcal{O}_X F^s\overline{D}^{(v'-s)}_{F,X} \subset \mathcal{O}_X F^s\overline{\mathscr{D}}^{(v)}_{F,X} \subset \overline{\mathscr{D}}^{(v)}_{F,X}.$$

Thus

$$\overline{D}^{(v')}_{F,X}M \subset \overline{D}^{(v')}_{F,X}F^s\overline{\mathscr{D}}^{(v)}_{F,X}M \subset \overline{\mathscr{D}}^{(v)}_{F,X}\overline{\mathscr{D}}^{(v)}_{F,X}M = \overline{\mathscr{D}}^{(v)}_{F,X}M.$$

Since v' was arbitrary, we see that $\mathcal{M} = \mathscr{D}_{F,X}M = \overline{\mathscr{D}}^{(v)}_{F,X}M$. This completes the first step of the proof.

Now reconsider the equation $\mathcal{M} = \mathcal{O}_X F\mathcal{M}$. Since M is coherent there is a coherent submodule M_1 of $\mathcal{M}$ such that $M \subset \mathcal{O}_X FM_1$. Then we see that

$$\mathcal{M} = \overline{\mathscr{D}}^{(v)}_{F,X}M = \overline{\mathscr{D}}^{(v)}_{F,X}\mathcal{O}_X FM_1 \subset \mathcal{O}_X F\overline{D}^{(v-1)}_{F,X}M_1 \subset \overline{D}^{(v-1)}_{F,X}M_1.$$

We may continue by descending induction on m to conclude that there is a coherent submodule M_{v+1} of $\mathcal{M}$ such that $\mathcal{M} = \overline{D}^{(-1)}_{F,X}M_{v+1} = \mathcal{O}_{F,X}M_{v+1}$. This completes the proof of the proposition. □

Lemma 15.3.2. — *Let X be a smooth W_n-scheme, and $\mathcal{M}$ a unit (resp. lfgu $\mathscr{D}_{F,X}$) module. Then for every integer $i \leq n$, the tensor product $\mathcal{M} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^i\mathbb{Z}$ is a unit (resp. lfgu) module.*

Proof. — Clear. □

Lemma 15.3.3. — *Let X be a smooth W_n-scheme, and assume that there exists a lifting F of the absolute Frobenius to X. Let $\mathcal{M}$ be a unit $\mathscr{D}_{F,X}$-module.*

(*i*) *$\mathcal{M}$ is locally finitely generated as an $\mathcal{O}_{F,X}$-module if and only if it is so as a $\mathscr{D}_{F,X}$-module.*

(*ii*) *If $\mathcal{M}$ is an lfgu $\mathscr{D}_{F,X}$-module, then there exists an $\mathcal{O}_X$-coherent submodule $M \subset \mathcal{M}$ such that the isomorphism $\phi_{\mathcal{M}} : F^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ sends M into F^*M, and M generates $\mathcal{M}$.*

Proof. — We prove (1). The only claim which is not immediate is that an lfgu module is locally finitely generated over $\mathcal{O}_{F,X}$. For this we proceed by induction on n. The case $n = 1$ is provided by Proposition 15.3.1. In general, we have an exact sequence of lfgu $\mathscr{D}_{F,X}$-modules

$$\mathcal{M} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^{n-1}\mathbb{Z} \to \mathcal{M} \to \mathcal{M} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to 0$$

The two outer terms are finitely generated over $\mathcal{O}_{F,X}$ by induction on n. Hence $\mathcal{M}$ is also finitely generated over $\mathcal{O}_{F,X}$. This proves (1).

The proof of (2) is the same as Theorem 6.1.3. □

Proposition 15.3.4. — *Let X be smooth W_n-scheme. The locally finitely generated unit left $\mathscr{D}_{F,X}$-modules form a thick subcategory of the category of quasi-coherent left $\mathscr{D}_{F,X}$-modules, which furthermore is closed under passing to quasi-coherent subobjects and quotients.*

Proof. — To show that the lfgu $\mathscr{D}_{F,X}$-modules form a thick subcategory of the category of all left $\mathscr{D}_{F,X}$-modules we have to show that this category is stable under taking kernels, cokernels, and extensions. By Lemma 15.1.1, we only have to show that these are finitely generated, and this is obvious in the case of a cokernel or an extension. For kernels it follows by the argument of [Lyu, Thm. 2.8], using Lemma 15.3.3. The same argument shows that any unit submodule of $\mathcal{M}$ is an lfgu module.

It remains to show that any quasi-coherent left $\mathscr{D}_{F,\mathcal{X}}$-submodule of $\mathcal{M}$ is necessarily a unit module. To this end, let $\mathcal{N}$ be a $\mathscr{D}_{F,\mathcal{X}}$-submodule of $\mathcal{M}$. Since $\mathcal{N}$ is a subobject of $\mathcal{M}$, we see that its structural morphism must be injective, whence we must also have an injective map $\mathcal{N} \to U(\mathcal{N})$. On the other hand, the inclusion $\mathcal{N} \to \mathcal{M}$ factors as

$$\mathcal{N} \to U(\mathcal{N}) \to U(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}.$$

Then $U(\mathcal{N})$ *is* a unit submodule of $\mathcal{M}$, and so by what we have already observed, it must be locally finitely generated. Since $U(\mathcal{N})$ is the union of the $\mathscr{D}_{F,X}$-modules $(F^*)^{-i}(\mathcal{N})$, $i \geq 0$, we must have $(F^*)^{-i}\mathcal{N} = (F^*)^{-i-1}\mathcal{N}$ for i sufficiently large. As $(F^*)^{-1}$ is an equivalence of categories, this implies that $\mathcal{N} \xrightarrow{\sim} (F^*)^{-1}\mathcal{N}$, whence also $F^*(\mathcal{N}) \xrightarrow{\sim} \mathcal{N}$. □

Lemma 15.3.5. — *Let X be a smooth W_n-scheme and $\mathcal{M}$ be a unit (resp. lfgu) $\mathscr{D}_{F,X}$-module. For every positive integer $i \leq n$ the groups $Tor_j^{\mathbb{Z}/p^n\mathbb{Z}}(\mathcal{M}, \mathbb{Z}/p^i\mathbb{Z})$ have a natural structure of unit (resp. lfgu) $\mathscr{D}_{F,X}$-modules.*

Proof. — Let $J^\bullet$ be a left resolution of $\mathbb{Z}/p^i\mathbb{Z}$ by finite free $\mathbb{Z}/p^n\mathbb{Z}$-modules. Then the Tor groups in (1) are computed by $(\mathcal{M}^\bullet \otimes_{\mathbb{Z}/p^n\mathbb{Z}} J^\bullet)^{Tot}$. Since this is a complex of $\mathscr{D}_{F,X}$-modules, this shows that these Tor groups have a natural structure of $\mathscr{D}_{F,X}$-modules. Since F^* is an exact functor we have

$$F^*Tor_j^{\mathbb{Z}/p^n\mathbb{Z}}(\mathcal{M}, \mathbb{Z}/p^i\mathbb{Z}) \xrightarrow{\sim} Tor_j^{\mathbb{Z}/p^n\mathbb{Z}}(F^*\mathcal{M}, \mathbb{Z}/p^i\mathbb{Z}) \xrightarrow{\sim} Tor_j^{\mathbb{Z}/p^n\mathbb{Z}}(\mathcal{M}, \mathbb{Z}/p^i\mathbb{Z})$$

so the Tor groups are unit $\mathscr{D}_{F,X}$-modules.

We now show that they are locally finitely generated if $\mathcal{M}$ is. For any choice of $i \leq n$, the long exact sequence, obtained by tensoring the exact sequence

$$0 \to \mathbb{Z}/p^{n-i}\mathbb{Z} \xrightarrow{\cdot p^i} \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z} \to 0$$

through by $\mathcal{M}$ over $\mathbb{Z}/p^n\mathbb{Z}$ yields an isomorphism

$$Tor_j^{\mathbb{Z}/p^i}(\mathcal{M}, \mathbb{Z}/p^i\mathbb{Z}) \xrightarrow{\sim} Tor_{j-1}^{\mathbb{Z}/p^n}(\mathcal{M}, \mathbb{Z}/p^{n-i}\mathbb{Z}),$$

for $j > 1$. Thus we see that it suffices to consider the case $j = 1$ (and i arbitrary). The same long exact sequence shows that $Tor_1^{\mathbb{Z}/p^n\mathbb{Z}}(\mathcal{M}, \mathbb{Z}/p^i\mathbb{Z})$ is a unit submodule of $\mathcal{M} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^{n-i}\mathbb{Z}$, and hence is lfgu by Proposition 15.3.4. □

15.4. — In the notation of (14.1), we denote by $D_u^{\bullet}(\mathscr{D}_{F,X})$ (resp. $D_u^b(\mathscr{D}_{F,X})^{\circ}$) the triangulated sub-category of $D_{qc}^{\bullet}(\mathscr{D}_{F,X})$ (resp. $D_{qc}^b(\mathscr{D}_{F,X})^{\circ}$) consisting of those complexes with whose cohomology sheaves are unit $\mathscr{D}_{F,X}$-modules. We define $D_{lfgu}^{\bullet}(\mathscr{D}_{F,X})$ (resp. $D_{lfgu}^b(\mathscr{D}_{F,X})^{\circ}$) in a similar way.

The following two results, combined with Proposition 14.8.1, will often allow us to reduce questions about complexes in $D_u^{\bullet}(\mathscr{D}_{F,X})$ to the analogous results for complexes in $D_u^{\bullet}(\mathcal{O}_{F,X})$ proved in §11..

Proposition 15.4.1. — *Let $j \leq n$ be a positive integer. Then the functor*

$$D^-(\mathscr{D}_{F,X}^{(v)}) \xrightarrow{\mathcal{M}^\bullet \mapsto \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}} D^-(\mathscr{D}_{F,X\otimes_{W_n}W_j}^{(v)})$$

restricts to functors

$$-\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z} : D_u^-(\mathscr{D}_{F,X}) \xrightarrow{\mathcal{M}^\bullet \mapsto \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}} D_u^-(\mathscr{D}_{F,X})$$

and

$$-\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z} : D_{lfgu}^-(\mathscr{D}_{F,X}) \xrightarrow{\mathcal{M}^\bullet \mapsto \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}} D_{lfgu}^-(\mathscr{D}_{F,X}).$$

Proof. — The only point which is not immediate is that the functors in question preserve the property of having unit (resp. lfgu) cohomology sheaves. To prove this we reduce by a standard spectral sequence argument to the case of a single unit (resp. lfgu) $\mathscr{D}_{F,X}$-module. The result then follows from Lemma 15.3.5. □

Lemma 15.4.2. — *Let $f : Y \to X$ be a map of smooth k-schemes. (So we put ourselves in the situation where $n = 1$.) The functors f_+, $f^!$ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ restrict to functors*

$$f^! : D_u(\mathscr{D}_{F,X}) \to D_u(\mathscr{D}_{F,Y})$$
$$f_+ : D_u(\mathscr{D}_{F,Y}) \to D_u(\mathscr{D}_{F,X})$$

and

$$\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} : D_u^-(\mathscr{D}_{F,X}) \times D_u^-(\mathscr{D}_{F,X}) \to D_u^-(\mathscr{D}_{F,X}).$$

The analogous results hold for complexes with lfgu cohomology sheaves.

Proof. — The proof will be given in the course of the proof of the following proposition. □

Proposition 15.4.3. — *Let X be a smooth k-scheme (i.e. $n = 1$). There is an equivalence of triangulated categories*

$$D_u^-(\mathscr{D}_{F,X}) \xrightarrow{\sim} D_u^-(\mathcal{O}_{F,X}),$$

given by viewing a complex of $\mathscr{D}_{F,X}$-modules as a complex of $\mathcal{O}_{F,X}$-modules. A quasi-inverse is given by the functor

$$D_u^-(\mathcal{O}_{F,X}) \xrightarrow{\mathcal{M}^\bullet \mapsto \mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}^\bullet} D_u^-(\mathscr{D}_{F,X}).$$

This equivalence respects the property of a complex having lfgu cohomology sheaves, and is compatible with the functors f_+, $f^!$ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$, where f is any map of smooth k-schemes.

Proof. — We begin by checking that the two functors are quasi-inverse. Let $\mathcal{N}^\bullet$ be in $D_u^-(\mathscr{D}_{F,X})$, and $\mathcal{M}^\bullet$ be in $D_u^-(\mathcal{O}_{F,X})$. We have to check that the natural map of complexes of $\mathscr{D}_{F,X}$-modules

$$\mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{N}^\bullet \to \mathcal{N}^\bullet$$

is a quasi-isomorphism, and that the natural map of complexes of $\mathcal{O}_{F,X}$-modules

$$\mathcal{M}^\bullet \to \mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}^\bullet$$

is a quasi-isomorphism. Since both claims can be checked on underlying $\mathcal{O}_{F,X}$-modules, it is enough to check the second claim. By dévissage, we can reduce to the case where $\mathcal{M}^\bullet$ is a single unit (resp. lfgu) module concentrated in degree 0. In this case the required result is contained in Proposition 15.2.3.

That the equivalence respects the property of a complex having lfgu cohomology sheaves follows from Lemma 15.3.1. Now Proposition 14.7.1 implies that the forgetful functor from $D_u^-(\mathscr{D}_{F,X})$ to $D_u^-(\mathcal{O}_{F,X})$ is compatible with $f^!$, and that the functor $\mathcal{M}^\bullet \mapsto \mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}^\bullet$ from complexes of $\mathcal{O}_{F,X}$-modules to complexes of $\mathscr{D}_{F,X}$-modules is compatible with the functor f_+. The equivalence of categories we have just proved then yields the first two claims in Lemma 15.4.2 as consequences of the analogous claims for $\mathcal{O}_{F,X}$-modules, which are proved in Theorem,5.8 (resp. Proposition 6.7 and Corollary 6.8.4] for complexes with lfgu cohomology sheaves). (Note that although at first it only implies them for complexes bounded above, since both $f^!$ and f_+ are of finite cohomological dimension, it then follows for complexes that might be unbounded above.) To see the final claim in Lemma 15.4.2, and the compatibility of our equivalence with $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$, we reduce as above to the analogous statement for $\mathcal{O}_{F,X}$-modules by using (14.7.2). The analogue of this claim for $\mathcal{O}_{F,X}$-complexes with unit (resp. lfgu) cohomology sheaves is given in Corollary 5.5.2 (resp. Corollary 6.4.1). □

15.5. — In this subsection, we deduce the analogues for unit $\mathscr{D}_{F,X}$-modules of some results of §5. The proofs are by reduction to the case of unit $\mathcal{O}_{F,X}$-modules, using the results of (15.4).

Proposition 15.5.1. — *Let $f : Y \to X$ be a map of smooth W_n-schemes. The functors f_+, $f^!$ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ restrict to functors*

$$f^! : D_u(\mathscr{D}_{F,X}) \to D_u(\mathscr{D}_{F,Y})$$
$$f_+ : D_u(\mathscr{D}_{F,Y}) \to D_u(\mathscr{D}_{F,X})$$

and

$$\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} : D^-_u(\mathscr{D}_{F,X}) \times D^-_u(\mathscr{D}_{F,X}) \to D^-_u(\mathscr{D}_{F,X}).$$

The analogous results hold for complexes with lfgu cohomology sheaves.

Proof. — Since the functors $f^!$ and f_+ are both of finite cohomological amplitude, a standard spectral sequence argument shows that is suffices to check this for complexes bounded above. Then we use Propositions 14.8.1 and 15.4.1 to reduce to the case $n = 1$. In this case the result follows from Proposition 15.4.2.

Above we have implicitly used (via Proposition 15.4.2) the second commutative diagram of Proposition 14.7.1, for which we gave a proof only in the case where f is allowable. One can reduce to this case as follows: let $\mathcal{M}^\bullet$ be in $D^-_u(\mathscr{D}_{F,Y})$. We show by induction on the dimension of the support Z of the cohomology sheaves of $\mathcal{M}^\bullet$ that $f_+\mathcal{M}^\bullet$ is in $D^-_u(\mathscr{D}_{F,X})$. Let $U \subset X$ be a dense open affine subscheme, such that $U \cap Z$ is dense in Z, and let j denote the inclusion of U in X. If $\mathcal{M}^\bullet$ is in $D^-(\mathscr{D}_{F,Y})$, then the cone of the natural morphism $\mathcal{M}^\bullet \to j_*\mathcal{M}^\bullet|_U$ is supported on $X\backslash U \cap Z$. Hence by induction, it is enough to prove that $f_+j_*\mathcal{M}^\bullet|_U$ is in $D^-_u(\mathscr{D}_{F,X})$. However Corollary 14.3.12 shows that this complex is quasi-isomorphic to $(f \circ j)_+\mathcal{M}^\bullet|_U$. Thus, we may replace X by U, in which case $X \to Y$ is allowable. □

15.5.2. — Let X be a smooth W_n-scheme, and $Y \subset X$ a closed subscheme. As in 5.9.2, we denote by $D_{u,Y}(\mathscr{D}_{F,X})$ the full triangulated subcategory of $D^-_u(\mathscr{D}_{F,X})$ consisting of complexes whose cohomology sheaves have support contained in Y.

Proposition 15.5.3. — *Let $f : Y \to X$ be a closed immersion of smooth W_n-schemes. The functor*

$$f_+ : D_u(\mathscr{D}_{F,Y}) \to D_u(\mathscr{D}_{F,X})$$

is fully faithful, and induces an equivalence of categories between $D^-_u(\mathscr{D}_{F,Y})$ and $D^-_{u,Y}(\mathscr{D}_{F,X})$. A quasi-inverse is given by the functor $f^!$.

Proof. — If $\mathcal{M}^\bullet$ is in $D^-_u(\mathscr{D}_{F,Y})$, then it is clear that $f_+\mathcal{M}^\bullet$ is in $D^-_{u,Y}(\mathscr{D}_{F,X})$ (it is in $D^-_u(\mathscr{D}_{F,X})$ by Proposition 15.5.1). Corollary 14.5.15 yields an adjunction morphism $\mathcal{M}^\bullet \to f^!f_+\mathcal{M}^\bullet$. Conversely, if $\mathcal{N}^\bullet$ is in $D^-_{u,Y}(\mathscr{D}_{F,X})$, then Corollary 14.5.15 yields a morphism $f_+f^!\mathcal{N}^\bullet \to \mathcal{N}^\bullet$. We will show that these two morphisms are isomorphisms. Since f_+ and $f^!$ are both of finite cohomological amplitude, it suffices to check this for complexes bounded above. Then we use Propositions 15.4.1 and 14.8.1 to reduce to the case $n = 1$. Proposition 15.4.3 then reduces us to showing the analogous result for complexes of $\mathcal{O}_{F,X}$-modules, which is given by Corollary 5.11.3. □

Corollary 15.5.4. — *Let X be a smooth W_n-scheme, and Y a closed subscheme. If $\mathcal{M}$ is in $D_u(\mathscr{D}_{F,X})$ then the complex $R\Gamma_Y(\mathcal{M}^\bullet)$ has unit cohomology sheaves. If Y is a smooth subscheme of X, then there is a canonical isomorphism*

$$R\Gamma_Y(\mathcal{M}^\bullet) \xrightarrow{\sim} f_+ f^! \mathcal{M}^\bullet.$$

In particular if j denotes the inclusion of $X \backslash Y$ in X, then we have an exact triangle

$$f_+ f^! \mathcal{M}^\bullet \to \mathcal{M}^\bullet \to j_+ j^! \mathcal{M}^\bullet \to f_+ f^! \mathcal{M}^\bullet[1]$$

Proof. — The proof of the first two claims is the same as that of Proposition 5.11.5, using Proposition 15.5.3 in place of Corollary 5.11.3. The final claim follows from the stated isomorphism. □

16. THE RIEMANN-HILBERT CORRESPONDENCE FOR UNIT $\mathscr{D}_{F,X}$-MODULES

16.1. — In this section we prove the Riemann-Hilbert correspondence alluded to in the title of the paper.

16.1.1. — We will begin by working in the étale site. We have the obvious analogue of all the theory developed in the last three sections, but with the Zariski topology replace by the étale topology $X_{\acute{e}t}$. In fact much of the étale variant of the theory follows from the Zariski version by étale descent. We refer to §7 for a more complete discussion of the analogous situation in the case of complexes of $\mathcal{O}_{F,X}$-modules. In particular we have functors

$$\pi_X^* : D_{qc}(\mathscr{D}_{F,X}) \to D_{qc}(\mathscr{D}_{F,X_{\acute{e}t}})$$

and

$$\pi_{X*} : D_{qc}(\mathscr{D}_{F,X_{\acute{e}t}}) \to D_{qc}(\mathscr{D}_{F,X}),$$

and these are equivalences of categories, each a quasi-inverse of the other. As the notation suggests, the functor π_{X*} is induced by an exact functor π_{X*} on the category of $\mathscr{D}_{F,X_{\acute{e}t}}$-modules. This is explained in 7.3.2, and uses the fact that $R\pi_{X*}$ has finite cohomological amplitude.

16.1.2. — We begin by defining the relevant functors from $D^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}})^\circ$ to the derived category of étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules. For technical reasons, it will be convenient to define a sequence of functors: for each positive integer $j \leq n$, and $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}})^\circ$, we write

$$\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet) = \underline{RHom}_{\mathscr{D}_{F,X_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{O}_{X_{\acute{e}t}} \otimes_{W_n} W_j)[d_X].$$

Replacing $\mathcal{M}^\bullet$ by a complex of locally free $\mathscr{D}_{F,X}$-modules, we see immediately that there is a canonical isomorphism

$$\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet), \tag{16.1.3}$$

where $\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}$ is regarded as an object of $D^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}\otimes_{W_n}W_j})^\circ$. Moreover, we have an exact triangle

$$\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet) \to \mathrm{Sol}_{\acute{e}t}{}^n(\mathcal{M}^\bullet) \to \mathrm{Sol}_{\acute{e}t}{}^{n-j}(\mathcal{M}^\bullet) \to \mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet)[1] \tag{16.1.4}$$

Sometimes we will simply write $\mathrm{Sol}_{\acute{e}t}$ in place of $\mathrm{Sol}_{\acute{e}t}{}^n$.

16.1.5. — Suppose that X is a smooth k-scheme (i.e. $n = 1$). Then the functor $\mathrm{Sol}_{\acute{e}t}$ is compatible with the functor $\mathrm{Sol}_{\acute{e}t}$ for complexes in $D^b_{lfgu}(\mathcal{O}_{F,X_{\acute{e}t}})$-modules, defined in §9, via the equivalence of categories given by Proposition 15.4.3: if $\mathcal{M}^\bullet$ is a complex in $D^b_{lfgu}(\mathcal{O}_{F,X_{\acute{e}t}})^\circ$, then we have a canonical isomorphism of complexes of étale sheaves

$$\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(\mathscr{D}_{F,X_{\acute{e}t}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X_{\acute{e}t}}} \mathcal{M}^\bullet),$$

as follows immediately from the definitions.

16.1.6. — We denote by $D^b(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ the bounded derived category of $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on $X_{\acute{e}t}$. We denote by $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ the full triangulated sub-category of $D^b(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ consisting of complexes having constructible cohomology sheaves and finite Tor dimension. A result of Deligne [**De**, p. 93] asserts that any such complex is represented by a finite length complex of flat, constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves.

Proposition 16.1.7. — *The functor* $\mathrm{Sol}_{\acute{e}t}{}^j$ *induces a functor*

$$\mathrm{Sol}_{\acute{e}t}{}^j : D^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}})^\circ \to D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^j\mathbb{Z}).$$

Proof. — Let $\mathcal{M}^\bullet$ be in $D^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}})^\circ$. We first show that $\mathrm{Sol}_{\acute{e}t}{}^j$ is bounded, with constructible cohomology sheaves. A dévissage argument using (16.1.3) and (16.1.4) reduces us to the case $j = n = 1$, and (16.1.5) then reduces us to the analogous statement for $\mathcal{O}_{F,X_{\acute{e}t}}$-modules. This is given by Proposition 9.8. It remains to show that $\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet)$ is of finite Tor dimension over $\mathbb{Z}/p^j\mathbb{Z}$. This is so if and only if $\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^j\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is a bounded complex. However, we have an isomorphism

$$\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^j\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}{}^1(\mathcal{M}^\bullet)$$

(as can be see by taking a resolution of $\mathbb{Z}/p\mathbb{Z}$ by flat $\mathbb{Z}/p^j\mathbb{Z}$-modules), and so the complex on the left is bounded, by what we have already proved. □

Proposition 16.1.8. — *Let $f : Y \to X$ be a map of smooth W_n-schemes, and j a positive integer $\leq n$. If $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}})^\circ$ there is a canonical isomorphism*

$$f^{-1}\mathrm{Sol}_{\acute{e}t}{}^j(\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}(f^!(\mathcal{M}^\bullet)).$$

Proof. — First note that if $\mathcal{M}^\bullet$ is in $D^b(\mathscr{D}_{F,X_{\acute{e}t}})$ and $\mathcal{N}^\bullet$ is in $D^b(\mathscr{D}_{F,X_{\acute{e}t}})$, then we have a natural morphism

$$f^{-1}\underline{RHom}_{\mathscr{D}_{F,X_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{RHom}_{\mathscr{D}_{F,Y_{\acute{e}t}}}(f^!\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet).$$

This can be obtained by replacing $\mathcal{N}^\bullet$ by a bounded below complex of injective $\mathscr{D}_{F,X_{\acute{e}t}}$-modules, and taking a resolution of $\mathscr{D}_{F,Y_{\acute{e}t}\to X_{\acute{e}t}}$ by $(\mathscr{D}_{F,Y_{\acute{e}t}}, f^{-1}\mathscr{D}_{F,X_{\acute{e}t}})$-bimodules, which are flat as right $f^{-1}\mathscr{D}_{F,X_{ét}}$-modules (compare Proposition 2.6). Applying this to an arbitrary object $\mathcal{M}^\bullet$ of $D^b_{lfgu}(\mathscr{D}_{F,X_{ét}})^\circ$ and the particular object $\mathcal{N}^\bullet = \mathcal{O}_{X_{ét}} \otimes_{W_n} W_j[d_X]$ yields a morphism

$$f^{-1}\mathrm{Sol}_{ét}{}^j(\mathcal{M}^\bullet) \to \mathrm{Sol}_{ét}{}^j(f^!\mathcal{M}^\bullet).$$

We have to show that this is an isomorphism. Using (16.1.3) and (16.1.4), together with a dévissage argument, we reduce to the case $j = n = 1$. Then (16.1.5) and Proposition 15.4.3 reduce us to the analogous statement for $\mathcal{O}_{F,X_{ét}}$-modules, which is provided by Proposition 9.3. □

Proposition 16.1.9. — *Suppose that $f : Y \to X$ is an allowable morphism (i.e. can be factored as a composite of an immersion and a smooth proper maps between smooth W_n-schemes), and that $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathscr{D}_{F,X_{ét}})^\circ$. Then there is a canonical isomorphism*

$$f_!\mathrm{Sol}_{ét}{}^j(\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{ét}{}^j(f_+\mathcal{M}^\bullet).$$

Proof. — To construct a morphism of the required sort, it suffices to construct it when f is either a proper map or an open immersion. In the former case, we get immediately an isomorphism of the required kind by applying Theorem 14.5.12. In the latter case, a simple argument using the map constructed at the beginning of the proof of Proposition 16.1.8 (which is an isomorphism in this case) and the isomorphism $f^!f_+\mathcal{M}^\bullet \xrightarrow{\sim} \mathcal{M}^\bullet$, shows that there is an isomorphism

$$f^{-1}\mathrm{Sol}_{ét}{}^j(f_+\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{ét}{}^j(\mathcal{M}^\bullet)$$

(compare with the proof of Proposition 9.5). By adjointness, the inverse of this morphism gives a morphism

$$f_!\mathrm{Sol}_{ét}{}^j(\mathcal{M}^\bullet) \to \mathrm{Sol}_{ét}{}^j(f_+\mathcal{M}^\bullet),$$

which we have to show is an isomorphism. By the same dévissage argument that we used in the proof of Proposition 16.1.8, we reduce to the case $j = n = 1$, and then, by Proposition 15.4.3, to the analogous statement for complexes of $\mathcal{O}_{F,X_{ét}}$ modules. This is given by Proposition 9.5.

We have now constructed an isomorphism of the required type, depending on the chosen factorisation of f as a composite of open immersions and proper maps. That our morphism is independent of this factorisation is proved as in §9.7. □

Proposition 16.1.10. — *Let $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ be in $D^b_{lfgu}(\mathscr{D}_{F,X_{ét}})^\circ$. There is a canonical isomorphism*

$$\mathrm{Sol}_{ét}{}^j(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^j\mathbb{Z}} \mathrm{Sol}_{ét}{}^j(\mathcal{N}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{ét}{}^j(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{X_{ét}}} \mathcal{N}^\bullet)[d_X]$$

Proof. — The construction of a natural morphism from the left hand side to the right hand side is easy and formally identical to that in Proposition 9.9. To prove that this is an isomorphism we reduce, as above, to the case $j = n = 1$ (using dévissage on both $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$), and then to the analogous statement for complexes of $\mathcal{O}_{F,X_{ét}}$-modules, which is provided by Proposition 9.9. □

16.2. — In this section we define the functor which will be quasi-inverse to $\mathrm{Sol}_{ét}{}^j$.

16.2.1. — For $\mathcal{L}^\bullet$ in $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$, and j a positive integer $\leq n$, we set

$$\mathrm{M}_{\acute{e}t}{}^j(\mathcal{L}^\bullet) = \underline{RHom}_{\mathbb{Z}/p^n\mathbb{Z}}(\mathcal{L}^\bullet, \mathcal{O}_{X_{\acute{e}t}} \otimes_{W_n} W_j)[d_X].$$

$\mathrm{M}_{\acute{e}t}{}^j$ can be computed by taking an injective resolution of $\mathcal{O}_{X_{\acute{e}t}} \otimes_{W_n} W_j$ by $\mathscr{D}_{F,X_{\acute{e}t}}$-modules, and therefore has the structure of a complex in $D^+(\mathscr{D}_{F,X_{\acute{e}t}})$. (Since $\mathscr{D}_{F,X_{\acute{e}t}}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$, any injective $\mathscr{D}_{F,X_{\acute{e}t}}$-module is also injective as a sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules.) Thus we obtain a functor

$$\mathrm{M}_{\acute{e}t}{}^j : D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z}) \to D^+(\mathscr{D}_{F,X_{\acute{e}t}}).$$

Just as for $\mathrm{Sol}_{\acute{e}t}{}^j$, we have a distinguished triangle

$$\mathrm{M}_{\acute{e}t}{}^j(\mathcal{L}^\bullet) \to \mathrm{M}_{\acute{e}t}{}^n(\mathcal{L}^\bullet) \to \mathrm{M}_{\acute{e}t}{}^{n-j}(\mathcal{L}^\bullet) \to \mathrm{M}_{\acute{e}t}{}^j(\mathcal{L}^\bullet)[1] \tag{16.2.2}$$

and an isomorphism

$$\mathrm{M}_{\acute{e}t}{}^j(\mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}) \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}{}^j(\mathcal{L}^\bullet), \tag{16.2.3}$$

where we regard $\mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}$ as belonging to $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^j\mathbb{Z})$.

16.2.4. — If $j = n = 1$, then the functor $\mathrm{M}_{\acute{e}t} = \mathrm{M}_{\acute{e}t}{}^1$ is compatible with the functor $\mathrm{M}_{\acute{e}t}$ defined in §10, via the isomorphism of Theorem 15.2.4. Indeed, this follows from the fact already noted, that if $\mathcal{M}$ is an injective $\mathscr{D}_{F,X_{\acute{e}t}}$-module, then it is in particular injective as an $\mathbb{Z}/p^n\mathbb{Z}$-module, and so can be used to compute the functor $\underline{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(-, \mathcal{M})$, now thought of as taking values in the derived category of complexes of $\mathcal{O}_{F,X_{\acute{e}t}}$-module.

Proposition 16.2.5. — *Let j be a positive integer $\leq n$. For $\mathcal{M}^\bullet$ in $\mathscr{D}^b_{lfgu}(\mathscr{D}_{F,X_{\acute{e}t}})$, there is a natural isomorphism*

$$\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z} \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}{}^j(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet))$$

For $\mathcal{L}^\bullet$ in $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ there is a natural isomorphism

$$\mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z} \xrightarrow{\sim} \mathrm{Sol}_{\acute{e}t}{}^j(\mathrm{M}_{\acute{e}t}(\mathcal{L}^\bullet)).$$

In particular the functors $\mathrm{M}_{\acute{e}t}$ and $\mathrm{Sol}_{\acute{e}t}$ are quasi-inverse anti-equivalences of categories.

Proof. — It is straightforward to construct the morphisms of the proposition (compare §11.1). We have to show that they are isomorphisms. To show that the first morphism is an isomorphism, we begin by applying the usual dévissage arguments which reduce us to the case $j = 1$. Then, since

$$\begin{aligned}\mathrm{M}_{\acute{e}t}{}^1(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet)) &\xrightarrow{\sim} \mathrm{M}_{\acute{e}t}{}^1(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^j\mathbb{Z}) \\ &\xrightarrow{\sim} \mathrm{M}_{\acute{e}t}{}^1(\mathrm{Sol}_{\acute{e}t}{}^1(\mathcal{M}^\bullet)) \xrightarrow{\sim} \mathrm{M}_{\acute{e}t}{}^1(\mathrm{Sol}_{\acute{e}t}(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})),\end{aligned}$$

we are reduced to the case $j = n = 1$, and thus to the analogous statement for complexes of $\mathcal{O}_{F,X}$-modules, which is provided by Theorem 11.3. The proof that the second morphism is an isomorphism is formally identical. □

Corollary 16.2.6. — *The functors* $\mathrm{M}_{ét}$ *and* $\mathrm{Sol}_{ét}$ *are quasi-inverse anti-equivalences of categories between* $D^b_{lfgu}(\mathscr{D}_{F,X_{ét}})^\circ$ *and* $D^b_{ctf}(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$, *and satisfy the following properties*

(*i*) $\mathrm{M}_{ét}$ *and* $\mathrm{Sol}_{ét}$ *interchange the functors* $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{X_{ét}}}$ *and* $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}}$ *(up to a shift).*

(*ii*) *If* $f : Y \to X$ *is a morphism of smooth* W_n*-schemes, then* $\mathrm{M}_{ét}$ *and* $\mathrm{Sol}_{ét}$ *interchange* $f^!$ *and* f^{-1}.

(*iii*) *If* f *is an allowable morphism then* $\mathrm{M}_{ét}$ *and* $\mathrm{Sol}_{ét}$ *interchange* f_+ *and* $f_!$

Proof. — This follows from Propositions 16.2.5, 16.1.10, 16.1.8 and 16.1.9. □

16.2.7. — To complete the construction of our Riemann-Hilbert correspondence, we descend the results above to the Zariski topology. More precisely we define a functor

$$\mathrm{Sol} : D^b_{lfgu}(\mathscr{D}_{F,X})^\circ \to D^b(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$$

by

$$\mathrm{Sol}(\mathcal{M}^\bullet) = \mathrm{Sol}_{ét}(\pi_X^* \mathcal{M}^\bullet)$$

and a functor

$$\mathrm{M} : D^b(X_{ét}, \mathbb{Z}/p^n\mathbb{Z}) \to D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$$

by

$$\mathrm{M}(\mathcal{L}^\bullet) = \pi_{X*}\mathrm{M}_{ét}(\mathcal{M}^\bullet).$$

Using Corollary 16.2.6 and the observations of (16.1.1), we immediately obtain the following result.

Corollary 16.2.8. — *The functors* Sol *and* M *are quasi-inverse anti-equivalences of categories. They exchange* $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ *and* $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}}$ *(up to a shift), and also* $f^!$ *and* f^{-1}, *for any smooth morphism of* W_n*-schemes. If furthermore* f *is an allowable morphism, then they exchange* f_+ *and* $f_!$.

17. AN EQUIVALENCE OF DERIVED CATEGORIES

17.1. — Let X be a smooth W_n-scheme. The goal of this section is to prove that the natural functor $D^b(\mu_{lfgu}(\mathscr{D}_{F,X}))^\circ \to D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ is an equivalence of categories. The Riemann-Hilbert correspondence of the previous section is a crucial ingredient in the proof.

Let $*$ be one of $\emptyset$, qc, u, or $lfgu$. For any abelian subcategory $\mathcal{A}$ of the category of $\mathscr{D}_{F,X}$-modules, we denote by $K^b_*(\mathcal{A})$ the full subcategory of the bounded homotopy category $K^b(\mathcal{A})$, consisting of complexes whose cohomology sheaves satisfy the condition $*$. We denote by $D^b_*(\mathcal{A})$ the corresponding derived category. We denote by $D^b_*(\mathcal{A})^\circ$ the full subcategory of $D^b_*(\mathcal{A})$ whose complexes have finite Tor dimension over $\mathcal{O}_X$.

With $*$ as above, we denote by μ_* the category of $\mathscr{D}_{F,X}$-modules satisfying the condition $*$.

We denote by $\mathrm{Ind} - \mu_{lfgu}$ the category of $\mathscr{D}_{F,X}$-modules which are direct limits of objects in μ_{lfgu}.

Proposition 17.1.1. — *Let $E^\bullet$ denote the residual complex of injective quasi-coherent $\mathcal{O}_{X_{ét}}$-modules resolving $\mathcal{O}_{X_{ét}}$. Then $E^\bullet$ is in a natural way a complex of unit $\mathscr{D}_{F,X}$-modules resolving the $\mathscr{D}_{F,X}$-module $\mathcal{O}_X$. The terms of this complex are in* $\mathrm{Ind} - \mu_{lfgu}$.

Proof. — Since the residual complex on the étale site is obtained by pulling back the residual complex on the Zariski site, it suffices to prove the analogous fact on the Zariski site. Thus in the course of the proof we let $E^\bullet$ denote the residual complex resolving $\mathcal{O}_X$ on the Zariski site of X, rather than the étale site.

Let $\mathcal{P}^r_X(1)$ be as in (14.4.4). We denote by p^r_1 and p^r_2 the two projections from $\mathcal{P}^r_X(1)$ to X. Both these maps are finite and flat and, using local co-ordinates, one checks easily that their fibres are complete intersections. It follows by [**Ha 1**, VI 5.3], that $p^{r*}_1 E^\bullet$ and $p^{r*}_2 E^\bullet$ are residual complexes on $\mathcal{P}^r_X(1)$, and they resolve $p^{r*}_1\mathcal{O}_X$ and $p^{r*}_2\mathcal{O}_X$ respectively. Thus, by [**Ha 1**, VI 1.1], the isomorphism of pointwise dualising complexes

$$p^{r*}_1\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_{\mathcal{P}^r_X(1)} \xrightarrow{\sim} p^{r*}_2\mathcal{O}_X$$

induces a unique isomorphism of residual complexes $p_1^{r*}E^\bullet \xrightarrow{\sim} p_2^{r*}E^\bullet$, compatible with the augmentation map $\mathcal{O}_X \to E^\bullet$. This collection of isomorphisms for $r \geq 1$, is compatible with pullback via the inclusions $\mathcal{P}_X^r(1) \to \mathcal{P}_X^{r+1}(1)$, and gives $E^\bullet$ the structure of a complex of $\mathscr{D}_X$-modules resolving the $\mathscr{D}_X$-module $\mathcal{O}_X$.

A similar argument shows that we have an isomorphism of $\mathcal{O}_X$-modules $F^*E^\bullet \to E^\bullet$ (compare with the proof of Theoremm 4.4.9). To check that this is a map of $\mathscr{D}_X$-modules we have to check that the diagram

$$\begin{array}{ccccccc} p_1^{r*}F^*E^\bullet & \xrightarrow{\sim} & F^*p_1^{r*}E^\bullet & \xrightarrow{\sim} & F^*p_2^{r*}E^\bullet & \xrightarrow{\sim} & p_2^{r*}F^*E^\bullet \\ \downarrow & & & & & & \downarrow \\ p_1^{r*}E^\bullet & & & \xrightarrow{\sim} & & & p_2^{r*}E^\bullet \end{array}$$

commutes. This follows from [**Ha 1**, VI 1.1] and the fact that the corresponding diagram, with $E^\bullet$ replaced by $\mathcal{O}_X$, commutes, since $\mathcal{O}_X$ is a $\mathscr{D}_{F,X}$-module.

We have proved that $E^\bullet$ is a complex of $\mathscr{D}_{F,X}$-modules resolving $\mathcal{O}_X$. To check that its terms are in $\mathrm{Ind} - \mu_{lfgu}$, we remark that by definition [**Ha 1**, IV, §3], the terms of $E^\bullet$ are (infinite) sums of terms of the form $\underline{H}_Y^i(\mathcal{O}_X)$, where Y is a closed subvariety of X. It is not hard to check that the $\mathscr{D}_{F,X}$-module structure on the terms of $E^\bullet$ defined above, is compatible with the $\mathscr{D}_{F,X}$-module structure on $\underline{H}_Y^i(\mathcal{O}_X)$ mentioned in Corollary 15.5.4, the point being that both arise by functoriality from the $\mathscr{D}_{F,X}$-module structure on $\mathcal{O}_X$. It follows by 15.5.4 that the terms of $E^\bullet$ are in $\mathrm{Ind} - \mu_{lfgu}$. □

Corollary 17.1.2. — *The functor $D^b(\mu_{lfgu})^\circ \to D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ is essentially surjective. If $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are in $D^b(\mu_{lfgu})^\circ$ the natural map*

$$\mathrm{Hom}_{D^b(\mu_{lfgu})}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \mathrm{Hom}_{D^b(\mathscr{D}_{F,X})}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$$

is surjective.

Proof. — Let $\mathcal{C}_X$ denote the category of constructible étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules on X, and let $D^b_{tf}(\mathcal{C}_X)$ be the full subcategory of $D^b(\mathcal{C}_X)$ consisting of complexes with finite Tor dimension over $\mathbb{Z}/p^n\mathbb{Z}$. The residual complex $E^\bullet$ of Proposition 17.1.1 is a complex of injective sheaves of $\mathcal{O}_X$-modules, and hence a complex of injective sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules. It follows that the functor M may be computed as $\pi_{X*}\underline{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(-, E^\bullet)$ and we have a commutative diagram

$$\begin{array}{ccc} D^b_{tf}(\mathcal{C}_X) & \longrightarrow & D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z}) \\ \downarrow{\scriptstyle \pi_{X*}\underline{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(-,E^\bullet)} & & \downarrow{\scriptstyle \mathrm{M}} \\ D^b_{lfgu}(\mathrm{Ind} - \mu_{lfgu})^\circ & \longrightarrow & D^b_{lfgu}(\mathscr{D}_{F,X})^\circ \end{array}$$

Using [**De**, p. 94], one easily checks that the top functor is essentially surjective, and induces a surjection on Hom's. (In fact it is even an equivalence - compare with §17.2 below). The right vertical functor is an equivalence by Corollary 16.2.8. The corollary now follows because the natural map $D^b(\mu_{lfgu}) \to D^b_{lfgu}(\mathrm{Ind} - \mu_{lfgu})$ is an equivalence of categories. This last claim is an easy exercise (compare with the proof of Lemma 17.2.3 (i) below). □

17.2. — To prove that the functor of (17.1) is an equivalence it remains to prove that it is faithful. For this we need some preparation.

Lemma 17.2.1. — *Let $\mathcal{M}$ be a unit $\mathscr{D}_{F,X}$-module.*

(*i*) *There exists a unique maximal subobject $L(\mathcal{M}) \subset \mathcal{M}$ which lies in* $\mathrm{Ind} - \mu_{lfgu}$.
(*ii*) *The assignment $\mathcal{M} \mapsto L(\mathcal{M})$ is functorial, and the functor L is left exact, and right adjoint to the inclusion* $\mathrm{Ind} - \mu_{lfgu} \to \mu_u$.

Proof. — If $\mathcal{M}_1, \mathcal{M}_2$ are two submodules of $\mathcal{M}$, which are in $\mathrm{Ind} - \mu_{lfgu}$, then $\mathcal{M}_1 + \mathcal{M}_2$ is unit, being the image of the map $\mathcal{M}_1 \oplus \mathcal{M}_2 \to \mathcal{M}$, and is locally finitely generated. Now (i) follows by an easy application of Zorn's lemma. The functoriality in (ii) also follows from the fact that the image of a map of unit modules in unit. The left exactness and right adjointness are clear. □

Recall from Corollary 15.1.6 that the category μ_u has enough injectives. Thus we may form the right derived functor of L, and we obtain a functor

$$RL : D^b(\mu_u) \to D^b(\mathrm{Ind} - \mu_{lfgu}).$$

The following lemma is crucial to proving the full faithfulness of the functor in Corollary 17.1.2.

Lemma 17.2.2. — *Objects in μ_{lfgu} are acyclic for L.*

Proof. — Let $\mathcal{M}$ be in μ_{lfgu}. Choose a resolution $\mathcal{M} \to \mathcal{M}^\bullet$ by injectives in μ_u. For $n \geq 0$, let $\mathcal{I}^n$ denote the image of the differential $\mathcal{M}^n \to \mathcal{M}^{n+1}$. To check that $\mathcal{M}$ is L-acyclic it suffices to check that the map $L(\mathcal{M}^n) \to L(I^n)$ is surjective.

Let $\mathcal{J} \subset \mathcal{I}^n$ be an lfgu $\mathscr{D}_{F,X}$-submodule. We denote by $\mathcal{M}^{\leq n}$ the brutal truncation. The complex $\mathcal{M}^{\leq n}$ has trivial cohomology outside degrees 0 and n, and corresponds to an $n+1$-extension of $\mathcal{I}^n$ by $\mathcal{M}$. Let $c \in \mathrm{Ext}^{n+1}_{\mu_u}(\mathcal{I}, \mathcal{M})$ be the class of this extension, and let $c_{\mathcal{J}}$ denote the image of c in $\mathrm{Ext}^{n+1}_{\mu_u}(\mathcal{J}, \mathcal{M})$. Since

$$\mathrm{Ext}^{n+1}_{\mu_u}(\mathcal{J}, \mathcal{M}) = \mathrm{Hom}_{D^b(\mu_u)}(\mathcal{J}[-n], \mathcal{M})$$

corollary 17.1.2 implies that there is an $n+1$-extension of lfgu $\mathscr{D}_{F,X}$ modules, corresponding to $c_{\mathcal{J}}$. This is a complex of lfgu $\mathscr{D}_{F,X}$-module $\mathcal{N}^\bullet$ whose terms are 0 in degrees outside $[0, n]$ and such that $H^i(\mathcal{N}^\bullet) = \mathcal{M}$ if $i = 0$, $\mathcal{J}$ if $i = n$, and is 0 otherwise. Since $\mathcal{M}^\bullet$ is a complex of injectives there exists a map of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}^{\leq n} & \longrightarrow & \mathcal{I}^n & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \theta^\bullet} & & \uparrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N}^\bullet & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \end{array}$$

Now consider the exact sequence

$$\mathrm{Hom}_{\mu_u}(\mathcal{J}, \mathcal{M}^n) \to \mathrm{Hom}_{\mu_u}(\mathcal{J}, \mathcal{I}^n) \xrightarrow{\delta} \mathrm{Ext}^{n+1}_{\mu_u}(\mathcal{J}, \mathcal{M}) \to 0.$$

By definition of $c_{\mathcal{J}}$ we have $\delta(\theta) = c_{\mathcal{J}}$, and if $\widetilde{\theta' : \mathcal{J}} \to \mathcal{I}^n$ denotes the natural inclusion, $\delta(\theta') = c_{\mathcal{J}}$. Thus $\theta' - \theta$ lifts to a map $\widetilde{(\theta' - \theta)} : \mathcal{J} \to \mathcal{M}^n$. This induces a map $\mathcal{N}^n \to \mathcal{M}^n$ which we also denote by $\widetilde{(\theta - \theta')}$. Now $(\theta^n + \widetilde{(\theta' - \theta)})(\mathcal{N}^n)$ is an lfgu $\mathscr{D}_{F,X}$-submodule of $\mathcal{M}^n$ which surjects on $\theta'(\mathcal{J})$. This proves the lemma. □

Lemma 17.2.3. — *Let $\mu_{L-\mathcal{A}C}$ denote the full subcategory of μ_u consisting of objects which are L-acyclic. The following functors are fully faithful:*

(*i*) $D^b(\mu_{lfgu}) \to D^b(\mu_{L-\mathcal{A}C})$.
(*ii*) $D^b(\mu_{L-\mathcal{A}C}) \to D^b(\mu_u)$
(*iii*) $D^b(\mu_u) \to D^b(\mu_{qc})$.

Proof. — Each of the three morphisms comes from a morphism of corresponding homotopy categories. If $K \to K'$ is one of these latter morphisms, then it is evidently fully faithful. We will apply the criterion of [**We**, 10.3.13], which says that the corresponding morphism of derived categories is fully faithful if either of the following two conditions hold:

(a) If $\mathcal{M}^\bullet \to \mathcal{N}^\bullet$ is a quasi-isomorphism in K' with $\mathcal{M}^\bullet$ in K, then there exists an $\mathcal{M}^{\bullet\prime}$ in K and a quasi-isomorphism $\mathcal{N}^\bullet \to \mathcal{M}^{\bullet\prime}$.

(b) If $\mathcal{N}^\bullet \to \mathcal{M}^\bullet$ is a quasi-isomorphism in K' with $\mathcal{M}^\bullet$ in K, then there exists an $\mathcal{M}^{\bullet\prime}$ in K and a quasi-isomorphism $\mathcal{M}^{\bullet\prime} \to \mathcal{N}^\bullet$

For (i) we apply (b): Take $\mathcal{N}^\bullet \to \mathcal{M}^\bullet$ in $K^b_{lfgu}(\mu_{L-\mathcal{A}C})$ with $\mathcal{M}^\bullet$ in $K^b(\mu_{lfgu})$. The adjunction map $L(\mathcal{N}^\bullet) \to \mathcal{N}^\bullet$ is a quasi-isomorphism because by Lemma 17.2.2, the terms of $\mathcal{N}^\bullet$ and its cohomology sheaves are acyclic for L. Now $L(\mathcal{N}^\bullet)$ has terms in Ind $-\ \mu_{lfgu}$ and lfgu cohomology sheaves. A simple argument shows that $L(\mathcal{N}^\bullet)$ contains a subcomplex $\mathcal{M}^{\bullet\prime}$ contained in $K^b(\mu_{lfgu})$ and quasi-isomorphic to $L(\mathcal{N}^\bullet)$.

For (ii), the full faithfulness follows from (a) and the fact that any complex in $K^b(\mu_u)$ has an injective resolution.

For (iii) we also apply criterion (a). If $\mathcal{M}^\bullet \to \mathcal{N}^\bullet$ is a quasi-isomorphism in $K^b(\mu_{qc})$, with $\mathcal{M}^\bullet$ in $K^\bullet(\mu_u)$, then using the exact functor U of Corollary 15.1.5, we get an adjunction map $\mathcal{N}^\bullet \to U(\mathcal{N}^\bullet)$, with $U(\mathcal{N}^\bullet)$ in $K^\bullet(\mu_u)$. Since $U(\mathcal{N}^\bullet)$ is quasi-isomorphic to $U(\mathcal{M}^\bullet) = \mathcal{M}^\bullet$, this adjunction map is a quasi-isomorphism. □

Corollary 17.2.4. — *The functor $D^b(\mu_{lfgu}) \to D^b_{qc}(\mathscr{D}_{F,X})$ is fully faithful. In particular, the functor $D^b(\mu_{lfgu}) \to D^b_{lfgu}(\mathscr{D}_{F,X})$ is fully faithful.*

Proof. — By Lemma 17.2.3, $D^b(\mu_{lfgu}) \to D^b(\mu_{qc})$ is fully faithful. Now the corollary follows from Bernstein's theorem which says that $D^b(\mu_{qc}) \to D^b_{qc}(\mathscr{D}_{F,X})$ is an equivalence of categories. □

Corollary 17.2.5. — *The natural functor $D^b(\mu_{lfgu})^\circ \to D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ is an equivalence of categories.*

Proof. — This follows from corollaries 17.1.2 and 17.2.4. □

APPENDIX A: DUALITY AND THE CARTIER OPERATOR

A. 1. — We recall some facts about duality for quasi-coherent sheaves on schemes.Let $f : Y \longrightarrow X$ be a morphism of finite type between Noetherian schemes which admit a dualising complex (in the sense of [**Ha 1**, V], and [**Con**, 3.1]); for example f could be a morphism between schemes of finite type over a field. There exits a functor $f^! : D_c^+(X) \longrightarrow D_c^+(Y)$ with the property that if f is proper, then $f^!$ is right adjoint to Rf_* in the sense of derived categories: for any two complexes $\mathcal{E}^\bullet$ in $D_{qc}^-(Y)$ and $\mathcal{F}^\bullet$ in $D_c^+(X)$ we have a natural isomorphism

$$Rf_* \underline{RHom}^\bullet_{\mathcal{O}_Y}(\mathcal{E}^\bullet, f^!\mathcal{F}^\bullet) \xrightarrow{\sim} \underline{RHom}^\bullet_{\mathcal{O}_X}(Rf_*\mathcal{E}^\bullet, \mathcal{F}^\bullet).$$

If we take global sections and then apply H^0, we obtain an isomorphism

$$\mathrm{Hom}_{D(\mathcal{O}_Y)}(\mathcal{E}^\bullet, f^!\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{O}_X)}(Rf_*\mathcal{E}^\bullet, \mathcal{F}^\bullet).$$

This adjointness yields a morphism $Rf_* f^! \mathcal{F}^\bullet \to \mathcal{F}^\bullet$, which we denote $tr_f \mathcal{F}^\bullet$. (In fact, in the development of the theory, the key point is to first construct this morphism in the special case when $\mathcal{F}^\bullet$ is a dualising complex.) This theorem is explained in [**Ha 1**, VII] and [**Con**, 3.4], where along with many other compatibilities it is shown that if $g : Z \to Y$ is a second morphism of finite type k-schemes then there is a natural isomorphism $(fg)^! \xrightarrow{\sim} g^! f^!$, which is compatible with traces if f and g are both proper.

This theory ought to be compatible with flat base-change, in the sense that if $u : X' \to X$ is a flat morphism of finite type k-schemes, giving rise to the cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{u'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{u} & X, \end{array}$$

then there should be natural isomorphisms

$$u'^* f^! \xrightarrow{\sim} f'^! u^*.$$

If furthermore f (and hence f') is proper, the diagram of natural transformations

$$\begin{array}{ccc} f'_*f'^!u^* & \xrightarrow{tr_{f'}} & u* \\ \downarrow\wr & & \uparrow \\ f'_*u'^*f^! & \xrightarrow{\sim} & u^*f_*f^! \end{array}$$

should commute. Such a result is proved in [**Ha 1**, VI, §5] and [**Con**, 3.2], in the case that the morphism u is residually stable (which is a very strong restriction). The case of a general flat base-change is treated in [**Ve**], using however different foundations. (See the remarks in the introduction of [**Con**] on the possibility of reconciling the approach of [**Ve**] (which is due to Deligne) with that of [**Ha 1**].)

A. 1.1. — In some situations the theory described above, which is constructed via residual complexes, can be made more explicit, and extended to more general complexes $\mathcal{F}^\bullet$. Suppose that $f : Y \to X$ is a regular morphism of Noetherian schemes admitting a dualising complex, in the sense that f may be factored as the composition of of a closed immersion $i : Y \to W$ which makes Y a local complete intersection in W, and a smooth morphism $p : W \to X$. Then $f^!\mathcal{O}_X$ is equal to a line bundle on Y placed in degree $d_{X/Y}$. (Thus such a morphism f is a special instance of a Gorenstein morphism, as defined in [**Ha 1**, p. 143-4]; note that this definition is more inclusive then the definition given subsequently in [**Ha 1**, V 9.7], in which the morphism is required to be flat. The remarks of this subsection provide a special case of the program outlined by Hartshorne for such morphisms.) We denote this line bundle by $\omega_{Y/X}$ (although of course it depends on the morphism f). If f is proper then we have the trace map $tr_f : Rf_*f^!\omega_{Y/X}[d_{X/Y}] \to \mathcal{O}_X$.

Now if $\mathcal{F}^\bullet$ is any complex in $D_c^+(\mathcal{O}_X)$ there is a natural isomorphism

$$f^!F^\bullet \xrightarrow{\sim} \omega_{Y/X} \otimes \mathbb{L}f^*F^\bullet[d_{Y/X}]. \tag{A.1.2}$$

(Note that since f is regular, $\mathbb{L}f^*$ has finite Tor-dimension, so the right hand side makes sense as a functor on $D_c^+(\mathcal{O}_X)$.) In the case that the morphism f is the closed immersion of a local complete intersection, this isomorphism follows from [**Ha 1**, III 6.9(a)], together with the compatibility of $f^!$ with the explicit duality for finite maps provided by $f^\flat$, as discussed in [**Con**, 3.3]. When f is smooth, it follows from [**Con**, 3.3]. In general, we may define (A.1.2) via a factorisation of f; its independence of the choice of such a factorisation follows from the compatibilities proved in [**Con**, 2.7].

One may use the right hand side of (A.1.2) to *define* $f^!\mathcal{F}^\bullet$ for any object $\mathcal{F}^\bullet$ of $D_{qc}^+(\mathcal{O}_X)$, and then in the case when f is proper, use the projection formula to define $tr_f\mathcal{F}^\bullet$ (as in [**Ha 1**, VII, §4] and [**Con**, 4.3]. A limit argument as in the proof of [**Con**, Thm. 4.3.1], then shows that the resulting morphism

$$Rf_*\underline{RHom}^\bullet_{\mathcal{O}_Y}(\mathcal{E}^\bullet, f^!\mathcal{F}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{O}_X}(Rf_*\mathcal{E}^\bullet, \mathcal{F}^\bullet)$$

is a quasi-isomorphism for any two complexes $\mathcal{E}^\bullet$ in $D_{qc}^-(\mathcal{O}_Y)$ and $\mathcal{F}^\bullet$ in $D_{qc}^+(\mathcal{O}_X)$. (Unlike in the situation considered there, we do not need to assume that $\mathcal{F}^\bullet$ is bounded with coherent cohomologies, since we are restricting ourselves to a finite-dimensional Noetherian situation, in which case Rf_* is of finite cohomological amplitude, and we may rely on the usual projection formula, without having to take recourse to the isomorphism of [**Con**, 4.3.2].) Furthermore this definition of tr_f is compatible with that defined via residual complexes for complexes in $D_c^b(\mathcal{O}_X)$, when one compares the two via isomorphism (A.1.2). (This compatibility in the smooth case is proved in [**Con**, Thm. 4.3.2]. In the case when f is the closed immersion of a local complete intersection, it follows from [**Ha 1**, III 6.9.(c)]. Note that although both trace maps are defined for complexes in $D_{qc}^+(\mathcal{O}_X)$, the necessary compatibility in the smooth case is only proved in [**Con**] for bounded complexes.)

The isomorphism (A.1.2), and the resulting trace map in the case that f is proper, is also compatible with flat base-change. To see this, one treats the case when f is smooth and the case when f is the closed immersion of a local complete intersection separately. For the first case the base-change result follows from [**Con**, Thm. 3.6.5]. In the case of the closed immersion of a local complete intersection, it follows from [**Ha 1**, III 6.3, 6.6(2)].

Let $g : Z \to Y$ be a second regular morphism, such that the composite fg is also regular. The isomorphism $(fg)^! \mathcal{O}_X \xrightarrow{\sim} g^! f^! \mathcal{O}_X \xrightarrow{\sim} g^! \omega_{Y/X} \xrightarrow{\sim} \omega_{Z/Y} \otimes_{\mathcal{O}_Z} g^* \omega_{Y/X}$ provides the sense in which formation of $\omega_{Y/X}$ is compatible with composition. Furthermore, we then have a commutative diagram

$$\begin{array}{ccc}
(fg)^! \mathcal{F}^\bullet & \xrightarrow{\sim} & \omega_{Z/X} \otimes_{\mathcal{O}_Z} \mathbb{L}(fg)^* \mathcal{F}^\bullet [d_{Z/X}] \\
 & & \downarrow \wr \\
\downarrow \wr & & \omega_{Z/Y} \otimes_{\mathcal{O}_Z} g^* \omega_{Y/X} \otimes_{\mathcal{O}_Z} \mathbb{L}g^* \mathbb{L}f^* \mathcal{F}^\bullet [d_{Z/Y}][d_{Y/X}] \\
 & & \downarrow \wr \\
g^! f^! \mathcal{F}^\bullet & \xrightarrow{\sim} & \omega_{Z/Y} \otimes_{\mathcal{O}_Z} \mathbb{L}g^* (\omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathbb{L}f^* \mathcal{F}^\bullet [d_{Y/X}])[d_{Z/Y}].
\end{array}$$

These compatibilities will be used in the discussion of the Cartier operator in A.2.

An important example of a regular morphism is one where X and Y are smooth of finite type over a field k and if f is a morphism of k-schemes. Factoring f as the graph morphism followed by a projection shows that f is regular. There is a canonical identification $\omega_{Y/X} = \omega_Y \otimes f^* \omega_X^{-1}$, where ω_X (respectively ω_Y) is the canonical bundle of X (respectively Y).

It would be useful to fully realise the program outlined in [**Ha 1**, III, p. 143-4] for Gorenstein morphisms. (These would include as a special case morphisms which are regular (in the above sense) locally on Y, which is a more flexible condition than the global factorisation that we have required.) The case of *flat* Gorenstein morphisms is covered by [**Con**].

A. 1.3. — If $f : Y \to X$ as above is in fact a finite morphism then the compatibility of duality, as defined via residual complexes, with the explicit duality for finite morphisms of [**Ha 1**, III, §6], yields a natural isomorphism

$$f^! \mathcal{F}^\bullet = f^\flat \mathcal{F}^\bullet = \underline{RHom}^\bullet_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{F}^\bullet).$$

Thus if f is the closed immersion of a local complete intersection then (A.1.2) specialises to an isomorphism

$$\underline{RHom}^{\bullet}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F}^{\bullet}) \xrightarrow{\sim} \omega_{Y/X} \otimes \mathbb{L}f^*\mathcal{F}^{\bullet}[d_{Y/X}]. \tag{A.1.4}$$

In this situation the line bundle $\omega_{Y/X}$ can be identified with the top exterior product of the conormal bundle of Y in X, and the isomorphism (A.1.4) is then explicitly constructed using Koszul complexes. More precisely, one uses Koszul complexes to construct an isomorphism

$$\underline{Ext}^{d_{X/Y}}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F}) \xrightarrow{\sim} \omega_{Y/X} \otimes_{\mathcal{O}_Y} f^*\mathcal{F}$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ (the fundamental local isomorphism of [**Ha 1**, III, §7] and [**Con**, 2.5]), and then the isomorphism (A.1.4) is deduced by the homological algebra of [**Ha 1**, I 7.4] and [**Con**, 2.1]. This explicit description of (A.1.4) is compatible with flat base-change; to see this, it is enough (by the construction of [**Ha 1**, I 7.4] and [**Con**, 2.1]) to observe that the property of being acyclic for the derived functors appearing on each side of (A.1.4) is preserved by flat base-change. But since locally both derived functors may be computed via Koszul complexes, this is clear.

Let us close this subsection by making a remark on notation. If I is an ideal cutting out a closed subscheme Y of X, and $f : Y \to X$ denotes the closed immersion, we will sometimes write $\mathcal{F}[I] = \underline{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F}) = H^0(f^{\flat}\mathcal{F})$ for the sheaf of sections of $\mathcal{F}$ annihilated by I. We regard $\mathcal{F}[I]$ as a sheaf of $\mathcal{O}_Y$-modules. Given this convention, we should write $f_*\mathcal{F}[I]$ if we wish to regard $\mathcal{F}[I]$ as a subsheaf of the sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules; however, no confusion should be caused if we sometimes abuse notation and denote this sheaf simply by $\mathcal{F}[I]$ also.

A. 2. — For the remainder of the appendix we assume that k is a field of characteristic p. Suppose that $f : Y \to X$ is a morphism of smooth k-schemes. For any integer r we may form the r^{th} relative Frobenius diagram of Y over X :

$$\begin{array}{ccccc} Y & \xrightarrow{F^{(r)}_{Y/X}} & Y^{(r)} & \xrightarrow{F^{r\prime}_X} & Y \\ & \searrow^{f} & \downarrow{\scriptstyle f^{(r)}} & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{F^r_X} & X. \end{array}$$

This is a commutative diagram defined so that the right square is cartesian, and the composite of the two upper horizontal arrows is equal to F^r_Y. (Note that the fibre product which we denote by $Y^{(r)}$ is frequently denoted by $Y^{(p^r)}$. Since we always deal with a fixed prime p, our notation should not cause any confusion.

In particular if we replace Y by X (for any smooth k-scheme X) and X by $\operatorname{Spec} k$, then we may write F^r_X as the composition of the r^{th} power of the relative Frobenius $F^r_{X/k} : X \to X^{(r)}$ (which is a morphism of k-schemes) and the base-change $F^r_k{}'$ of the r^{th} power of the Frobenius morphism $F^r_k : \operatorname{Spec} k \to \operatorname{Spec} k$. The morphism $F^r_{X/k}$ is a finite flat map (since X is smooth over k), but the morphism F^r_k need not be so (it is finite flat if and only if k has finite degree over k^p). However it is an affine morphism, and k is free over k^p, which implies that $F^r_k{}'$ is an affine map such that $F^r_{X/k*}\mathcal{O}_{X'}$ is free over $\mathcal{O}_X$. Thus $F^r_X = F^r_{X/k} F^{r\prime}_k$ is affine, with the property that $F^r_{X*}\mathcal{O}_{X'}$ is locally free over $\mathcal{O}_X$.

Now return to the situation of a morphism $f : Y \to X$ between smooth k-schemes. Since X and Y are of finite type over k, we may find a subfield k_1 of k which is finitely generated over $\mathbb{F}_p$, smooth schemes X_1 and Y_1 over k_1, and a morphism $f_1 : Y_1 \to X_1$ such that the schemes X and Y and the morphism $f : Y \to X$ are obtained by base-changing the schemes X_1 and Y_1 and the morphism $f_1 : X_1 \to Y_1$ from the field k_1 to the field k.

Since the morphism $F^r_{X_1}$ and $F^r_{Y_1}$ are finite flat (k_1 being finite over k_1^p) we see that the morphism $F^{(r)}_{Y_1/X_1}$ is finite (although it is not flat in general). Since the morphism $F^{(r)}_{Y/X}$ is obtained by base-changing from k_1 to k, it is also finite.

Now consider the isomorphisms

$$f^!\mathcal{O}_X \xrightarrow{\sim} F^{(r)!}_{Y/X} f^{(r)!}\mathcal{O}_X \xrightarrow{\sim} F^{(r)!}_{Y/X} f^{(r)!} F^{r*}_X \mathcal{O}_X \xrightarrow{\sim} F^{(r)!}_{Y/X} F^{r\prime *}_X f^!\mathcal{O}_X$$

Since $F^{(r)}_{Y/X}$ is finite, we may apply adjointness of $F^{(r)}_{Y/X*}$ and $F^{(r)!}_{Y/X}$ and obtain a map

$$F^{(r)}_{Y/X*} f^!\mathcal{O}_X \longrightarrow F^{r\prime *}_X f^!\mathcal{O}_X.$$

Since X, Y are smooth, we may rewrite this (after shifting) as

$$\mathcal{C}^{(r)}_{Y/X} : F^{(r)}_{Y/X*}\omega_{Y/X} \longrightarrow F^{r' *}_X \omega_{Y/X}.$$

We refer to this morphism as the r^{th} relative Cartier operator of Y over X.

A. 2.1. — Suppose that $f : Y \to X$ is a smooth morphism of k-schemes. Then the r^{th} relative Cartier operator coincides with the restriction to top forms of the r^{th} iterate of the usual Cartier operator (as described for example in [**DI**]). For in this case $F^{(r)}_{Y/X}$ is a finite morphism of smooth X-schemes, and so $\mathcal{C}^{(r)}_{Y/X} : F^{(r)}_{Y/X}\omega_{Y/X} \to F^{r' *}_X \omega_{Y/X}$ is given by the usual trace map on top forms.

A. 2.2. — Suppose that $f : Y \to X$ is a closed immersion of smooth k-schemes. Then the relative Cartier operator is calculated in the last paragraph of page 96 of [**Lyu**] (although not under this name). In this subsection we make several useful computations involving the r^{th} relative Frobenius diagram of the closed immersion f, which in particular make apparent the connection with the calculations of [**Lyu**].

Let M be an $\mathcal{O}_X$-module. The isomorphism $f^!M \xrightarrow{\sim} F^{(r)!}_{Y/X} f^{(r)!}M$ yields a morphism

$$\underline{Ext}^{d_{X/Y}}(f_*\mathcal{O}_Y, M) \to \underline{Ext}^{d_{X/Y^{(r)}}}(f^{(r)}_*\mathcal{O}_{Y^{(r)}}, M)[I];$$

this is an edge map of the spectral sequence

$$\underline{Ext}^p_{\mathcal{O}_{Y^{(r)}}}(F^{(r)}_{Y/X*}\mathcal{O}_Y, \underline{Ext}^q_{\mathcal{O}_X}(f^{(r)}_*\mathcal{O}_{Y^{(r)}}, M)) \Longrightarrow \underline{Ext}^{p+q}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, M).$$

We will describe this edge map in terms of the fundamental local isomorphism.

Let $a_1, \ldots, a_s$ be a regular sequence of sections which (locally) define Y. Then $a_1^q, \ldots, a_s^q$ define $Y^{(r)}$. We may compute $\underline{Ext}^{d_{X/Y}}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, M)$ as the top cohomology of the complex $K^\bullet(\mathbf{a}, M)$. (We are following the notational conventions of [**Con**, 1.3].) This is the complex whose n^{th} term is equal to $\underline{Hom}(\Lambda^n \mathcal{O}_X^s, M)$, and whose n^{th} differential is given by the formula (letting $e_1, \dots e_s$ denoting the standard basis of $\mathcal{O}_X^s$)

$$d^n\alpha(e_{i_1} \wedge \cdots e_{i_{n+1}}) = \sum_{j=1}^{n+1} (-1)^{j+1} a_{i_j} \alpha(e_{i_1} \wedge \cdots \wedge \hat{e_{i_j}} \wedge \cdots \wedge e_{i_{p+1}}).$$

Similarly, we may compute $\underline{Ext}_{\mathcal{O}_X}^{d_{X/Y^{(r)}}}(f^{(r)}\mathcal{O}_{Y^{(r)}}, M)$ as the top cohomology of $K^\bullet(\mathbf{a}^q, M)$. Given this, it is not hard to see that the above edge homomorphism is given by the taking the top cohomology of the map of complexes

$$\iota : K^\bullet(\mathbf{a}, M) \to K^\bullet(\mathbf{a}^q, M)$$

defined in degree n by the formula

$$\iota(\alpha)(e_{i_1} \wedge \cdots \wedge e_{i_n}) = a_1^{q-1} \cdots a_s^{q-1} \alpha(e_{i_1} \wedge \cdots \wedge e_{i_n}).$$

Thus the following diagram (in which the horizontal arrows are given by the fundamental local isomorphism and the left hand vertical arrow is the above edge map) commutes:

$$\begin{array}{ccc}
\underline{Ext}_{\mathcal{O}_X}^{d_{Y/X}}(f_*\mathcal{O}_Y, M) & \xrightarrow{\sim} & \omega_{Y/X} \otimes_{\mathcal{O}_Y} f^*M \\
\downarrow & & \downarrow {\scriptstyle (da_1 \wedge \cdots \wedge da_s)^{-1} \otimes m \mapsto a_1^{q-1} \cdots a_s^{q-1} (da_1^q \wedge \cdots \wedge da_s^q)^{-1} \otimes m} \\
\underline{Ext}_{\mathcal{O}_X}^{d_{X/Y^{(r)}}}(f_*^{(r)}\mathcal{O}_{Y^{(r)}}, M)[I] & \longrightarrow & (\omega_{Y^{(r)}/X} \otimes_{\mathcal{O}_{Y^{(r)}}} f^{(r)*}M)[I] \\
 & & \downarrow \wr \\
 & & (F_X^{r\prime *}\omega_{Y/X} \otimes_{\mathcal{O}_{Y^{(r)}}} f^{(r)*}M)[I].
\end{array}$$

In the particular case that $M = \mathcal{O}_X$, we deduce that the composite of the right hand vertical arrows is the map

$$\omega_{Y/X} \to F_{Y/X}^{(r)!} F_X^{r\prime *} \omega_{Y/X}$$

which induces

$$\mathcal{C}_{Y/X}^{(r)} : F_{Y/X*}^{(r)} \omega_{Y/X} \to F_X^{r\prime *} \omega_{Y/X}$$

by adjointness. Thus $\mathcal{C}_{Y/X}^{(r)}$ is given by the formula

$$(da_1 \wedge \cdots \wedge da_s)^{-1} \mapsto a_1^{q-1} \cdots a_s^{q-1} \otimes (da_1 \wedge \cdots \wedge da_s)^{-1}.$$

There is one more construction in the context of the closed immersion f that we will explain. Suppose that N is an $\mathcal{O}_Y$-module. Then we may tensor $\mathcal{C}_{Y/X}^{(r)}$ with $F_X^{r\prime *}N$ to obtain a morphism

$$\begin{aligned}
F_{Y/X*}^{(r)}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} F_Y^{r*} N) &\xrightarrow{\sim} F_{Y/X*}^{(r)} \omega_{Y/X} \otimes_{\mathcal{O}_{Y^{(r)}}} F_X^{r\prime *} N \\
&\longrightarrow F_X^{r\prime *} \omega_{Y/X} \otimes_{\mathcal{O}_{Y^{(r)}}} F_X^{r\prime *} N \xrightarrow{\sim} F_X^{r\prime *}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N).
\end{aligned}$$

Adjointness yields a morphism

$$\omega_{Y/X} \otimes_{\mathcal{O}_Y} F_Y^{r*} N \to (F_X^{r\prime *}(\omega_{Y/X} \otimes_{\mathcal{O}_Y} N))[I].$$

The above formula for $\mathcal{C}^{(r)}_{Y/X}$ yields the explicit formula

$$(da_1 \wedge \cdots \wedge da_s)^{-1} \otimes m \mapsto a_1^{q-1} \cdots a_s^{q-1} \otimes (da_1 \wedge \cdots \wedge da_s)^{-1} \otimes m$$

for this morphism. This map is an isomorphism (as follows either from the isomorphism [**Ha 1**, III 6.9(a)], with $F^\bullet = \omega_{Y^{(r)}/X} = F_X^{r\prime *}\omega_{Y/X}$ and $G^\bullet = F_X^{r\prime *}N$, or from the calculations at the bottom of page 96 and the top of page 97 of [**Lyu**]).

A. 2.3. — The following list of properties uniquely determines the relative Cartier operator:

(i) Étale morphisms: If $f : Y \to X$ is étale, so that $\omega_{Y/X} = \mathcal{O}_Y$ and $F^{(r)}_{Y/X}$ is an isomorphism of schemes, $\mathcal{C}^{(r)}_{Y/X} : F^{(r)}_{Y/X*}\mathcal{O}_Y \to \mathcal{O}_{Y^{(r)}}$ is the inverse of the isomorphism $\mathcal{O}_{Y^{(r)}} \to F^{(r)}_{Y/X*}\mathcal{O}_Y$ defined by the isomorphism $F^{(r)}_{Y/X}$. (This follows from (A.2.1).)

(ii) Products: If Z is a smooth k-scheme, we may form the product morphism

$$f \times \mathrm{id}_Z : Y \times Z \to X \times Z$$

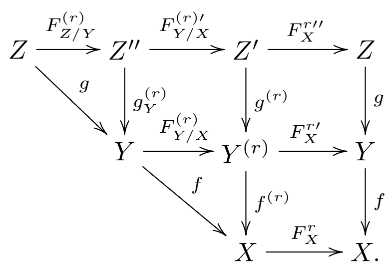
(all the products being taken over $\mathbb{F}_q$). Then there is a natural identification of $(Y \times Z)^{(r)}$ with $Y^{(r)} \times Z$.

Letting p_1 (respectively p_1') denote the projection of the product $Y \times Z$ (respectively $(Y \times Z)^{(r)} \xrightarrow{\sim} Y^{(r)} \times Z$) onto its first factor, there is a canonical identification $\omega_{Y\times Z/X\times Z} = p_1^*\omega_{Y/X}$. Then $\mathcal{C}^{(r)}_{Y\times Z/X\times Z}$ is given by

$$F^r_{Y\times Z/X\times Z*}\omega_{Y\times Z/X\times Z} \xrightarrow{\sim} (F^{(r)}_{Y/X} \times \mathrm{id}_Z)_* p_1^*\omega_{Y/X} \xrightarrow{\sim} {p_1'}^* F^r{}_{Y/X_*}\omega_{Y/X}$$
$$\xrightarrow{p_1'^*\mathcal{C}^{(r)}_{Y/X}} {p_1'}^* {F_X^r}'^*\omega_{Y/X} \xrightarrow{\sim} ({F_X^r}' \times F_Z^r)^* p_1^*\omega_{Y/X} \xrightarrow{\sim} {F^r}'_{X\times Z}{}^*\omega_{Y\times Z/X\times Z}.$$

(This follows from the compatibility of duality with flat base-change.)

(iii) Composition: Suppose that $f : Y \to X$ and $g : Z \to Y$ are two morphisms of smooth k-schemes. Then we may amalgamate the relative Frobenius diagrams of Y over X, of Z over X and of Z over Y into the following diagram:



The relative Cartier operator

$$\mathcal{C}^{(r)}_{Z/X} : F^{(r)}_{Z/X*}\omega_{Z/X} \to {F_X^r}''^*\omega_{Z/X}$$

is determined by the relative Cartier operators $\mathcal{C}^{(r)}_{Y/X}, \mathcal{C}^{(r)}_{Z/Y}$, as follows

$$
\begin{aligned}
F^{(r)}_{Z/X*}\omega_{Z/X} &\xrightarrow{\sim} F^{(r)\prime}_{Y/X*}F^{(r)}_{Z/Y*}(\omega_{Z/Y} \otimes g^*\omega_{Y/X}) \\
&\xrightarrow{\sim} F^{(r)\prime}_{Y/X*}F^{(r)}_{Z/Y*}(\omega_{Z/Y} \otimes F^{(r)*}_{Z/Y} g^{(r)*}_{Y}\omega_{Y/X}) \\
&\xrightarrow{\sim} F^{(r)\prime}_{Y/X*}(F^{(r)}_{Z/Y*}\omega_{Z/Y} \otimes g^{(r)*}_{Y}\omega_{Y/X}) \\
&\xrightarrow{F^{(r)\prime}_{Y/X*}(\mathcal{C}^{(r)}_{Z/Y}\otimes 1)} F^{(r)\prime}_{Y/X*}(F^{r\,\prime *}_{Y}\omega_{Z/Y} \otimes g^{(r)*}_{Y}\omega_{Y/X}) \\
&\xrightarrow{\sim} F^{(r)\prime}_{Y/X*}(F^{(r)\prime *}_{Y/X}F^{r\,\prime\prime *}_{X}\omega_{Z/Y} \otimes g^{(r)*}_{Y}\omega_{Y/X}) \\
&\xrightarrow{\sim} F^{r\,\prime\prime *}_{X}\omega_{Z/Y} \otimes F^{(r)\prime}_{Y/X*}g^{(r)*}_{Y}\omega_{Y/X} \\
&\xrightarrow{\sim} F^{r\prime\prime *}_{X}\omega_{Z/Y} \otimes g^{(r)*}F^{(r)}_{Y/X*}\omega_{Y/X} \\
&\xrightarrow{1\otimes g^{(r)*}\mathcal{C}^{(r)}_{Y/X}} F^{r\prime\prime *}_{X}\omega_{Z/Y} \otimes g^{(r)*}F^{r\prime *}_{X}\omega_{Y/X} \\
&\xrightarrow{\sim} F^{r\,\prime\prime *}_{X}\omega_{Z/Y} \otimes F^{r\,\prime\prime *}_{X}g^*\omega_{Y/X} \\
&\xrightarrow{\sim} F^{r\prime\prime *}_{X}(\omega_{Z/Y} \otimes g^*\omega_{Y/X}).
\end{aligned}
$$

(This follows from the compatibility of duality with composition.)

(iv) Embedding a divisor: If $f : Y \to X$ is the closed immersion of a divisor, let $\mathcal{O}_X(Y)$ denote the invertible sheaf corresponding to Y (the inverse of the ideal sheaf of Y). Then $Y^{(r)}$ is the divisor corresponding to the invertible sheaf $\mathcal{O}_X(Y)^{\otimes q}$. There is a natural isomorphism $\omega_{Y/X} = f^*\mathcal{O}_X(Y) = \mathcal{O}_X(Y)/\mathcal{O}_X$, and

$$
\begin{aligned}
F^{r\,\prime *}_{X}\omega_{Y/X} \xrightarrow{\sim} F^{r\,\prime *}_{X}f^*\mathcal{O}_X(Y) \xrightarrow{\sim} f^{(r)*}F^{r\,*}_{X}\mathcal{O}_X(Y)& \\
\xrightarrow{\sim} f^{(r)*}\mathcal{O}_X(Y^{(r)}) \xrightarrow{\sim} \mathcal{O}_X(Y^{(r)})/\mathcal{O}_X&.
\end{aligned}
$$

The map $F^{(r)}_{Y/X}$ is the closed immersion of Y into $Y^{(r)}$. In this situation, the relative Cartier isomorphism

$$\mathcal{C}^{(r)}_{Y/X} : F^{(r)}_{Y/X*}(\mathcal{O}_X(Y)/\mathcal{O}_X) \to \mathcal{O}_X(Y^{(r)})/\mathcal{O}_X$$

is the map induced by the inclusion

$$\mathcal{O}_X(Y) \to \mathcal{O}_X(Y)^{\otimes q} = \mathcal{O}_X(Y^{(r)}).$$

(This follows from (A.2.2), if one works in terms of a local equation for Y.)

(v) Mapping $\mathbb{G}_m$ to a point: In the particular case of $Y = \mathbb{G}_m = \operatorname{Spec} k\,[t, t^{-1}]$, the differential dt/t forms a global basis for the invertible sheaf $\omega_{\mathbb{G}_m}$, allowing us to identify this sheaf with $\mathcal{O}_{\mathbb{G}_m}$, and under this identification the Cartier operator becomes the trace of the Frobenius $F^r_{Y/k} : \mathcal{O}_{\mathbb{G}_m} \to \mathcal{O}_{\mathbb{G}_m}$. (This follows from (A.2.1).)

Note that *(i)* – *(v)* determine the Cartier operator for any morphism $Y \longrightarrow X$ of smooth k schemes. Indeed any such map is the composite of a closed immersion and a smooth map (use the usual factorisation via the graph), and any smooth map factors étale locally into a composite of projections $\mathbb{G}_m \times X \longrightarrow X$.

APPENDIX B: HOMOLOGICAL ALGEBRA.

B. 1. — Let X be a topological space equipped with a sheaf of commutative rings $\mathcal{A}$, let $\mathcal{A}'$ and $\mathcal{A}''$ be two sheaves of commutative $\mathcal{A}$-algebras on X, equipped with a morphism of $\mathcal{A}$-algebras $\mathcal{A}' \to \mathcal{A}''$. Let $\mathcal{B}$ be a (not necessarily commutative) sheaf of $\mathcal{B}$-algebras on X, and write $\mathcal{B}' = \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{B}$, $\mathcal{B}'' = \mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{B}$.

Proposition B. 1.1. — *In the situation of section (B.1), suppose that $\mathcal{A}''$ is flat over $\mathcal{A}$. Let $\mathcal{M}^\bullet$ be a bounded above complex of left $\mathcal{B}'$-modules, let $\mathcal{N}^\bullet$ be bounded complex of left $\mathcal{B}'$-modules of finite $\mathcal{A}'$-Tor-dimension, and suppose that the complex of $\mathcal{A}'$-modules $\underline{RHom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$ is bounded. Then there is a natural morphism in the derived category of complexes of $\mathcal{A}''$-modules*

$$\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \underline{RHom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \longrightarrow \underline{RHom}^\bullet_{\mathcal{B}''}(\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{M}^\bullet, \mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{N}^\bullet).$$

(Note that our assumptions on $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ guarantee that all the derived functors appearing in the source and target of this morphism are well-defined.)

Proof. — In order to compute $\underline{RHom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$, we replace $\mathcal{N}^\bullet$ by an injective resolution. Since the former complex is assumed to be bounded, we may in fact replace $\mathcal{N}^\bullet$ by one of the finite truncations of this injective resolution. Thus we may assume that $\mathcal{N}^\bullet$ is of finite length, and that the complex $\underline{Hom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$ computes $\underline{RHom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$.

The given morphism $\mathcal{A}' \to \mathcal{A}''$ allows us to regard $\mathcal{A}''$ as an $\mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{A}'$-module. Let $\mathcal{P}^\bullet$ be a left resolution of $\mathcal{A}''$ by flat $\mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{A}'$-modules. (Such a resolution can be constructed by the technique of [**Ha 1**, II 1.2].) Since $\mathcal{A}''$ is assumed to be flat over A, we see that $\mathcal{P}^\bullet$ is flat as a complex of sheaves of $\mathcal{A}'$-modules. If $\mathcal{F}^\bullet$ is a bounded above complex of $\mathcal{B}'$-modules then we regard $\mathcal{P}^\bullet \otimes_{\mathcal{A}} \mathcal{F}^\bullet$ as a complex of $\mathcal{B}''$-modules via the $\mathcal{A}''$-action on $\mathcal{P}^\bullet$ and the $\mathcal{B}$-action on $\mathcal{F}^\bullet$. This computes the derived functor $\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{F}^\bullet$.

There is a natural $\mathcal{A}'$-linear morphism

$$\underline{Hom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{Hom}^\bullet_{\mathcal{B}''}(\mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{M}^\bullet, \mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{N}^\bullet),$$

which induces a $\mathcal{A}''$-linear morphism

$$\mathcal{A}'' \otimes_{\mathcal{A}'} \underline{Hom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{Hom}^\bullet_{\mathcal{B}''}(\mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{M}^\bullet, \mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{N}^\bullet) \tag{B.1.2}$$

Let $\mathcal{I}^\bullet$ be a bounded below injective resolution of $\mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{N}^\bullet$ (which exists because our assumption on $\mathcal{N}^\bullet$ implies that this complex has only finitely many non-vanishing cohomology sheaves). Then combining the augmentations $\mathcal{P}^\bullet \to \mathcal{A}''$ and $\mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{N}^\bullet \to \mathcal{I}^\bullet$ with the map (B.1.2) we may construct a map

$$\mathcal{P}^\bullet \otimes_{\mathcal{A}'} \underline{Hom}^\bullet_{\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \underline{Hom}^\bullet(\mathcal{P}^\bullet \otimes_{\mathcal{A}'} \mathcal{M}^\bullet, \mathcal{I}^\bullet),$$

which yields the natural morphism of the proposition upon passing to the derived category. □

Proposition B. 1.3. — *In the situation of section (B.1), suppose that $f : Y \to X$ is a map of topological spaces, with Y Noetherian of finite dimension, and also that $\mathcal{A}''$ admits a left resolution by sheaves of $\mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{A}'$-modules which are locally free as $\mathcal{A}'$-modules. If $\mathcal{M}^\bullet$ is any bounded above complex of left $f^{-1}\mathcal{B}'$-modules, then there is a natural isomorphism in the derived category of complexes of left $\mathcal{B}''$-modules*

$$\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} Rf_*\mathcal{M}^\bullet \xrightarrow{\sim} Rf_*(f^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}'} \mathcal{M}^\bullet).$$

(Note that since Y is assumed to be Noetherian of finite dimension, by Grothendieck's vanishing theorem the functor Rf_* has finite cohomological amplitude [**Ha 2**, III 2.7], *and so the derived functors appearing on either side of this map are well-defined.)*

Proof. — If $\mathcal{F}$ is a sheaf of left $f^{-1}\mathcal{B}'$-modules on Y, then Godement's canonical resolution yields a right resolution of $f^{-1}\mathcal{B}'$ by flasque $f^{-1}\mathcal{B}'$-modules. Since Y is Noetherian of finite dimension, say d, the d^{th} truncation of this complex is again a flasque resolution of $\mathcal{F}$. Thus we have a canonical construction of a bounded length flasque resolution of any sheaf of left $f^{-1}\mathcal{B}'$-modules. Applying this construction to the members of $\mathcal{M}^\bullet$, we may assume that $\mathcal{M}^\bullet$ is a bounded above complex of flasque sheaves. Then $f_*\mathcal{M}^\bullet$ computes $Rf_*\mathcal{M}^\bullet$ (even though $\mathcal{M}^\bullet$ may not be bounded below, again because f_* has finite cohomological amplitude).

Let $\mathcal{P}^\bullet$ be the left resolution of $\mathcal{A}''$ by $\mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{A}'$-modules which are locally free as $\mathcal{A}'$-modules, whose existence we have assumed. Then $f^{-1}\mathcal{P}^\bullet$ is similarly a left resolution of $f^{-1}\mathcal{A}''$ by $f^{-1}\mathcal{A}'' \otimes_{f^{-1}\mathcal{A}} f^{-1}\mathcal{A}'$-modules which are locally free as $f^{-1}\mathcal{A}'$-modules. Thus $f^{-1}\mathcal{P}^\bullet \otimes_{f^{-1}\mathcal{A}'} \mathcal{M}^\bullet$ is a complex of flasque sheaves (since it is locally a direct sum of flasque sheaves, and Y is Noetherian). We regard this complex as a complex of sheaves of left $f^{-1}\mathcal{B}''$-modules, via the $f^{-1}\mathcal{A}''$-action on $f^{-1}\mathcal{P}^\bullet$ and the $f^{-1}\mathcal{B}$-action on $\mathcal{M}^\bullet$. The projection formula yields an isomorphism of $\mathcal{B}''$-modules

$$\mathcal{P}^\bullet \otimes_{\mathcal{A}'} f_*\mathcal{M}^\bullet \xrightarrow{\sim} f_*(f^{-1}\mathcal{P}^\bullet \otimes_{f^{-1}\mathcal{A}'} f_*\mathcal{M}^\bullet).$$

Passing to the derived category, this yields the desired isomorphism. □

B. 1.4. — In the situation of (B.1), suppose both that $\mathcal{A}''$ is flat over $\mathcal{A}$, and that $\mathcal{A}''$ admits a resolution by $\mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{A}'$-modules which are free as left $\mathcal{A}'$-modules. Let $j : U \to X$ be the immersion of an open subset U of X, and assume that U is Noetherian of finite dimension.

Let $\mathcal{M}^\bullet$ be a bounded above complex of sheaves of left $j^{-1}\mathcal{B}'$-modules on U, and let $\mathcal{N}^\bullet$ be a bounded complex of sheaves of left $\mathcal{B}'$-modules on X which is of finite $\mathcal{A}'$-Tor-dimension. Suppose furthermore that $\underline{RHom}^\bullet_{\mathcal{B}'}(j_!\mathcal{M}^\bullet,\mathcal{N}^\bullet)$ is bounded (in which case $\underline{RHom}^\bullet_{j^{-1}\mathcal{B}'}(\mathcal{M}^\bullet, j^{-1}\mathcal{N}^\bullet)$ is also bounded, since it is obtained by pulling back the former complex by j).

We may form the following diagram:

$$\begin{array}{ccc}
\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \underline{RHom}^\bullet_{\mathcal{B}'}(j_!\mathcal{M}^\bullet,\mathcal{N}^\bullet) & \xrightarrow[\text{(B.1.1)}]{} & \underline{RHom}^\bullet_{\mathcal{B}''}(\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} j_!\mathcal{M}^\bullet, \mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{N}^\bullet) \\
\downarrow \wr & & \downarrow \wr \\
\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} Rj_*\underline{RHom}^\bullet_{j^{-1}\mathcal{B}'}(\mathcal{M}^\bullet, j^{-1}\mathcal{N}^\bullet) & & \underline{RHom}^\bullet_{\mathcal{B}''}(j_!(\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{M}^\bullet), \mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{N}^\bullet) \\
\text{(B.1.3)} \downarrow \wr & & \downarrow \wr \\
Rj_*(j^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{j^{-1}\mathcal{A}'} \underline{RHom}^\bullet_{j^{-1}\mathcal{B}'}(\mathcal{M}^\bullet, j^{-1}\mathcal{N}^\bullet) & & Rj_*\underline{RHom}^\bullet_{j^{-1}\mathcal{B}''}(j^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{j^{-1}\mathcal{A}} \mathcal{M}^\bullet, j^{-1}(\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \mathcal{N}^\bullet)) \\
\text{(B.1.1)} \downarrow & \nearrow \sim & \\
Rj_*\underline{RHom}^\bullet_{j^{-1}\mathcal{B}''}(j^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{j^{-1}\mathcal{A}} \mathcal{M}^\bullet, j^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{j^{-1}\mathcal{A}'} j^{-1}\mathcal{N}^\bullet). & &
\end{array}$$

We leave it to the reader to check that this commutes.

Proposition B. 2. — *Let $f : Y \to X$ be a morphism of topological spaces with Y Noetherian of finite dimension, and let $\mathcal{B}$ be a (not necessarily commutative) sheaf of rings on X containing the sheaf of rings $\mathcal{A}$ in its centre. Then if $\mathcal{M}^\bullet$ is a bounded above complex of left $f^{-1}\mathcal{B}$-modules and $\mathcal{N}^\bullet$ a bounded below complex of sheaves of left $f^{-1}\mathcal{B}$-modules on Y, there is a natural morphism in the derived category of $\mathcal{A}$-modules*

$$Rf_*\underline{RHom}^\bullet_{f^{-1}\mathcal{B}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) \to \underline{RHom}^\bullet_{\mathcal{B}}(Rf_*\mathcal{M}^\bullet, Rf_*\mathcal{N}^\bullet).$$

Proof. — As in the proof of Proposition B.1.3, we may and do assume that $\mathcal{M}^\bullet$ is a complex of flasque left $f^{-1}\mathcal{B}$-modules, and we may also assume that $\mathcal{N}^\bullet$ is a complex, bounded below, of injective left $f^{-1}\mathcal{B}$-modules. There is a natural morphism of sheaves of $\mathcal{A}$-modules

$$f_*\underline{Hom}^\bullet_{f^{-1}\mathcal{B}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet) \to \underline{Hom}^\bullet_{\mathcal{B}}(f_*\mathcal{M}^\bullet, f_*\mathcal{N}^\bullet). \tag{B.2.1}$$

Since $\mathcal{N}^\bullet$ is a complex of injectives, the complex $\underline{Hom}^\bullet_{f^{-1}\mathcal{B}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$ computes $\underline{RHom}^\bullet_{f^{-1}\mathcal{B}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$. Furthermore, again using the fact that the members of $\mathcal{N}^\bullet$ are injective, this is a complex of flasque sheaves. Since $\mathcal{M}^\bullet$ is a complex of flasque sheaves, the complex $f_*\mathcal{M}^\bullet$ represents $Rf_*\mathcal{M}^\bullet$. Similarly, the complex $f_*\mathcal{N}^\bullet$ computes $Rf_*\mathcal{N}^\bullet$. Finally, since f_* is right-adjoint to the exact functor f^{-1}, we see that $f_*\mathcal{N}^\bullet$ is a complex of injective left-$\mathcal{B}$-modules, and so $\underline{Hom}^\bullet_{\mathcal{B}}(f_*\mathcal{M}^\bullet, f_*\mathcal{N}^\bullet)$ computes $\underline{RHom}^\bullet_{\mathcal{B}}(Rf_*\mathcal{M}^\bullet, Rf_*\mathcal{N}^\bullet)$. Putting all these remarks together, we see that (B.2.1) provides the natural transformation of the proposition upon passing to the derived category. □

B. 3. — Let us return to the situation of (B.1). Suppose furthermore both that $\mathcal{A}''$ is flat over $\mathcal{A}$, and that $\mathcal{A}''$ admits a resolution by $\mathcal{A}'' \otimes_{\mathcal{A}} \mathcal{A}'$-modules which are free as left $\mathcal{A}'$-modules. Let $f : Y \to X$ be a morphism of topological spaces with Y Noetherian of finite dimension.

Let $\mathcal{M}^\bullet$ be a bounded above complex and $\mathcal{N}^\bullet$ a bounded complex of sheaves of left $f^{-1}\mathcal{B}'$-modules on Y. Suppose furthermore that $\mathcal{N}^\bullet$ is of finite $f^{-1}\mathcal{A}'$-Tor-dimension, that $Rf_*\mathcal{N}^\bullet$ is of finite $\mathcal{A}'$-Tor-dimension, and that

$$\underline{RHom}^\bullet_{f^{-1}\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$$

and

$$\underline{RHom}^\bullet_{\mathcal{B}'}(Rf_*\mathcal{M}^\bullet, Rf_*\mathcal{N}^\bullet)$$

are both bounded complexes. Then we may form the following diagram

$$\begin{array}{ccc}
\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} Rf_*\underline{RHom}^\bullet_{f^{-1}\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) & \xrightarrow{\text{(B.2)}} & \\
\downarrow{\scriptstyle (B.1.3)} & & \\
Rf_*(f^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}'} \underline{RHom}^\bullet_{f^{-1}\mathcal{B}'}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)) & & \mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} \underline{RHom}^\bullet_{\mathcal{B}'}(Rf_*\mathcal{M}^\bullet, Rf_*\mathcal{N}^\bullet) \\
\downarrow{\scriptstyle (B.1.1)} & & \downarrow{\scriptstyle (B.1.1)} \\
Rf_*\underline{RHom}^\bullet_{f^{-1}\mathcal{B}''}(f^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}'} \mathcal{M}^\bullet, f^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}'} \mathcal{N}^\bullet) & & \underline{RHom}^\bullet_{\mathcal{B}''}(\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} Rf_*\mathcal{M}^\bullet, \mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} Rf_*\mathcal{N}^\bullet) \\
\downarrow{\scriptstyle (B.2)} & \xleftarrow{\text{(B.1.3)}} & \\
\underline{RHom}^\bullet_{\mathcal{B}''}(Rf_*(f^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}'} \mathcal{M}^\bullet), Rf_*(f^{-1}\mathcal{A}'' \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}'} \mathcal{N}^\bullet)) & & .
\end{array}$$

We leave it as an exercise for the reader to check that it commutes.

B. 4. — Although the previous results were phrased in the context of sheaves on a topological space, it is clear that they extend to more general situations; for example, to the étale site of a scheme.

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