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LOCALLY ANALYTIC VECTORS IN REPRESENTATIONS OF LOCALLY p-ADIC ANALYTIC GROUPS

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Introduction

Recent years have seen the emergence of a new branch of representation theory: the theory of representations of locally p-adic analytic groups on locally convex p-adic topological vector spaces (or "locally analytic representation theory", for short). Examples of such representations are provided by finite dimensional algebraic representations of p-adic reductive groups, and also by smooth representations of such groups (on p-adic vector spaces). One might call these the "classical" examples of such representations. One of the main interests of the theory (from the point of view of number theory) is that it provides a setting in which one can study p-adic completions of the classical representations [6], or construct "p-adic interpolations" of them (for example, by defining locally analytic analogues of the principal series, as in [23], or by constructing representations via the cohomology of arithmetic quotients of symmetric spaces, as in [9]).

Locally analytic representation theory also plays an important role in the analysis of p-adic symmetric spaces; indeed, this analysis provided the original motivation for its development. The first "non-classical" examples in the theory were found by Morita, in his analysis of the p-adic upper half-plane (the p-adic symmetric space attached to $\mathrm{GL}_2(\mathbb{Q}_p)$) [17], and further examples were found by Schneider and Teitelbaum in their analytic investigations of the p-adic symmetric spaces of $\mathrm{GL}_n(\mathbb{Q}_p)$ (for arbitrary n) [22]. Motivated in part by the desire to understand these examples, Schneider and Teitelbaum have recently initiated a systematic study of locally analytic representation theory [22, 23, 24, 25, 27]. In particular, they have introduced the important notions of admissible and strongly admissible locally

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analytic representations of a locally p-adic analytic group (as well as the related notion of admissible continuous representations of such a group).

The goal of this memoir is to provide the foundations for the locally analytic representation theory that is required in the papers [8, 9, 10]. In the course of writing those papers we have found it useful to adopt a particular point of view on locally analytic representation theory: namely, we regard a locally analytic representation as being the inductive limit of its subspaces of analytic vectors (of various "radii of analyticity"), and we use the analysis of these subspaces as one of the basic tools in our study of such representations. Thus in this memoir we present a development of locally analytic representation theory built around this point of view. Some of the material that we present is entirely new (for example, the notion of essentially admissible representation, which plays a key role in [8], and the results of chapter 7, which are used in [9]); other parts of it can be found (perhaps up to minor variations) in the papers of Schneider and Teitelbaum cited above, or in the thesis of Feaux de Lacroix [12]. We have made a deliberate effort to keep our exposition reasonably self-contained, and we hope that this will be of some benefit to the reader.

0.1. Locally analytic vectors and locally analytic representations

We will now give a more precise description of the view-point on locally analytic representation theory that this memoir adopts, and that we summarized above.

Let L be a finite extension of $\mathbb{Q}_p$, and let K be an extension of L, complete with respect to a discrete valuation extending the discrete valuation of L.[1] We let G denote a locally L-analytic group, and consider representations of G by continuous operators on Hausdorff locally convex topological K-vector spaces. Our first goal is to define, for any such representation V, the topological K-vector space V_{la} of locally analytic vectors in V. As a vector space, V_{la} is defined to be the subspace of V consisting of those vectors v for which the orbit map $g \mapsto gv$ is a locally analytic V-valued function on G. The non-trivial point in the definition is to equip V_{la} with an appropriate topology. In the references [22,27], the authors endow this space (which they denote by V_{an}, rather than V_{la}) with the topology that it inherits as a closed subspace of the space $\mathcal{C}^{\mathrm{la}}(G, V)$ of locally analytic V-valued functions on G (each vector $v \in V_{\mathrm{la}}$ being identified with the corresponding orbit map $o_v : G \to V$). We have found it advantageous to endow V_{la} with a finer topology, with respect to which it is exhibited as a locally convex inductive limit of Banach spaces. (In some important situations — for example, when V is of compact type, or is a Banach space equipped with an admissible continuous G-action — we prove that the topology that we consider coincides with that considered by Schneider and Teitelbaum.)

Suppose first that $\mathbb{G}$ is an affinoid rigid analytic group defined over L, and that G is the group of L-valued points of $\mathbb{G}$. If W is a Banach space equipped with a

[1] In the main body of the text, in contrast to this introduction, the field K of coefficients is assumed merely to be spherically complete with respect to a valuation extending the discrete valution on L. Nevertheless, several of our results require the additional hypothesis that K be discretely valued, and so we have imposed this hypothesis on K throughout the introducion in order to simplify the discussion. In the main body of the text, when a given result requires for its validity that K be discretely valued, we have always indicated this requirement in the statement of the result (with one exception: it is assumed throught section 6.5 that K is discretely valued, and so this is not explictly mentioned in the statements of the results appearing in that section).

representation of G, then we say that this representation is $\mathbb{G}$-analytic if for each $w \in W$ the orbit map $o_w : G \to W$ given by w extends to a W-valued rigid analytic function on $\mathbb{G}$. For any G-representation V, we define $V_{\mathbb{G}-\mathrm{an}}$ to be the locally convex inductive limit over the inductive system of G-equivariant maps $W \to V$, where W is a Banach space equipped with a $\mathbb{G}$-analytic action of G.

We now consider the case of an arbitrary locally L-analytic group G. Recall that a chart $(\phi, H, \mathbb{H})$ of G consists of an open subset H of G, an affinoid space $\mathbb{H}$ isomorphic to a closed ball, and a locally analytic isomorphism $\phi : H \xrightarrow{\sim} \mathbb{H}(L)$. If H is furthermore a subgroup of G, then the fact that $\mathbb{H}(L)$ is Zariski dense in $\mathbb{H}$ implies that there is at most one rigid analytic group structure on $\mathbb{H}$ inducing the given group structure on H. If such a group structure exists, we refer to the chart $(\phi, H, \mathbb{H})$ as an analytic open subgroup of G. We will typically suppress reference to the isomorphism ϕ, and so will speak of an analytic open subgroup H of G, letting $\mathbb{H}$ denote the corresponding rigid analytic group, and identifying H with the group of points $\mathbb{H}(L)$.

For any G-representation on a Hausdorff convex K-vector space V, and any analytic open subgroup H of $\mathbb{G}$, we can define as above the space $V_{\mathbb{H}-\mathrm{an}}$ of $\mathbb{H}$-analytic vectors in V. (If we ignore questions of topology, then $V_{\mathbb{H}-\mathrm{an}}$ consists of those locally analytic vectors with "radius of analyticity" bounded below by H.) We define V_{la} to be the locally convex inductive limit over all locally analytic open subgroups H of G of the spaces $V_{\mathbb{H}-\mathrm{an}}$.

The representation V of G is said to be locally analytic if V is barrelled, and if the natural map $V_{\mathrm{la}} \to V$ is a bijection. If V is an LF-space (we recall the meaning of this, and some related, functional analytic terminology in section 1.1 below), then we can show that if this map is a bijection, it is in fact a topological isomorphism. Thus given a locally analytic representation of G on an LF-space V, we may write $V \xrightarrow{\sim} \varinjlim_n V_{\mathbb{H}_n-\mathrm{an}}$, where H_n runs over a cofinal sequence of analytic open subgroups of G.

The category of admissible locally analytic G-representations, introduced in [27], admits a useful description from this point of view. We show that a locally analytic G-representation on a Hausdorff convex K-vector space V is admissible if and only if V is an LB-space, such that for each analytic open subgroup H of G, the space $V_{\mathbb{H}-\mathrm{an}}$ admits a closed H-equivariant embedding into a finite direct sum of copies of the space $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)$ of rigid analytic functions on $\mathbb{H}$.

Recall that in [27], a locally analytic G-representation V is defined to be admissible if and only if V is of compact type, and if the dual space V' is a coadmissible module under the action of the ring $\mathcal{D}^{\mathrm{an}}(H, K)$ of locally analytic distributions on H, for some (or equivalently, every) compact open subgroup H of G. For this definition to make sense (that is, for the notion of a coadmissible $\mathcal{D}^{\mathrm{an}}(H, K)$-module to be defined), the authors must prove that the ring $\mathcal{D}^{\mathrm{an}}(H, K)$ is a Fréchet-Stein algebra, in the sense of [27, def., p. 8]. This result [27, thm. 5.1] is the main theorem of that reference.

In order to establish our characterization of admissible locally analytic representations, we are led to give an alternative proof of this theorem, and an alternative description of the Fréchet-Stein structure on $\mathcal{D}^{\mathrm{la}}(H, K)$, which is more in keeping with our point of view on locally analytic representations. While the proof of [27] relies on the methods of [16], we rely instead on the methods used in [1] to prove the coherence of the sheaf of rings $\mathcal{D}^{\dagger}$.

We also introduce the category of essentially admissible locally analytic G-representations. To define this category, we must assume that the centre Z of G is topologically finitely generated. (This is a rather mild condition, which is satisfied, for example, if G is the group of L-valued points of a reductive linear algebraic group over L.) Supposing that this is so, we let $\hat{Z}$ denote the rigid analytic space parameterizing the locally analytic characters of Z, and let $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ denote the Fréchet-Stein algebra of K-valued rigid analytic functions on $\hat{Z}$.

Let V be a convex K-vector space of compact type equipped with a locally analytic G-representation, and suppose that V may be written as a union $V = \varinjlim_n V_n$, where each V_n is a Z-invariant BH-subspace of V. The $\mathcal{D}^{\mathrm{la}}(H, K)$-action on the dual space V' then extends naturally to an action of the completed tensor product algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$. Our proof of the fact that $\mathcal{D}^{\mathrm{la}}(H, K)$ is Fréchet-Stein generalizes to show that this completed tensor product is also Fréchet-Stein. We say that V is an essentially admissible locally analytic representation of G if, furthermore, V' is a coadmissible module with respect to this Fréchet-Stein algebra, for some (or equivalently, any) compact open subgroup H of G.

It is easy to show, using the characterization of admissible locally analytic representations described above, that any such locally analytic representation of G is essentially admissible. Conversely, if V is any essentially admissible locally analytic representation of G, and if χ is a K-valued point of $\hat{Z}$, then the closed subspace V^{χ} of V on which Z acts through χ is an admissible locally analytic representation of G. The general theory of Fréchet-Stein algebras [27, §3] implies that the category of essentially admissible locally analytic G-representations is abelian.

Since the category of coadmissible $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-modules is (anti)equivalent to the category of coherent rigid analytic sheaves on the rigid analytic space $\hat{Z}$, one may think of a coadmissible $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module as being a "coherent sheaf" of coadmissible $\mathcal{D}^{\mathrm{la}}(H, K)$-modules on $\hat{Z}$ (or, better, a "coherent cosheaf"). Thus, roughly speaking, one may regard an essentially admissible locally analytic G-representation V as being a family of admissible locally analytic G-representations parameterized by the space $\hat{Z}$, whose fibre (or, better, "cofibre") over a point $\chi \in \hat{Z}(K)$ is equal to V^{χ}.

The category of essentially admissible locally analytic representations provides the setting for the Jacquet module construction for locally analytic representations that is the subject of the papers [8] and [10]. These functors are in turn applied in [9] to construct "eigenvarieties" (generalizing the eigencurve of [7]) that p-adically interpolate systems of eigenvalues attached to automorphic Hecke eigenforms on reductive groups over number fields.

Let us point out that functional analysis currently provides the most important technical tool in the theory of locally analytic representations. Indeed, since continuous p-adic valued functions on a p-adic group are typically not locally integrable for Haar measure (unless they are locally constant), there is so far no real analogue in this theory of the harmonic analysis which plays such an important role in the theory of smooth representations (although one can see some shades of harmonic analysis in the theory: the irreducibility result of [23, thm. 6.1] depends for its proof on Fourier analysis in the non-compact picture, and Hecke operators make an appearance in the construction of the Jacquet module functor of [8]). Thus one relies on softer functional analytic methods to make progress. This memoir is no exception; it relies almost entirely on such methods.

0.2. The organization of the memoir

A more detailed summary of the memoir now follows, preceding chapter by chapter.

In chapter 1 we develop the non-archimedean functional analysis that we will require in the rest of the memoir. Section 1.1 is devoted to recalling various pieces of terminology that we will need, and to proving some results for which we could not find references in the literature. None of the results are difficult, and most or all are presumably well-known to experts. In section 1.2 we recall the theory of Fréchet-Stein algebras developed in [23, §3], and prove some additional results that we will require in our applications of this theory.

In chapter 2 we recall the basics of non-archimedean function theory. Section 2.1 recalls the basic definitions regarding spaces of continuous, rigid analytic, and locally analytic functions with values in locally convex K-vector spaces, and establishes some basic properties of these spaces that we will require. Section 2.2 introduces the corresponding spaces of distributions. In section 2.3 we recall the definition and basic properties of the restriction of scalars functor, in both the rigid analytic and the locally analytic setting.

In chapter 3 we present our construction of the space of locally analytic vectors attached to a representation of a non-archimedean locally L-analytic group G.

After some preliminaries in sections 3.1 and 3.2, in section 3.3 we suppose that G is the group of points of an affinoid rigid analytic group defined over L, and define the space of analytic vectors. In section 3.4 we extend this construction to certain non-affinoid rigid analytic groups.

In section 3.5, we return to the situation in which G is a locally L-analytic group, and construct the space of locally analytic vectors attached to any G-representation. In section 3.6 we recall the notion of locally analytic representation, and also introduce the related notion of analytic representation, and establish some basic properties of such representations.

Chapter 4 begins by recalling, in section 4.1, the notion of smooth and locally finite vectors in a G-representation. The main point of this section is to prove some simple facts about representations in which every vector is smooth or locally finite.

In section 4.2 we assume that G is the group of L-valued points of a connected reductive linear algebraic group $\mathbb{G}$ over L. For any finite dimensional algebraic representation W of $\mathbb{G}$ over K, and for any G-representation V, we define the space $V_{W-\mathrm{alg}}$ of locally W-algebraic vectors in V, and study some of its basic properties. As in [18], the representation V is said to be locally algebraic if every vector of V is locally W-algebraic for some representation W of $\mathbb{G}$. We prove that any irreducible locally algebraic representation is isomorphic to the tensor product of a smooth representation of G and a finite dimensional algebraic representation of $\mathbb{G}$ (first proved in [18]).

One approach to analyzing representations V of G, the importance of which has been emphasized by Schneider and Teitelbaum, is to pass to the dual space V', and to regard V' as a module over an appropriate ring of distributions on G. The goal of chapter 5 is to recall this approach, and to relate it to the view-point of chapter 3.

In section 5.1 we prove some simple forms of Frobenius reciprocity, and apply these to obtain a uniform development of the dual point of view for continuous, analytic, and locally analytic representations. In section 5.2 we recall the description of algebras of analytic distributions via appropriate completions of universal enveloping algebras. In section 5.3 we use this description, together with the methods of [1, §3], to present a new construction of the Fréchet-Stein structure on the ring $\mathcal{D}^{\rm la}(H,K)$ of locally analytic distributions on any compact open subgroup H of G. In fact, we prove a slightly more general result, which implies not only that $\mathcal{D}^{\rm la}(H,K)$ is a Fréchet-Stein algebra, but also that the completed tensor product $A \,\hat{\otimes}_K\, \mathcal{D}^{\rm la}(H,K)$ is Fréchet-Stein, for a fairly general class of commutative Fréchet-Stein K-algebras A, namely, those possessing a Fréchet-Stein structure each of whose transition maps admits a certain kind of good integral model (as specified in definition 5.3.21).

In chapter 6 we study the various admissibility conditions that have arisen so far in locally analytic representation theory.

In section 6.1 we present our alternative definition of the category of admissible locally analytic G-representations, and prove that it is equivalent to the definition presented in [27].

In section 6.2, we recall the notion of strongly admissible locally analytic G-representation (introduced in [23]) and also the notion of admissible continuous G-representation (introduced in [25]). We prove that any strongly admissible locally analytic G-representation is an admissible G-representation, and also that if V is an admissible continuous G-representation, then $V_{\rm la}$ is a strongly admissible locally analytic G-representation. (Both of these results were originally established in [27].)

In section 6.3 we study admissible smooth and admissible locally algebraic representations. The main result concerning smooth representations is that a locally analytic smooth representation is admissible as a locally analytic representation if and only if it is admissible as a smooth representation, and that it is then necessarily equipped with its finest convex topology. (This was first proved in [27].)

In section 6.4 we suppose that the centre Z of G is topologically finitely generated. We are then able to define the category of essentially admissible locally analytic G-representations. As already indicated, these are locally analytic G-representations on a space V of compact type whose dual space V' is a coadmissible $\mathcal{C}^{\rm an}(\hat{Z},K) \,\hat{\otimes}_K\, \mathcal{D}^{\rm la}(H,K)$-module, for one (or equivalently any) compact open subgroup H of G. (The results of section 5.3 imply that $\mathcal{C}^{\rm an}(\hat{Z},K) \,\hat{\otimes}_K\, \mathcal{D}^{\rm la}(H,K)$ is a Fréchet-Stein algebra.)

In section 6.5 we introduce some simple notions related to invariant open lattices in G-representations. In particular, we say that a G-invariant open lattice M is admissible if the (necessarily smooth) action of G induced on $M/\pi M$ (where π is a uniformizer of the ring of integers of K) is admissible. We characterize strongly admissible locally analytic representations as being those essentially admissible locally analytic representations that contain a separated open H-invariant admissible lattice, for some (or equivalently, every) compact open subgroup H of G.

In chapter 7, we discuss the representations of groups of the form $G \times \Gamma$, where G is a locally L-analytic group and Γ is a Hausdorff locally compact topological group that admits a countable basis of neighbourhoods of the identity consisting of open subgroups. (The motivating example is a group of the form $\mathbb{G}(\mathbb{A}_f)$, where $\mathbb{G}$ is a reductive group defined over some number field F, $\mathbb{A}_f$ denotes the ring of finite adèles of F, and L is the completion of F at some finite prime.) In section 7.1 we introduce the notion of a strictly smooth representation of Γ on a Hausdorff convex K-vector space V. This is a useful strengthening of the notion of a smooth representation in the context of topological group actions, defined by requiring that the natural map $\varinjlim_{H} V^H \to V$ be a topological isomorphism. (Here the locally convex inductive limit is indexed by the open subgroups H of Γ.) In section 7.2 we extend the various notions of admissibility introduced in chapter 6 to the context of $G \times \Gamma$ representations. Briefly, a topological representation of $G \times \Gamma$ on a Hausdorff convex K-vector space V is said to be admissible locally analytic, essentially admissible locally analytic, or admissible continuous, if the Γ-action on V is strictly smooth, and if the closed subspace of invariants V^H has the corresponding property as a G-representation, for each open subgroup H of Γ. The section closes by considering locally algebraic representations in this context.

0.3. Terminology, notation, and conventions

We describe our terminological and notational conventions.

0.3.1. Throughout the memoir we fix a finite extension L of the field of p-adic numbers $\mathbb{Q}_p$ (for some fixed prime p), and let $\mathrm{ord}_L : L^\times \to \mathbb{Q}$ denote the discrete valuation on L, normalized so that $\mathrm{ord}_L(p) = 1$. We also fix an extension K of L, spherically complete with respect to a a non-archimedean valuation $\mathrm{ord}_K : K^\times \to \mathbb{R}$ extending ord_L. We let $\mathcal{O}_L$ and $\mathcal{O}_K$ denote the rings of integers of L and K respectively. We also let $|-|$ denote the absolute value on K induced by the valuation ord_K.

0.3.2. A topological K-vector space V is called locally convex if there is a neighbourhood basis of the origin consisting of $\mathcal{O}_K$-submodules. Let us remark that the theory of locally convex vector spaces over the non-archimedean field K can be developed in a fashion quite similar to that of locally convex real or complex topological vector spaces. In particular, the hypothesis that K is spherically complete ensures that the Hahn-Banach theorem holds for convex K-vector spaces [13]. Thus we will on occasion cite a result in the literature dealing with real or complex convex spaces and then apply the corresponding non-archimedean analogue, leaving it to the reader to check that the archimedean proof carries over to the non-archimedean setting. The lecture notes of Schneider [20] present a concise development of the foundations of non-archimedean functional analysis.

We will often abbreviate the phrase "locally convex topological K-vector space" to "convex K-vector space".

0.3.3. The category of convex K-vector spaces and continuous K-linear maps is additive, and all morphisms have kernels, cokernels, images, and coimages. We say that a continuous K-linear map $\phi : V \to W$ in this category is strict if its image is equal to its coimage. Concretely, this means that the quotient of V modulo the kernel of ϕ embeds as a topological subspace of W. In particular, a surjection of convex K-vector spaces is strict if and only if it is an open mapping.

We will say that a sequence of continuous K-linear maps of convex K-vector spaces is exact if it is exact as a sequence of K-linear maps, and if furthermore all the maps are strict.

0.3.4. Unless otherwise stated, any finite dimensional K-vector space is regarded as a convex K-vector space, by equipping it with its unique Hausdorff topology.

0.3.5. If V and W are two K-vector space we let $\mathrm{Hom}(V,W)$ denote the K-vector space of K-linear maps from V to W. The formation of $\mathrm{Hom}(V,W)$ is contravariantly functorial in V and covariantly functorial in W.

0.3.6. An important case of the discussion of (0.3.5) occurs when $W = K$. The space $\mathrm{Hom}(V,K)$ is the dual space to V, which we denote by $\check{V}$. The formation of $\check{V}$ is contravariantly functorial in V.

Passing to the transpose yields a natural injection $\mathrm{Hom}(V,W) \to \mathrm{Hom}(\check{W},\check{V})$.

0.3.7. If V and W are two convex K-vector spaces, then we let $\mathcal{L}(V,W)$ denote the K-vector subspace $\mathrm{Hom}(V,W)$ consisting of continuous K-linear maps. As with $\mathrm{Hom}(V,W)$, the formation of $\mathcal{L}(V,W)$ is contravariantly functorial in V and covariantly functorial in W.

The space $\mathcal{L}(V,W)$ admits various convex topologies. The two most common are the strong topology (that is, the topology of uniform convergence on bounded subsets of V), and the weak topology (that is, the topology of pointwise convergence). We let $\mathcal{L}_b(V,W)$, respectively $\mathcal{L}_s(V,W)$, denote the space $\mathcal{L}(V,W)$ equipped with the strong, respectively weak, topology. The formation of $\mathcal{L}_b(V,W)$ and $\mathcal{L}_s(V,W)$ are again both functorial in V and W.

0.3.8. An important case of the discussion of (0.3.7) occurs when $W = K$. The space $\mathcal{L}(V,K)$ is the topological dual space to V, which we will denote by V'. The Hahn-Banach theorem shows that if V is Hausdorff, then the functionals in V' separate the elements of V. The formation of V' is contravariantly functorial in V. We let V'_b and V'_s denote V equipped with the strong, respectively weak, topology. The formation of either of these convex spaces is contravariantly functorial in V.

Passing to the transpose yields the maps $\mathcal{L}(V,W) \to \mathcal{L}(W'_b,V'_b)$ and $\mathcal{L}(V,W) \to \mathcal{L}(W'_s,V'_s)$, which are injective when W is Hausdorff (since the elements of W' then separate the elements of W).

There are natural K-linear maps $V \to (V'_s)'$ and $V \to (V'_b)'$, which are injective when V is Hausdorff. The first is then in fact always a bijection, while the second is typically not. If it is, then V is called semi-reflexive.

There is a refinement of the notion of semi-reflexivity. Namely, we say that the Hausdorff convex K-vector space V is reflexive if the map $V \to (V'_b)'$ is a topological isomorphism, when the target is endowed with its strong topology.

0.3.9. If V and W are two convex K-vector spaces then there are (at least) two natural locally convex topologies that can be placed on the tensor product $V \otimes_K W$. The inductive tensor product topology is universal for separately continuous bilinear maps of convex K-vector spaces $V \times W \to U$, while the projective tensor product topology is universal for jointly continuous bilinear maps $V \times W \to U$. We let $V \otimes_{K,\iota} W$ denote $V \otimes_K W$ equipped with its inductive tensor product topology, and $V \otimes_{K,\pi} W$ denote $V \otimes_K W$ equipped with its projective tensor product topology. (This is the notation of [20, §17].) We denote the completions of these two convex K-vector spaces by $V \hat{\otimes}_{K,\iota} W$ and $V \hat{\otimes}_{K,\pi} W$ respectively. The identity map on $V \otimes_K W$ induces a natural continuous bijection $V \otimes_{K,\iota} W \to V \otimes_{K,\pi} W$, which extends to a continuous linear map $V \hat{\otimes}_{K,\iota} W \to V \hat{\otimes}_{K,\pi} W$.

In many contexts, the bijection $V \otimes_{K,\iota} W \to V \otimes_{K,\pi} W$ is a topological isomorphism. This is the case, for example, if W is finite dimensional, or if V and W are Fréchet spaces. (See proposition 1.1.31 below for an additional such case.) In these situations, we write simply $V \otimes_K W$ and $V \hat{\otimes}_K W$ to denote the topological tensor product and its completion.

0.3.10. If G is a group then we let e denote the identity element of G.

0.3.11. If V is a K-vector space and G is a group, then we will refer to a left action of G on V by K-linear endomorphisms of V simply as a G-action on V, or as a G-representation on V. (Thus by convention our group actions are always left actions.)

If V is furthermore a topological K-vector space, and G acts on V through continuous endomorphisms (or equivalently, topological automorphisms) of V, we will refer to the G-action as a topological G-action on V.[2]

Finally, if V is a topological K-vector space and G is a topological group, then we have the notion of a continuous G-action on V: this is an action for which the action map $G \times V \to V$ is continuous. (Such an action is thus in particular a topological G-action on V.)

One can also consider an intermediate notion of continuity for the action of a topological group G on a topological K-vector space V, in which one asks merely that the action map $G \times V \to V$ be separately continuous. We remind the reader that if V is a barrelled convex K-vector space and if G is locally compact then this *a priori* weaker condition in fact implies the joint continuity of the G-action. (Indeed, since G is locally compact, we see that the action of a neighbourhood of the identity in G is pointwise bounded, and thus equicontinuous, since V is barrelled. The claim is then a consequence of lemma 3.1.1 below.)

0.3.12. If V and W are two K-vector spaces each equipped with a G-action, then the K-vector space $\mathrm{Hom}(V, W)$ is equipped with a G-action, defined by the condition that $g(\phi(v)) = (g\phi)(gv)$, for $g \in G$, $\phi \in \mathrm{Hom}(V, W)$ and $v \in V$. An element of $\mathrm{Hom}(V, W)$ is fixed under this G-action precisely if it is G-equivariant. We let $\mathrm{Hom}_G(V, W)$ denote the subspace of $\mathrm{Hom}(V, W)$ consisting of G-fixed elements; it is thus the space of G-equivariant K-linear maps from V to W.

0.3.13. A particular case of the discussion of (0.3.12) occurs when V is a K-vector space equipped with a G-action, and $W = K$, equipped with the trivial G-action. The resulting G-action on $\check{V}$ is called the contragredient G-action, and is characterized by the condition $\langle g\check{v}, gv\rangle = \langle \check{v}, v\rangle$ for any elements $\check{v} \in \check{V}$ and $v \in V$. In other words, it is the action obtained on $\check{V}$ after converting the transpose action of G on $\check{V}$ (which is a right action) into a left action.

If W is a second K-vector space equipped with a G-action, then the natural injection $\mathrm{Hom}(V, W) \to \mathrm{Hom}(\check{W}, \check{V})$ is G-equivariant, if we endow the target with the G-action of (0.3.12) induced by the contragredient G-action on each of $\check{W}$ and $\check{V}$.

0.3.14. If V and W are two convex K-vector spaces each equipped with a topological G-action then $\mathcal{L}(V, W)$ is a G-invariant subspace of $\mathrm{Hom}(V, W)$. We let $\mathcal{L}_G(V, W)$ denote the subspace of G-invariant elements; this is then the space of continuous G-equivariant K-linear maps from V to W.

Since the formation of $\mathcal{L}_b(V, W)$ and $\mathcal{L}_s(V, W)$ is functorial in V and W, we see that the G-action on either of these convex spaces is topological.

[2] This is perhaps not the best imaginable terminology, but provides a succinct substitute for the more accurate, but somewhat lengthy, expression, "an action of G by topological automorphisms of V".

0.3.15. A particular case of the discussion of (0.3.14) occurs when V is a convex K-vector space equipped with a topological G-action, and $W = K$ equipped with the trivial G-action. Thus we see that V' is G-invariant with respect to the contragredient action of G on $\check{V}$, and that the contragredient action of G on either V'_b or V'_s is again topological.

Furthermore, if W is a second convex K-vector space equipped with a topological G-action, then the natural maps $\mathcal{L}(V, W) \to \mathcal{L}(W'_b, V'_b)$ and $\mathcal{L}(V, W) \to \mathcal{L}(W'_s, V'_s)$ are G-equivariant. (Here the G-action on the target of either of these maps is defined as in (0.3.14) using the contragredient G-action on W' and V'.)

0.3.16. We will frequently be considering groups or K-vector spaces equipped with additional topological structures, and consequently we will sometimes use the adjective "abstract" to indicate that a group or vector space is not equipped with any additional such structure. Similarly, we might write that a K-linear map between convex K-vector spaces is an isomorphism of abstract K-vector spaces if it is a K-linear isomorphism that is not necessarily a topological isomorphism.

0.3.17. A crucial point in non-archimedean analysis is the distinction between locally analytic and rigid analytic spaces (and hence between locally analytic and rigid analytic functions). Naturally enough, in the theory of locally analytic representations it is the notion of locally analytic function that comes to the fore. However, any detailed investigation of locally analytic phenomena ultimately reduces to the consideration of functions given by power series, and hence to the consideration of algebras of rigid analytic functions. We refer to [28] and [3] for the foundations of the locally and rigid analytic theories respectively.

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Chapter 1. Non-archimedean functional analysis

1.1. Functional analytic preliminaries

In this section we recall some functional analytic terminology, and establish some simple results that we will require. We begin with a discussion of FH-spaces, BH-spaces, spaces of LF-type, spaces of LB-type, LF-spaces, LB-spaces, and spaces of compact type. Our discussion of these notions owes quite a lot to that of [12, §1]

Definition 1.1.1. Let V be a Hausdorff locally convex topological K-vector space. We say that V is an FH-space if it admits a complete metric that induces a locally convex topology on V finer than than its given topology. We refer to the topological vector space structure on V induced by such a metric as a latent Fréchet space structure on V.

If V admits a latent Fréchet space structure that can be defined by a norm (that is, a latent Banach space structure), then we say that V is a BH-space.

Proposition 1.1.2. (i) *An FH-space V admits a unique latent Fréchet space structure. We let $\overline{V}$ denote V equipped with its latent Fréchet space structure.*

(ii) *If $f : V_1 \to V_2$ is a continuous morphism of FH-spaces then $f : \overline{V}_1 \to \overline{V}_2$ is a continuous morphism of Fréchet spaces.*

Proof. Suppose that V_1 and V_2 are FH-spaces and that $f : V_1 \to V_2$ is a continuous map between them, and let $\overline{V}_1$ and $\overline{V}_2$ denote *some* latent Fréchet space structure on each of these spaces. The identity map then induces a continuous bijection

$$\overline{V}_1 \times \overline{V}_2 \to V_1 \times V_2.$$

Since V_2 is Hausdorff, the graph Γ_f of f is closed in $V_1 \times V_2$, and so also in $\overline{V}_1 \times \overline{V}_2$. The closed graph theorem now shows that $f : \overline{V}_1 \to \overline{V}_2$ is continuous. This proves (ii). To prove (i), take $V_1 = V_2 = V$, let $\overline{V}_1$ and $\overline{V}_2$ be two latent Fréchet space structures on V, and let f be the identity map from V to itself. □

Proposition 1.1.3. (i) *If V is an FH-space* (*respectively a BH-space*) *and W is a closed subspace of V then W is also an FH-space* (*respectively a BH-space*), *and $\overline{W}$ is a closed subspace of $\overline{V}$.*

(ii) *If $V_1 \to V_2$ is a surjective map of Hausdorff topological K-vector spaces and V_1 is an FH-space* (*respectively a BH-space*) *then V_2 is an FH-space* (*respectively a BH-space*).

Proof. (i) The identity map from V to itself induces a continuous map $\overline{V} \to V$. The preimage of W under this map is a closed subspace of $\overline{V}$, which thus equips W with a latent Fréchet space structure, which is in fact a latent Banach space structure if $\overline{V}$ is a Banach space.

(ii) Let W denote the kernel of the composite $\overline{V}_1 \to V_1 \to V_2$, which is a continuous map. Since V_2 is Hausdorff we see that W is closed in $\overline{V}_1$, and so the map $\overline{V}_1/W \to V_2$ is a continuous bijection whose source is either a Fréchet space or a Banach space, depending on our hypothesis on V_1. This equips V_2 with either a latent Fréchet space structure or a latent Banach space structure. □

Definition 1.1.4. If V is a Hausdorff topological K-vector space, an FH-subspace (respectively a BH-subspace) W of V is a subspace W of V that becomes an FH-space (respectively a BH-space) when equipped with its subspace topology.

Proposition 1.1.5. *If W_1 and W_2 are FH-subspaces* (*respectively BH-subspaces*) *of the Hausdorff topological K-vector space V then so are $W_1 \bigcap W_2$ and $W_1 + W_2$.*

Proof. Let W denote the kernel of the morphism

$$\overline{W}_1 \oplus \overline{W}_2 \stackrel{(w_1,w_2)\mapsto w_1+w_2}{\longrightarrow} V. \tag{1.1.6}$$

Since this is a continuous morphism into the Hausdorff space V, we see that W is a closed subspace of $\overline{W}_1 \oplus \overline{W}_2$, and so is either a Fréchet space or a Banach space, depending on our hypothesis on W_1 and W_2.

The continuous morphism

$$\overline{W}_1 \oplus \overline{W}_2 \stackrel{(w_1,w_2)\mapsto w_1}{\longrightarrow} V$$

induces a continuous bijection from W to $W_1 \bigcap W_2$, and thus equips $W_1 \bigcap W_2$ with either a latent Fréchet space structure or a latent Banach space structure. The morphism (1.1.6) induces a continuous bijection from $(\overline{W}_1 \oplus \overline{W}_2)/W$ to $W_1 + W_2$, and thus equips $W_1 + W_2$ with either a latent Fréchet space structure or a latent Banach space structure. This proves the proposition. □

Proposition 1.1.7. *If $f : V_1 \to V_2$ is a continuous map of Hausdorff topological K-vector spaces and if W is an FH-subspace (respectively a BH-subspace) of V_1, then $f(W)$ is an FH-subspace (respectively a BH-subspace) of V_2.*

Proof. This follows from proposition 1.1.3 (ii). $\square$

Proposition 1.1.8. *Let V be a Hausdorff topological K-vector space, and let W be a finite dimensional K-vector space. If U is an FH-subspace (respectively a BH-subspace) of V then $U \otimes_K W$ is an FH-subspace (respectively a BH-subspace) of $V \otimes_K W$. Conversely, if T is an FH-subspace (respectively a BH-subspace) of $V \otimes_K W$, then there is an FH-subspace (respectively a BH-subspace) U of V such that $T \subset U \otimes_K W$.*

Proof. Let U be an FH-subspace or a BH-subspace of V. The latent Fréchet or Banach space structure $\overline{U}$ on U induces a latent Fréchet or Banach space structure $\overline{U} \otimes_K W$ on $U \otimes_K W$, proving the first assertion.

Suppose now that T is an arbitrary FH-subspace or BH-subspace of $V \otimes_K W$. What we have just proved shows that $T \otimes_K \check{W}$ is either an FH-subspace or a BH-subspace of $V \otimes_K W \otimes_K \check{W}$. Let U be the image of $T \otimes_K \check{W}$ under the natural map $V \otimes_K W \otimes_K \check{W} \to V$, given by contracting W against $\check{W}$. Proposition 1.1.7 shows that U is either an FH-subspace or a BH-subspace of V, and it is immediately seen that there is an inclusion $T \subset U \otimes_K W$. $\square$

Definition 1.1.9. We say that a Hausdorff convex K-vector space V is of LF-type (respectively of LB-type) if we may write $V = \bigcup_{n=1}^{\infty} V_n$, where $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ is an increasing sequence of FH-subspaces (respectively BH-subspaces) of V.

Proposition 1.1.10. *If V is a Hausdorff convex K-vector space of LF-type, say $V = \bigcup_{n=1}^{\infty} V_n$, for some increasing sequence $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ of FH-subspaces of V, then any FH-subspace of V is contained in V_n for some n.*

Proof. This follows from [5, prop. 1, p. I.20]. $\square$

If V is a Hausdorff locally convex topological K-vector space, if A is a bounded $\mathcal{O}_K$-submodule of V, and if V_A denotes the vector subspace of V spanned by A, then the gauge of A defines a norm on V_A, with respect to which the map $V_A \to V$ becomes continuous. If A is furthermore closed and semi-complete, then V_A is complete with respect to this norm [5, cor., p. III.8] (see also [20, lem. 7.17]), and hence is a BH-subspace of V. Conversely, if W is a BH-subspace of V, then the image of the unit ball of $\overline{W}$ is a bounded $\mathcal{O}_K$-submodule A of V, and the map $\overline{W} \to V$ factors as $\overline{W} \to V_B \to V$ for any bounded $\mathcal{O}_K$-submodule B of V containing A. (Here V_B is equipped with its norm topology.) In particular, if V is semi-complete, then the spaces V_A, as A ranges over all closed bounded $\mathcal{O}_K$-submodules of V, are cofinal in the directed set of all BH-subspaces of V.

Proposition 1.1.11. *If V is a semi-complete locally convex Hausdorff K-vector space, and if A is a bounded subset of V, then there is a BH-subspace W of V containing A, such that A is bounded when regarded as a subset of $\overline{W}$. If furthermore V is either Fréchet or of compact type (as defined below in definition* 1.1.16 (ii)), *then W can be chosen so that the topologies induced on A by V and $\overline{W}$ coincide.*

Proof. We begin by replacing A by the closed $\mathcal{O}_K$-submodule of V that it generates; this is then a closed bounded $\mathcal{O}_K$-submodule of V. The first claim then follows directly from the preceding discussion.

In the case where V is a Fréchet space, the second claim of the theorem follows from [20, lem. 20.5]. Suppose now that V is of compact type, and write $V \xrightarrow{\sim} \varinjlim_n V_n$, where the transition maps $V_n \to V_{n+1}$ are injective and compact maps between Banach spaces. Since V is complete and of LB-type, the first claim, together with proposition 1.1.10, shows that we may find an integer n and a bounded subset A_n of V_n such that A is equal to the image of A_n under the injection $V_n \to V$. (Alternatively, apply [20, lem. 16.9 (ii)], which in any case is used in the proof that V is complete [20, prop. 16.10 (i)].) Since the injection $V_n \to V_{n+1}$ is compact, the image A_{n+1} of A_n in V_{n+1} is c-compact (being closed, since it is the preimage of the closed subset A of V). The map $A_{n+1} \to A$ is thus a continuous bijection from the c-compact $\mathcal{O}_K$-module A_{n+1} to the $\mathcal{O}_K$-module A, and so is a topological isomorphism. Thus taking $\overline{W} = V_{n+1}$ proves the second claim in the case where V is of compact type. □

If V is a Hausdorff convex K-vector space, then there is a continuous bijection $\varinjlim_W \overline{W} \to V$, where W runs over all BH-subspaces of V. The spaces for which this map is a topological isomorphism are said to be ultrabornological [5, ex. 20, p. III.46]. In particular, semi-complete bornological Hausdorff convex spaces are ultrabornological [5, cor., p. III.12]. Since the property of being either barrelled or bornological is preserved under the passage to locally convex inductive limits, we see that any ultrabornological Hausdorff convex K vector space is necessarily both barrelled and bornological.

Suppose that V is a convex K-vector space described as a locally convex inductive limit $V = \varinjlim_{i\in I} V_i$, and that U is a closed subspace of V. If U_i denotes the preimage of U in V_i, for each $i \in I$, then the natural map $\varinjlim_{i\in I} U_i \to U$ is a continuous bijection, but in general it is not a topological isomorphism.

Proposition 1.1.12. *Suppose that V is a Hausdorff ultrabornological convex K-vector space, and write $V = \varinjlim_W \overline{W}$, where W ranges over all BH-subspaces of V. If U is a closed subspace of V that is also ultrabornological, then the natural map $\varinjlim_W \overline{U \bigcap W} \to U$ is a topological isomorphism.*

Proof. Since U is ultrabornological, by assumption, it is the inductive limit of its BH-subspaces. Each of these subspaces is a BH-subspace of V, and so $U \bigcap W$ ranges over all BH-subspaces of U, as W ranges over all BH-subspaces of V. □

The following definition provides a relative version of the notion of BH-space.

Definition 1.1.13. A continuous linear map between Hausdorff topological K-vector spaces $U \to V$ is said to be BH if it admits a factorization of the form $U \to \overline{W} \to V$, where $\overline{W}$ is a K-Banach space.

Equivalently, $U \to V$ is a BH-map if its image lies in a BH-subspace W of V, and if it lifts to a continuous map $U \to \overline{W}$.

Obviously, any map whose source or target is a K-Banach space is a BH-map.

Lemma 1.1.14. *Any compact map between locally convex Hausdorff topological K-vector spaces is a BH-map.*

Proof. Given a compact map $U \to V$, there is by definition an open $\mathcal{O}_K$-submodule of U whose image in V has c-compact (and in particular bounded and complete) closure. If we denote this closure by A, we may factor $U \to V$ into continuous maps $U \to V_A \to V$. Since A is closed, bounded, and complete the above discussion shows that V_A is a Banach space, and so the map $U \to V$ is a BH-map. (This argument is taken from [20, p. 101]; note that we are using the notation and results discussed prior to the statement of proposition 1.1.11.) $\square$

Lemma 1.1.15. *If V is a semi-complete convex Hausdorff K-vector space, then a map $U \to V$ is a BH-map if and only if some open neighbourhood of the origin in U has bounded image in V.*

Proof. Since Banach spaces contain bounded open neighbourhoods of the origin, it is clear that any BH-map $U \to V$ has the stated property. Conversely, suppose that some open neighbourhood of U maps into a bounded subset A of V. We may as well suppose that A is a closed $\mathcal{O}_K$-submodule of V. Then $U \to V$ factors through the natural map $V_A \to V$, and so is a BH-map. (We are using the notation and results discussed prior to the statement of proposition 1.1.11.) $\square$

Definition 1.1.16. (i) We say that a locally convex topological K-vector space is an LF-space (respectively, LB-space) if it is isomorphic to the locally convex inductive limit of a sequence of K-Fréchet spaces (respectively K-Banach spaces).

(ii) We say that a Hausdorff locally convex topological K-vector space is of compact type if it is isomorphic to the locally convex inductive limit of a sequence of convex K-vector spaces in which the transition maps are compact.

Let V be a Hausdorff LF-space (respectively LB-space), and write $V \xrightarrow{\sim} \varinjlim_n V_n$, where $\{V_n\}$ is an inductive sequence of K-Fréchet spaces (respectively K-Banach spaces). If we let W_n denote the quotient of V_n by the kernel of the natural map $V_n \to V$, then W_n is again a K-Fréchet space (respectively K-Banach space), since V is Hausdorff by assumption. The transition maps between the spaces V_n induce transition maps between the spaces W_n, which thus form an inductive sequence, and the natural map $\varinjlim_n V_n \to \varinjlim_n W_n$ is an isomorphism. Hence V may be written as the inductive limit of a sequence of K-Fréchet spaces (respectively K-Banach spaces) with injective transition maps. In particular V is of LF-type (respectively, LB-type). In fact, proposition 1.1.10 shows that the Hausdorff LF-spaces (respectively LB-spaces) are precisely the Hausdorff ultrabornological convex spaces of LF-type (respectively LB-type). (To see the LF-case of this assertion, use the fact that Fréchet spaces are ultrabornological.)

Any convex K-vector space V of compact type is in fact an LB-space (as follows from Lemma 1.1.14). Write $V \xrightarrow{\sim} \varinjlim_n V_n$, where $\{V_n\}$ is an inductive sequence of K-Banach spaces with compact transition maps. As in the preceding paragraph, let W_n denote the quotient of V_n by the kernel of the natural map $V_n \to V$. The transition maps in the inductive sequence $\{W_n\}$ are again compact, and thus any compact type convex K-vector space may in fact be written as the locally convex inductive limit of an inductive sequence of K-Banach spaces having compact injective transition maps. Conversely, the locally convex inductive limit of any inductive sequence of K-Banach spaces having compact injective transition maps is Hausdorff [20, lem. 16.9], and so is of compact type. We refer to [20, prop. 16.10] and [23, p. 445] for a fuller discussion of spaces of compact type. Of particular importance is the fact that any such space is complete and reflexive, and that passage to the strong dual induces an anti-equivalence of categories between the category of compact convex K-vector spaces and the category of nuclear K-Fréchet spaces.

We recall the following version of the open mapping theorem.

Theorem 1.1.17. *A continuous surjection $V \to W$ from a convex Hausdorff LF-space to a convex Hausdorff space is strict if and only if W is an LF-space.*

Proof. The forward implication is a special case of [5, cor., p. II.34]. To prove the converse implication, write $V \xrightarrow{\sim} \varinjlim_n V_n$ as an inductive limit of a sequence of Fréchet spaces, and let $V \to W$ be a strict surjection. For each value of n, let W_n denote the quotient of the Fréchet space V_n by the kernel of the composite $V_n \to V \to W$; then W_n is again a Fréchet space. The surjection from V to W factors as

$$V \to \varinjlim_n W_n \to W,$$

and the second morphism is a continuous bijection. Since the composite of both morphisms is a strict surjection, by assumption, so is this second morphism. Thus it is a topological isomorphism, and hence W is an LF-space, as claimed. □

In particular, a continuous bijection between LF-spaces is necessarily a topological isomorphism.

Proposition 1.1.18. *If V is a Hausdorff convex K-vector space that is both an LB-space and normable then V is a Banach space.*

Proof. Write $V \xrightarrow{\sim} \varinjlim_n V_n$. We may and do assume, without loss of generality, that the transition morphisms $V_n \to V_{n+1}$ are injective. Passing to duals yields a continuous bijection

$$V'_b \to \varprojlim_n (V'_n)_b. \tag{1.1.19}$$

(Recall that the subscript b denotes that each dual is equipped with its strong topology.) Since V is normable, its dual V'_b is a Banach space. Similarly, each dual $(V'_n)_b$ is a Banach space, and so $\varprojlim_n (V'_n)_b$ is a Fréchet space. Thus the open mapping theorem shows that (1.1.19) is an isomorphism.

Let $p_n : V'_b \to (V'_n)_b$ denote the natural map. Let U denote the unit ball in V'_b, and U_n the unit ball in $(V'_n)_b$ (with respect to some choices of norms on these Banach spaces). Taking into account the fact that (1.1.19) is an isomorphism, together with the definition of the projective limit topology, we see that we may find some n and some $\alpha \in K^\times$ such that

$$p_n^{-1}(\alpha U_n) \subset U. \tag{1.1.20}$$

Since U contains no non-trivial subspace of V'_b, we see from (1.1.20) that p_n has a trivial kernel, and so is injective. Thus (1.1.20) shows that in fact p_n is a topological embedding.

Let $\hat{V}$ denote the Banach space obtained by completing V. The natural map $\hat{V}'_b \to V'_b$ is an isomorphism, and so the natural map $\hat{V}'_b \to (V'_n)_b$ is a topological embedding. We deduce from [5, cor. 3, p. IV.30] that the injection

$$V_n \to \hat{V} \tag{1.1.21}$$

is a topological embedding, and hence has a closed image. Since (1.1.21) also has dense image (since $V' \to V'_n$ is injective), it is a topological isomorphism. We conclude that the natural map $V \to \hat{V}$ is a topological isomorphism, as required. □

Proposition 1.1.22. *If V is a Hausdorff convex K-vector space that is both an LF-space and semi-complete, then writing $V \xrightarrow{\sim} \varinjlim_n V_n$ as the inductive limit of a sequence of Fréchet spaces, we have that for any convex K-vector space W, the natural map $\mathcal{L}_b(V,W) \to \varprojlim_n \mathcal{L}_b(V_n,W)$ is a topological isomorphism.*

Proof. Since $V \xrightarrow{\sim} \varinjlim_n V_n$, we see that for any convex K-vector space W the natural map $\mathcal{L}_b(V,W) \to \varprojlim_n \mathcal{L}_b(V_n,W)$ is certainly a continuous bijection. Propositions 1.1.10 and 1.1.11 show that it is in fact a topological isomorphism, as claimed. □

Corollary 1.1.23. *If V is a semi-complete locally convex Hausdorff LB-space, and if W is a Fréchet space, then $\mathcal{L}_b(V,W)$ is a Fréchet space.*

Proof. If we write $V = \varinjlim_n V_n$ as the locally convex inductive limit of a sequence of Banach spaces, then proposition 1.1.22 shows that $\mathcal{L}_b(V,W) \xrightarrow{\sim} \varprojlim_n \mathcal{L}_b(V_n,W)$. Since a projective limit of a sequence of Fréchet spaces is again a Fréchet space, it suffices to prove the corollary in the case when V is a Banach space. Since W is a Fréchet space, it is obvious that $\mathcal{L}_b(V_n,W)$ has a countable basis of neighbourhoods of the origin. That it is complete follows from [20, prop. 7.16]. □

Proposition 1.1.24. *If V is a semi-complete locally convex Hausdorff LB-space, and if W is a Fréchet space, then any morphism $V \to W$ is a BH-morphism.*

Proof. Suppose given such a morphism. By lemma 1.1.15, we must show that there is an open neighbourhood of the origin of V that has bounded image in W. The transpose of the given map induces a continuous map

$$W'_b \to V'_b. \tag{1.1.25}$$

Corollary 1.1.23 shows that V'_b is a Fréchet space. Thus [5, lem. 1, p. IV.26] implies that some neighbourhood of the origin of W'_b maps into a bounded subset of V'_b. Since V is bornological (being an LB-space), any bounded subset of V'_b is equicontinuous. Thus (1.1.25) maps the pseudo-polar of some bounded $\mathcal{O}_K$-submodule of W into the pseudo-polar of an open lattice in V. From [20, prop. 13.4] we conclude that the original map $V \to W$ maps some open lattice in V into a bounded $\mathcal{O}_K$-submodule of W, and we are done. □

We now establish some results pertaining to topological tensor products.

Proposition 1.1.26. *Suppose that U, V and W are Hausdorff locally convex K-vector spaces, with U bornological, and V and W complete. If we are given an injective continuous map $V \to W$, then the induced map $U \hat{\otimes}_{K,\pi} V \to U \hat{\otimes}_{K,\pi} W$ is again injective.*

Proof. Since U is bornological, the double duality map $U \to (U'_b)'_b$ is a topological embedding [20, lem. 9.9], which by [20, cor. 17.5 (ii)] induces a commutative diagram

$$\begin{tikzcd} U \hat{\otimes}_{K,\pi} V \arrow[r] \arrow[d] & U \hat{\otimes}_{K,\pi} W \arrow[d] \\ (U'_b)'_b \hat{\otimes}_{K,\pi} V \arrow[r] & (U'_b)'_b \hat{\otimes}_{K,\pi} W \end{tikzcd}$$

in which the vertical arrows are again topological embeddings. Thus it suffices to prove the proposition with U replaced by $(U'_b)'_b$. Propositions 7.16 and 18.2 of [20] now yield a commutative diagram

$$\begin{tikzcd} (U'_b)'_b \hat{\otimes}_{K,\pi} V \arrow[r] \arrow[d] & (U'_b)'_b \hat{\otimes}_{K,\pi} W \arrow[d] \\ \mathcal{L}(U'_b, V) \arrow[r] & \mathcal{L}(U'_b, W), \end{tikzcd}$$

in which the vertical arrows identify each of the topological tensor products with the subspace of the corresponding spaces of linear maps consisting of completely continuous maps. These arrows are thus injective, and so is the lower horizontal arrow (being induced by the injection $V \to W$). Consequently the upper horizontal arrow is also injective, as claimed. $\square$

Corollary 1.1.27. *If $U_0 \to U_1$ and $V_0 \to V_1$ are injective continuous maps of K-Banach spaces, then the induced map $U_0 \hat{\otimes}_K V_0 \to U_1 \hat{\otimes}_K V_1$ is injective.*

Proof. Banach spaces are bornological and complete. Thus this follows by applying proposition 1.1.26 to the pair of maps $U_0 \hat{\otimes}_K V_0 \to U_0 \hat{\otimes}_K V_1$ and $U_0 \hat{\otimes}_K V_1 \to U_1 \hat{\otimes}_K V_1$. $\square$

Proposition 1.1.28. *If V and W are two nuclear K-Fréchet spaces then the completed topological tensor product $V \hat{\otimes}_K W$ is again a nuclear K-Fréchet space.*

Proof. The completed tensor product of two Fréchet spaces is again a Fréchet space. Propositions 19.11 and 20.4 of [20] show that if V and W are nuclear, then so is $V \hat{\otimes}_K W$. $\square$

Proposition 1.1.29. *If V and W are two Fréchet spaces, each described as a projective limit of a sequence of Fréchet spaces, $V \xrightarrow{\sim} \varprojlim_n V_n$ and $W \xrightarrow{\sim} \varprojlim_n W_n$, then there is a natural isomorphism $V \hat{\otimes}_K W \xrightarrow{\sim} \varprojlim_n V_n \hat{\otimes}_K W_n$.*

Proof. The functoriality of the formation of projective tensor products and of completions shows that there are natural maps

$$V \otimes_K W \to V \hat{\otimes}_K W \to \varprojlim_n V_n \hat{\otimes}_K W_n.$$

A consideration of the definition of the projective topology on $V \otimes_K W$, and hence of the topology on the Fréchet space $V \hat{\otimes}_K W$, shows that the second of these maps is a topological isomorphism. $\square$

Lemma 1.1.30. *If V and W are two convex K-vector spaces expressed as locally convex inductive limits, say $V \xrightarrow{\sim} \varinjlim_{i\in I} V_i$ and $W \xrightarrow{\sim} \varinjlim_{j\in J} W_j$, then there is a natural isomorphism $\varinjlim_{(i,j)\in I\times J} V_i \otimes_{K,\iota} W_j \xrightarrow{\sim} V \otimes_{K,\iota} W$.*

Proof. For each $(i,j) \in I \times J$, the functoriality of the formation of inductive tensor products yields a continuous map $V_i \otimes_{K,\iota} W_j \to V \otimes_{K,\iota} W$. Passing to the inductive limit yields a continuous map $\varinjlim_{(i,j)\in I\times J} V_i \otimes_{K,\iota} W_j \to V \otimes_{K,\iota} W$, which is in fact a bijection (since on the level of abstract K-vector spaces, the formation of tensor products commutes with passing to inductive limits). A consideration of the universal property that defines the inductive tensor product topology shows that this map is a topological isomorphism. □

Proposition 1.1.31. *If V and W are two semi-complete LB-spaces, then the natural bijection $V \otimes_{K,\iota} W \to V \otimes_{K,\pi} W$ is a topological isomorphism.*

Proof. This follows from the argument used to prove [5, thm. 2, p. IV.26]. Corollary 1.1.23 replaces the citation of "IV, p. 23, prop. 3" in that argument, while proposition 1.1.24 replaces the citation of "lemma 1". □

Not only is the preceding proposition proved in the same way as [5, thm. 2, p. IV.26], it generalizes that result. Indeed, the strong dual of a reflexive Fréchet space is a complete LB-space (as follows from propositions 2 and 4 of [5, pp. IV.22–23]).

As was already mentioned in (0.3.9), when the hypotheses of the preceding proposition apply, we write simply $V \otimes_K W$ to denote the tensor product of V and W, equipped with its inductive (or equivalently, its projective) tensor product topology. We write $V \hat{\otimes}_K W$ to denote the completion of $V \otimes_K W$.

Proposition 1.1.32. *Let V and W be convex K-vector spaces of compact type, and let $\varinjlim_n V_n \xrightarrow{\sim} V$ and $\varinjlim_n W_n \xrightarrow{\sim} W$ be expressions of V and W respectively as the locally convex inductive limit of Banach spaces, with injective and compact transition maps.*

(i) *There is a natural isomorphism $\varinjlim_n V_n \hat{\otimes}_K W_n \xrightarrow{\sim} V \hat{\otimes}_K W$. In particular, $V \hat{\otimes}_K W$ is of compact type.*

(ii) *The compact space $V \hat{\otimes}_K W$ is strongly dual to the nuclear Fréchet space $V'_b \hat{\otimes}_K W'_b$.*

Proof. We begin with (i). Consider the commutative diagram

$$\begin{tikzcd} \varinjlim_n V_n \otimes_K W_n \arrow[r] \arrow[d] & V \otimes_K W \arrow[d] \\ \varinjlim_n V_n \hat{\otimes}_K W_n \arrow[r] & V \hat{\otimes}_K W. \end{tikzcd} \tag{1.1.33}$$

Lemma 1.1.30 implies that the top horizontal arrow is a topological isomorphism. By definition, the right-hand vertical arrow is a topological embedding that identifies its target with the completion of its source. We will show that the same is true of the left-hand vertical arrow. This will imply that the bottom horizontal is also a topological isomorphism, as required.

The fact that the composite of the top horizontal arrow and the right-hand vertical arrow in (1.1.33) is a topological embedding implies that the left-hand vertical arrow is also a topological embedding. It clearly has dense image. From corollary 1.1.27 and [20, lem. 18.12] we see that $\varinjlim_n V_n \hat{\otimes}_K W_n$ is the inductive limit of a sequence of Banach spaces with compact and injective transition maps, hence is of compact type, and so in particular is complete. Thus the left-hand vertical arrow of (1.1.33) does identify its target with the completion of its source, and the discussion of the preceding paragraph shows that (i) is proved.

In order to prove part (ii), note that the strong duals V'_b and W'_b are nuclear Fréchet spaces (and so in particular are reflexive). It follows from [20, prop. 20.13] that there is a topological isomorphism $V \hat{\otimes}_K W \xrightarrow{\sim} (V'_b \hat{\otimes}_K W'_b)'_b$. Part (ii) follows upon appealing to the reflexivity of the convex space $V \hat{\otimes}_K W$. (Alternatively, proposition 1.1.28 shows that $V'_b \hat{\otimes}_K W'_b$ is again a nuclear Fréchet space. This also provides an alternative proof that $V \hat{\otimes}_K W$ is of compact type.) $\square$

Lemma 1.1.34. *If U and V are convex K-vector spaces, with U barrelled, and if A and B are bounded $\mathcal{O}_K$-submodules of U and V respectively, then $A \otimes_{\mathcal{O}_K} B$ is a bounded $\mathcal{O}_K$-submodule of $U \otimes_{K,\imath} V$.*

Proof. To prove the lemma, it suffices to show that for any separately continuous K-bilinear pairing $U \times V \to W$, the induced map $U \otimes_{K,\imath} V \to W$ takes $A \otimes_{\mathcal{O}_K} B$ to a bounded subset of W. The given pairing realizes B as a pointwise bounded set of continuous maps $U \to W$. Since U is barrelled, it follows that in fact B induces an equicontinuous set of maps $U \to W$. Thus $A \otimes_{\mathcal{O}_K} B$ does indeed have bounded image in W, as required. $\square$

Proposition 1.1.35. *If U, V and W are convex K-vector spaces, with U barrelled, then the natural isomorphism of abstract K-vector spaces* $\mathrm{Hom}(U \otimes_K V, W) \xrightarrow{\sim} \mathrm{Hom}(U, \mathrm{Hom}(V, W))$ (*expressing the adjointness of* $\otimes$ *and* Hom) *induces a continuous bijection of convex K-vector spaces*

$$\mathcal{L}_b(U \otimes_{K,\imath} V, W) \to \mathcal{L}_b(U, \mathcal{L}_b(V, W)).$$

Proof. It follows from [5, prop. 6, p. III.31] that the adjointness of $\otimes$ and Hom induces an isomorphism of abstract K-vector spaces

$$\mathcal{L}(U \otimes_{K,\imath} V, W) \xrightarrow{\sim} \mathcal{L}(U, \mathcal{L}_b(V, W)).$$

It remains to be shown that this map is continuous, when both the source and the target are equipped with their strong topology. This follows from lemma 1.1.34. $\square$

We end this section with a presentation of some miscellaneous results and definitions.

Proposition 1.1.36. *Let V and W be Hausdorff convex K-vector spaces, and suppose that every bounded subset of W'_b is equicontinuous as a set of functionals on W. (This holds if W is barrelled or bornological, for example.) Then passing to the transpose induces a topological embedding of convex spaces $\mathcal{L}_b(V, W) \to \mathcal{L}_b(W'_b, V'_b)$.*

Proof. As observed in (0.3.8), passing to the transpose always induces an injection $\mathcal{L}_b(V, W) \to \mathcal{L}_b(W'_b, V'_b)$. We must show that under the assumptions of the proposition this is a topological embedding.

Let B' be bounded in W'_b and let U' be a neighbourhood of zero in V'_b. Write $\mathfrak{m}$ to denote the maximal ideal of $\mathcal{O}_K$. Then by definition of the strong topology on V', there exists a bounded subset B of V such that $U' \supset \{v' \in V' \mid v'(B) \subset \mathfrak{m}\}$. A neighbourhood basis of the origin in $\mathcal{L}_b(W'_b, V'_b)$ is thus given by the sets $S'_{B,B'} = \{\phi' : W'_b \to V'_b \mid \phi'(b')(B) \subset \mathfrak{m}$ for all $b' \in B'\}$, as B and B' range over the bounded subsets of V and W'_b respectively. If $S_{B,B'}$ denotes the preimage of $S'_{B,B'}$ in $\mathcal{L}(V, W)$, then $S_{B,B'} = \{\phi : V \to W \mid b'(\phi(B)) \subset \mathfrak{m}$ for all $b' \in B'\}$.

If B is a bounded subset of V and if U is a neighbourhood of zero in W, then write $T_{B,U} = \{\phi : V \to W \mid \phi(B) \subset U\}$. As B runs over all bounded subsets of V and U runs over all neighbourhoods of zero in W the sets $T_{B,U}$ form a neighbourhood basis of zero in $\mathcal{L}_b(V, W)$. Thus we must compare the collections of subsets $T_{B,U}$ and $S_{B,B'}$ of $\mathcal{L}(V, W)$. Since W is locally convex, we may restrict our attention to those $T_{B,U}$ for which U is an open $\mathcal{O}_K$-submodule of W.

If U is an open $\mathcal{O}_K$-submodule of W and we set $B'_U = \{w' \in W' \mid w'(U) \subset \mathfrak{m}\}$ then B'_U is a bounded subset of W'_s, and the theory of pseudo-polars (see [20, cor. 13.5]) shows that $U = \{w \in W \mid b'(w) \in \mathfrak{m}$ for all $b' \in B'_U\}$. We conclude that $T_{B,U} = S_{B,B'_U}$.

Conversely, if B' is an arbitrary bounded subset of W'_b then by assumption B' is an equicontinuous set of functionals on W, and so there exists an open $\mathcal{O}_K$-submodule U in W such that $b'(U) \subset \mathfrak{m}$ for every $b' \in B'$. Thus $B' \subset B'_U$, and so $S_{B,B'} \supset S_{B,B'_U} = T_{B,U}$. This result, together with that of the preceding paragraph, shows that $\mathcal{L}_b(V, W) \to \mathcal{L}_b(W'_b, V'_b)$ is a topological embedding, as required. □

Proposition 1.1.36 is an extension of the well-known fact that for any convex K-vector space W satisfying the hypothesis of the proposition, the double duality map $W \to (W'_b)'_b$ is a topological embedding. (See [5, p. IV.15], or [20, lem. 9.9].)

Definition 1.1.37. If V is a locally convex topological K-vector space, then we define the bounded-weak topology on V' to be the finest locally convex topology on V' which induces on each bounded subset of V'_b a topology coarser than that induced by V'_s. We let V'_{bs} denote V' equipped with the bounded-weak topology.

Proposition 1.1.38. *Suppose given locally convex spaces V and W, with W barrelled and complete, and a continuous linear map $V \to W$. If this map has the property that it maps any bounded subset of V into a c-compact subset of W, then its transpose induces a continuous map $W'_{bs} \to V'_b$.*

Proof. By [19, prop. 5, p. 153], we see that the transpose map $W'_b \to V'_b$ also takes bounded subsets into c-compact subsets.

Let B be a bounded subset of W'_b. Since W is barrelled, the classes of equicontinuous, strongly bounded, and weakly bounded subsets of W' coincide. Also, any weakly closed weakly bounded subset of W', being also equicontinuous, is in fact weakly c-compact.

We see in particular that the weak closure of a strongly bounded set is again strongly bounded, and thus that any strongly bounded subset of W' is contained in a strongly bounded subset that is weakly c-compact. Let B be such a subset of W'. The image of B is then c-compact with respect to the topology on V'_s. In particular it is closed in V'_s, and so also in V'_b. On the other hand this image is contained in a c-compact subset of V'_b, and thus is itself a c-compact subset of V'_b. The topologies on the image of B induced by V'_b and V'_s thus coincide, and so the transpose map

$W'_s \to V'_b$ is continuous when restricted to B. By definition, we conclude that it yields a continuous map $W'_{bs} \to V'_b$, as claimed. $\square$

Definition 1.1.39. We say that a Hausdorff convex K-vector space V is hereditarily complete if any Hausdorff quotient of V is complete.

Fréchet spaces are hereditarily complete, as are spaces of compact type. (In fact, the strong dual to any reflexive Fréchet space is hereditarily complete, by [19, cor., p. 114] and the discussion on page 123 of this reference.)

Proposition 1.1.40. *Any closed subspace of a hereditarily complete Hausdorff convex K-vector space is again hereditarily complete.*

Proof. Let $V \to W$ be a closed embedding, with W a hereditarily complete convex K-vector space. If U is a closed subspace of V, then V/U is a closed subspace of W/U. Since W/U is complete, by assumption, the same if true of V/U. Thus V is also hereditarily complete. $\square$

Proposition 1.1.41. *Let V be convex K-vector space of compact type, and write $V \xrightarrow{\sim} \varinjlim_n V_n$, where $\{V_n\}$ is a sequence of Banach spaces with compact injective transition maps. Suppose given a closed subspace $U_n \subset V_n$ for each n, such that for all n the preimage of U_{n+1} under the map $V_n \to V_{n+1}$ is equal to U_n. Then $U = \varinjlim_n U_n$ is again of compact type, and the natural map $U \to V$ is a closed embedding.*

Proof. For each n write $W_n = V_n/U_n$. It is immediate from our assumptions that for each n, the maps $U_n \to U_{n+1}$ and $W_n \to W_{n+1}$ are compact and injective. Thus we obtain the diagram of continuous K-linear maps

$$0 \to \varinjlim_n U_n \to \varinjlim_n V_n \to \varinjlim_n W_n \to 0,$$

which is short exact as a diagram of abstract K-vector spaces, and in which each of the spaces appearing is of compact type. Theorem 1.1.17 implies that the maps in this diagram are in fact strict, and so in particular that $\varinjlim_n U_n \to \varinjlim_n V_n$ is a closed embedding, as claimed. $\square$

1.2. Fréchet-Stein algebras

In this section we recall some basic aspects of the theory of Banach and Fréchet K-algebras, and especially the notion of Fréchet-Stein structure, introduced in [27].

We adopt the convention that all modules are left modules.

Definition 1.2.1. (i) A locally convex topological K-algebra is a locally convex topological K-vector space A equipped with a K-algebra structure, such that the multiplication map $A \times A \to A$ is jointly continuous. (Equivalently, such that the multiplication map induces a continuous morphism $A \otimes_{K,\pi} A \to A$.)

If A is a K-Banach space (respectively, a K-Fréchet space), then we will refer to A as a K-Banach algebra (respectively, a K-Fréchet algebra).

(ii) If A is a locally convex topological K-algebra, then a locally convex topological A-module is a locally convex topological K-vector space M equipped with an A-module structure (compatible with its K-vector space structure), such that the multiplication map $A \times M \to M$ is jointly continuous. (Equivalently, such that the multiplication map induces a continuous morphism $A \otimes_{K,\pi} M \to M$.)

If M is a K-Banach space (respectively, a K-Fréchet space), then we will refer to M as an A-Banach module (respectively, an A-Fréchet module).

(iii) If A is a locally convex topological K-algebra, and M is a locally convex topological A-module, then we say that M is finitely generated over A (as a topological module) if there exists a strict surjection of A-modules $A^n \to M$ for some $n \geq 0$.

Note that in part (iii), when we say that a topological module is finitely generated, we require that it be presentable topologically (not just algebraically) as a quotient of some finite power of A.

We remark that it follows from [3, prop. 1.2.1/2] that if A is a Banach algebra in the sense defined above, then A admits a norm with respect to which it becomes a Banach algebra in the usual sense [3, def. 3.7.1/1].

Lemma 1.2.2. *If A is a locally convex topological K-algebra, and M is a locally convex topological A-module, then there is a unique locally convex topological A-module structure on the completion $\hat{M}$ of M, such that the natural morphism $M \to \hat{M}$ becomes A-linear.*

Proof. Clear. □

Lemma 1.2.3. *Let $A \to B$ be a continuous homomorphism of locally convex topological K-algebras, and let M be a locally convex topological A-module. If $B \otimes_A M$ is endowed with the quotient topology obtained by regarding it as a quotient of $B \otimes_{K,\pi} M$, then it becomes a locally convex topological B-module.*

Proof. Clear. □

In the context of lemma 1.2.3, we will always regard $B \otimes_A M$ as a topological B-module by endowing with it the quotient topology induced from $B \otimes_{K,\pi} M$. We let $B \hat{\otimes}_A M$ denote the completion of the locally convex B-module $B \otimes_A M$.

Proposition 1.2.4. *Let A be a Noetherian K-Banach algebra.*

(i) *Every finitely generated A-module has a unique structure of K-Banach space making it an A-Banach module.*

(ii) *If $f : M \to N$ is an A-linear morphism between finitely generated A-modules, then f is continuous and strict with respect to the Banach space structures on M and N given by* (i).

Proof. Part (i) is a restatement of [3, props. 3.7.3/3]. Part (ii) is a restatement of [3, prop. 3.7.3/2, cor. 3.7.3/5]. □

This proposition shows in particular that if A is Noetherian, then the forgetful functor from the category of finitely generated A-Banach modules to the category of finitely generated abstract A-modules is an equivalence of categories. In particular, since the latter category is abelian (A being Noetherian), the same is true of the former.

Proposition 1.2.5. *Let $A \to B$ be a continuous homomorphism of locally convex topological K-algebras, and suppose that B is hereditarily complete* (*in the sense of definition* 1.1.39). *If M is a finitely generated locally convex topological A-module,* *then the natural map $B \otimes_A M \to B \hat{\otimes}_A M$ is surjective, and $B \hat{\otimes}_A M$ is a finitely generated topological B-module.*

Proof. By assumption there is a strict surjection $A^n \to M$ for some $n \geq 0$, and hence a strict surjection $B^n \to B \otimes_A M$. Since B, and hence B^n, is hereditarily complete, the quotient of B^n by the closure of the kernel of this surjection is complete. Thus $B \otimes_A M \to B \hat{\otimes}_A M$ is surjective, as claimed. □

Note that if A and B are Noetherian Banach algebras, then [3, prop. 3.7.3/6] shows that the map $B \otimes_A M \to B \hat{\otimes}_A M$ is in fact an isomorphism.

Definition 1.2.6. Let A be a locally convex topological K-algebra. A weak Fréchet-Stein structure on A consists of the following data:

(i) A sequence of locally convex topological K-algebras $\{A_n\}_{n \geq 1}$, such that each A_n is hereditarily complete (in the sense of definition 1.1.39).

(ii) For each $n \geq 1$, a continuous K-algebra homomorphism $A_{n+1} \to A_n$, which is a BH-map of convex K-vector spaces (in the sense of definition 1.1.13).

(iii) An isomorphism of locally convex topological K-algebras $A \xrightarrow{\sim} \varprojlim_n A_n$ (the projective limit being taken with respect to the maps of (ii)), such that each of the induced maps $A \to A_n$ has dense image.

We say that A is a weak Fréchet-Stein K-algebra if it admits a weak Fréchet-Stein structure.

Note that if A admits a weak Fréchet-Stein structure, then it is the projective limit of a sequence of Hausdorff convex K-vector spaces with BH-transition maps, and so is a Fréchet space. Thus a weak Fréchet-Stein algebra is, in particular, a Fréchet algebra.

Suppose given a sequence of locally convex topological K-algebras $\{A_n\}_{n \geq 1}$ satisfying conditions (i) and (ii) of definition 1.2.6, and an isomorphism of locally convex topological K-algebras $A \xrightarrow{\sim} \varprojlim_n A_n$. If we let B_n denote the closure of the image of A in A_n, then the projective system $\{B_n\}_{n \geq 1}$ also satisfies conditions (i) (by proposition 1.1.40) and (ii) (since clearly the BH-map $A_{n+1} \to A_n$ induces a BH-map $B_{n+1} \to B_n$). The natural map $A \xrightarrow{\sim} \varprojlim_n B_n$ is also an isomorphism, and by construction it satisfies condition (iii) of definition 1.2.6. Thus A is a weak Fréchet-Stein K-algebra, and the projective system $\{B_n\}_{n \geq 1}$ forms a weak Fréchet-Stein structure on A.

We say that two weak Fréchet-Stein structures $A \xrightarrow{\sim} \varprojlim_n A_n$ and $A \xrightarrow{\sim} \varprojlim_n B_n$ on A are equivalent if the projective systems $\{A_n\}_{n \geq 1}$ and $\{B_n\}_{n \geq 1}$ are isomorphic in the category of projective systems of topological K-algebras.

Proposition 1.2.7. *If A is a locally convex topological K-algebra, then any two weak Fréchet-Stein structures on A are equivalent.*

Proof. Let $A \xrightarrow{\sim} \varprojlim_n A_n$ and $A \xrightarrow{\sim} \varprojlim_n B_n$ be two weak Fréchet-Stein structures on A. Since each transition map $A_{n+1} \to A_n$ is assumed to be a BH-map with dense image, we may factor each map as $A_{n+1} \to V_n \to A_n$, where V_n is a Banach space, and each map has dense image. Similarly, we may factor each map $B_{n+1} \to B_n$ as $B_{n+1} \to W_n \to B_n$, where W_n is a Banach space, and each map has dense image.

Thus we obtain a pair of projective systems of Banach spaces $\{V_n\}_{n\geq 1}$ and $\{W_n\}_{n\geq 1}$, each with dense transition maps, the first being equivalent to $\{A_n\}_{n\geq 1}$ and the second being equivalent to $\{B_n\}_{n\geq 1}$. Since the projective limit of each of $\{V_n\}_{n\geq 1}$ and $\{W_n\}_{n\geq 1}$ is the same Fréchet space A, they are necessarily equivalent. It follows that $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ are equivalent as projective systems of convex K-vector spaces. Since A has dense image in each of the topological algebras A_n and B_n, they are then necessarily equivalent as projective systems of topological K-algebras. $\square$

Definition 1.2.8. Let A be a weak Fréchet-Stein algebra, and let $A \xrightarrow{\sim} \varprojlim_n A_n$ be a choice of a weak Fréchet-Stein structure on A.

If M is a locally convex topological A-module, then we say that M is coadmissible (with respect to the given weak Fréchet-Stein structure on A) if we may find the following data:

(i) A sequence $\{M_n\}_{n\geq 1}$, such that M_n is a finitely generated locally convex topological A_n-module for each $n \geq 1$.

(ii) An isomorphism of topological A_n-modules $A_n \hat{\otimes}_{A_{n+1}} M_{n+1} \xrightarrow{\sim} M_n$ for each $n \geq 1$.

(iii) An isomorphism of topological A-modules $M \xrightarrow{\sim} \varprojlim_n M_n$. (Here the projective limit is taken with respect to the transition maps $M_{n+1} \to M_n$ induced by the isomorphisms of (ii).)

We will refer to such a projective sequence $\{M_n\}_{n\geq 1}$ as above as an $\{A_n\}_{n\geq 1}$-sequence (for the coadmissible module M).

If A is equipped with a weak Fréchet-Stein structure $A \xrightarrow{\sim} \varprojlim_n A_n$, then since each A_n is hereditarily complete, proposition 1.2.5 implies that for any finitely generated locally convex topological A_{n+1}-module M_n, the completed tensor product $A_n \hat{\otimes}_{A_{n+1}} M_n$ is finitely generated. Thus hypothesis (ii) of definition 1.2.8 is reasonable. Note that it also implies that M_n is Hausdorff, for each $n \geq 1$.

If we are in the situation of definition 1.2.8, then for any $n \geq 1$, we may construct a commutative diagram

$$\begin{array}{ccc} A_{n+1}^m & \longrightarrow & A_n^m \\ \downarrow & & \downarrow \\ M_{n+1} & \longrightarrow & M_n \end{array}$$

(for an appropriately chosen value of $m \geq 0$), in which the vertical arrows are strict surjections. Since the upper horizontal arrow is BH, by assumption, the same is true of the lower horizontal arrow. Thus $M \xrightarrow{\sim} \varprojlim_n M_n$ is isomorphic to the projective limit of Hausdorff locally convex spaces under BH-transition maps, and so is a Fréchet space.

Let us remark that the isomorphism $A \xrightarrow{\sim} \varprojlim_n A_n$ witnesses A as a coadmissible topological A-module.

Proposition 1.2.9. *If A is a weak Fréchet-Stein algebra, and if M is a locally convex topological A-module that is coadmissible with respect to one choice of weak Fréchet-Stein structure on A, then M is coadmissible with respect to any such choice.*

Proof. Let $A \xrightarrow{\sim} \varprojlim_{n\geq 1} A_n$ and $A \xrightarrow{\sim} \varprojlim_{n\geq 1} B_n$ be two weak Fréchet-Stein structures on A. Proposition 1.2.7 shows that we may find strictly increasing functions ϕ and ψ mapping $\mathbb{N}$ to itself, such that for each $n \geq 1$, the map $A \to A_n$ (respectively $A \to B_n$) factors through the map $A \to B_{\phi(n)}$ (respectively $A \to A_{\psi(n)}$).

Suppose that $\{M_n\}_{n\geq 1}$ is an $\{A_n\}_{n\geq 1}$-sequence realizing M as a coadmissible module for the weak Fréchet-Stein structure on A induced by the projective sequence $\{A_n\}_{n\geq 1}$. For each $n \geq 1$, define $N_n := B_n \hat{\otimes}_{A_{\psi(n)}} M_{\psi(n)}$. Proposition 1.2.5 shows that N_n is a finitely generated B_m-module. Since $\psi(n+1) \geq \psi(n)$, the transition map $M_{\psi(n+1)} \to M_{\psi(n)}$ induces a continuous B_{n+1}-linear map $N_{n+1} \to N_n$. This in turn induces a continuous B_n-linear map $B_n \hat{\otimes}_{B_{n+1}} N_{n+1} \to N_n$, which sits in the commutative diagram

$$\begin{tikzcd}
B_n \hat{\otimes}_{B_{n+1}} N_{n+1} \arrow[rr] \arrow[d, equal] & & N_n \arrow[ddd, equal] \\
B_n \hat{\otimes}_{B_{n+1}} (B_{n+1} \hat{\otimes}_{A_{\psi(n+1)}} M_{\psi(n+1)}) \arrow[d, "\sim"] & & \\
B_n \hat{\otimes}_{A_{\psi(n+1)}} M_{\psi(n+1)} \arrow[d, "\sim"] & & \\
B_n \hat{\otimes}_{A_{\psi(n)}} (A_{\psi(n)} \hat{\otimes}_{A_{\psi(n+1)}} M_{\psi(n+1)}) \arrow[rr] & & B_n \hat{\otimes}_{A_{\psi(n)}} M_{\psi(n)}.
\end{tikzcd}$$

By assumption, the natural map $A_{\psi(n)} \hat{\otimes}_{A_{\psi(n+1)}} M_{\psi(n+1)} \to M_{\psi(n)}$ is an isomorphism, and thus the same is true of the lower, and so also of the upper, horizontal arrow in this diagram. Altogether, we see that the sequence of topological modules $\{N_n\}_{n\geq 1}$ satisfies conditions (i) and (ii) of definition 1.2.8.

The projection $M \to M_{\psi(n)}$ induces a continuous A-linear map $M \to N_n$, for each $n \geq 1$ (compatible with the transition map $N_{n+1} \to N_n$), and consequently there is a continuous A-linear map $M \to \varprojlim_n N_n$. We will prove that this is a topological isomorphism, and hence show that $\{N_n\}_{n\geq 1}$ also satisfies condition (iii) of definition 1.2.8, and thus is a $\{B_n\}_{n\geq 1}$-sequence realizing M as a coadmissible module with respect to the weak Fréchet-Stein structure on M induced by the projective sequence $\{B_n\}_{n\geq 1}$.

Note that there is the obvious natural map $M_{\psi(n)} \to N_n$. On the other hand, since $\psi(\phi(n)) \geq n$ for any $n \geq 1$, we see that for any such value of n there is a natural map $N_{\phi(n)} = B_{\phi(n)} \hat{\otimes}_{A_{\psi(\phi(n))}} M_{\psi(\phi(n))} \to A_n \hat{\otimes}_{A_{\psi(\phi(n))}} M_{\psi(\phi(n))} \to M_n$ (the first map being induced by the fact that $A \to A_n$ factors through $A \to B_{\phi(n)}$, and the second map by the transition function $M_{\psi(\phi(n))} \to M_n$). Thus the projective systems $\{M_n\}_{n\geq 1}$ and $\{N_n\}_{n\geq 1}$ are equivalent, and in particular the natural map $M \to \varprojlim_n N_n$ is a topological isomorphism. This completes the proof of the proposition. □

Definition 1.2.10. Let A be a locally convex topological K-algebra. A Fréchet-Stein structure on A is a weak Fréchet-Stein structure $A \xrightarrow{\sim} \varprojlim_n A_n$ on A, such that for each $n \geq 1$, the algebra A_n is a left Noetherian K-Banach algebra, and the transition map $A_{n+1} \to A_n$ is right flat.

We say that A is a Fréchet-Stein algebra if it admits a Fréchet-Stein structure.

The notion of Fréchet-Stein structure and Fréchet-Stein algebra is introduced in [27, def., p. 7]. The notion of weak Fréchet-Stein structure is a useful generalization, which provides additional flexibility when it comes to constructing and analyzing Fréchet-Stein algebras and their coadmissible modules.

Suppose that $A \xrightarrow{\sim} \varprojlim_n A_n$ is a Fréchet-Stein structure on the locally convex topological K-algebra A. If $M \xrightarrow{\sim} \varprojlim_n M_n$ is a coadmissible A-module, where $\{M_n\}_{n\geq 1}$ is an $\{A_n\}_{n\geq 1}$-sequence, then each M_n is a Hausdorff quotient of $A_n^{r_n}$, for some $r_n \geq 0$, and hence is an A_n-Banach module. The remark following the proof of proposition 1.2.5 implies that the natural map $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism. Thus the notion of coadmissibility for topological A-modules as defined above coincides with that defined in [27, p. 152].

The following theorem incorporates the most important results of [27, §3].

Theorem 1.2.11. *Let A be a Fréchet-Stein algebra, and let $A \xrightarrow{\sim} \varprojlim_n A_n$ be a choice of weak Fréchet-Stein structure on A.*

(i) *If M is a coadmissible A-module, and if $\{M_n\}_{n\geq 1}$ is an $\{A_n\}_{n\geq 1}$-sequence for which there is a topological isomorphism $M \xrightarrow{\sim} \varprojlim_n M_n$, then for each value of n, the natural map $A_n \hat{\otimes}_A M \to M_n$ is an isomorphism. Consequently, the natural map $A \to \varprojlim_n A_n \hat{\otimes}_A M$ is an isomorphism.*

(ii) *The full subcategory of the category of topological A-modules consisting of coadmissible A-modules is closed under passing to finite direct sums, closed submodules, and Hausdorff quotient modules, and is abelian.*

Proof. Let $A \xrightarrow{\sim} \varprojlim_m B_n$ be a Fréchet-Stein structure on A; such a structure exists, by assumption. Following the proof of proposition 1.2.9, we use proposition 1.2.7 to choose strictly increasing functions ϕ and ψ mapping $\mathbb{N}$ to itself, such that for each $n \geq 1$, the map $A \to A_n$ (respectively $A \to B_n$) factors through the map $A \to B_{\phi(n)}$ (respectively $A \to A_{\psi(n)}$). For each $n \geq 1$, we define $N_n := B_n \hat{\otimes}_{A_{\psi(n)}} M_{\psi(n)}$. As in the proof of proposition 1.2.9, we find that $\{N_n\}_{n\geq 1}$ is a $\{B_n\}_{n\geq 1}$-sequence realizing M as a coadmissible module with respect to the Fréchet-Stein structure on A induced by the projective sequence $\{B_n\}$. It follows from [27, cor. 3.1] that for any $n \geq 1$, the natural map $B_n \otimes_A M \to N_n$ is a topological isomorphism. (Actually, that result as stated shows only that this map is a continuous B_n-linear bijection between Hausdorff finitely generated topological B_n-modules. However, such a map is necessarily a topological isomorphism, since B_n is a Banach algebra.) Since N_n is in fact complete, the source of this isomorphism is isomorphic to $B_n \hat{\otimes}_A M$. Thus the natural map $B_n \hat{\otimes}_A M \to N_n = B_n \hat{\otimes}_{A_{\psi(n)}} M_{\psi(n)}$ is an isomorphism for any $n \geq 1$. Applying this with n replaced by $\phi(n)$, we obtain the required isomorphism

$$
\begin{aligned}
&A_n \hat{\otimes}_A M \xrightarrow{\sim} A_n \hat{\otimes}_{B_{\phi(n)}} (B_{\phi(n)} \hat{\otimes}_A M) \\
&\quad \xrightarrow{\sim} A_n \hat{\otimes}_{B_{\phi(n)}} (B_{\phi(n)} \hat{\otimes}_{A_{\psi(\phi(n))}} M_{\psi(\phi(n))}) \xrightarrow{\sim} A_n \hat{\otimes}_{A_{\psi(\phi(n))}} M_{\psi(\phi(n))} \xrightarrow{\sim} M_n
\end{aligned}
$$

(the final isomorphism following from our assumption that $\{M_n\}$ is an $\{A_n\}$-sequence). This proves (i).

Part (ii) is a restatement of [27, cor. 3.4 (i),(ii)]. $\square$

Note that part (ii) of the preceding result implies in particular that any finitely presented A-module admits a unique topology with respect to which it becomes a coadmissible topological A-module, and that continuous morphisms between coadmissible topological A-modules are strict. It is furthermore shown in [27] that any A-linear map between coadmissible locally convex topological A-modules is necessarily continuous (and hence strict), or equivalently, that the forgetful functor from the category of coadmissible locally convex topological A-modules to the category of abstract A-modules is fully faithful. (See the remark following the proof of [27, lem. 3.6].)

Definition 1.2.12. We say that a topological K-algebra A is a nuclear Fréchet algebra if there exists a projective system $\{A_n\}_{n\geq 1}$ of K-Banach algebras with compact transition maps, and an isomorphism of topological K-algebras $A \xrightarrow{\sim} \varprojlim_n A_n$.

It follows from [20, cor. 16.6] that any topological K-algebra A satisfying the conditions of the preceding definition is in particular a nuclear Fréchet space.

If we let B_n denote the closure of the image of A under the natural map $A \to A_n$, for each $n \geq 1$, the space B_n is a Banach subalgebra of A_n, and the natural map $A \to \varprojlim_n B_n$ is an isomorphism. Thus, in definition 1.2.12, it is no loss of generality to require that A have dense image in each of the topological K-algebras A_n. Since compact type maps are BH-maps (by lemma 1.1.14), and since Banach spaces are hereditarily complete, we find that a nuclear Fréchet algebra is in particular a weak Fréchet-Stein algebra.

Lemma 1.2.13. *The completed tensor product of a pair of nuclear Fréchet algebras over K is again a nuclear Fréchet algebra.*

Proof. If $A \xrightarrow{\sim} \varprojlim_n A_n$ and $B \xrightarrow{\sim} \varprojlim_n B_n$ are two nuclear Fréchet algebras, written as projective limits, with compact transition maps, of K-Banach algebras, then propositions 1.1.29 and [20, lem. 18.12] together yield a topological isomorphism $A \hat{\otimes}_K B \xrightarrow{\sim} \varprojlim_n A_n \hat{\otimes}_K B_n$, where the transition maps are again compact. This proves the lemma. $\square$

Proposition 1.2.14. *Let A be a nuclear Fréchet algebra, and write $A \xrightarrow{\sim} \varprojlim_n A_n$ as in definition* 1.2.12*, with the further hypothesis that A has dense image in each A_n. Let A^{op} denote the opposite algebra to A. If V is a compact type convex K-vector space, then the following structures on V are equivalent:*

(i) *An A-module structure for which the multiplication map $A \times V \to V$ is separately continuous.*

(ii) *A topological A^{op}-module structure on the dual space V_b'.*

(iii) *An isomorphism $V \xrightarrow{\sim} \varinjlim_n V_n$, where $\{V_n\}$ is an inductive sequence of K-Banach spaces, in which each V_n is endowed with the structure of a topological A_n-module, and such that the transition maps $V_n \to V_{n+1}$ are compatible with the map $A_{n+1} \to A_n$.*

Proof. Proposition 1.1.35 shows that giving a separately continuous bilinear map

$$A \times V \to V \tag{1.2.15}$$

is equivalent to giving a continuous map $A \to \mathcal{L}_b(V, V)$, and (taking into account that A and V'_b are both nuclear Fréchet spaces, so that a separately continuous bilinear map

$$A \times V'_b \to V'_b \tag{1.2.16}$$

is automatically jointly continuous) that giving a jointly continuous bilinear map as in (1.2.16) is equivalent to giving a map $A \to \mathcal{L}_b(V'_b, V'_b)$. Proposition 1.1.36, and the fact that V is reflexive, shows that the natural map $\mathcal{L}_b(V, V) \to \mathcal{L}_b(V'_b, V'_b)$ is a topological isomorphism. Thus passing to duals shows that the existence of the separately continuous bilinear map (1.2.15) is equivalent to the existence of the jointly continuous bilinear map (1.2.16). It is straightforward to check that (1.2.15) induces an A-module structure on V if and only if (1.2.16) induces an A^{op}-module structure on V'_b. Thus (i) and (ii) are equivalent.

If (iii) holds, then each V_n is in particular a topological A-module, and the transition maps $V_n \to V_{n+1}$ are compatible with the A-module structure on source and target. Passing to the inductive limit in n, we find that V is an A-module, and that the map $A \times V \to V$ describing this module structure is separately continuous. Thus (iii) implies (i).

We now turn to showing that (i) implies (iii). Suppose to begin with that V is any compact type convex K-vector space, and consider the space $\mathcal{L}_b(A, V)$. By [20, cor. 18.8] there is an isomorphism $\mathcal{L}_b(A, V) \xrightarrow{\sim} A'_b \hat{\otimes}_K V$, and so by proposition 1.1.32 (i) we see that $\mathcal{L}_b(A, V)$ is of compact type. By proposition 1.1.10, any map $A \to V$ factors through a map $A_n \to V_n$ for some $n \geq 1$, and hence there is a continuous bijection

$$\varinjlim_n \mathcal{L}_b(A_n, V_n) \to \mathcal{L}_b(A, V).$$

(In fact this is a topological isomorphism, since its source and target are both LB-spaces.) The image of $\mathcal{L}_b(A_n, V_n)$ in $\mathcal{L}_b(A, V)$ is an A_n-invariant BH-subspace of $\mathcal{L}_b(A, V)$. The right regular representation of A_n on $\mathcal{L}_b(A_n, V_n)$ makes the latter a topological A_n-Banach module, and thus $\mathcal{L}_b(A, V)$ satisfies the condition of (iii).

Now suppose that V satisfies condition (i). Proposition 1.1.35 also yields a continuous injection

$$V \to \mathcal{L}_b(A, V). \tag{1.2.17}$$

Evaluation at $1 \in A$ gives a splitting of the injection (1.2.17), and hence this map is in fact a closed embedding. If we give $\mathcal{L}_b(A, V)$ the A-module structure induced by the right regular representation of A, then (1.2.17) is furthermore a map of A-modules. The discussion of the preceding paragraph shows that V is thus a closed A-submodule of an A-module satisfying condition (iii). Thus V also satisfies condition (iii). More precisely, if we let V_n denote the preimage in $\mathcal{L}_b(A_n, V_n)$ of V, so that V_n is a closed A-submodule of $\mathcal{L}_b(A_n, V_n)$, then V_n is in fact an A_n-submodule of $\mathcal{L}_b(A_n, V_n)$ (because A is dense in A_n), and $V \xrightarrow{\sim} \varinjlim_n V_n$. $\square$

Chapter 2. Non-archimedean function theory

2.1. Continuous, rigid analytic, and locally analytic functions

In this section we recall the basic notions of non-archimedean function theory.

Definition 2.1.1. If X is a set and V is a K-vector space, we let $\mathcal{F}(X, V)$ denote the K-vector space of V-valued functions on X.

The formation of $\mathcal{F}(X, V)$ is evidently covariantly functorial in V and contravariantly functorial in X. In particular, for each $x \in X$ we obtain a natural map $\mathcal{F}(X, V) \to \mathcal{F}(x, V) \xrightarrow{\sim} V$, that we denote by ev_x. (Of course, this is simply the evaluation map at x.)

Definition 2.1.2. If X is a Hausdorff topological space and if V is a Hausdorff locally convex topological K-vector space, then we let $\mathcal{C}(X, V)$ denote the K-vector space of continuous V-valued functions on G, equipped with the (Hausdorff locally convex) topology of uniform convergence on compact sets.

The vector space $\mathcal{C}(X, V)$ is a subspace of $\mathcal{F}(X, V)$, and the formation of $\mathcal{C}(X, V)$ is covariantly functorial in V and contravariantly functorial in X. In particular, if $x \in X$ then the restriction of ev_x to $\mathcal{C}(X, V)$ is a continuous map to V, which we denote by the same symbol ev_x.

Multiplying K-valued functions by vectors in V induces a jointly continuous bilinear map $\mathcal{C}(X, K) \times V \to \mathcal{C}(X, V)$, and hence a continuous map $\mathcal{C}(X, K) \otimes_{K,\pi} V \to \mathcal{C}(X, V)$. The example of [20, pp. 111–112] shows that if X is compact and if V is complete, then this map induces an isomorphism

$$\mathcal{C}(X, K) \hat{\otimes}_{K,\pi} V \xrightarrow{\sim} \mathcal{C}(X, V). \tag{2.1.3}$$

Proposition 2.1.4. *If X is a compact topological space, and if V is a Fréchet space (respectively a Banach space), then $\mathcal{C}(X, V)$ is again a Fréchet space (respectively a Banach space).*

Proof. Since V is in particular assumed to be complete, the isomorphism (2.1.3) shows that $\mathcal{C}(X, V)$ is the completed tensor product of a Banach space and a Fréchet space (respectively a Banach space), and hence is itself a Fréchet space (respectively a Banach space). (One can also prove this statement directly, by using a choice of metric or norm defining the topology on V to construct a sup metric or sup norm on $\mathcal{C}(X, V)$, which defines the topology on $\mathcal{C}(X, V)$.) $\square$

In particular, if X is compact then $\mathcal{C}(X, K)$ is a Banach space.

Corollary 2.1.5. *Let X be a locally compact topological space that can be written as a disjoint union of compact open subsets, and let V be a Fréchet space.*

(i) *The convex space $\mathcal{C}(X, V)$ is barrelled.*

(ii) *If X is furthermore σ-compact, then $\mathcal{C}(X, V)$ is a Fréchet space.*

Proof. Let $X = \coprod_i X_i$ be the hypothesized decomposition of X into a disjoint union of compact open subsets. Then we obtain a natural isomorphism $\mathcal{C}(X, V) \xrightarrow{\sim} \prod_i \mathcal{C}(X_i, V)$. Proposition 2.1.4 implies in particular that each of the spaces $\mathcal{C}(X_i, V)$ is barrelled. Thus $\mathcal{C}(X, V)$ is isomorphic to a product of barrelled spaces, hence is itself barrelled [5, cor., p. IV.14].

If X is σ-compact, then the collection $\{X_i\}$ is countable. Thus (again by proposition 2.1.4) $\mathcal{C}(X,V)$ is isomorphic to a product of a countable number of Fréchet spaces, and so is itself a Fréchet space. $\square$

Let X be a compact topological space. If V is a Hausdorff convex K-vector space and if W is a BH-subspace of V, then the image of the natural map $\mathcal{C}(X,\overline{W}) \to \mathcal{C}(X,V)$ is a BH-subspace of V. (We refer to definition 1.1.1 for the notion of a BH-subspace of V. Recall in particular that if W is a BH-subspace of V then $\overline{W}$ denotes the latent Banach space structure on W.)

Proposition 2.1.6. *If X is a compact topological space, and if V is either a Fréchet space or a space of compact type, then any BH-subspace of $\mathcal{C}(X,V)$ is contained in the image of the natural map $\mathcal{C}(X,\overline{W}) \to \mathcal{C}(X,V)$, for some BH-subspace W of V. In particular, the natural map $\varinjlim_{W} \mathcal{C}(X,\overline{W}) \to \mathcal{C}(X,V)$ (where the locally convex inductive limit is taken over all BH-subspaces W of V) is a continuous bijection. If V is a Fréchet space, then this map is even a topological isomorphism.*

Proof. Since V is in particular assumed to be complete, the same is true of $\mathcal{C}(X,V)$. Thus any BH-subspace of $\mathcal{C}(X,V)$ is contained in the subspace generated by some closed bounded $\mathcal{O}_K$-submodule of $\mathcal{C}(X,V)$. Any such set is in turn contained in the set $\mathcal{C}(X,A)$, for some closed bounded $\mathcal{O}_K$-submodule A of V. Proposition 1.1.11 shows that we may find a BH-subspace W of V containing A, such that A is bounded when regarded as a subset of $\overline{W}$ and such that the topology induced on A by $\overline{W}$ agrees with the topology induced on A by V. We may therefore regard $\mathcal{C}(X,A)$ as a bounded subset of $\mathcal{C}(X,\overline{W})$. The first claim of the proposition follows.

If U ranges over the directed set of all BH-subspaces of $\mathcal{C}(X,V)$, then the natural map $\varinjlim_{U} \overline{U} \to \mathcal{C}(X,V)$ is a continuous bijection. If V is a Fréchet space, then proposition 2.1.4 implies that $\mathcal{C}(X,V)$ is also, and so in particular is ultrabornological. Thus in this case, the preceding natural map is even a topological isomorphism. The result of the preceding paragraph shows that the natural map $\varinjlim_{W} \mathcal{C}(X,\overline{W}) \to \varinjlim_{U} \overline{U}$ is a also a topological isomorphism. The remaining claims of the proposition now follow. $\square$

If V is a convex space of compact type, so that in particular V is of LB-type, then the preceding proposition implies that the convex space $\mathcal{C}(X,V)$ is also of LB-type.

Proposition 2.1.7. *If V and W are two Hausdorff convex K-vector spaces, and if X is a non-empty Hausdorff topological space, then the natural map*

$$\mathcal{L}_b(V,W) \to \mathcal{L}_b(\mathcal{C}(X,V),\mathcal{C}(X,W))$$

(*induced by the functoriality of the formation of $\mathcal{C}(X,-)$*) *is a topological embedding.*

Proof. If C is a compact subset of X and U is a neighbourhood of zero in V (respectively W) then let $S_{C,U}$ denote the subset of $\mathcal{C}(X,V)$ (respectively $\mathcal{C}(X,W)$) defined by $S_{C,U} = \{f \mid f(C) \subset U\}$. This set is a neighbourhood of zero in $\mathcal{C}(X,V)$ (respectively $\mathcal{C}(X,W)$), and such sets form a neighbourhood basis of zero, if C is allowed to run over all compact subsets of X and U to run over all neighbourhoods of zero in V (respectively W). In light of this it is straightforward to check that a subset B of $\mathcal{C}(X,V)$ is bounded if and only if for every compact subset C of X, the subset $\{f(c) \mid f \in B, c \in C\}$ is a bounded subset of V.

If B is bounded subset of $\mathcal{C}(X,V)$ and $S_{C,U}$ is a neighbourhood of zero in $\mathcal{C}(X,W)$ of the type described in the preceding paragraph, let $T_{B,C,U}$ denote the subset of $\mathcal{L}(\mathcal{C}(X,V),\mathcal{C}(X,W))$ defined by $T_{B,C,U} = \{\phi \mid \phi(B) \subset S_{C,U}\}$. This set is a neighbourhood of zero in $\mathcal{L}_b(\mathcal{C}(X,V),\mathcal{C}(X,W))$, and such sets form a neighbourhood basis of zero if B is allowed to run over all bounded subsets B of $\mathcal{C}(X,V)$, C over all compact subsets of X, and U over all neighbourhoods of zero in W.

Fixing such a B, C, and U, let D denote the subset of V defined by $D = \{f(c) \mid f \in B, c \in C\}$. As observed above, D is a bounded subset of V. The preimage of $T_{B,C,U}$ under the natural map $\mathcal{L}(V,W) \to \mathcal{L}(\mathcal{C}(X,V),\mathcal{C}(X,W))$ is equal to $\{\phi \mid \phi(D) \subset U\}$. Since D is bounded in V and U is open in W this a neighbourhood of zero in $\mathcal{L}_b(V,W)$. Thus we have proved that the natural map $\mathcal{L}_b(V,W) \to \mathcal{L}_b(\mathcal{C}(X,V),\mathcal{C}(X,W))$ is continuous. Since X is non-empty this map is also evidently injective, and so it remains to prove that any open subset of $\mathcal{L}_b(V,W)$ can be obtained by pulling back an open subset of $\mathcal{L}_b(\mathcal{C}(X,V),\mathcal{C}(X,W))$.

Let D be an arbitrary bounded subset of V, and define B to be that subset of $\mathcal{C}(X,V)$ consisting of the constant functions corresponding to the elements of D. In this case we see that for any non-empty compact subset C of X, the set D is recovered as the set $\{f(c) \mid f \in B, c \in C\}$. Thus our preceding calculations show that for any neighbourhood U of zero in W, the preimage of $T_{B,C,U}$ in $\mathcal{L}(V,W)$ is precisely equal to $\{\phi \mid \phi(D) \subset U\}$. Since sets of this type form a neighbourhood basis of zero in $\mathcal{L}_b(V,W)$, we find that the map $\mathcal{L}_b(V,W) \to \mathcal{L}_b(\mathcal{C}(X,W),\mathcal{C}(X,W))$ is a topological embedding, as claimed. $\square$

Definition 2.1.8. If V is a Hausdorff convex K-vector space, then let $c_0(\mathbb{N},V)$ denote the space of sequences in V that converge to zero, equipped with the (locally convex Hausdorff) topology of uniform convergence.

If $\hat{\mathbb{N}} = \mathbb{N} \bigcup \{\infty\}$ denotes the one-point compactification of the discrete topological space $\mathbb{N}$, then we may regard $c_0(\mathbb{N},V)$ as the closed subspace of $\mathcal{C}(\hat{\mathbb{N}},V)$ consisting of those functions that vanish at infinity.

Definition 2.1.9. Let $\mathbb{X}$ be an affinoid rigid analytic space defined over L.

(i) We let $\mathcal{C}^{\mathrm{an}}(\mathbb{X},K)$ denote the K-Banach algebra of K-valued rigid analytic functions defined on $\mathbb{X}$.

(ii) If V is a K-Banach space then we define the K-Banach space $\mathcal{C}^{\mathrm{an}}(\mathbb{X},V)$ of V-valued rigid analytic functions on $\mathbb{X}$ to be the completed tensor product $\mathcal{C}^{\mathrm{an}}(\mathbb{X},K) \,\hat{\otimes}\, V$.

The formation of $\mathcal{C}^{\mathrm{an}}(\mathbb{X},V)$ is evidently covariantly functorial in V and contravariantly functorial in $\mathbb{X}$.

Proposition 2.1.10. *If $\mathbb{X}$ is an affinoid rigid analytic space defined over L, then there is an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{X},K) \xrightarrow{\sim} c_0(\mathbb{N},K)$.*

Proof. This is standard. $\square$

Definition 2.1.11. If V is a Hausdorff locally convex topological K-vector space and $\mathbb{X}$ is an affinoid rigid analytic space defined over L, we define the locally convex space $\mathcal{C}^{\mathrm{an}}(\mathbb{X},V)$ of V-valued rigid analytic functions on $\mathbb{X}$ to be the locally convex inductive limit of Banach spaces

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) := \varinjlim_{W} \mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{W}),$$

where W runs over the directed set of all BH-subspaces W of V.

The formation of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is covariantly functorial in V (use proposition 1.1.7) and contravariantly functorial in $\mathbb{X}$. Note that if V is a K-Banach space then definitions 2.1.9 and 2.1.11 yield naturally isomorphic objects, since V then forms a final object in the directed set of all BH-subspaces of V.

If W is a BH-subspace of V, then the continuous injection $\overline{W} \to V$ induces a natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{W}) = \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_K \overline{W} \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_{K,\pi} V$.

Passing to the inductive limit in W we obtain a natural map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_{K,\pi} V. \tag{2.1.12}$$

Proposition 2.1.13. *Let V be a Hausdorff convex space, and let $\mathbb{X}$ be an affinoid rigid analytic space over L.*

(i) *The map* (2.1.12) *is injective.*

(ii) *If V is furthermore a Fréchet space, then the map* (2.1.12) *is a topological isomorphism.*

Proof. Let $\hat{V}$ denote the completion of V. If W is a BH-subspace of V, then the injection $\overline{W} \to V \to \hat{V}$ yields an injection $c_0(\mathbb{N}, \overline{W}) \to c_0(\mathbb{N}, \hat{V})$. Passing to the locally convex inductive limit over all BH-subspaces W of V, we obtain an injection

$$\varinjlim_{W} c_0(\mathbb{N}, \overline{W}) \to c_0(\mathbb{N}, \hat{V}). \tag{2.1.14}$$

If we choose an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \xrightarrow{\sim} c_0(\mathbb{N}, K)$, as in proposition 2.1.10, then the isomorphism (2.1.3) — applied to the one-point compactification of $\mathbb{N}$ — allows us to rewrite this injection as an injection

$$\varinjlim_{W} \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_K \overline{W} \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_{K,\pi} V.$$

This is the map (2.1.12), and so we have proved part (i) of the proposition.

If V is a Fréchet space, then proposition 2.1.6 shows that the map (2.1.14) is a topological isomorphism, and hence that (2.1.12) is a topological isomorphism in this case. This proves part (ii) of the proposition. □

Note that the preceding result shows that $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is Hausdorff, since it injects continuously into the Hausdorff space $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_{K,\pi} V$.

Since the proof identifies $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ with $c_0(\mathbb{N}, V)$ when V is Fréchet, we also see that if V is Fréchet then $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is again Fréchet. On the other hand, if V is of LB-type, then proposition 1.1.10, together with the definition of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ as an inductive limit, shows that $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is an LB-space.

We require the notion of a relatively compact morphism of rigid analytic spaces. Although this can be defined more generally [3, §9.6.2], the following special case of this definition will suffice for our purposes.

Definition 2.1.15. We say that a morphism $\mathbb{X} \to \mathbb{Y}$ of affinoid rigid analytic varieties over L is relatively compact if it fits into a commutative diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r] \arrow[d] & \mathbb{Y} \arrow[d] \\ \mathbb{B}_n(r_1) \arrow[r] & \mathbb{B}_n(r_2), \end{tikzcd}$$

for some $r_1 < r_2$, in which the right-hand vertical arrow is a closed immersion, and the lower horizontal arrow is the natural open immersion. (Here for any n and r we let $\mathbb{B}_n(r) := \{(x_1, \dots, x_n) \,|\, |x_i| \leq r \text{ for all } i = 1, \dots, n\}$ denote the n-dimensional closed ball of radius r centred at the origin of $\mathbb{A}^n$.)

Proposition 2.1.16. *If $\mathbb{X} \to \mathbb{Y}$ is a relatively compact morphism of affinoid rigid analytic spaces over L, then the induced map $\mathcal{C}^{\mathrm{an}}(\mathbb{Y}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is compact.*

Proof. The diagram whose existence is guaranteed by definition 2.1.15 yields a diagram of morphisms of Banach spaces

$$\begin{tikzcd} \mathcal{C}^{\mathrm{an}}(\mathbb{B}(r_2), K) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{B}(r_1), K) \arrow[d] \\ \mathcal{C}^{\mathrm{an}}(\mathbb{Y}, K) \arrow[r] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K), \end{tikzcd}$$

in which the left-hand vertical arrow is surjective, and hence open. Thus to prove the proposition, it suffices to show that the upper morphism is compact. This is standard, and is in any case easily checked. $\square$

We will need to consider topological vector-space valued analytic functions on rigid analytic spaces that are not necessarily affinoid. The various classes of spaces encompassed by the following definition are suitable for our purposes.

Definition 2.1.17. Let $\mathbb{X}$ be a rigid analytic space defined over L.

(i) We say that $\mathbb{X}$ is σ-affinoid if there is an increasing sequence $\mathbb{X}_1 \subset \mathbb{X}_2 \subset \cdots \subset \mathbb{X}_n \subset \cdots$ of affinoid open subsets of $\mathbb{X}$ such that $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$, with $\{\mathbb{X}_n\}_{n \geq 1}$ forming an admissible cover of $\mathbb{X}$.

(ii) We say that $\mathbb{X}$ is strictly σ-affinoid if there is an increasing sequence $\mathbb{X}_1 \subset \mathbb{X}_2 \subset \cdots \subset \mathbb{X}_n \subset \cdots$ of admissible affinoid open subsets of $\mathbb{X}$ such that each of these inclusions is relatively compact, such that $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$, with $\{\mathbb{X}_n\}_{n \geq 1}$ forming an admissible cover of $\mathbb{X}$.

(iii) We say that $\mathbb{X}$ is quasi-Stein if there is an increasing sequence $\mathbb{X}_1 \subset \mathbb{X}_2 \subset \cdots \subset \mathbb{X}_n \subset \cdots$ of admissible affinoid open subsets of $\mathbb{X}$ such that $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$, with $\{\mathbb{X}_n\}_{n \geq 1}$ forming an admissible cover of $\mathbb{X}$, and such that for each $n \geq 1$, the map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_{n+1}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$ induced by the inclusion $\mathbb{X}_n \subset \mathbb{X}_{n+1}$ has dense image. (This definition is due to Kiehl [15].)

(iv) We say that $\mathbb{X}$ is strictly quasi-Stein if there is an increasing sequence $\mathbb{X}_1 \subset \mathbb{X}_2 \subset \cdots \subset \mathbb{X}_n \subset \cdots$ of admissible affinoid open subsets of $\mathbb{X}$ such that each of these inclusions is relatively compact, such that $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$, with $\{\mathbb{X}_n\}_{n \geq 1}$ forming an admissible cover of $\mathbb{X}$, and such that for each $n \geq 1$, the map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_{n+1}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$ induced by the inclusion $\mathbb{X}_n \subset \mathbb{X}_{n+1}$ has dense image.

Evidently any strictly σ-affinoid or quasi-Stein rigid analytic space is also σ-affinoid, and any strictly quasi-Stein rigid analytic space is quasi-Stein. The basic example of a σ-affinoid rigid analytic space that we have in mind is that of an open ball, which may be written as a union of an increasing sequence of closed balls. Of course, an open ball is even strictly quasi-Stein.

If $\mathbb{X}$ is σ-affinoid, and if $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$ is an admissible open cover of $\mathbb{X}$ satisfying the conditions of definition 2.1.17 (i), then any admissible affinoid open subset $\mathbb{Y}$ of $\mathbb{X}$ is contained in $\mathbb{X}_n$ for some n. Thus the sequence of admissible affinoid open subsets $\{\mathbb{X}_n\}$ of $\mathbb{X}$ is cofinal in the directed set of all admissible affinoid open subsets of $\mathbb{X}$.

Definition 2.1.18. Suppose that $\mathbb{X}$ is a σ-affinoid rigid analytic space defined over L. If V is a Hausdorff locally analytic convex K-vector space, then we define the convex K-vector space $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ of analytic V-valued functions on $\mathbb{X}$ to be the projective limit $\varprojlim_{\mathbb{Y}} \mathcal{C}^{\mathrm{an}}(\mathbb{Y}, V)$, where $\mathbb{Y}$ runs over all admissible affinoid open subsets of $\mathbb{X}$.

Since $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is defined as the projective limit of Hausdorff locally convex spaces, it is again Hausdorff locally convex. The formation of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is covariantly functorial in V and contravariantly functorial in $\mathbb{X}$. The remarks preceding definition 2.1.18 show that the projective limit that appears in this definition may be taken over any increasing sequence $\{\mathbb{X}_n\}$ which witnesses the fact that $\mathbb{X}$ is σ-affinoid.

If V is a Fréchet space, then each $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, V)$ is a Fréchet space (by proposition 2.1.13 (ii)), and thus $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ is a Fréchet space, being the projective limit of a sequence of Fréchet spaces. In particular, $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is a Fréchet algebra. If $\mathbb{X}$ is strictly σ-affinoid, then proposition 2.1.16 shows that $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is the projective limit of a sequence of Banach spaces equipped with compact transition maps. Thus if $\mathbb{X}$ is strictly σ-affinoid, then $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is a nuclear Fréchet algebra. If $\mathbb{X}$ is quasi-Stein, then $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is a Fréchet-Stein algebra (if $\{\mathbb{X}_n\}_{n\geq 1}$ is an admissible covering of $\mathbb{X}$ satisfying the conditions of Definition 2.1.17 (iii), then the isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \xrightarrow{\sim} \varprojlim_{n} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$ provides the necessary weak Fréchet-Stein structure), and if $\mathbb{X}$ is strictly quasi-Stein, then $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is a nuclear Fréchet-Stein algebra.

Proposition 2.1.19. *Let $\mathbb{X}$ be a σ-affinoid rigid analytic space over L. If V is a K-Fréchet space, then there is a natural isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_K V$.*

Proof. Write $X = \bigcup_{n\geq 1} \mathbb{X}_n$, where $\mathbb{X}_1 \subset \cdots \mathbb{X}_n \subset \mathbb{X}_{n+1} \cdots$ is an increasing sequence of admissible open affinoid subsets of $\mathbb{X}$, forming an admissible open cover of $\mathbb{X}$. Definition 2.1.18, together with proposition 2.1.13 (iii), yields an isomorphism

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) = \varprojlim_{n} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, V) \xrightarrow{\sim} \varprojlim_{n} (\mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K) \hat{\otimes}_K V).$$

The claimed isomorphism is now provided by proposition 1.1.29. $\square$

We now discuss the evaluation of rigid analytic functions on $\mathbb{X}$ at the L-valued points of $\mathbb{X}$.

Proposition 2.1.20. *If $\mathbb{X}$ is a σ-affinoid rigid analytic space over L, if V is any Hausdorff convex K-vector space, and if we write $X := \mathbb{X}(L)$, then the evaluation at the points of X yields a natural continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}(X, V)$. If, furthermore, we may find an admissible cover $\mathbb{X} := \bigcup_{n=1}^{\infty} \mathbb{X}_n$ satisfying the conditions of definition* 2.1.17 (i), *with the additional property that $\mathbb{X}_n(L)$ is Zariski dense in $\mathbb{X}_n$ for each $n \geq 1$, then this map is injective.*

Proof. We first suppose that $\mathbb{X}$ is affinoid, and that V is a Banach space. Then by definition $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) = \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes} V$. If we take into account the natural isomorphism provided by (2.1.3), then we that the continuous map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \to \mathcal{C}(X, K) \tag{2.1.21}$$

induced by evaluating rigid analytic functions on points in X yields the continuous map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}(X, V) \tag{2.1.22}$$

of the proposition. The naturality of (2.1.22) is evident. Furthermore, by definition X is Zariski dense in $\mathbb{X}$ if and only if the map (2.1.21) is injective, in which case the map (2.1.22) is also injective, by corollary 1.1.27, since (taking into account the isomorphism (2.1.3)) it is obtained from (2.1.21) by taking the completed tensor product with V over K.

Now consider the case where $\mathbb{X}$ is affinoid but V is arbitrary. The preceding paragraph gives a continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{W}) \to \mathcal{C}(X, \overline{W})$ for each BH-subspace W of V, which is injective if X is Zariski dense in $\mathbb{X}$. Composing this with the continuous injection $\mathcal{C}(X, \overline{W}) \to \mathcal{C}(X, V)$ yields a continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{W}) \to \mathcal{C}(X, V)$, which is injective if X is Zariski dense in $\mathbb{X}$. Finally, taking the inductive limit over all BH-subspaces W yields the required continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}(X, V)$, which is injective if X is Zariski dense in $\mathbb{X}$. Its naturality is clear.

Now suppose that $\mathbb{X}$ is σ-affinoid, and write $\mathbb{X} = \bigcup_{n=1}^{\infty}$, with $\{\mathbb{X}_n\}$ an admissible cover of $\mathbb{X}$ by an increasing sequence of admissible affinoid open subsets of $\mathbb{X}$. Set $X_n := \mathbb{X}_n(L)$, so that $X = \bigcup_{n=1}^{\infty} X_n$. Each X_n is a compact open subset of X, and the sequence X_n is cofinal in the directed set of compact subsets of X. Thus $\mathcal{C}(X, V) \xrightarrow{\sim} \varprojlim_n \mathcal{C}(X_n, V)$, and the map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}(X, V)$ under consideration is the projective limit of the maps $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, V) \to \mathcal{C}(X_n, V)$. The preceding paragraph shows that these maps are continuous, and thus so is their projective limit. If X_n is Zariski dense in $\mathbb{X}_n$ for each value of n, then each of the maps $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, V) \to \mathcal{C}(X_n, V)$ is injective. Passing to the projective limit over n, we find that the map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}(X, V)$ is injective, as required. □

Proposition 2.1.23. *Let V be a Hausdorff convex K-vector space and let W be a closed subspace of V. Let $\mathbb{X}$ be a σ-affinoid rigid analytic space defined over L, for which we may find an admissible cover $\mathbb{X} := \bigcup_{n=1}^{\infty} \mathbb{X}_n$ satisfying the conditions of definition* 2.1.17 (i), *with the additional property that $\mathbb{X}_n(L)$ is Zariski dense in $\mathbb{X}_n$ for each $n \geq 1$. Then the diagram*

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W) \arrow[r] \arrow[d, "(2.1.20)"] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \arrow[d, "(2.1.20)"] \\
\mathcal{C}(X, W) \arrow[r] & \mathcal{C}(X, V)
\end{tikzcd}$$

is Cartesian on the level of abstract vector spaces. If V (and hence W) is a Fréchet space, then the horizontal arrows of this diagram are closed embeddings, and the diagram is Cartesian on the level of topological vector spaces.

Proof. Let us suppose first that V is a Fréchet space, that $\mathbb{X}$ is affinoid, and that $X := \mathbb{X}(L)$ is Zariski dense in $\mathbb{X}$. We must show that the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ identifies $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W)$ with the closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V)$ consisting of functions that are W-valued when evaluated at points of X.

Consider the exact sequence $0 \to W \to V \to V/W \to 0$, in which the inclusion is a closed embedding and the surjection is strict. After taking the completed tensor product of this exact sequence with $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$, and applying the isomorphisms provided by proposition 2.1.13 (ii), we obtain an exact sequence

$$0 \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, W) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, V/W) \to 0$$

having the same properties. (To see that the sequence obtained from $0 \to W \to V \to V/W \to 0$ by taking the completed tensor product with $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ over K is exact, note that the isomorphism of proposition 2.1.10, together with the isomorphism (2.1.3) applied to the one-point compactification of $\mathbb{N}$, shows that the resulting sequence is topologically isomorphic to the sequence

$$0 \to c_0(\mathbb{N}, W) \to c_0(\mathbb{N}, V) \to c_0(\mathbb{N}, V/W) \to 0,$$

which is easily seen to be exact.) Thus to prove the proposition, it suffices to observe that (since X is Zariski dense in $\mathbb{X}$) an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V/W)$ that vanishes at every point of X is necessarily the zero element.

Now suppose that V is an arbitrary Hausdorff convex K-vector space (and continue to suppose that $\mathbb{X}$ is affinoid). If U runs through the members of the directed set of BH-subspaces of V then $U \bigcap W$ runs through the members of the directed set of BH-subspaces of W. Applying the result already proved, we find that the diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{U \bigcap W}) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{U}) \arrow[d] \\
\mathcal{C}(X, \overline{U \bigcap W}) \arrow[r] & \mathcal{C}(X, \overline{U})
\end{tikzcd}$$

is Cartesian. The diagram

$$\begin{tikzcd}
\mathcal{C}(X, \overline{U \bigcap W}) \arrow[r] \arrow[d] & \mathcal{C}(X, \overline{U}) \arrow[d] \\
\mathcal{C}(X, W) \arrow[r] & \mathcal{C}(X, V)
\end{tikzcd}$$

is also evidently Cartesian, and so we conclude that the diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{U \bigcap W}) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}, \overline{U}) \arrow[d] \\
\mathcal{C}(X, W) \arrow[r] & \mathcal{C}(X, V)
\end{tikzcd}$$

is Cartesian. Passing to the inductive limit over U yields the proposition. (Note that because we pass to a locally convex inductive limit, we cannot conclude that the diagram of the proposition is Cartesian as a diagram of topological vector spaces.)

Finally, if we assume that $\mathbb{X}$ is σ-affinoid, admitting an admissible cover $\mathbb{X} = \bigcup \mathbb{X}_n$, where $\{\mathbb{X}_n\}$ is an increasing sequence of admissible affinoid open subspaces for which $X_n := \mathbb{X}_n(L)$ is Zariski dense in $\mathbb{X}_n$ for each n, then the diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{an}}(\mathbb{X}, W) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \arrow[d] \\ \mathcal{C}(X, W) \arrow[r] & \mathcal{C}(X, V) \end{tikzcd}$$

is obtained as a projective limit over n of the diagrams

$$\begin{tikzcd} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, W) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, V) \arrow[d] \\ \mathcal{C}(X_n, W) \arrow[r] & \mathcal{C}(X_n, V). \end{tikzcd}$$

These diagrams are Cartesian by what has already been proved, and so the same is true of their projective limit. Since furthermore the projective limit of closed embeddings is a closed embedding, we see that if V and W are Fréchet spaces, then the horizontal maps in this diagram are closed embeddings, and that in this case the diagram is Cartesian even on the level of topological K-vector spaces. $\square$

Proposition 2.1.24. *If $\mathbb{X}$ is a σ-affinoid rigid analytic space defined over L, and if each of V and W is a Fréchet space, then the natural map*

$$\mathcal{L}_b(V, W) \to \mathcal{L}_b(\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V), \mathcal{C}^{\mathrm{an}}(\mathbb{X}, W))$$

(induced by the functoriality of the construction of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, -)$) is continuous.

Proof. We prove the proposition first in the case where $\mathbb{X}$ is affinoid. If we fix an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \xrightarrow{\sim} c_0(\mathbb{N}, K)$, as we may do, by proposition 2.1.10, then (2.1.3) and proposition 2.1.13 (ii) together yield an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \xrightarrow{\sim} c_0(\mathbb{N}, V)$, functorial in V. The proposition, in the case where $\mathbb{X}$ is affinoid, now follows from proposition 2.1.7 (applied to $X = \hat{\mathbb{N}}$).

Suppose now that X is σ-affinoid, and write $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$ as the union of an increasing sequence of admissible open affinoid subspaces. Since by definition $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W) := \varprojlim_n \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, W)$, there is a natural isomorphism

$$\mathcal{L}_b(\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V), \mathcal{C}^{\mathrm{an}}(\mathbb{X}, W)) \xrightarrow{\sim} \varprojlim_n \mathcal{L}_b(\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W), \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, W)).$$

Thus it suffices to show that, for each $n \geq 1$, the natural map $\mathcal{L}_b(V, W) \to \mathcal{L}_b(\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V), \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, W))$ is continuous. If we factor this map as $\mathcal{L}_b(V, W) \to \mathcal{L}_b(\mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, V), \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, W)) \to \mathcal{L}_b(\mathcal{C}^{\mathrm{an}}(\mathbb{X}, V), \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, W))$, then its continuity is a consequence of the result of the preceding paragraph. $\square$

Suppose now that X is a locally L-analytic manifold. (We always assume that such an X is paracompact, and hence strictly paracompact, in the sense of [23, p. 446], by a theorem of Schneider [21, Satz 8.6].) Let $\{X_i\}_{i\in I}$ be a partition of X into a disjoint union of charts X_i; we refer to such a partition as an analytic partition of X. By definition each X_i is equipped with an identification $\phi_i : X_i \xrightarrow{\sim} \mathbb{X}_i(L)$, where $\mathbb{X}_i$ is a rigid analytic closed ball defined over L. The collection of all analytic partitions of X forms a directed set (under the relation of refinement).

Let $\{X_i\}_{i\in I}$ be an analytic partition of X, and let $\{X_j\}_{j\in J}$ be an analytic partition of X that refines it. Let $\sigma : J \to I$ be the map describing the refinement, so that $\mathbb{X}_j \subset \mathbb{X}_{\sigma(j)}$ for each $j \in J$. If each of these inclusions is a relatively compact morphism, then we say that the refinement is relatively compact.

The following definition is taken from [12, 2.1.10]. Another account appears in [23, p. 5].

Definition 2.1.25. Let V be a Hausdorff convex K-vector space and X be a locally L-analytic manifold.

We define the convex K-vector space $\mathcal{C}^{\mathrm{la}}(X,V)$ of locally analytic V-valued functions on X to be the locally convex inductive limit $\varinjlim_{\{X_i,W_i\}_{i\in I}} \prod_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, \overline{W}_i)$, taken over the directed set of collections of pairs $\{X_i, W_i\}_{i\in I}$, where $\{X_i\}_{i\in I}$ is an analytic partition of X, and each W_i is a BH-subspace of V.

The formation of $\mathcal{C}^{\mathrm{la}}(X,V)$ is covariantly functorial in V (by proposition 1.1.7) and contravariantly functorial in X.

If $\{X_i\}_{i\in I}$ is any partition of X into disjoint open subsets, then the natural map $\mathcal{C}^{\mathrm{la}}(X,V) \to \prod_{i\in I} \mathcal{C}^{\mathrm{la}}(X_i,V)$ is an isomorphism [12, 2.2.4].

If X is compact, then any analytic partition of X is finite, and so we may replace the variable BH-subspaces W_i by a BH-subspace W containing each of them. Thus taking first the inductive limit in W, we obtain a natural isomorphism $\mathcal{C}^{\mathrm{la}}(X,V) \xrightarrow{\sim} \varinjlim_{\{X_i\}_{i\in I}} \prod_i \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, V)$, where the inductive limit is taken over the directed set of analytic partitions $\{X_i\}_{i\in I}$ of X. Alternatively, we may first take the inductive limit with respect to the analytic partitions of X, and so obtain a natural isomorphism $\varinjlim_{W} \mathcal{C}^{\mathrm{la}}(X,\overline{W}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(X,V)$, where the inductive limit is taken over all BH-subspaces W of V.

Maintaining the assumption that X is compact, choose a sequence $(\{X_i\}_{i\in I_n})_{n\geq 1}$ of analytic partitions of X that is cofinal in the directed set of all such analytic partitions, and such that for each $n \geq 1$, the partition $\{X_i\}_{i\in I_{n+1}}$ is a relatively compact refinement of the partition $\{X_i\}_{i\in I_n}$. Proposition 2.1.16 then shows that the natural map $\prod_{i\in I_n} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \to \prod_{i\in I_{n+1}} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K)$ is a compact map. Passing to the locally convex inductive limit, we find that $\mathcal{C}^{\mathrm{la}}(X,K)$ is of compact type.

If X is arbitrary, then we see from the preceding paragraph that $\mathcal{C}^{\mathrm{la}}(X,K)$ is a product of spaces of compact type. Thus $\mathcal{C}^{\mathrm{la}}(X,K)$ is reflexive (hence barrelled) and complete, being a product of reflexive and complete spaces.

Proposition 2.1.26. *If V is a Hausdorff convex K-vector space and X is a locally L-analytic manifold, then evaluation at points of X induces a continuous injection $\mathcal{C}^{\mathrm{la}}(X,V) \to \mathcal{C}(X,V)$, that is natural, in the sense that it is compatible with the functorial properties of its source and target. Furthermore, this injection has dense image.*

Proof. If $\{X_i\}_{i\in I}$ is an analytic partition of X then proposition 2.1.20 yields a continuous injection $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, V) \to \mathcal{C}(X_i, V)$, and so a continuous injection

$$\prod_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, V) \to \prod_{i\in I} \mathcal{C}(X_i, V) \xrightarrow{\sim} \mathcal{C}(X, V).$$

Passing to the inductive limit yields the required map of the proposition. Its naturality is clear.

To see that the image is dense, note that locally constant functions are contained in $\mathcal{C}^{\mathrm{la}}(X, V)$, and are dense in $\mathcal{C}(X, V)$. □

This result shows that $\mathcal{C}^{\mathrm{la}}(X, V)$ is Hausdorff, since it admits a continuous injection into the Hausdorff space $\mathcal{C}(X, V)$.

Proposition 2.1.27. *If X is a locally L-analytic manifold, if V is a Hausdorff convex K-vector space, and if W is a closed subspace of V, then the diagram*

$$\begin{tikzcd} \mathcal{C}^{\mathrm{la}}(X, W) \arrow[r] \arrow[d, "(2.1.26)"] & \mathcal{C}^{\mathrm{la}}(X, V) \arrow[d, "(2.1.26)"] \\ \mathcal{C}(X, W) \arrow[r] & \mathcal{C}(X, V) \end{tikzcd}$$

is Cartesian on the level of abstract K-vector spaces.

Proof. We begin with an observation that relates the BH-subspaces of V and W (which was already made in the proofs of each of propositions 1.1.12 and 2.1.23): If U is a BH-subspace of V, then since W is closed in V, the intersection $U \bigcap W$ is a BH-subspace of W. On the other hand, any BH-subspace of W is also a BH-subspace of V. Thus as U ranges over all BH-subspaces of V, the intersections $U \bigcap W$ range over all BH-subspaces of W. Furthermore, the induced map $\overline{U \bigcap W} \to \overline{U}$ on latent Banach spaces is a closed embedding.

Now, let $\{X_i\}_{i\in I}$ be an analytic partition of X, and for each $i \in I$, let U_i be a BH-subspace of V. For each $i \in I$ we see by proposition 2.1.23 that the diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, \overline{U_i \bigcap W}) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, \overline{U}_i) \arrow[d] \\ \mathcal{C}(X_i, W) \arrow[r] & \mathcal{C}(X_i, V) \end{tikzcd}$$

is Cartesian as a diagram of abstract K-vector spaces. Taking the product over all $i \in I$, taking into account the isomorphism $\mathcal{C}(X, W) \xrightarrow{\sim} \prod_{i\in I} \mathcal{C}(X_i, W)$, and passing to the locally convex inductive limit over all collections of pairs $\{(X_i, U_i)\}_{i\in I}$ (taking into account as well the observation of the preceding paragraph), we deduce the proposition. □

Proposition 2.1.28. *If V is a Hausdorff locally convex topological K-vector space, and if X is a compact locally L-analytic manifold, then there is a continuous injection $\mathcal{C}^{\mathrm{la}}(X, V) \to \mathcal{C}^{\mathrm{la}}(X, K) \hat{\otimes}_{K,\pi} V$. If V is of compact type, then it is even a topological isomorphism, and so in particular $\mathcal{C}^{\mathrm{la}}(X, V)$ is again a space of compact type.*

Proof. If $\{X_i\}_{i\in I}$ is an analytic partition of X (necessarily finite, since X is compact), then (2.1.12) yields a natural map

$$\prod_{i\in I}\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i,V)\to\prod_{i\in I}(\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i,K)\,\hat{\otimes}_{K,\pi}V)$$
$$\xrightarrow{\sim}(\prod_{i\in I}\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i,K))\,\hat{\otimes}_{K,\pi}V\to\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi}V.$$

Passing to the locally convex inductive limit over all analytic partitions, we obtain a map $\mathcal{C}^{\mathrm{la}}(X,V)\to\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi}V$. To see that it is injective, we may compose it with the natural map $\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi}V\to\mathcal{C}(X,K)\,\hat{\otimes}_{K,\pi}V\xrightarrow{\sim}\mathcal{C}(X,\hat{V})$ (where $\hat{V}$ denotes the completion of V, and the isomorphism is provided by (2.1.3)). The map that we obtain is then seen to be equal to the composite $\mathcal{C}^{\mathrm{la}}(X,V)\to\mathcal{C}(X,V)\to\mathcal{C}(X,\hat{V})$, where the first arrow is the injection of proposition 2.1.26, and the second arrow is induced by the embedding $V\to\hat{V}$. Since this latter composite is injective, we conclude that the map $\mathcal{C}^{\mathrm{la}}(X,V)\to\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi}V$ that we have constructed is also injective.

Suppose now that V is of compact type, and write $V=\varinjlim_n V_n$ as the inductive limit of a sequence of Banach spaces, with injective and compact transition maps. Choose a cofinal sequence $(\{X_i\}_{i\in I_m})_{m\geq 1}$ of analytic partitions of X, such that each partition provides a relatively compact refinement of the partition that precedes it. We may compute $\mathcal{C}^{\mathrm{la}}(X,V)$ as the locally convex inductive limit $\varinjlim_{\{X_i\}_{i\in I_m},n}(\prod_{i\in I_m}\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i,K))\,\hat{\otimes}_K V_n$. Part (i) of proposition 1.1.32 now shows that the injection $\mathcal{C}^{\mathrm{la}}(X,V)\to\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_K V$ is a topological isomorphism, and that $\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_K V$ is of compact type, as claimed. □

The final statement of the preceding proposition is a restatement of [23, lem. 2.1]. The discussion that follows proposition 2.2.10 below shows that if V is complete and of LB-type, then the map of the preceding proposition is bijective.

Corollary 2.1.29. *If $W\to V$ is a closed embedding of Hausdorff convex K-vector spaces of compact type, and if X is a compact locally L-analytic manifold, then the natural injection $\mathcal{C}^{\mathrm{la}}(X,W)\to\mathcal{C}^{\mathrm{la}}(X,V)$ is a closed embedding.*

Proof. Propositions 2.1.27 and 2.1.28 together imply that the source and image of this injection are both spaces of compact type. By theorem 1.1.17, it is necessarily a closed embedding. □

Proposition 2.1.30. *If V is a Hausdorff convex space of LB-type, and if X is a compact locally L-analytic manifold, then $\mathcal{C}^{\mathrm{la}}(X,V)$ is an LB-space.*

Proof. Write $V=\bigcup nV_n$ as a union of BH-subspaces. Choose a cofinal sequence $(\{X_i\}_{i\in I_m})_{m\geq 1}$ of analytic partitions of X. We may compute $\mathcal{C}^{\mathrm{la}}(X,V)$ as the locally convex inductive limit $\varinjlim_{\{X_i\}_{i\in I_m},n}(\prod_{i\in I_m}\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i,K))\,\hat{\otimes}_K\overline{V}_n$. Thus it is an LB-space. □

Proposition 2.1.31. *If X is a compact locally analytic L-manifold and if V and W are two Hausdorff convex K-vector spaces of compact type, then the natural map $\mathcal{L}_b(V,W)\to\mathcal{L}_b(\mathcal{C}^{\mathrm{la}}(X,V),\mathcal{C}^{\mathrm{la}}(X,W))$, induced by the functoriality of the construction of $\mathcal{C}^{\mathrm{la}}(X,–)$, is a continuous map of convex K-vector spaces.*

Proof. Since V and W are of compact type, proposition 2.1.28 shows that the same is true of $\mathcal{C}^{\mathrm{la}}(X,V)$ and $\mathcal{C}^{\mathrm{la}}(X,W)$, and that furthermore, the latter spaces are isomorphic to $\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_K V$ and $\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_K V$ respectively. From [20, prop. 20.9] we deduce isomorphisms $V'_b\,\hat{\otimes}_{K,\pi} W \xrightarrow{\sim} \mathcal{L}_b(V,W)$ and

$$(\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_K V)'_b\,\hat{\otimes}_{K,\pi}\,\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi} W \xrightarrow{\sim} \mathcal{L}_b(\mathcal{C}^{\mathrm{la}}(X,V),\mathcal{C}^{\mathrm{la}}(X,W)).$$

Taking into account the isomorphism $(\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_K V)'_b \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(X,K)'_b\,\hat{\otimes}_{K,\pi} V'_b$ yielded by proposition 1.1.32 (ii), we find that the natural map $\mathcal{L}_b(V,W) \to \mathcal{L}_b(\mathcal{C}^{\mathrm{la}}(X,V),\mathcal{C}^{\mathrm{la}}(X,W))$ that is under consideration may be rewritten as a map

$$\begin{aligned}(2.1.32)\quad V'_b\,\hat{\otimes}_{K,\pi} W &\to \mathcal{C}^{\mathrm{la}}(X,K)'_b\,\hat{\otimes}_{K,\pi} V'_b\,\hat{\otimes}_{K,\pi}\,\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi} W \\ &\xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(X,K)'_b\,\hat{\otimes}_{K,\pi}\,\mathcal{C}^{\mathrm{la}}(X,K)\,\hat{\otimes}_{K,\pi} V'_b\,\hat{\otimes}_{K,\pi} W.\end{aligned}$$

This is immediately seen to be obtained from the map

$$(2.1.33)\qquad K \to \mathcal{C}^{\mathrm{la}}(X,K)'_b\,\hat{\otimes}_{K,\pi}\,\mathcal{C}^{\mathrm{la}}(X,K)$$

(obtained by taking $V = W = K$ in (2.1.32), which sends $1 \in K$ to the element of the tensor product corresponding to the identity map on $\mathcal{C}^{\mathrm{la}}(X,K)$) by tensoring through with $V'_b\,\hat{\otimes}_{K,\pi} W$, and then completing. Since (2.1.33) is certainly continuous, we conclude that the same is true of (2.1.32). $\square$

2.2. Distributions

In this section we recall the various types of distributions that are relevant to non-archimedean function theory.

Definition 2.2.1. If X is a Hausdorff topological space, then we let $\mathcal{D}(X,K)$ denote the dual space to the convex K-vector space $\mathcal{C}(X,K)$. This is the space of K-valued measures on X.

Definition 2.2.2. If $\mathbb{X}$ is a σ-affinoid rigid analytic space defined over L, then we let $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$ denote the dual space to the convex K-vector space $\mathcal{C}^{\mathrm{an}}(\mathbb{X},K)$. This is the space of K-valued analytic distributions on $\mathbb{X}$.

Definition 2.2.3. If X is a locally L-analytic manifold, then we let $\mathcal{D}^{\mathrm{la}}(X,K)$ denote the dual space to the convex space of compact type $\mathcal{C}^{\mathrm{la}}(X,K)$. This is the space of K-valued locally analytic distributions on X.

In these definitions we have not specified any particular topology on the dual spaces under consideration. As with the dual space to any convex space, they admit various locally convex topologies. Frequently we will endow these spaces with their strong topologies, in which case we add a subscript 'b' to emphasize this. If X is a compact topological space X then $\mathcal{D}(X,K)_b$ is a Banach space, if $\mathbb{X}$ is an affinoid rigid analytic space over L then $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)_b$ is also a Banach space, if $\mathbb{X}$ is a strictly σ-affinoid rigid analytic space then $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)_b$ is of compact type, if X is a locally L-analytic space then $\mathcal{D}^{\mathrm{la}}(X,K)_b$ is reflexive, and if X is furthermore compact then $\mathcal{D}^{\mathrm{la}}(X,K)_b$ is a nuclear Fréchet space. (The third and fifth claims follow from the fact that the dual of a nuclear Fréchet space is a space of compact type, and conversely. The fourth follows from the fact that $\mathcal{D}^{\mathrm{la}}(X,K)$ is dual to the reflexive space $\mathcal{C}^{\mathrm{la}}(X,K)$.)

If μ is an element of $\mathcal{D}(X,K)$ and f is an element of $\mathcal{C}(X,K)$, then it is sometimes suggestive to write $\int_X f(x)d\mu(x)$ to indicate the evaluation of the functional μ on f. Similar notation can be used for the evaluation of elements of $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$ (respectively $\mathcal{D}^{\mathrm{la}}(X,K)$) on elements of $\mathcal{C}^{\mathrm{an}}(\mathbb{X},K)$ (respectively $\mathcal{C}^{\mathrm{la}}(X,K)$) for σ-affinoid rigid analytic spaces $\mathbb{X}$ (respectively locally analytic spaces X) over L.

If X is a Hausdorff topological space and x is an element of X then the functional ev_x defines an element of $\mathcal{D}(X,K)$, which we denote (as usual) by δ_x. If we let $K[X]$ denote the vector space spanned on X over K then the map $x \mapsto \delta_x$ gives a map $K[X] \to \mathcal{D}(X,K)$, which is easily seen to be an embedding (since a continuous function on X can assume arbitrary values at a finite collection of points of X). The image of this map is weakly dense in $\mathcal{D}(X,K)$ (since a continuous function that vanishes at every point of X vanishes), but is typically not strongly dense (since $\mathcal{C}(X,K)$ is typically not reflexive).

Similarly, if $\mathbb{X}$ is a rigid analytic space over L, if $X = \mathbb{X}(L)$, and if x is an element of X, then ev_x defines an element δ_x of $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$. If X is Zariski dense in $\mathbb{X}$ then the elements δ_x are weakly dense in $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$ (since by assumption a rigid analytic function which vanishes at every point of X vanishes on $\mathbb{X}$), but are typically not strongly dense in $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$ (since $\mathcal{C}^{\mathrm{an}}(X,K)$ is typically not reflexive). However, if $\mathbb{X}$ is strictly σ-affinoid, then the nuclear Fréchet space $\mathcal{C}^{\mathrm{an}}(\mathbb{X},K)$ is reflexive, and so the elements $\delta_x \in \mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$ corresponding to the functionals ev_x are both weakly and strongly dense in $\mathcal{D}^{\mathrm{an}}(\mathbb{X},K)$. Similarly, if X is a locally L-analytic manifold then $\mathcal{C}^{\mathrm{la}}(X,K)$ is reflexive, and so the elements δ_x defined by the functionals ev_x are both weakly and strongly dense in $\mathcal{D}^{\mathrm{la}}(X,K)$. (This is [23, lem. 3.1] .)

Lemma 2.2.4. *If X is a Hausdorff topological space, then the map $X \to \mathcal{D}(X,K)_s$, given by $x \mapsto \delta_x$, is continuous.*

Proof. Let ξ denote the topology on X induced by this injection. By definition of the weak topology, for any $x \in X$, a basis of ξ-neighbourhoods of x is provided by sets of the form $\{y \in X \mid |f(x) - f(y)| < \epsilon\}$, where f is an element of $\mathcal{C}(X,K)$, and ϵ a positive real number. Thus the topology ξ is coarser than the given topology on X, and the proposition follows. $\square$

If X is a locally L-analytic manifold, then the continuous injection $\mathcal{C}^{\mathrm{la}}(X,K) \to \mathcal{C}(X,K)$ of proposition 2.1.26 induces a continuous map

$$\mathcal{D}(X,K)_b \to \mathcal{D}^{\mathrm{la}}(X,K)_b. \tag{2.2.5}$$

Also, the association of δ_x to x induces an injective map

$$X \to \mathcal{D}^{\mathrm{la}}(X,K)_b. \tag{2.2.6}$$

Proposition 2.2.7. *Let X be a locally L-analytic manifold.*

(i) *The map* (2.2.5) *is injective, with dense image.*
(ii) *The map* (2.2.6) *is a topological embedding.*

Proof. Since the map of proposition 2.1.26 has dense image, the map (2.2.5) is injective. Since the map (2.2.5) evidently contains all the elements δ_x ($x \in X$) in its image (they are the images of the corresponding elements δ_x in its source), it also has dense image. This proves (i).

We turn to proving (ii). Let ξ denote the given topology on X, and let ξ' denote the topology induced on X by regarding it as a subspace of $\mathcal{D}^{\mathrm{la}}(X,K)$ via the map (2.2.6). Any locally analytic function $f \in \mathcal{C}^{\mathrm{la}}(X,K)$ induces a continuous functional on $\mathcal{D}^{\mathrm{la}}(X,K)$, which thus restricts to a ξ'-continuous function on X. From the definition of (2.2.6), this is exactly the function f again. Thus we see in particular that any ξ-locally constant function on X is ξ'-continuous. Since a basis of ξ-open subsets of X can be cut out via locally constant functions, we see that the topology ξ' is finer than ξ.

Since $\mathcal{C}^{\mathrm{la}}(X,K)$ is reflexive, any closed and bounded subset of $\mathcal{C}^{\mathrm{la}}(X,K)$ is c-compact. Also, $\mathcal{C}(X,K)$ is complete and barrelled (the latter by corollary 2.1.5). The hypotheses of proposition 1.1.38 are thus satisfied by the map of proposition 2.1.26, and we infer from that proposition that the transpose map (2.2.5) factors as $\mathcal{D}(X,K)_b \to \mathcal{D}(X,K)_{bs} \to \mathcal{D}^{\mathrm{la}}(X,K)_b$. We will show that the map $x \mapsto \delta_x$ induces a continuous map $X \to \mathcal{D}(X,K)_{bs}$, and thus that (2.2.6) is continuous. This will imply that ξ is finer than ξ', and so prove (ii).

Let $X = \coprod_i X_i$ be a partition of X into a disjoint union of compact open subsets. It suffices to show that the map $X_i \to \mathcal{D}(X,K)_{bs}$, given by $x \mapsto \delta_x$, is continuous for each i. The image of X_i in $\mathcal{D}(X,K)$ is bounded as a subset of $\mathcal{D}(X,K)_b$ (since X_i is compact), and so by definition of the bounded-weak topology, the topology on this image induced by $\mathcal{D}(X,K)_{bs}$ is coarser than the weak topology. Thus it suffices to show that the map $X_i \to \mathcal{D}(X,K)_s$ is continuous. This map factors as $X_i \to \mathcal{D}(X_i,K)_s \to \mathcal{D}(X,K)_s$, and so by lemma 2.2.4 we are done. □

If V is any Hausdorff convex K-vector space, then by composing elements of $\mathcal{L}(\mathcal{D}^{\mathrm{la}}(X,K)_b, V)$ with the map (2.2.6), we obtain a K-linear map

$$\mathcal{L}(\mathcal{D}^{\mathrm{la}}(X,K)_b, V) \to \mathcal{C}(X,V). \tag{2.2.8}$$

Proposition 2.2.9. *The map* (2.2.8) *is continuous, when the source is given its strong topology.*

Proof. If we let $\{X_i\}_{i\in I}$ denote a partition of X into compact open subsets, then the isomorphisms $\bigoplus_{i\in I} \mathcal{D}^{\mathrm{la}}(X_i,K)_b \xrightarrow{\sim} \mathcal{D}^{\mathrm{la}}(X,K)_b$ and $\prod_{i\in I} \mathcal{C}(X_i,V) \xrightarrow{\sim} \mathcal{C}(X,V)$ allow us to write (2.2.8) as the product of the maps $\mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X_i,K)_b, V) \to \mathcal{C}(X_i,V)$. Hence we may restrict our attention to the case where X is compact. If $\hat{V}$ denotes the completion of V, we may embed (2.2.8) into the commutative diagram

$$\begin{tikzcd} \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X,K)_b, V) \arrow[r] \arrow[d] & \mathcal{C}(X,V) \arrow[d] \\ \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X,K)_b, \hat{V}) \arrow[r] & \mathcal{C}(X,\hat{V}), \end{tikzcd}$$

in which the vertical arrows are embeddings. Thus we may assume in addition that V is complete. Taking into account [20, cor. 18.8] and the isomorphism (2.1.3), we may rewrite (2.2.8) (in the case where X is compact and V is complete) as the completed tensor product with V of the natural map $\mathcal{C}^{\mathrm{la}}(X,K) \to \mathcal{C}(X,K)$. Thus it is continuous, as claimed. □

Proposition 2.2.10. *There is a continuous linear map*

$$\mathcal{C}^{\mathrm{la}}(X,V) \to \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X,K)_b, V),$$

uniquely determined by the fact that its composite with (2.2.8) *is the map of proposition* 2.1.26. *If furthermore V is of LB-type, then this map is a bijection.*

Proof. The required map is constructed in [23, thm. 2.2] and the discussion that follows the proof of that theorem, and by that theorem is shown to be bijective if V is of LB-type. We will give another description of this map, which shows that it is continuous. Let $\{X_i\}_{i\in I}$ be an analytic partition of X, and let $\{W_i\}_{i\in I}$ be a collection of BH-subspaces of V. For each $i \in I$, the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \to \mathcal{C}^{\mathrm{la}}(X_i, K)$ and the map $\overline{W}_i \to V$ together induce a continuous map

$$\begin{aligned}\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, \overline{W}_i) = \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \mathbin{\hat{\otimes}}_K \overline{W}_i \to \mathcal{C}^{\mathrm{la}}(X_i, K) \mathbin{\hat{\otimes}}_K \overline{W}_i \\ \xrightarrow{\sim} \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X_i, K), \overline{W}_i) \to \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X_i, K), V)\end{aligned}$$

(the isomorphism following from [20, cor. 18.8]). Taking the product of these maps over all $i \in I$, we obtain a continuous map

$$\begin{aligned}\prod_i \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, \overline{W}_i) \to \prod_i \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X_i, K), V) \\ \xrightarrow{\sim} \mathcal{L}_b(\bigoplus_{i\in I} \mathcal{D}^{\mathrm{la}}(X_i, K), V) \xrightarrow{\sim} \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X, K), V).\end{aligned}$$

Passing to the inductive limit over all such collections of pairs $\{(X_i, W_i)\}_{i\in I}$, we obtain a continuous map

$$\mathcal{C}^{\mathrm{la}}(X, K) \to \mathcal{L}_b(\mathcal{D}^{\mathrm{la}}(X, K), V). \tag{2.2.12}$$

If $f \in \mathcal{C}^{\mathrm{la}}(X, K)$, then its image $\phi \in \mathcal{L}(\mathcal{D}^{\mathrm{la}}(X, K)_b, V)$ is characterized by the property $\phi(\delta_x) = f(x)$, and so we see that (2.2.12) coincides with the map constructed in [23], and that the composite of this map with (2.2.8) is the map of proposition 2.1.26, as claimed. □

The preceding result relates to proposition 2.1.28, in the following way. Suppose that V is complete. Then, taking into account the isomorphism of [20, cor. 18.8], the map of proposition 2.2.10 may be rewritten as a map

$$\mathcal{C}^{\mathrm{la}}(X, V) \to \mathcal{C}^{\mathrm{la}}(X, K) \mathbin{\hat{\otimes}}_{K,\pi} V.$$

This is easily checked to be equal to the map of proposition 2.1.28. Thus if V is complete and of LB-type, then we see that the map of proposition 2.1.28 is a bijection.

2.3. Change of field

In this brief section we discuss two "change of field" functors.

If F is a finite extension of K, then the valuation on K extends in a unique fashion to a spherically complete non-archimedean valuation on F. We can now define an "extension of scalars" functor as follows. If V is any locally convex K-vector space V, then the tensor product $F \otimes_K V$ is naturally a locally convex F-vector space. Furthermore, it is Hausdorff, normable, metrizable, complete, barrelled, bornological, an LF-space, and LB-space, or of compact type, if and only if the same is true of V. (If F is an arbitrary extension of K, spherically complete with respect to a non-archimedean valuation extending that on K, then we can define an extension of scalars functor by sending V to $F \otimes_{K,\pi} V$. However, this functor is not so easily studied if F is of infinite degree over K.)

If V is any Hausdorff locally convex K-vector space, then given any Hausdorff topological space X, any σ-affinoid rigid analytic space $\mathbb{X}$ over L, or any locally L-analytic space X, there are natural isomorphisms $F \otimes_K \mathcal{C}(X, V) \xrightarrow{\sim} \mathcal{C}(X, F \otimes_K V)$, $F \otimes_K \mathcal{C}^{\mathrm{an}}(\mathbb{X}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{X}, F \otimes_K V)$, and $F \otimes_K \mathcal{C}^{\mathrm{la}}(X, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(X, F \otimes_K V)$.

Suppose now that E is a closed subfield of L (so that there are inclusions $\mathbb{Q}_p \subset E \subset L$). If $\mathbb{X}$ is a rigid analytic space over E, then we let $\mathbb{X}_{/L}$ denote the corresponding rigid analytic space over L, obtained by extending scalars. If A is an affinoid E-Banach algebra, then $(\mathrm{Sp}\, A)_{/L} = \mathrm{Sp}(L \otimes_E A)$. Extension of scalars defines a functor from the category of rigid analytic spaces over E to the category of rigid analytic spaces over L. This functor admits a right adjoint, referred to as "restriction of scalars". If $\mathbb{X}$ is a rigid analytic space over L, then we let $\mathrm{Res}^L_E \mathbb{X}$ denote the rigid analytic space over E obtained by restriction of scalars.

The right adjointness property of restriction of scalars implies that for any rigid analytic space $\mathbb{X}$ over L, there is a natural map

$$(\mathrm{Res}^L_E \mathbb{X})_{/L} \to \mathbb{X}. \tag{2.3.1}$$

If L is a a Galois extension of E, then there is furthermore a natural isomorphism $(\mathrm{Res}^L_E \mathbb{X})_{/L} \xrightarrow{\sim} \prod_\tau \mathbb{X}^{(\tau)}$, where τ ranges over the elements of the Galois group of L over E, and $\mathbb{X}^{(\tau)}$ denotes the rigid analytic space over L obtained by twisting $\mathbb{X}$ via the automorphism $\tau : L \xrightarrow{\sim} L$. (The map (2.3.1) is obtained by projecting onto the factor corresponding to the identity automorphism.)

We may similarly define an extension of scalars functor from the category of locally E-analytic spaces to the category of locally L-analytic spaces, and a corresponding right adjoint restriction of scalars functor. Note that if $\mathbb{X}$ is a rigid analytic space over L, then $\mathrm{Res}^L_E \mathbb{X}(E) = \mathbb{X}(L)$, and hence that if X is a locally L-analytic space, then $\mathrm{Res}^L_E X$ has the same underlying topological space as X. Thus restriction of scalars may equally well be regarded as the forgetful functor, "regard a locally L-analytic space as a locally E-analytic space".

If X is a locally L-analytic space, and we first restrict and then extend scalars, then there is a natural map

$$(\mathrm{Res}^L_E X)_{/L} \to X. \tag{2.3.2}$$

Chapter 3. Continuous, analytic, and locally analytic vectors

3.1. Regular representations

In this section we study the left and right regular actions of a group on the function spaces associated to it. We begin with a lemma.

Lemma 3.1.1. *Let V be a topological K-vector space and G a topological group, and consider a map*

$$G \times V \to V \tag{3.1.2}$$

such that each $g \in G$ acts as a linear transformation of V. Then the map (3.1.2) *is continuous if and only if it satisfies the following conditions:*

(i) *for each $v \in V$ the map $g \mapsto gv$ from G to V induced by* (3.1.2) *is continuous at the identity $e \in G$;*

(ii) *for each $g \in G$ the linear transformation $v \mapsto gv$ of V induced by* (3.1.2) *is continuous* (*that is, the G-action on V is topological*)*;*

(iii) *the map* (3.1.2) *is continuous at the point* $(e, 0)$ *of $G \times V$.*

Proof. It is clear that these conditions are necessary, and we must show that they are also sufficient. We begin by showing that if v is an element of V, then the map (3.1.2) is continuous at the point $(e, v) \in G \times V$.

Let M be a neighbourhood of zero in V, and choose a neighbourhood M' of zero such that $M' + M' \subset M$. By (iii), we may find a neighbourhood H of e in G and a neighbourhood M'' of zero in V such that $gM'' \subset M'$ if $g \in H$. Since the map $g \mapsto gv$ is continuous at the element e, by (i), we may also find a neighbourhood H' of e contained in H such that $H'v \subset v + M'$. Thus we see that the image of $H' \times (v+M'')$ is contained in $v+M$, proving that (3.1.2) is continuous at the point $(e, v) \in G \times V$.

Now let (g, v) be an arbitrary element of $G \times V$, and let M be neighbourhood of zero in V. Replacing v by gv in the discussion of the preceding paragraph, we may find an open subset H' of $e \in G$ and a neighbourhood M'' of zero in V such that the image of $H' \times (gv + M'')$ is contained in $gv + M$. Since the action of g on V is continuous, by (ii), we may find a neighbourhood M''' of zero such that $gM''' \subset M''$. Now $H'g$ is a neighbourhood of g and the image of $H'g \times (v + M''')$ under (3.1.2) is contained in $gv + M$. This proves that (3.1.2) is continuous at the point $(g, v) \in G \times V$. $\square$

Note that condition (iii) of the preceding lemma holds if a neighbourhood of e in G acts equicontinuously on V. Suppose, for example, that G is a locally compact abelian group, and V is a barrelled convex K-vector space equipped with a G-action for which the action map (3.1.2) is separately continuous. Any relatively compact neighbourhood of e then acts in a pointwise bounded fashion on V, and hence (since V is barrelled) acts equicontinuously on V. Thus all three conditions of lemma 3.1.1 are satisfied, and (3.1.2) is jointly continuous.

Corollary 3.1.3. *If V is a convex K-vector space equipped with a topological action of the topological group G, and if H is an open subgroup of G such that the resulting H-action on V is continuous, then the G-action on V is continuous.*

Proof. By assumption, condition (ii) of lemma 3.1.1 is satisfied for G, and conditions (i) and (iii) are satisfied with H in place of G. But these latter two conditions depend only on an arbitrarily small neighbourhood of the identity $e \in G$, and so they hold for G since they hold for H. Lemma 3.1.1 now implies that the G-action on V is continuous. $\square$

Lemma 3.1.4. *If V is a topological K-vector space equipped with a continuous action of a topological group G, then any compact subset of G acts equicontinuously on V.*

Proof. Let C be a compact subset of G, and let M be a neighbourhood of zero in V. If $g \in C$, then there exists a neighbourhood U_g of g in G, and a neighbourhood M_g of zero in V, such that $U_g M_g \subset M$. A finite number of the U_g suffice to cover the compact set C. If we let M' denote the intersection of the corresponding finite collection of neighbourhoods M_g, we obtain a neighbourhood of zero with the property that $CM' \subset M$. This proves the lemma. $\square$

Suppose now that G is an abstract group and that V is an abstract K-vector space. The left and right regular actions of G on itself are the maps $G \times G \to G$ defined as $(g_1, g_2) \mapsto g_2^{-1} g_1$ and $(g_1, g_2) \mapsto g_1 g_2$ respectively. (We regard these as right actions of G on itself.) They commute one with the other. Via functoriality of the formation of the K-vector space of functions $\mathcal{F}(G, V)$, we obtain (left) actions of G on the space $\mathcal{F}(G, V)$, which we again refer to as the left and right regular actions. These actions also commute with one another.

If G is a topological group (respectively the L-valued points of a σ-affinoid rigid analytic group $\mathbb{G}$ defined over L, respectively a locally L-analytic group) and if V is a Hausdorff locally convex topological K-vector space, then we similarly may consider the left and right regular representations of G on $\mathcal{C}(G, V)$ (respectively on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, respectively on $\mathcal{C}^{\mathrm{la}}(G, V)$).

Proposition 3.1.5. *If G is a locally compact topological group and V is a Hausdorff locally convex K-vector space, then the left and right regular actions of G on $\mathcal{C}(G, V)$ are both continuous.*

Proof. We will show that the right regular action is continuous. The proof of the analogous result for the left regular action proceeds along identical lines.

If C is a compact subset of G and M is a neighbourhood of zero in V, let $U_{C,M}$ denote the subset of $\mathcal{C}(G, V)$ consisting of functions ϕ such that $\phi(C) \subset M$. As C runs over all compact subsets of G and M runs over the neighbourhoods of zero in V, the sets $U_{C,M}$ form a basis of neighbourhoods of zero in $\mathcal{C}(G, V)$. If H is a relatively compact open subset of G then CH is a compact subset of G, and the right regular action of H on $\mathcal{C}(G, V)$ takes $U_{CH,M}$ to $U_{C,M}$. Thus we see that H acts equicontinuously on $\mathcal{C}(G, V)$. In particular, since G is covered by such H, condition (ii) of lemma 3.1.1 holds. Choosing H so that it contains a neighbourhood of $e \in G$, we see that condition (iii) of that lemma also holds. It remains to be shown that condition (i) of lemma 3.1.1 holds.

Let ϕ be an element of $\mathcal{C}(G, V)$, C a compact subset of G, and M a neighbourhood of zero in V. We will find a relatively compact neighbourhood H of the identity in G such that $H(\phi + U_{CH,M}) \subset \phi + U_{C,M}$. This will show that the right regular action is continuous at the point (e, ϕ) of $G \times \mathcal{C}(G, V)$, verifying condition (i).

Since ϕ is continuous and C is compact, we may find an open neighbourhood H of e such that $\phi(g) - \phi(gh) \in M$ for any $g \in C$ and $h \in H$. (This is a variation on the fact that for a function on a compact uniform space, continuity implies uniform continuity.) Since G is locally compact, we may choose H to be relatively compact. Thus $h\phi \subset \phi + U_{C,M}$ for every $h \in H$, and so $H(\phi + U_{CH,M}) \subset \phi + U_{C,M}$, as required. $\square$

Proposition 3.1.6. *Suppose that $\mathbb{G}$ is a rigid analytic group defined over L, which is furthermore σ-affinoid as a group, in the sense that $\mathbb{G} = \bigcup_{n=1}^{\infty} \mathbb{G}_n$, where each $\mathbb{G}_n$ is an admissible affinoid open subgroup of $\mathbb{G}$. If we write $G = \mathbb{G}(L)$, then for any Hausdorff convex K-vector space V, the left and right regular actions of G on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ are continuous.*

Proof. If we write $G = \bigcup_{n=1}^{\infty} G_n$, with $G_n := \mathbb{G}_n(L)$ open in G for each $n \geq 1$, then

$\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) = \varprojlim_n \mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, V)$, and so it suffices to show that G_n acts continuously on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, V)$ for each $n \geq 1$. This reduces us to the case where $\mathbb{G}$ is affinoid. We suppose this to be the case for the remainder of the proof.

Since $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is barrelled, the discussion following lemma 3.1.1 shows that it suffices to show that the G-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is separately continuous. The definition of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ as an inductive limit then allows us to reduce first to the case where V is a Banach space. Finally, the definition of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ for a Banach space V allows us to reduce to the case where $V = K$. The proof then proceeds along the same lines as that of the preceding proposition. (In place of the fact that a continuous function on a compact space is uniformly continuous, one should use the fact that an analytic function on an affinoid space is Lipschitz [3, prop. 7.2.1/1].) $\square$

3.2. The orbit map and continuous vectors

Now suppose that V is an abstract K-vector space equipped with an action of an abstract group G. Each element v of V gives rise to a function $o_v \in \mathcal{F}(G, V)$, defined by $o_v(g) = gv$; we refer to o_v as the orbit map of the vector v. Formation of the orbit map yields a K-linear embedding $o : V \to \mathcal{F}(G, V)$, which is a section to the map $\mathrm{ev}_e : \mathcal{F}(G, V) \to V$.

Lemma 3.2.1. *The map $o : V \to \mathcal{F}(G, V)$ is G-equivariant when $\mathcal{F}(G, V)$ is equipped with the right regular G-action. Thus the restriction of* ev_e *to $o(V)$ is a G-equivariant isomorphism when $o(V)$ is endowed with the right regular G-action.*

Proof. This lemma is an immediate consequence of the fact that o is a section to ev_e, together with the formula $o_v(gg') = o_{g'v}(g)$. $\square$

There is a convenient characterization of the image of o. First note that, in addition to the left and right regular actions of G on $\mathcal{F}(G, V)$, we obtain another action of G induced by its action on V and the functoriality of $\mathcal{F}(G, V)$ in V, which we refer to as the pointwise action of G. This commutes with both the left and right regular actions. Since the left and right regular actions also commute one with the other, we obtain an action of $G \times G \times G$ on $\mathcal{F}(G, V)$, the first factor acting via the pointwise action, the second via the left regular action, and the third via the right regular action. Explicitly, if $(g_1, g_2, g_3) \in G \times G \times G$ and $\phi \in \mathcal{F}(V, G)$, then

$$((g_1, g_2, g_3) \cdot \phi)(g') = g_1 \cdot \phi(g_2^{-1} g' g_3). \tag{3.2.2}$$

Let $\Delta_{1,2} : G \to G \times G \times G$ denote the map $g \mapsto (g, g, 1)$. Then $\Delta_{1,2}$ induces a G-action on $\mathcal{F}(G, V)$ which commutes with the right regular action of G. Explicitly, this action is defined by

$$(g \cdot \phi)(g') = g \cdot \phi(g^{-1} g'). \tag{3.2.3}$$

We first explain how to untwist the $\Delta_{1,2}(G)$-action on $\mathcal{F}(G, V)$.

Lemma 3.2.4. *If we define a K-linear map $\mathcal{F}(G, V) \to \mathcal{F}(G, V)$ by sending a function ϕ to the function $\tilde{\phi}$ defined by $\tilde{\phi}(g) = g^{-1} \cdot \phi(g)$, then this map is an isomorphism. It is furthermore G-equivariant, provided we equip its source with the $\Delta_{1,2}(G)$-action and its target with the left regular G-action.*

Proof. This is easily checked by the reader. $\square$

We observe for later that we could also consider the $\Delta_{1,3}(G)$-action on $\mathcal{F}(G,V)$, and that this action is made isomorphic to the right regular G-action on $\mathcal{F}(G,V)$ via the isomorphism $\phi \mapsto (\tilde{\phi} : g \mapsto g\phi(g))$. For now, we require the following corollary.

Corollary 3.2.5. *The image of o is equal to the space $\mathcal{F}(G,V)^{\Delta_{1,2}(G)}$ of $\Delta_{1,2}(G)$-fixed vectors in $\mathcal{F}(G,V)$.*

Proof. An element ϕ of $\mathcal{F}(G,V)$ is fixed under the action (3.2.3) if and only if the element $\tilde{\phi}$ that it is sent to under the isomorphism of lemma 3.2.4 is fixed under the left regular action of G; equivalently, if and only if $\tilde{\phi}$ is constant. The explicit formula relating ϕ and $\tilde{\phi}$ shows that $\tilde{\phi}$ is constant if and only if $\phi(g) = g\phi(e) = o_{\phi(e)}(g)$ for all $g \in G$. $\square$

If V is any K-vector space (not necessarily equipped with a G-action), we may consider the double function space $\mathcal{F}(G, \mathcal{F}(G,V))$. If we endow $\mathcal{F}(G,V)$ with its right regular G-action, then the preceding constructions apply, with $\mathcal{F}(G,V)$ in place of V. There is a natural isomorphism $\imath : \mathcal{F}(G \times G, V) \xrightarrow{\sim} \mathcal{F}(G, \mathcal{F}(G,V))$, defined by $(\imath\phi)(g_1)(g_2) = \phi(g_1, g_2)$, and we will reinterpret the preceding constructions (applied to $\mathcal{F}(G, \mathcal{F}(G,V))$) in terms of this isomorphism.

Let $\alpha : G \times G \to G$ denote the map $\alpha : (g_1, g_2) \mapsto g_2 g_1$ and $\beta : G \to G \times G$ denote the map $\beta : g \mapsto (e, g)$. These maps induce maps $\alpha^* : \mathcal{F}(G,V) \to \mathcal{F}(G \times G, V)$ and $\beta^* : \mathcal{F}(G \times G, V) \to \mathcal{F}(G,V)$, and we obtain the following commutative diagrams:

$$\begin{tikzcd} \mathcal{F}(G,V) \arrow[d, "\alpha^*"] \arrow[dr, "o"] & \\ \mathcal{F}(G \times G, V) \arrow[r, "\imath"] & \mathcal{F}(G, \mathcal{F}(G,V)) \end{tikzcd} \tag{3.2.6}$$

and

$$\begin{tikzcd} \mathcal{F}(G \times G, V) \arrow[r, "\imath"] \arrow[dr, "\beta^*"'] & \mathcal{F}(G, \mathcal{F}(G,V)) \arrow[d, "\mathrm{ev}_e"] \\ & \mathcal{F}(G,V). \end{tikzcd} \tag{3.2.7}$$

(The commutativity of each diagram is easily checked by the reader.)

Suppose now that G is a locally compact topological group and that V is a Hausdorff convex K-vector space equipped with a topological G-action. We may apply the above considerations with $\mathcal{C}(G,V)$ in place of $\mathcal{F}(G,V)$.

Definition 3.2.8. We define the convex K-vector space V_{con} of continuous vectors in V to be the closed subspace of $\Delta_{1,2}(G)$-invariant elements of $\mathcal{C}(G,V)$, equipped with the continuous G-action induced by the right regular action of G on $\mathcal{C}(G,V)$.

The formation of V_{con} is evidently covariantly functorial in V and contravariantly functorial in G.

Proposition 3.2.9. *The restriction of the map* ev_e *to* V_{con} *is a continuous* G*-equivariant injection of* V_{con} *into* V*. Its image consists of the subspace of vectors* v *in* V *for which the orbit map* o_v *is continuous.*

Proof. Since the map $\mathrm{ev}_e : \mathcal{C}(G, V) \to V$ is continuous, so is its restriction to the closed subspace V_{con} of $\mathcal{C}(G, V)$. It is G-equivariant by lemma 3.2.1. By corollary 3.2.5, an element $\phi \in \mathcal{C}(G, V)$ is fixed under the action of $\Delta_{1,2}(G)$ if and only if it is of the form o_v for some $v \in V$. Thus an element v of V lies in $\mathrm{ev}_e(V_{\mathrm{con}})$ if and only if o_v is continuous. □

Now suppose that the G-action on V is continuous, so that V_{con} maps continuously and bijectively onto V. The map $v \mapsto o_v$ then yields an embedding $o : V \to \mathcal{C}(G, V)$ of K-vector spaces, which is a section to the map $\mathrm{ev}_e : \mathcal{C}(G, V) \to V$.

Proposition 3.2.10. *If* V *is a Hausdorff convex* K*-vector space, equipped with a continuous* G*-action, then the map* o *is continuous and* G*-equivariant, when* $\mathcal{C}(G, V)$ *is equipped with the right regular* G*-action, and restricts to a topological isomorphism of* V *onto the closed subspace* V_{con} *of* $\mathcal{C}(G, V)$*.*

Proof. Lemma 3.2.1 and corollary 3.2.5 taken together imply that the map o is G-equivariant, with image equal to V_{con}. To show that o is continuous, it suffices to show that for each compact subset C of G and each neighbourhood M of zero in V there is a neighbourhood M' of zero in V such that for all $v \in M'$, the function o_v takes C into M. In other words, regarding C as fixed for a moment, for every neighbourhood M of zero, there should be a neighbourhood M' of zero such that $CM' \subset M$. This is precisely the condition that the action of C be equicontinuous. Since C is compact, lemma 3.1.4 shows that it does act equicontinuously. Thus the map o is indeed continuous, and provides a continuous inverse to the the continuous map $V_{\mathrm{con}} \to V$ induced by ev_e. □

Lemma 3.2.11. *The isomorphism of lemma* 3.2.4 *induces a topological isomorphism of* $\mathcal{C}(G, V)$ *onto itself.*

Proof. We begin by noting that since the G-action on V is assumed to be continuous, the function $\tilde{\phi} : G \to V$ is continuous if the function ϕ is. (Here we are using the notation introduced in the statement of lemma 3.2.4.)

Fix a compact subset C of G and an open neighbourhood M of zero in V. Since C acts equicontinuously on V, by lemma 3.1.4, we may find an open neighbourhood M' of zero such that $CM' \subset M$. Then the map $\phi \mapsto \tilde{\phi}$ takes $\{\phi \,|\, \phi(C) \subset M'\}$ into the set $\{\tilde{\phi} \,|\, \tilde{\phi(C)} \subset M\}$. Thus $\phi \mapsto \tilde{\phi}$ is continuous. It is equally straightforward to check that its inverse is continuous, and thus the lemma is proved. □

Similarly, one may show that the map $\phi \mapsto (\tilde{\phi} : g \mapsto g\phi(g))$ induces an isomorphism between $\mathcal{C}(G, V)$ and $\mathcal{C}(G, V)$. (As observed following the proof of lemma 3.2.4, this map intertwines the $\Delta_{1,3}(G)$-action and the right regular action on its source and target.)

The following results will be required in subsequent sections.

Lemma 3.2.12. *Let* G *be a group and* V *be a Hausdorff convex* K*-vector space equipped with a topological* G*-action. If* W *is an* FH*-subspace of* V *that is invariant under the* G*-action on* V *then the topological* G*-action on* W *induces a topological* G*-action on* $\overline{W}$*.*

Proof. This follows from proposition 1.1.2 (ii). □

Definition 3.2.13. If G is a topological group and V is a Hausdorff convex K-vector space equipped with a topological G-action then we say that a G-invariant FH-subspace W of V is continuously invariant if the induced G-action on $\overline{W}$ is continuous.

Proposition 3.2.14. *Let G be a compact topological group and V be a Hausdorff convex K-vector space equipped with a topological G-action. If W is an FH-subspace of V then among all the continuously G-invariant FH-subspaces of W is one that contains all the others, which we denote by $\mathrm{co}_G(W)$. If W is a BH-subspace of V, then so is $\mathrm{co}_G(W)$.*

Proof. The continuous injection $\overline{W} \to V$ induces a continuous injection $\mathcal{C}(G, \overline{W}) \to \mathcal{C}(G, V)$. Define $\overline{U}$ to be the preimage under this map of the closed subspace V_{con} of $\mathcal{C}(G, V)$. Then $\overline{U}$ is a closed subspace of $\mathcal{C}(G, \overline{W})$, and thus a Fréchet space. By construction it is invariant under the right regular action of G on $\mathcal{C}(G, \overline{W})$, and thus endowed with a continuous G-action. We have continuous G-equivariant injections $\overline{U} \to V_{\mathrm{con}} \to V$; let U denote the image of $\overline{U}$ under the composite of these maps.

By construction, U is a continuously G-invariant FH-subspace of W. Furthermore, U is equal to the set of vectors $w \in W$ for which $Gw \subset W$ and $o_w : G \to \overline{W}$ is continuous, and so U contains any continuously G-invariant FH-subspace of W. Thus we set $\mathrm{co}_G(W) := U$. To complete the proof of the proposition, note that if $\overline{W}$ is in fact a Banach space, then so also is $\overline{U}$, and so U is a BH-subspace of V. $\square$

The following result is also useful.

Proposition 3.2.15. *If G is a compact topological group and V is a semi-complete Hausdorff convex K-vector space equipped with a continuous G-action, then any BH-subspace of V is contained in a G-invariant BH-subspace.*

Proof. The discussion preceding proposition 1.1.11 shows that it suffices to prove that any bounded subset of V may be embedded in a G-invariant bounded subset of V. Equivalently, if A is a bounded subset of V, we must show that the set GA of all G-translates of elements of A is bounded. This follows from lemma 3.1.4, which shows that G acts equicontinuously. $\square$

3.3. Analytic vectors

Let $\mathbb{G}$ be an affinoid rigid analytic group, and suppose that the group of L-rational points $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$. Let V be a Hausdorff convex K-vector space equipped with a topological G-action. Our goal in this section is to define the convex space of $\mathbb{G}$-analytic vectors in V.

We begin by supposing that V is a K-Banach space equipped with a topological action of G. We can repeat the discussion that begins section 3.2 in the context of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. In particular, the functoriality of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ in V induces a topological action of G on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ that we refer to as the pointwise action of G, which commutes with both the left and right regular actions. Since the left and right regular actions also commute one with the other, we obtain an action of $G \times G \times G$ on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, the first factor acting via the pointwise action, the second via the left regular action, and the third via the right regular action. As in section 2.1 we let $\Delta_{1,2} : G \to G \times G \times G$ denote the map $g \mapsto (g, g, 1)$. Then $\Delta_{1,2}$ induces a topological G-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ that commutes with the right regular action of G.

Definition 3.3.1. If V is a K-Banach space equipped with a topological action of G, then define the Banach space $V_{\mathbb{G}-\mathrm{an}}$ of $\mathbb{G}$-analytic vectors in V to be the closed subspace of $\Delta_{1,2}(G)$-invariant elements of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, equipped with the right regular action of G.

The formation of $V_{\mathbb{G}-\mathrm{an}}$ is clearly covariantly functorial in V and contravariantly functorial in $\mathbb{G}$. Note that the G-action on $V_{\mathbb{G}-\mathrm{an}}$ is continuous, since the right regular G-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is so (by proposition 3.1.6).

Proposition 2.1.20 yields a continuous injection

$$\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}(G, V) \tag{3.3.2}$$

which is compatible with the $G \times G \times G$-action on source and target.

Proposition 3.3.3. *The subspace $V_{\mathbb{G}-\mathrm{an}}$ of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is equal to the preimage under* (3.3.2) *of the closed subspace V_{con} of $\mathcal{C}(G, V)$. Evaluation at e yields a continuous injection $V_{\mathbb{G}-\mathrm{an}} \to V$. The image of $V_{\mathbb{G}-\mathrm{an}}$ in V consists of the subspace of those vectors v for which the orbit map o_v is* (*the restriction to G of*) *an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$.*

Proof. This is an immediate consequence of the definitions. □

Proposition 3.3.4. *Let V be a K-Banach space. If the Banach space $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is equipped with its right regular G-action, then the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, obtained by applying proposition* 3.3.3 *to $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, is an isomorphism.*

Proof. We consider the rigid analytic analogue of the diagrams (3.2.6) and (3.2.7). Namely, let $\alpha : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ denote the rigid analytic map $\alpha : (g_1, g_2) \mapsto g_2 g_1$, and let $\beta : \mathbb{G} \to \mathbb{G} \times \mathbb{G}$ denote the rigid analytic map $\beta : g \mapsto (e, g)$. Then α is faithfully flat, β is a closed immersion, and the composite $\alpha \circ \beta$ is equal to the identity map from $\mathbb{G}$ to itself.

Passing to spaces of rigid analytic functions, we obtain morphisms of Banach spaces $\alpha^* : \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}, K)$ and $\beta^* : \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)$, the first of which is a closed embedding, and the second of which is a strict surjection. There is a natural isomorphism

$$\imath : \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}, K) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \mathbin{\hat{\otimes}} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) = \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)).$$

Via this isomorphism, the action of $\Delta_{1,2}(G)$ on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K))$ induces an action of $\Delta_{1,2}(G)$ on $\mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}, K)$, given explicitly by the formula

$$(g \cdot \phi)(g_1, g_2) = \phi(g^{-1} g_1, g_2 g).$$

As G is dense in $\mathbb{G}$, we deduce that α^* identifies $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)$ with the $\Delta_{1,2}(G)$-invariant subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}, K)$. Also, since ev_e is equal to the composite $\beta^* \circ \imath^{-1}$, and since $\beta^* \circ \alpha^*$ is the identity on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)$, we conclude that $\mathrm{ev}_e : \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)_{\mathbb{G}-\mathrm{an}} = \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K))^{\Delta_{1,2}(G)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)$ is an isomorphism. Tensoring this isomorphism through by V, and completing, we conclude similarly that $\mathrm{ev}_e : \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is an isomorphism, as required. □

Proposition 3.3.5. *If $W \to V$ is a G-equivariant closed embedding of K-Banach spaces equipped with topological G-actions then the diagram*

$$\begin{tikzcd} W_{\mathbb{G}-\mathrm{an}} \arrow[r] \arrow[d, "(3.3.3)"] & V_{\mathbb{G}-\mathrm{an}} \arrow[d, "(3.3.3)"] \\ W \arrow[r] & V \end{tikzcd}$$

is Cartesian. In particular, the induced map $W_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is a closed embedding of Banach spaces.

Proof. The closed embedding $W \to V$ induces a closed embedding of Banach spaces $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, W) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, which by proposition 2.1.23 identifies $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, W)$ with the subspace of functions in $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ that are W-valued. Passing to $\Delta_{1,2}(G)$-invariants yields the proposition. $\square$

Corollary 3.3.6. *If V is a K-Banach space equipped with a topological G-action then the natural map $(V_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is an isomorphism.*

Proof. Proposition 3.3.4 shows that the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is an isomorphism. Taking this into account, the corollary follows by applying proposition 3.3.5 to the closed embedding $V_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. $\square$

Proposition 3.3.7. *For any K-Banach space V the continuous injective map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}(G, V)$ of proposition* 2.1.20 *induces an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \xrightarrow{\sim} \mathcal{C}(G, V)_{\mathbb{G}-\mathrm{an}}$. (The analytic vectors of the target are computed with respect to the right regular G-action.)*

Proof. The functoriality of the construction of the space of analytic vectors yields a map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}(G, V)_{\mathbb{G}-\mathrm{an}}$, which when composed with the inverse of the isomorphism of proposition 3.3.4 yields a map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}(G, V)_{\mathbb{G}-\mathrm{an}}$. We will show this map to be an isomorphism.

Consider the commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \arrow[r, "\sim", "\alpha^*"'] \arrow[dd] & \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}, V)^{\Delta_{1,2}(G)} \arrow[r, "\sim"] \arrow[dd] & (\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V))^{\Delta_{1,2}(G)} \arrow[d] \\ & & (\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes} \mathcal{C}(G, V))^{\Delta_{1,2}(G)} \arrow[d] \\ \mathcal{C}(G, V) \arrow[r, "\sim", "\alpha^*"'] & \mathcal{C}(G \times G, V)^{\Delta_{1,2}(G)} \arrow[r, "\sim"] & (\mathcal{C}(G, K) \hat{\otimes} \mathcal{C}(G, V))^{\Delta_{1,2}(G)}, \end{tikzcd}$$

in which α^* denotes the map induced on function spaces by the map $\alpha : (g_1, g_2) \mapsto g_2 g_1$, and all of whose horizontal arrows are isomorphisms. By definition, the space $\mathcal{C}(G, V)_{\mathbb{G}-\mathrm{an}}$ is equal to the space $(\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes} \mathcal{C}(G, V))^{\Delta_{1,2}(G)}$, and this diagram shows that this space may be identified with those V-valued functions on $G \times G$ that are of the form $(g_1, g_2) \mapsto f(g_2 g_1)$ for some $f \in \mathcal{C}(G, V)$, and that are analytic with respect to the variable g_1. By restricting such a function to $G \times e \subset G \times G$, we see that f must be an analytic function, and thus that the map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}(G, V)_{\mathbb{G}-\mathrm{an}}$ is a continuous bijection. Since its source and target are Banach spaces, the open mapping theorem shows that it is an isomorphism, as claimed. $\square$

Corollary 3.3.8. *If V is a K-Banach space equipped with a topological G-action then the natural map $(V_{\mathrm{con}})_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is an isomorphism.*

Proof. By construction there is a G-equivariant closed embedding $V_{\mathrm{con}} \to \mathcal{C}(G, V)$, and so by propositions 3.3.5 and 3.3.7 a Cartesian diagram

$$\begin{tikzcd} (V_{\mathrm{con}})_{\mathbb{G}-\mathrm{an}} \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \arrow[d] \\ V_{\mathrm{con}} \arrow[r] & \mathcal{C}(G, V). \end{tikzcd}$$

The definition of V_{con} then implies that $(V_{\mathrm{con}})_{\mathbb{G}-\mathrm{an}}$ maps isomorphically onto the subspace of $\Delta_{1,2}(G)$-invariants in $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$; that is, $V_{\mathbb{G}-\mathrm{an}}$. ☐

The next two lemmas are included for later reference. For both lemmas, we assume given an open affinoid subgroup $\mathbb{H}$ of $\mathbb{G}$, equal to the Zariski closure of its group $H := \mathbb{H}(L)$ of L-valued points. If V is a K-Banach space equipped with a topological H-action, then we may restrict the left and right regular action of G on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ to actions of H. Combining these actions with the pointwise action of H arising from the action of H on V, we obtain an action of $H \times H \times H$ on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ (the first factor acting via the pointwise action, the second factor by the left regular representation, and the third factor by the right regular representation). The formation of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ with its $H \times H \times H$-action is obviously functorial in V.

Lemma 3.3.9. *If V is a K-Banach space equipped with a topological H-action, then the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V_{\mathbb{H}-\mathrm{an}})^{\Delta_{1,2}(H)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)^{\Delta_{1,2}(H)}$ is an isomorphism.*

Proof. Directly from the definitions, this is the map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta_{1,2}(H)})^{\Delta_{1,2}(H)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)^{\Delta_{1,2}(H)} \tag{3.3.10}$$

induced by restricting the evaluation map $\mathrm{ev}_e : \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V) \to V$ to $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta_{1,2}(H)}$. There is a canonical isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{H}, V)$, which induces an isomorphism

$$\begin{aligned} &\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta_{1,2}(H)})^{\Delta_{1,2}(H)} \\ &\qquad \xrightarrow{\sim} \{f \in \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{H}, V) \mid f(h_1 g, h_2) = f(g, h_2 h_1) \text{ and} \\ &\qquad\qquad f(g, h_1 h_2) = h_1 f(g, h_2) \text{ for all } g \in G, h_1, h_2 \in H\}. \end{aligned}$$

We also have the isomorphism

$$\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)^{\Delta_{1,2}(H)} \xrightarrow{\sim} \{f \in \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \mid f(hg) = hf(g) \text{ for all } g \in G, h \in H\}.$$

With respect to these descriptions of its source and target, we may reinterpret the map (3.3.10) as the map

$$\begin{aligned} &\{f \in \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{H}, V) \mid f(h_1 g, h_2) = f(g, h_2 h_1) \text{ and } f(g, h_1 h_2) = h_1 f(g, h_2) \text{ for all} \\ &\quad g \in G, h_1, h_2 \in H\} \to \{f \in \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \mid f(hg) = hf(g) \text{ for all } g \in G, h \in H\}, \end{aligned}$$

given by restricting to $\mathbb{G} \times e \subset \mathbb{G} \times \mathbb{H}$. Described this way, it is immediate that (3.3.10) induces an isomorphism between its source and target. ☐

Lemma 3.3.11. *The evaluation map* $\mathrm{ev}_e : \mathcal{C}^{\mathrm{an}}(\mathbb{H}, K) \to K$ *induces an isomorphism* $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \mathcal{C}^{\mathrm{an}}(\mathbb{H}, K))^{\Delta_{1,2}(H)} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)$.

Proof. This is proved in an analogous manner to the previous lemma. □

Proposition 3.3.12. *If U and V are two K-Banach spaces, each equipped with a topological G-action, then the map $U_{\mathbb{G}-\mathrm{an}} \hat{\otimes}_K V_{\mathbb{G}-\mathrm{an}} \to U \hat{\otimes}_K V$ (obtained by taking the completed tensor product of the natural continuous injections $U_{\mathbb{G}-\mathrm{an}} \to U$ and $V_{\mathbb{G}-\mathrm{an}} \to V$) factors through the natural continuous injection $(U \hat{\otimes}_K V)_{\mathbb{G}-\mathrm{an}} \to U \hat{\otimes}_K V$ (where $U \hat{\otimes}_K V$ is equipped with the diagonal G-action).*

Proof. If ϕ_1 is an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, U)$ and ϕ_2 is an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, then the map $g \mapsto \phi_1(g) \otimes \phi_2(g)$ yields an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, U \hat{\otimes}_K V)$. Thus we obtain a continuous bilinear map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, U) \times \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, U \hat{\otimes}_K V)$, which induces a continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, U) \hat{\otimes} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, U \hat{\otimes}_K V)$. This restricts to a map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{G}, U)^{\Delta_{1,2}(G)} \hat{\otimes}_K \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)^{\Delta_{1,2}(G)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, U \hat{\otimes}_K V)^{\Delta_{1,2}(G)}$$

(where the G-action on $U \hat{\otimes}_K V$ is taken to be the diagonal action). Taking into account definition 3.3.1, this proves the proposition. □

We now allow V to be an arbitrary convex K-vector space equipped with a topological action of G, and extend definition 3.3.1 to this more general context.

Definition 3.3.13. If V is a locally convex Hausdorff topological K-vector space equipped with a topological action of G, define the convex K-vector space $V_{\mathbb{G}-\mathrm{an}}$ of $\mathbb{G}$-analytic vectors in V to be the locally convex inductive limit $V_{\mathbb{G}-\mathrm{an}} = \varinjlim_{W} \overline{W}_{\mathbb{G}-\mathrm{an}}$, where W runs over all the G-invariant BH-subspaces of V, and each $\overline{W}$ is equipped with the G-action provided by lemma 3.2.12.

The formation of $V_{\mathbb{G}-\mathrm{an}}$ (as a convex K-vector space with topological G-action) is covariantly functorial in V (by proposition 1.1.7) and contravariantly functorial in $\mathbb{G}$. Since $V_{\mathbb{G}-\mathrm{an}}$ is defined as the locally convex inductive limit of a family of Banach spaces, it is both barrelled and bornological.

Proposition 3.3.3 provides for each G-invariant BH-subspace W of V a continuous injection $\overline{W}_{\mathbb{G}-\mathrm{an}} \to \overline{W}$, which we may compose with the continuous injection $\overline{W} \to V$ to obtain a continuous injection

$$\overline{W}_{\mathbb{G}-\mathrm{an}} \to V. \tag{3.3.14}$$

Taking the limit over all such W then yields a continuous injection

$$V_{\mathbb{G}-\mathrm{an}} \to V. \tag{3.3.15}$$

As a particular consequence, we conclude that $V_{\mathbb{G}-\mathrm{an}}$ is Hausdorff.

The following result justifies the designation of $V_{\mathbb{G}-\mathrm{an}}$ as the space of $\mathbb{G}$-analytic vectors in V.

Theorem 3.3.16. *If V is a Hausdorff convex K-vector space equipped with a topological G-action then there is a (uniquely determined) continuous injection $V_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ such that the diagram*

$$\begin{tikzcd} V_{\mathbb{G}-\mathrm{an}} \arrow[r, "(3.3.15)"] \arrow[d] & V \arrow[d, "o"] \\ \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \arrow[r] & \mathcal{F}(G, V) \end{tikzcd} \tag{3.3.17}$$

is Cartesian on the level of abstract vector spaces. Thus the image of the continuous injection (3.3.15) *contains precisely those vectors $v \in V$ for which the orbit map o_v is* (*the restriction to G of*) *an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$.*

Proof. Since the map (3.3.15) and the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{F}(G, V)$ are injective (the latter since we have assumed that G is Zariski dense in $\mathbb{G}$), there is clearly at most one such map which makes (3.3.17) commute.

If W is a G-invariant BH-subspace of V, then by construction there is a closed embedding $\overline{W}_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$. Composing with the injection $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W}) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, and then passing to the inductive limit over all $\overline{W}$, yields a continuous injection $V_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ that makes (3.3.17) commute. It remains to be shown that (3.3.17) is Cartesian.

Let V_1 denote the subspace of V consisting of those vectors $v \in V$ for which o_v is given by an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. We must show that the image of (3.3.15) is equal to V_1.

The commutativity of (3.3.17) shows that image of (3.3.15) lies in V_1. Conversely, let v be a vector in V_1. The orbit map $o_v : G \to V$ is thus given by an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$, and so by an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$ for some BH-subspace W of V. The continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W}) \to \mathcal{C}(G, \overline{W}) \to \mathcal{C}(G, V)$ is G-equivariant, if we endow all the spaces appearing with the right regular G-action. Let $\overline{W}_1$ denote the preimage in $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$ of the G-invariant closed subspace V_{con} of $\mathcal{C}(G, V)$; then $\overline{W}_1$ is a G-invariant closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$. Let W_1 denote the image of $\overline{W}_1$ under the continuous map $\overline{W}_1 \to V_{\mathrm{con}} \xrightarrow{\mathrm{ev}_e} V$; by construction W_1 is a G-invariant BH-subspace of V. Again by construction, W_1 is contained in V_1 and contains the element v. We claim that the natural map $(\overline{W}_1)_{\mathbb{G}-\mathrm{an}} \to \overline{W}_1$ is an isomorphism. Granting this, it follows that v is contained in the image of $(\overline{W}_1)_{\mathbb{G}-\mathrm{an}}$ in V_1 and the commutativity of (3.3.17) is proved.

We now prove the claim. Observe that by proposition 3.3.4 the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$ is an isomorphism. Our claim follows upon applying proposition 3.3.5 to the closed embedding $\overline{W}_1 \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$. □

Proposition 3.3.18. *If V is a Fréchet space equipped with a topological G-action, then the map $V_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ provided by proposition* 3.3.16 *is a closed embedding. In particular, $V_{\mathbb{G}-\mathrm{an}}$ is again a Fréchet space.*

Proof. Proposition 2.1.13 (ii) shows that $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is a Fréchet space, and the proof of that proposition, together with propositions 2.1.6 and 3.2.15, shows that, as W ranges over all G-invariant BH-subspaces of V, the images of the maps $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W}) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ are cofinal in the directed set of all BH-subspaces of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. Since the closed subspace $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)^{\Delta_{1,2}(G)}$ of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is again a Fréchet space, and hence ultrabornological, we deduce from proposition 1.1.12 that the natural map $\varinjlim_{W} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})^{\Delta_{1,2}(G)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)^{\Delta_{1,2}(G)}$ is an isomorphism. This proves the proposition. □

Proposition 3.3.19. *If V is a Hausdorff convex K-vector space equipped with a topological G-action then the action of G on $V_{\mathbb{G}-\mathrm{an}}$ is continuous.*

Proof. If W is a G-invariant BH-subspace of V then the action of G on $\overline{W}_{\mathbb{G}-\mathrm{an}}$ is continuous. This implies that the action of G on $V_{\mathbb{G}-\mathrm{an}}$ is separately continuous. Since $V_{\mathbb{G}-\mathrm{an}}$ is barrelled, the G-action on $V_{\mathbb{G}-\mathrm{an}}$ is continuous, as claimed. □

The following result shows that in definition 3.3.13 we may restrict our attention to continuously G-invariant BH-subspaces of V.

Proposition 3.3.20. *If V is a Hausdorff convex K-vector space equipped with a topological G-action then the natural map $\varinjlim_{U} \overline{U}_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$, in which the inductive limit is taken over all continuously G-invariant BH-subspaces U of V, is an isomorphism.*

Proof. Let W be a G-invariant BH-subspace of V. Then the image U of $\overline{W}_{\mathrm{con}}$ in W is a continuously G-invariant BH-subspace of V that is contained in W. Since $\overline{U}$ is isomorphic to $\overline{W}_{\mathrm{con}}$ by construction, corollary 3.3.8 shows that the map $\overline{U}_{\mathbb{G}-\mathrm{an}} \to \overline{W}_{\mathbb{G}-\mathrm{an}}$ induced by the inclusion $U \subset W$ is an isomorphism. This proves the proposition. $\square$

Corollary 3.3.21. *If V is a Hausdorff convex K-vector space of LF-type* (*respectively of LB-type*)*, and if V is equipped with a topological G-action, then $V_{\mathbb{G}-\mathrm{an}}$ is an LF-space* (*respectively an LB-space.*)*.*

Proof. Suppose $V = \bigcup_{n=1}^{\infty} V_n$, where $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ is an increasing sequence of FH-subspaces of V. Proposition 1.1.10 shows that the sequence $\{V_n\}_{n\geq 1}$ is cofinal in the directed set of FH-subspaces of V. Applying proposition 3.2.14, define $W_n := \mathrm{co}_G(V_n)$; the sequence $\{W_n\}_{n\geq 1}$ is then cofinal in the directed set of continuously G-invariant FH-subspaces of V. Consider now the continuous injections $\varinjlim_{U} \overline{U}_{\mathbb{G}-\mathrm{an}} \to \varinjlim_{W} \overline{W}_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$, the first inductive limit being taken over all continuously G-invariant BH-subspaces U of V, and the second over all continuously G-invariant FH-subspaces W of V. Since, by Proposition 3.3.20, the composite of these injections is an isomorphism, so is each of them separately; in particular, the second map is. Since, as was already noted, the sequence $\{W_n\}_{n\geq 1}$ is cofinal among the W, we obtain an isomorphism $\varinjlim_{n} (\overline{W}_n)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{G}-\mathrm{an}}$. Proposition 3.3.18 shows that $(\overline{W}_n)_{\mathbb{G}-\mathrm{an}}$ is a Fréchet space for each value of n, and thus that $V_{\mathbb{G}-\mathrm{an}}$ is an LF-space.

If in fact each of the V_n is a BH-subspace of V_n, then so is each W_n. Thus $(\overline{W}_n)_{\mathbb{G}-\mathrm{an}}$ is a Banach space for each value of n, and in this case we see that $V_{\mathbb{G}-\mathrm{an}}$ is even an LB-space. $\square$

Proposition 3.3.22. *If V is a Hausdorff convex K-vector space equipped with a topological G-action then the continuous injection $(V_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is in fact a topological isomorphism.*

Proof. Let W range over all G-invariant BH-subspace of V. The functoriality of the formation of $\mathbb{G}$-analytic vectors induces a commutative diagram

$$\begin{tikzcd}
\varinjlim_{W} (\overline{W}_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \arrow[r] \arrow[d] & (V_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \arrow[d] \\
\varinjlim_{W} \overline{W}_{\mathbb{G}-\mathrm{an}} \arrow[r] & V_{\mathbb{G}-\mathrm{an}},
\end{tikzcd}$$

in which the inductive limits are taken over all such W. Corollary 3.3.6 implies that the left-hand vertical map is an isomorphism, while by definition the lower horizontal arrow is an isomorphism. Since the other arrows are continuous injective maps, we see that the right-hand vertical arrow is also a topological isomorphism, as claimed. □

Proposition 3.3.23. *Let V be a Hausdorff convex K-vector space equipped with a topological G-action, and let W be a G-invariant closed subspace of V. If W_1 denotes the preimage of W in $V_{\mathbb{G}-\mathrm{an}}$ under the natural map $V_{\mathbb{G}-\mathrm{an}} \to V$ of* (3.3.15) (*so that W_1 is a G-invariant closed subspace of $V_{\mathbb{G}-\mathrm{an}}$*)*, then the map $W_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$* (*induced by the functoriality of the formation of analytic vectors*) *induces a continuous bijection of $W_{\mathbb{G}-\mathrm{an}}$ onto W_1. If furthermore $V_{\mathbb{G}-\mathrm{an}}$ is either a Fréchet space or of compact type, then this continuous bijection is a topological isomorphism.*

Proof. The first statement of the corollary follows from propositions 2.1.23 and theorem 3.3.16.

To prove the second, we first observe that proposition 3.3.22 yields an isomorphism $(V_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{G}-\mathrm{an}}$, and so the first statement of the proposition, applied to the closed embedding $W_1 \to V_{\mathbb{G}-\mathrm{an}}$, induces a continuous bijection $(W_1)_{\mathbb{G}-\mathrm{an}} \to W_1$. As $V_{\mathbb{G}-\mathrm{an}}$ is either a Fréchet space or a space of compact type, the same is true of its closed subspace W_1, and hence by corollary 3.3.21, $(W_1)_{\mathbb{G}-\mathrm{an}}$ is an LF-space. This bijection is thus a topological isomorphism. This isomorphism fits into the sequence of continuous injections $(W_1)_{\mathbb{G}-\mathrm{an}} \to W_{\mathbb{G}-\mathrm{an}} \to W_1 \to W$ (the first of these being induced by the third and functoriality of the formation of $\mathbb{G}$-analytic vectors). We deduce that the map $W_{\mathbb{G}-\mathrm{an}} \to W_1$ is also a topological isomorphism, as required. □

Proposition 3.3.24. *If V is either a Fréchet space or a convex space of compact type, then there is a natural isomorphism $\mathcal{C}(G,V)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$.* (*Here $\mathcal{C}(G,V)$ and $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$ are regarded as G-representations via the right regular G-action.*)

Proof. Propositions 2.1.6 implies that as W ranges over all BH-subspaces of V, the images of the maps $\mathcal{C}(G,\overline{W}) \to \mathcal{C}(G,V)$ are cofinal in the set of all BH-subspaces of $\mathcal{C}(G,V)$. This gives the first and last members of the sequence of isomorphisms

$$\mathcal{C}(G,V)_{\mathrm{an}} \xrightarrow{\sim} \varinjlim_{W} \mathcal{C}(G,\overline{W})_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} \varinjlim_{W} \mathcal{C}^{\mathrm{an}}(\mathbb{G},\overline{W}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G},V),$$

the middle isomorphism being provided by proposition 3.3.7. □

Corollary 3.3.25. *If V is either a Fréchet space or a convex space of compact type, then the natural injection $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$ is a topological isomorphism.* (*Here $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$ is regarded as a G-representation via the right regular G-action.*)

Proof. Proposition 3.3.24 shows that the injection $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$ is obtained by passing to $\mathbb{G}$-analytic vectors in the injection $\mathcal{C}(G,V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}(G,V)$. The corollary follows from proposition 3.3.22. □

Corollary 3.3.26. *If V is either a Fréchet space or a convex space of compact type, then there is a natural isomorphism $\mathcal{C}^{\mathrm{la}}(G,V)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$.* (*Here $\mathcal{C}^{\mathrm{la}}(G,V)$ and $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$ are regarded as G-representations via the right regular G-action.*)

Proof. If we pass to $\mathbb{G}$-analytic vectors in the sequence of continuous injections $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V) \to \mathcal{C}^{\mathrm{la}}(G,V) \to \mathcal{C}(G,V)$ we obtain a sequence of continuous injections $\mathcal{C}^{\mathrm{an}}(\mathbb{G},V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{la}}(G,V)_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}(G,V)_{\mathbb{G}-\mathrm{an}}$. As we observed in the proof of corollary 3.3.25, the composite of these bijections is an isomorphism, and thus each of these bijections is itself an isomorphism. Applying proposition 3.3.24 we see that $\mathcal{C}^{\mathrm{la}}(G,V)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$, as required. $\square$

Proposition 3.3.27. *If V is a Hausdorff convex K-vector space equipped with a topological G-action, and if there is a G-equivariant isomorphism $\varinjlim_n V_n \xrightarrow{\sim} V$, where $\{V_n\}_{n\geq 1}$ is an inductive sequence of Hausdorff LB-spaces, each equipped with a topological G-action, admitting injective G-equivariant transition maps, then there is a natural isomorphism $\varinjlim_n (V_n)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{G}-\mathrm{an}}$.*

Proof. It follows from proposition 1.1.10 that any BH-subspace of V lies in the image of the map $V_n \to V$ for some sufficiently large value of n. This yields the proposition. $\square$

3.4. Analytic vectors continued

In this section we suppose that $\mathbb{G}$ is a rigid analytic group defined over L, which is σ-affinoid as a group, in the sense that $\mathbb{G}$ admits an admissible cover $\mathbb{G} = \bigcup_{n=1}^{\infty} \mathbb{G}_n$, where $\{\mathbb{G}_n\}_{n\geq 1}$ is an increasing sequence of admissible affinoid open subgroups of $\mathbb{G}$. We write $G := \mathbb{G}(L)$, and for each n, write $G_n := \mathbb{G}_n(L)$, so that $G = \bigcup_{n=1}^{\infty} G_n$. We also assume that G_n is Zariski dense in $\mathbb{G}_n$ for each $n \geq 1$, and thus that G is Zariski dense in $\mathbb{G}$.

Definition 3.4.1. If V is a Hausdorff convex K-vector space equipped with a topological G-action, then we define the convex K-vector space $V_{\mathbb{G}-\mathrm{an}}$ of $\mathbb{G}$-analytic vectors in V to be the projective limit $\varprojlim_n V_{\mathbb{H}-\mathrm{an}}$, where the projective limit is taken over all admissible affinoid open subgroups $\mathbb{H}$ of $\mathbb{G}$.

If $\mathbb{H}_1 \subset \mathbb{H}_2$ is an inclusion of admissible affinoid open subgroups of $\mathbb{G}$, then the natural map $V_{\mathbb{H}_2-\mathrm{an}} \to V_{\mathbb{H}_1-\mathrm{an}}$ is an injection (since when composed with the natural injection $V_{\mathbb{H}_1-\mathrm{an}} \to V$ it yields the natural injection $V_{\mathbb{H}_2-\mathrm{an}} \to V$). Thus the transition maps in the projective system of definition 3.4.1 are all injections. The remarks preceding definition 2.1.18 show that we may in fact restrict this projective limit to be taken over the subgroups $\mathbb{G}_n$ of $\mathbb{G}$.

The formation of $V_{\mathbb{G}-\mathrm{an}}$ is evidently covariantly functorial in V and contravariantly functorial in $\mathbb{G}$.

Proposition 3.4.2. *There is a natural continuous G-action on $V_{\mathbb{G}-\mathrm{an}}$, as well as a natural continuous G-equivariant injection $V_{\mathbb{G}-\mathrm{an}} \to V$.*

Proof. We regard $V_{\mathbb{G}-\mathrm{an}}$ as the projective limit $\varprojlim_n V_{\mathbb{G}_n-\mathrm{an}}$. If $m \leq n$ then the continuous G_n-action on $V_{\mathbb{G}_n-\mathrm{an}}$ provided by proposition 3.3.19 restricts to a continuous G_m-action on $V_{\mathbb{G}_n-\mathrm{an}}$. Passing to the projective limit, we obtain a continuous G_m-action on $V_{\mathbb{G}-\mathrm{an}}$. Since G is the union of its open subgroups G_m, we obtain a continuous G-action on $V_{\mathbb{G}-\mathrm{an}}$.

The G-equivariant injection $V_{\mathbb{G}-\mathrm{an}} \to V$ is obtained by composing the natural projection $V_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}_n-\mathrm{an}}$ (for some choice of n) with the continuous injection $V_{\mathbb{G}_n-\mathrm{an}} \to V$. (The resulting continuous injection is obviously independent of the choice of n.) $\square$

Theorem 3.4.3. *If V is a Hausdorff convex K-vector space equipped with a topological G-action then there is a (uniquely determined) continuous injection $V_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ such that the diagram*

$$\begin{tikzcd} V_{\mathbb{G}-\mathrm{an}} \arrow[r, "(3.4.2)"] \arrow[d] & V \arrow[d, "o"] \\ \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \arrow[r] & \mathcal{F}(G, V) \end{tikzcd}$$

is Cartesian on the level of abstract vector spaces. In particular, the image of the continuous injection (3.4.2) *contains precisely those vectors $v \in V$ for which the orbit map o_v is (the restriction to G of) an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$.*

Proof. For each value of n, theorem 3.3.16 yields a diagram of continuous maps

$$\begin{tikzcd} V_{\mathbb{G}_n-\mathrm{an}} \arrow[r] \arrow[d] & V \arrow[d, "o"] \\ \mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, V) \arrow[r] & \mathcal{F}(G_n, V), \end{tikzcd}$$

Cartesian on the level of abstract K-vector spaces. These diagrams are compatible with respect to the maps $\mathbb{G}_n \to \mathbb{G}_{n+1}$, and passing to the projective limit in n yields the diagram of the theorem. □

Proposition 3.4.4. *If V is a Fréchet space, then the map $V_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ provided by theorem* 3.4.3 *is a closed embedding. In particular, $V_{\mathbb{G}-\mathrm{an}}$ is again a Fréchet space (since the remark following definition* 2.1.18 *shows that $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ is a Fréchet space).*

Proof. Proposition 3.3.18 shows that the natural map $V_{\mathbb{G}_n-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, V)$ is a closed embedding, for each $n \geq 1$. Since the projective limit of closed embeddings is a closed embedding, the result follows. □

Corollary 3.4.5. *If V is a K-Fréchet space equipped with a topological action of G, then there exists a natural G-equivariant topological isomorphism $V_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} (\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K V)^{\Delta_{1,2}(G)}$.*

Proof. Proposition 2.1.19 yields a natural topological isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K V$. The claim now follows from proposition 3.4.4. □

Corollary 3.4.6. *If $\mathbb{G}$ is strictly σ-affinoid, and if V is a nuclear Fréchet space equipped with a topological G-action, then $V_{\mathbb{G}-\mathrm{an}}$ is again a nuclear Fréchet space.*

Proof. Since $\mathbb{G}$ is strictly σ-affinoid, the space $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)$ is a nuclear Fréchet space. Proposition 1.1.28 and corollary 3.4.5 show that $V_{\mathbb{G}-\mathrm{an}}$ is a closed subspace of a nuclear Fréchet space, and hence is again a nuclear Fréchet space. □

Corollary 3.4.7. *If V is a K-Fréchet space equipped with a topological G-action, then the natural map $(V_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is a topological isomorphism.*

Proof. Since V is a Fréchet space, the same is true of $V_{\mathbb{G}-\mathrm{an}}$, by proposition 3.4.4. Corollary 3.4.5 thus yields an isomorphism

$$(3.4.8) \quad (V_{\mathbb{G}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} (\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K V_{\mathbb{G}-\mathrm{an}})^{\Delta_{1,2}(G)} = ((\varprojlim_n \mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, K)) \hat{\otimes}_K (\varprojlim_n V_{\mathbb{G}_n-\mathrm{an}}))^{\Delta_{1,2}(G)}.$$

Proposition 3.3.18 shows that each $V_{\mathbb{G}_n-\mathrm{an}}$ is a Fréchet space, and we obtain isomorphisms

$$(3.4.9)\quad ((\varprojlim_n \mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, K)) \hat{\otimes}_K (\varprojlim_n V_{\mathbb{G}_n-\mathrm{an}}))^{\Delta_{1,2}(G)} \xrightarrow{\sim} (\varprojlim_n (\mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, K) \hat{\otimes}_K V_{\mathbb{G}_n-\mathrm{an}}))^{\Delta_{1,2}(G)} \xrightarrow{\sim} \varprojlim_n (\mathcal{C}^{\mathrm{an}}(\mathbb{G}_n, K) \hat{\otimes}_K V_{\mathbb{G}_n-\mathrm{an}})^{\Delta_{1,2}(G_n)} \xrightarrow{\sim} \varprojlim_n V_{\mathbb{G}_n-\mathrm{an}} = V_{\mathbb{G}-\mathrm{an}}.$$

(The first isomorphism is provided by proposition 1.1.29, the second is evident, and the third is provided by the definition of $(V_{\mathbb{G}_n-\mathrm{an}})_{\mathbb{G}_n-\mathrm{an}}$, together with propositions 2.1.13 (ii), 3.3.18, and 3.3.22.) Composing the isomorphism provided by (3.4.9) with that provided by (3.4.8) establishes the corollary. □

Proposition 3.4.10. *Let V be a Hausdorff convex K-vector space equipped with a topological G-action, and let W be a G-invariant closed subspace of V. If W_1 denotes the preimage of W in $V_{\mathbb{G}-\mathrm{an}}$ under the natural map $V_{\mathbb{G}-\mathrm{an}} \to V$ (a G-invariant closed subspace of $V_{\mathbb{G}-\mathrm{an}}$), then the map $W_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ (induced by functoriality of the formation of analytic vectors) induces a continuous bijection of $W_{\mathbb{G}-\mathrm{an}}$ onto W_1. If furthermore $V_{\mathbb{G}-\mathrm{an}}$ is a Fréchet space, then this map is even a topological isomorphism.*

Proof. The first claim follows from theorem 3.4.3 and proposition 2.1.23. The second claim is proved in an analogous manner to the second claim of proposition 3.3.23, by appealing to proposition 3.4.4 and corollary 3.4.7. □

Proposition 3.4.11. *If G_n has finite index in G for some (equivalently, all) $n \geq 1$ (or equivalently, if G is compact), and if V is either a K-Fréchet space or a convex K-vector space of compact type, then there is a natural isomorphism $\mathcal{C}(G, V)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. (Here the spaces $\mathcal{C}(G, V)$ and $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ are regarded as G-representations via the right regular G-action.)*

Proof. Note that G_n is both open in G and compact (being the L-analytic points of the open affinoid subgroup $\mathbb{G}_n$ of $\mathbb{G}$), and so has finite index in G if and only if G is itself compact. Note also that this latter condition makes no reference to n, and thus that the former condition hold for one value of n if only if holds for all of them. This verifies the equivalences claimed in the statement of the proposition.

Suppose now that G_n is finite index in G. The decomposition $G = \coprod_{g \in G/G_n} g\, G_n$ yields a G_n-equivariant isomorphism $\mathcal{C}(G, V) \xrightarrow{\sim} \prod_{g \in G/G_n} \mathcal{C}(g\, G_n, V)$, and hence (since the formation of analytic vectors obviously commutes with the formation of finite direct products) an isomorphism

$$\mathcal{C}(G, V)_{\mathbb{G}_n-\mathrm{an}} \xrightarrow{\sim} \prod_{g \in G/G_n} \mathcal{C}(g\, G_n, V)_{\mathbb{G}_n-\mathrm{an}} \xrightarrow{\sim} \prod_{g \in G/G_n} \mathcal{C}^{\mathrm{an}}(g\, \mathbb{G}_n, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\coprod_{g \in G/G_n} g\, \mathbb{G}_n, V).$$

(For each $g \in G/G_n$, we have written $g\, \mathbb{G}_n$ to denote the admissible open affinoid subset of $\mathbb{G}$ obtained by translating $\mathbb{G}_n$ on the left by the coset representative g. The second isomorphism is provided by proposition 3.3.24.) The admissible open affinoid subsets $\coprod_{g\in G/G_n} g\,\mathbb{G}_n$, for $n \geq 1$, form a cofinal collection of admissible open affinoid subsets of $\mathbb{G}$. Thus, passing to the projective limit over $n \geq 1$, we obtain the required isomorphism $\mathcal{C}(G,V)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$. $\square$

Proposition 3.4.12. *If V is a Hausdorff convex K-vector space equipped with a topological action of G, and if $\mathbb{H}$ is an affinoid rigid analytic subgroup of $\mathbb{G}$ that is normalized by G (so that G acts naturally on $V_{\mathbb{H}-\mathrm{an}}$), then the natural map $(V_{\mathbb{H}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is a topological isomorphism.*

Proof. Recall that $\mathbb{G}$ may be written as an increasing union of a sequence of affinoid open subgroups $\{\mathbb{G}_n\}_{n\geq 1}$. We may choose this sequence of subgroups so that $\mathbb{H} = \mathbb{G}_1$. If $n \geq 1$, then there are continuous injections $V_{\mathbb{G}_n-\mathrm{an}} \to V_{\mathbb{H}-\mathrm{an}} \to V$, and hence continuous injections $(V_{\mathbb{G}_n-\mathrm{an}})_{\mathbb{G}_n-\mathrm{an}} \to (V_{\mathbb{H}-\mathrm{an}})_{\mathbb{G}_n-\mathrm{an}} \to V_{\mathbb{G}_n-\mathrm{an}}$. Proposition 3.3.22 shows that the composite of these maps is a topological isomorphism, and thus in particular so is the second of these, the map

$$(V_{\mathbb{H}-\mathrm{an}})_{\mathbb{G}_n-\mathrm{an}} \to V_{\mathbb{G}_n-\mathrm{an}}. \tag{3.4.13}$$

Passing to the projective limit of the maps (3.4.13) as n tends to infinity, the proposition follows. $\square$

Proposition 3.4.14. *Suppose that $\mathbb{H}$ is a rigid analytic affinoid group defined over L, containing $\mathbb{G}$ as a rigid analytic subgroup, and assume that $H := \mathbb{H}(L)$ normalizes $\mathbb{G}$. If V is a Hausdorff convex K-vector space equipped with a topological H-action, then the natural map $(V_{\mathbb{G}-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to V_{\mathbb{H}-\mathrm{an}}$ is a topological isomorphism.*

Proof. Since $\mathbb{G}$ is contained in $\mathbb{H}$, we have natural continuous injections $V_{\mathbb{H}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}} \to V$. As H normalizes $\mathbb{G}$, the H-action on V lifts to a H-action on $V_{\mathbb{G}-\mathrm{an}}$, and both injections are H-equivariant. Passing to $\mathbb{H}$-analytic vectors yields continuous injections $(V_{\mathbb{H}-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to (V_{\mathbb{G}-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to V_{\mathbb{H}-\mathrm{an}}$. Proposition 3.3.22 shows that the composite of these two maps is a topological isomorphism. The same is thus true of each of these maps separately. $\square$

Corollary 3.4.15. *Suppose that $\mathbb{J} \subset \mathbb{G}$ is an inclusion of σ-affinoid groups, which factors as $\mathbb{J} \subset \mathbb{G}_1 \subset \mathbb{G}$, and such that G normalizes $\mathbb{J}$. If V is a Hausdorff convex K-vector space equipped with a topological G-action then the natural map $(V_{\mathbb{J}-\mathrm{an}})_{\mathbb{G}-\mathrm{an}} \to V_{\mathbb{G}-\mathrm{an}}$ is a topological isomorphism.*

Proof. By proposition 3.4.14, the natural map $(V_{\mathbb{J}-\mathrm{an}})_{\mathbb{G}_n-\mathrm{an}} \to V_{\mathbb{G}_n-\mathrm{an}}$ is an isomorphism for each $n \geq 1$. Passing to the projective limit in n yields the corollary. $\square$

3.5. Locally analytic vectors

In this section we let G denote a locally L-analytic group. We begin by introducing the notion of an analytic open subgroup of G.

Suppose that $(\phi, H, \mathbb{H})$ is a chart of G — thus H is a compact open subset of G, $\mathbb{H}$ is an affinoid rigid analytic space over L isomorphic to a closed ball, and ϕ is an isomorphism $\phi : H \xrightarrow{\sim} \mathbb{H}(L)$ — with the additional property that H is a subgroup of G. Since H is Zariski dense in $\mathbb{H}$, there is at most one rigid analytic group structure on $\mathbb{H}$ giving rise to the group structure on H. If such a rigid analytic group structure exists on $\mathbb{H}$, we will refer to the chart $(\phi, H, \mathbb{H})$ as an analytic open subgroup of G. Usually, we will suppress the isomorphism ϕ, and simply refer to an analytic open subgroup H of G, and write $\mathbb{H}$ to denote the corresponding rigid analytic group determined by H.

The analytic open subgroups of G form a directed set in an obvious fashion: if $H' \subset H$ is an inclusion of open subgroups of G each of which is equipped with the structure of an analytic open subgroup of G, then we say that it is an inclusion of analytic open subgroups if it lifts to a rigid analytic map $\mathbb{H}' \to \mathbb{H}$. (Since H' and H are Zariski dense in $\mathbb{H}'$ and $\mathbb{H}$ respectively, such a lift is uniquely determined if it exists, and is automatically a homomorphism of rigid analytic groups.) Forgetting the chart structure yields an order-preserving map from the directed set of analytic open subgroups of G to the set of all open subgroups of G. Since the group structure on G is locally analytic by assumption, the image of this map is cofinal in the directed set of all open subgroups of G. (For example, see [28, LG 4 §8]; standard open subgroups — in the terminology of that reference — provide examples of analytic open subgroups.)

If H is an analytic open subgroup of G and g is an element of G, then multiplication by g induces an isomorphism $H \xrightarrow{\sim} gH$. Thus gH is naturally a chart of G. Just as we let $\mathbb{H}$ denote the rigid analytic structure induced on H by its structure as a chart of G, we will let $g\mathbb{H}$ denote the rigid analytic space induced on gH by its structure as a chart of G. (In fact, if $\phi : H \xrightarrow{\sim} \mathbb{H}(L)$ describes the chart structure on H, then the chart structure on gH is describe by the map $\phi' : gH \xrightarrow{\sim} \mathbb{H}(L)$ defined as $\phi' : gh \mapsto \phi(h)$. Thus the rigid analytic structure on gH is obtained not by changing the rigid analytic space $\mathbb{H}$, but rather by changing the map ϕ. However, it is easier and more suggestive in the exposition to suppress the maps ϕ and ϕ', and to instead use the notation $g\mathbb{H}$ to denote the rigid analytic structure on gH.) If g ranges over a set of right coset representatives of H in G, then the collection of charts $\{g\mathbb{H}\}_{g\in G/H}$ forms an analytic partition of G. If G is compact, then the set of such analytic partitions obtained by allowing H to run over all analytic open subgroups of G is cofinal in the set of all analytic partitions of G.

We will use a similar notation for the conjugate of an analytic open subgroup. If H is an analytic open subgroup of G, and g is an element of G, then conjugation by g induces an isomorphism $H \xrightarrow{\sim} gHg^{-1}$, and so puts a natural chart structure on gHg^{-1}, making gHg^{-1} an analytic open subgroup of G. We let $g\mathbb{H}g^{-1}$ denote the underlying rigid analytic group. (Again, if $\phi : H \xrightarrow{\sim} \mathbb{H}(L)$ describes the chart structure on H, then the chart structure on gHg^{-1} is describe by the map $\phi' : gHg^{-1} \xrightarrow{\sim} \mathbb{H}(L)$ defined as $\phi' : ghg^{-1} \mapsto \phi(h)$. Thus, as in the case of cosets, the rigid analytic group structure on gHg^{-1} is obtained not by changing the rigid analytic group $\mathbb{H}$, but rather by changing the map ϕ. However, just as in the case of cosets, it is easier and more suggestive in the exposition to suppress the maps ϕ and ϕ', and to instead use the notation $g\mathbb{H}g^{-1}$ to denote the rigid analytic structure on gHg^{-1}.)

If V is a Hausdorff locally convex topological K-vector space equipped with a topological action of G then for each analytic open subgroup H of G we may form the convex K-vector space $V_{\mathbb{H}-\mathrm{an}}$ of $\mathbb{H}$-analytic vectors in V, equipped with its H-invariant continuous injection $V_{\mathbb{H}-\mathrm{an}} \to V$. Since the formation of this space and injection is functorial in H these spaces form an inductive system: if $H' \subset H$ is an inclusion of analytic open subgroups of G then there is a continuous H'-equivariant injection $V_{\mathbb{H}-\mathrm{an}} \to V_{\mathbb{H}'-\mathrm{an}}$, compatible with the injections of each of the source and target into V. Passing to the locally convex inductive limit we obtain a continuous injection $\varinjlim_{H} V_{\mathbb{H}-\mathrm{an}} \to V$.

Lemma 3.5.1. *Let V be a Hausdorff convex K-vector space equipped with a topological action of G. If g is an element of G and H is an analytic open subgroup of G (so that gHg^{-1} is a second such subgroup) then the automorphism of V induced by the action of g lifts in a unique fashion to an isomorphism $V_{\mathbb{H}-\mathrm{an}} \xrightarrow{\sim} V_{g\mathbb{H}g^{-1}-\mathrm{an}}$.*

Proof. The meaning of the lemma is that there is a unique way to fill in the top horizontal arrow of the following diagram so as to make it commute:

$$\begin{tikzcd} V_{\mathbb{H}-\mathrm{an}} \arrow[r] \arrow[d] & V_{g\mathbb{H}g^{-1}-\mathrm{an}} \arrow[d] \\ V \arrow[r, "g\cdot"] & V. \end{tikzcd}$$

The uniqueness is clear, since the vertical arrows are injections. To see the existence, let $\phi : H \to gHg^{-1}$ denote the conjugation map $h \mapsto ghg^{-1}$. Then the continuous linear map from V to itself given by the action of g is equivariant with respect to ϕ. The functoriality of the construction of analytic vectors shows that the action of g on V lifts to a map $V_{\mathbb{H}-\mathrm{an}} \to V_{g\mathbb{H}g^{-1}-\mathrm{an}}$. $\square$

Corollary 3.5.2. *In the preceding situation there is an action of G on the inductive system $\{V_{\mathbb{H}-\mathrm{an}}\}$, and hence on the inductive limit $\varinjlim_{H} V_{\mathbb{H}-\mathrm{an}}$, uniquely determined by the condition that the continuous injection $\varinjlim_{H} V_{\mathbb{H}-\mathrm{an}} \to V$ be G-equivariant.*

Proof. The uniqueness is clear, since all the transition maps in the inductive system $\{V_{\mathbb{H}-\mathrm{an}}\}$ are injective. Lemma 3.5.1 shows that if g is an element of G and H is any analytic open subgroup of G then the action of g on V lifts to a map $V_{\mathbb{H}-\mathrm{an}} \to V_{g\mathbb{H}g^{-1}-\mathrm{an}}$. By functoriality of the construction of analytic vectors, these combine to yield the required action of G on the inductive system $\{V_{\mathbb{H}-\mathrm{an}}\}$. $\square$

Definition 3.5.3. Suppose that V is a locally convex Hausdorff K-topological vector space equipped with a topological action of G. We define the locally convex space V_{la} of locally analytic vectors in V to be the locally convex inductive limit

$$V_{\mathrm{la}} := \varinjlim_{H} V_{\mathbb{H}-\mathrm{an}},$$

where H runs over all the analytic open subgroups of $\mathbb{G}$ (and $\mathbb{H}$ denotes the rigid analytic group corresponding to H). If we wish to emphasize the role of the locally analytic group G then we will write $V_{G-\mathrm{la}}$ in place of V_{la}.

The space V_{la} is equipped with a continuous injection into V, and thus is Hausdorff. Lemma 3.5.1 shows that V_{la} is equipped with a continuous action of G with respect to which its injection into V is G-equivariant. The construction of V_{la} (as a locally convex topological K-vector space with G-action) is obviously contravariantly functorial in V and covariantly functorial in G.

Theorem 3.5.7 below justifies our designation of V_{la} as the space of locally analytic vectors of V. (See the remarks following that theorem for a comparison of our definition of the space of locally analytic vectors with the definition given in [22, §3] and [27, p. 38].)

Lemma 3.5.4. *If H is an open subgroup of G and V is a Hausdorff convex K-vector space equipped with a topological G-action then the natural H-equivariant morphism $V_{H-\mathrm{la}} \xrightarrow{\sim} V_{G-\mathrm{la}}$ is an isomorphism.*

Proof. This follows immediately from the fact that the directed set of analytic open subgroups of G that are contained in H is cofinal in the directed set of analytic open subgroups of G. $\square$

Proposition 3.5.5. *If V is a Hausdorff convex K-vector space equipped with a topological action of G then the continuous injection $(V_{\mathrm{la}})_{\mathrm{la}} \to V_{\mathrm{la}}$ is a topological isomorphism.*

Proof. If we fix an analytic open subgroup H of G and an H-invariant BH-subspace W of V, then the image of $\overline{W}_{\mathbb{H}-\mathrm{an}}$ in V_{la} is an H-invariant BH-subspace of V_{la}, and corollary 3.3.6 shows that the natural map $(\overline{W}_{\mathbb{H}-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to \overline{W}_{\mathbb{H}-\mathrm{an}}$ is an isomorphism. Taking the limit over all such W and H we obtain continuous injections

$$\varinjlim_{W,H} (\overline{W}_{\mathbb{H}-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to (V_{\mathrm{la}})_{\mathrm{la}} \to V_{\mathrm{la}} = \varinjlim_{W,H} \overline{W}_{\mathbb{H}-\mathrm{an}}.$$

Our preceding remarks imply that the composite of these injections is a topological isomorphism, and consequently the second injection is also a topological isomorphism. $\square$

Proposition 3.5.6. *If V is a Hausdorff convex K-vector space of LF-type (respectively LB-type), then V_{la} is an LF-space (respectively an LB-space).*

Proof. Let $\{H_n\}_{n\geq 1}$ denote a cofinal sequence of of analytic open subgroups of $\mathbb{G}$. Then $V_{\mathrm{la}} = \varinjlim_n V_{\mathbb{H}_n-\mathrm{an}}$. Corollary 3.3.21 shows that each $V_{\mathbb{H}_n-\mathrm{an}}$ is either an LF-space or an LB-space (depending on the hypothesis on V), and thus the same is true of V_{la}. $\square$

Theorem 3.5.7. *If V is a Hausdorff convex K-vector space equipped with a topological action of G then there is a (uniquely determined) continuous injection $V_{\mathrm{la}} \to \mathcal{C}^{\mathrm{la}}(G,V)$ such that the diagram*

(3.5.8)
$$\begin{tikzcd} V_{\mathrm{la}} \arrow[r] \arrow[d] & V \arrow[d, "o"] \\ \mathcal{C}^{\mathrm{la}}(G,V) \arrow[r] & \mathcal{F}(G,V) \end{tikzcd}$$

is Cartesian on the level of abstract vector spaces. Thus V_{la} maps via a continuous bijection onto $\mathcal{C}^{\mathrm{la}}(G,V)^{\Delta_{1,2}(G)}$, and the image of the continuous injection $V_{\mathrm{la}} \to V$ is equal to the subspace of V consisting of those vectors $v \in V$ for which o_v is an element of $\mathcal{C}^{\mathrm{la}}(G,V)$.

Proof. Since the natural maps $V_{\mathrm{la}} \to V$ and $\mathcal{C}^{\mathrm{la}}(G,K) \to \mathcal{F}(G,V)$ are injective, there is at most one map $V_{\mathrm{la}} \to V$ making (3.5.8) commute. If H is an analytic open subgroup of G then theorem 3.3.16 shows that the association of $o_{v|H}$ to any $v \in V$ yields a continuous map $V_{\mathbb{H}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H},V)$. From this we deduce that the association of $o_{v|gH}$ to $v \in V$ induces a continuous map $V_{\mathbb{H}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(g\mathbb{H},V)$ for any $g \in G$. Indeed, multiplication on the left by g^{-1} induces an analytic isomorphism $g\mathbb{H} \to \mathbb{H}$, and thus an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V)$. The continuous action of g on V induces an isomorphism $\mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V)$. The composite

$$V_{\mathbb{H}-\mathrm{an}} \xrightarrow{v \mapsto o_{v|\mathbb{H}}} \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V) \xrightarrow{v \mapsto gv} \mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V)$$

is thus a continuous map from $V_{\mathbb{H}-\mathrm{an}}$ to $\mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V)$ which is immediately seen to be equal to the map $v \mapsto o_{v|g\mathbb{H}}$. If we take the product of these maps as g ranges over a set of right coset representatives for $g \in G$, we obtain a continuous map $V_{\mathbb{H}-\mathrm{an}} \to \prod_{g \in G/H} \mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V)$, given by $g \mapsto gv$. Composing with the natural map $\prod_{g \in G/H} \mathcal{C}^{\mathrm{an}}(g\mathbb{H}, V) \to \mathcal{C}^{\mathrm{la}}(G, V)$ (recall that $\{g\mathbb{H}\}_{g \in G/H}$ forms an analytic partition of G) and then passing to the inductive limit over all analytic open subgroups H of G, we obtain the continuous map $V_{\mathrm{la}} \to \mathcal{C}^{\mathrm{la}}(G, V)$ which makes (3.5.8) commutes.

We turn to showing that (3.5.8) is Cartesian. If v is an element of V for which o_v lies in $\mathcal{C}^{\mathrm{la}}(G, V)$ then in particular there is an analytic open subgroup H of G such that $o_{v|H}$ lies in $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$. Theorem 3.3.16 shows that v then lies in the image of $V_{\mathbb{H}-\mathrm{an}}$, and so in particular in the image of V_{la}. This proves that (3.5.8) is Cartesian. $\square$

In the papers [22] and [27], the space of locally analytic vectors in a continuous representation of G on a Banach space V is *defined* to be the closed subspace $\mathcal{C}^{\mathrm{la}}(G, V)^{\Delta_{1,2}(G)}$ of $\mathcal{C}^{\mathrm{la}}(G, V)$. (In these references, the authors use the terminology "analytic vectors" rather than "locally analytic vectors", and write V_{an} rather than V_{la}.) Thus the topology imposed on the space V_{la} in these references is in general coarser than the topology that we impose on V_{la}.

Proposition 3.5.9. *If V is a convex space of compact type equipped with a topological G-action, then the map $V_{\mathrm{la}} \to \mathcal{C}^{\mathrm{la}}(G, V)$ of theorem* 3.5.7 *is a closed embedding. In particular, V_{la} is again of compact type.*

Proof. Let H be a compact open subgroup of G. It is observed in [22, §3] that the natural map $\mathcal{C}^{\mathrm{la}}(H, V)^{\Delta_{1,2}(H)} \to \mathcal{C}^{\mathrm{la}}(G, V)^{\Delta_{1,2}(G)}$ is a topological isomorphism. Thus, replacing G by H if necessary, we may suppose without loss of generality that G is compact. Proposition 2.1.28 then shows that $\mathcal{C}^{\mathrm{la}}(G, V)$ is of compact type, while proposition 3.5.6 shows that V_{la} is an LB-space. The image of V_{la} under the map of theorem 3.5.7 is a closed subspace of $\mathcal{C}^{\mathrm{la}}(G, V)$, and thus is of compact type. Theorem 1.1.17 now shows that V_{la} maps isomorphically onto this subspace. $\square$

Proposition 3.5.10. *Let V be a Hausdorff convex K-vector space equipped with a topological G-action, and let W be a G-invariant closed subspace of V. Let W_1 denote the preimage of W in V_{la} under the natural map $V_{\mathrm{la}} \to V$, which is a G-invariant closed subspace of V_{la}. Then the map $W_{\mathrm{la}} \to V_{\mathrm{la}}$* (*induced by functoriality of the formation of locally analytic vectors*) *induces a continuous bijection of W_{la} onto W_1. If V_{la} is of compact type, then this bijection is even a topological isomorphism.*

Proof. The first statement follows from proposition 2.1.27 and theorem 3.5.7.

We now prove the second statement, assuming that V_{la} is of compact type. Proposition 3.5.5 implies that the natural map $(V_{\mathrm{la}})_{\mathrm{la}} \to V_{\mathrm{la}}$ is a topological isomorphism. The first statement of the proposition, applied to the closed embedding

$W_1 \to V_{\mathrm{la}}$, yields a continuous bijection $(W_1)_{\mathrm{la}} \to W_1$. As V_{la} is assumed to be of compact type, its closed subspace W_1 is also of compact type. Proposition 3.5.9 implies that $(W_1)_{\mathrm{la}}$ is again of compact type, and thus that this continuous bijection is in fact a topological isomorphism. This isomorphism fits into the sequence of continuous injections $(W_1)_{\mathrm{la}} \to W_{\mathrm{la}} \to W_1 \to W$ (the first of these being induced by the third and functoriality of the formation of locally analytic vectors). We conclude that the second of these maps is also a topological isomorphism, as required. □

Proposition 3.5.11. *If G is compact, and if V is either a K-Fréchet space or a convex K-vector space of compact type, then there is a natural isomorphism $\mathcal{C}(G,V)_{\mathrm{la}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(G,V)$. (Here the spaces $\mathcal{C}(G,V)$ and $\mathcal{C}^{\mathrm{la}}(G,V)$ are regarded as G-representations via the right regular G-action.)*

Proof. If H is an analytic open subgroup of G then it is of finite index in G, and we may write $G = \coprod_{i\in I} g_i H$, where I is some finite index set. Then $\mathcal{C}(G,V) \xrightarrow{\sim} \prod_{i\in I} \mathcal{C}(g_i H, V)$, and so proposition 3.3.24 shows that

$$\mathcal{C}(G,V)_{\mathbb{H}-\mathrm{an}} \xrightarrow{\sim} \prod_{i\in I} \mathcal{C}(g_i H, V)_{\mathbb{H}-\mathrm{an}} \xrightarrow{\sim} \prod_{i\in I} \mathcal{C}^{\mathrm{an}}(g_i \mathbb{H}, V).$$

As H ranges over all analytic open subgroups of G, the partitions $\{g_i H\}_{i\in I}$ are cofinal in the set of all analytic partitions of G. Thus we see that

$$\mathcal{C}(G,V)_{\mathrm{la}} = \varinjlim_{H} \mathcal{C}(G,V)_{\mathbb{H}-\mathrm{an}} \xrightarrow{\sim} \varinjlim_{H} \prod_{i\in I} \mathcal{C}^{\mathrm{an}}(g_i\mathbb{H}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(G,V). \quad \square$$

The next result is originally due to Feaux de Lacroix [12, bei. 3.1.6].

Proposition 3.5.12. *If G is compact, then for any Hausdorff convex K-vector space V, the natural map $\mathcal{C}^{\mathrm{la}}(G,V)_{\mathrm{la}} \to \mathcal{C}^{\mathrm{la}}(G,V)$ is a topological isomorphism. (Here $\mathcal{C}^{\mathrm{la}}(G,V)$ is regarded as a G-representation via the right regular G-action.)*

Proof. Since G is compact, there is a natural isomorphism

$$\varinjlim_{W} \mathcal{C}^{\mathrm{la}}(G, \overline{W}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(G,V),$$

where W runs over all BH-subspaces of V. The functoriality of the formation of locally analytic vectors gives rise to the commutative diagram

$$\begin{tikzcd} \varinjlim_{W} \mathcal{C}^{\mathrm{la}}(G,\overline{W})_{\mathrm{la}} \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{la}}(G,V)_{\mathrm{la}} \arrow[d] \\ \varinjlim_{W} \mathcal{C}^{\mathrm{la}}(G,\overline{W}) \arrow[r, "\sim"] & \mathcal{C}^{\mathrm{la}}(G,V). \end{tikzcd} \tag{3.5.13}$$

Propositions 3.5.5 and 3.5.11 together show that for each W, the natural map $\mathcal{C}^{\mathrm{la}}(G,\overline{W})_{\mathrm{la}} \to \mathcal{C}^{\mathrm{la}}(G,\overline{W})$ is a topological isomorphism. The left-hand vertical arrow of (3.5.13) is thus a topological isomorphism. Since the lower horizontal arrow is also an isomorphism, and the other arrows are continuous injections, we see that the right-hand vertical arrow is a topological isomorphism, as required. □

Proposition 3.5.14. *If V is a Hausdorff convex K-vector space equipped with a topological G-action, and if there is a G-equivariant isomorphism $\varinjlim_n V_n \xrightarrow{\sim} V$, where $\{V_n\}_{n\geq 1}$ is an inductive sequence of Hausdorff LB-spaces, each equipped with a topological G-action, admitting injective G-equivariant transition maps, then there is a natural isomorphism $\varinjlim_n (V_n)_{\mathrm{la}} \xrightarrow{\sim} V_{\mathrm{la}}$.*

Proof. It follows from proposition 1.1.10 that any BH-subspace of V lies in the image of the map $V_n \to V$ for some sufficiently large value of n. This yields the proposition. □

Proposition 3.5.15. *If U and V are Hausdorff LB-spaces, each equipped with a topological action of G, for which U_{la} and V_{la} are spaces of compact type, then the natural map $U_{\mathrm{la}} \hat{\otimes}_K V_{\mathrm{la}} \to U \hat{\otimes}_K V$ factors as a composite of continuous G-equivariant maps*

$$U_{\mathrm{la}} \hat{\otimes}_K V_{\mathrm{la}} \to (U \hat{\otimes}_K V)_{\mathrm{la}} \to U \hat{\otimes}_K V,$$

where the second arrow is the natural injection. (*Here $U \hat{\otimes}_K V$ is equipped with the diagonal action of G.*)

Proof. If $\{H_n\}_{n\geq 1}$ denotes a cofinal sequence of analytic open subgroups of G, then by definition,

$$U_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_{n, W_1} (\overline{W}_1)_{\mathbb{H}_n-\mathrm{an}},$$

where for a given value of n, we let W_1 run over all H_n-invariant BH-subspaces of U. By assumption, U_{la} is of compact type, and in particular is an LB-space. Taking into account Proposition 1.1.10, we thus see that we may find an increasing sequence of BH-subspaces $\{W_{1,n}\}_{n\geq 1}$ of U, with $W_{1,n}$ being H_n-invariant, such that

$$U_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_n (\overline{W}_{1,n})_{\mathbb{H}_n-\mathrm{an}}.$$

Similarly, we may find an increasing sequence of BH-subspaces $\{W_{2,n}\}_{n\geq 1}$ of V, with $W_{2,n}$ being H_n-invariant, such that

$$V_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_n (\overline{W}_{2,n})_{\mathbb{H}_n-\mathrm{an}}.$$

Proposition 1.1.32 yields an isomorphism

$$U_{\mathrm{la}} \hat{\otimes}_K V_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_n (\overline{W}_{1,n})_{\mathbb{H}_n-\mathrm{an}} \hat{\otimes}_K (\overline{W}_{2,n})_{\mathbb{H}_n-\mathrm{an}}. \tag{3.5.16}$$

Proposition 3.3.12 yields a map

$$(\overline{W}_{1,n})_{\mathbb{H}_n-\mathrm{an}} \hat{\otimes}_K (\overline{W}_{2,n})_{\mathbb{H}_n-\mathrm{an}} \to (\overline{W}_{1,n} \hat{\otimes}_K \overline{W}_{2,n})_{\mathbb{H}_n-\mathrm{an}}$$

for each n, while functoriality of the formation of analytic vectors yields a map

$$(\overline{W}_{1,n} \hat{\otimes}_K \overline{W}_{2,n})_{\mathbb{H}_n-\mathrm{an}} \to (U \hat{\otimes}_K V)_{\mathbb{H}_n-\mathrm{an}}$$

for each n. Altogether we obtain a map

$$\varinjlim_n (\overline{W}_{1,n})_{\mathbb{H}_n-\mathrm{an}} \hat{\otimes}_K (\overline{W}_{2,n})_{\mathbb{H}_n-\mathrm{an}} \to (U \hat{\otimes}_K V)_{\mathrm{la}}. \tag{3.5.17}$$

Composing (3.5.17) with (3.5.16) yields a map $U_{\mathrm{la}} \hat{\otimes}_K V_{\mathrm{la}} \to (U \hat{\otimes}_K V)_{\mathrm{la}}$. Composing this map with the injection $(U \hat{\otimes}_K V)_{\mathrm{la}} \to U \hat{\otimes}_K V$, yields the map $U_{\mathrm{la}} \hat{\otimes}_K V_{\mathrm{la}} \to U \hat{\otimes}_K V$. Thus we have obtained the required factorization. □

3.6. Analytic and locally analytic representations

In this section we recall the definition of a locally analytic representation of a locally L-analytic group on a convex K-vector space, and establish some basic properties of this notion. We begin by introducing the related notion of an analytic representation of the group of L-valued points of a σ-affinoid rigid analytic group.

Definition 3.6.1. Let $\mathbb{G}$ be a σ-affinoid rigid analytic group defined over L, which is σ-affinoid as a group, in the sense already considered in section 3.4, namely, that $\mathbb{G}$ admits an admissible cover $\mathbb{G} = \bigcup_{n=1}^{\infty} \mathbb{G}_n$, where $\{\mathbb{G}_n\}_{n\geq 1}$ is an increasing sequence of admissible affinoid open subgroups of $\mathbb{G}$. Suppose furthermore that this cover of $\mathbb{G}$ can be chosen so that $G_n := \mathbb{G}_n(L)$ is Zariski dense in $\mathbb{G}_n$ for each $n \geq 1$, so that $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$. A $\mathbb{G}$-analytic representation of G then consists of a barrelled Hausdorff convex K-vector space V equipped with a topological G-action, having the property that the natural injection $V_{\mathbb{G}-\mathrm{an}} \to V$ is in fact a bijection.

The justification for this terminology is provided by theorems 3.3.16 and 3.4.3. The requirement that V be barrelled is made so as to ensure that the G-action on V is continuous. (Indeed, since the G-action on $V_{\mathbb{G}-\mathrm{an}}$ is continuous, we see that the G-action on V is certainly separately continuous. Since V is barrelled, it follows that the G-action on V is continuous.)

If V is an arbitrary Hausdorff convex K-vector space equipped with a topological G-action, and if $\mathbb{G}$ is affinoid, then $V_{\mathbb{G}-\mathrm{an}}$ is barrelled, and so proposition 3.3.22 shows that $V_{\mathbb{G}-\mathrm{an}}$ affords a $\mathbb{G}$-analytic representation of G. If V is a K-Fréchet space, then for any σ-affinoid $\mathbb{G}$, corollary 3.4.7 shows that $V_{\mathbb{G}-\mathrm{an}}$ affords a $\mathbb{G}$-analytic representation of G.

Let us remark that the notion of $\mathbb{G}$-analytic representation is closely related to the notion of a uniformly analytic ("gleichmässig analytisch") representation of G defined in [12, def. 3.1.5]. Indeed, any such representation of G is $\mathbb{H}$-analytic for some sufficiently small analytic open subgroup H of G.

Proposition 3.6.2. *Let $\mathbb{G}$ be a σ-affinoid rigid analytic group satisfying the hypotheses of Definition* 3.6.1*, let V be a Hausdorff convex K-vector space equipped with a $\mathbb{G}$-analytic representation, and let W be a closed G-invariant subspace of V. If W* (*respectively V/W*) *is barrelled, then W* (*respectively V/W*) *is again an analytic representation of $\mathbb{G}$.*

Proof. Proposition 3.4.10 shows that the natural map $W_{\mathbb{G}-\mathrm{an}} \to W$ is a bijection. On the other hand, functoriality of the formation of $\mathbb{G}$-analytic vectors induces a commutative diagram

$$\begin{tikzcd} V_{\mathbb{G}-\mathrm{an}} \arrow[r] \arrow[d] & (V/W)_{\mathbb{G}-\mathrm{an}} \arrow[d] \\ V \arrow[r] & V/W. \end{tikzcd}$$

Since the bottom horizontal arrow is surjective, while both vertical arrows are injective, we see that if the left-hand vertical arrow is a bijection, the same is true of the right-hand vertical arrow. $\square$

Theorem 3.6.3. *If either $\mathbb{G}$ is a σ-affinoid rigid analytic group satisfying the hypothese of Definition* 3.6.1*, and V is a Fréchet space equipped with a $\mathbb{G}$-analytic representation of G, or if $\mathbb{G}$ is in fact an affinoid rigid analytic group such that $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$, and V is a Hausdorff LF-space equipped with a $\mathbb{G}$-analytic representation of G, then the natural map $V_{\mathbb{G}-\mathrm{an}} \to V$ is a G-equivariant topological isomorphism.*

Proof. Under the first (respectively second) set of assumptions, Proposition 3.4.4 (respectively Corollary 3.3.21) shows that $V_{\mathbb{G}-\mathrm{an}}$ is also a Fréchet space (respectively an LF-space). Thus the map $V_{\mathbb{G}-\mathrm{an}} \to V$ is a continuous G-equivariant bijection between Fréchet spaces (respectively LF-spaces), and so an isomorphism, by theorem 1.1.17. $\square$

The preceding theorem is the analogue for analytic representations of proposition 3.2.10. Taking into account theorem 3.3.16, it shows that if V is equipped with a $\mathbb{G}$-analytic representation of G, and if V and $\mathbb{G}$ satisfy one of the two possible hypotheses of the theorem, then there is a *continuous* orbit map $o : V \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$.

The next result provides a variant in the rigid analytic setting of the untwisting map of lemma 3.2.4.

Lemma 3.6.4. *If $\mathbb{G}$ is an affinoid rigid analytic group such that $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$, and if V is a $\mathbb{G}$-analytic representation of G on a Hausdorff LF-space, then there is a natural isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ which intertwines the $\Delta_{1,2}(G)$-action on its source and the left regular G-action on its target.*

Proof. The natural map $V_{\mathbb{G}-\mathrm{an}} \to V$ is a topological isomorphism, by theorem 3.6.3. Proposition 3.3.21 and its proof then show that V may be written as a G-equivariant inductive limit (with injective transition maps) of Fréchet spaces equipped with a $\mathbb{G}$-analytic action of G, say $V \xrightarrow{\sim} \varinjlim_n V_n$. There is then a corresponding isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V_n)$ (since any map from a Banach space to V factors through some V_n). Thus it suffices to prove the lemma with V replaced by each V_n in turn, and thus we assume for the remainder of the proof that V is a Fréchet space.

Proposition 2.1.13 (ii) gives an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K V$, while proposition 3.3.18 gives a closed embedding $V \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. Together, these give a closed embedding

$$(3.6.5) \quad \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \overset{(2.1.13)}{\xrightarrow{\sim}} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K V \xrightarrow{\mathrm{id}\,\hat{\otimes}(3.3.18)} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \overset{\mathrm{id}\,\hat{\otimes}\,(2.1.13)}{\xrightarrow{\sim}} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K V.$$

Now the antidiagonal embedding $a : \mathbb{G} \to \mathbb{G} \times \mathbb{G}$ (defined by $a : g \mapsto (g, g^{-1})$) induces a continuous map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \hat{\otimes}_K \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G} \times \mathbb{G}, K) \xrightarrow{a^*} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K).$$

Composing $a^* \hat{\otimes} \mathrm{id}$ with (3.6.5), and with the inverse of the isomorphism of proposition 2.1.13 (ii), yields a continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$.

From its construction, one easily checks that the automorphism of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ constructed in the preceding paragraph takes a function ϕ to the function $\tilde{\phi} : g \mapsto g^{-1}\phi(g)$. That is, we have shown that the automorphism of $\mathcal{F}(G, V)$ provided by lemma 3.2.4 leaves $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ invariant. Similar considerations show that the inverse of this automorphism likewise leaves $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ invariant. Thus the automorphism of $\mathcal{F}(G, V)$ provided by lemma 3.2.4 restricts to a topological automorphism of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. Lemma 3.2.4 shows that this automorphism intertwines the action of $\Delta_{1,2}(G)$ and the left regular G-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$. □

Similarly, we may find an isomorphism from $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$ to itself that intertwines the $\Delta_{1,3}(G)$-action and the right regular G-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V)$.

Proposition 3.6.6. *Let $\mathbb{G}$ be a σ-affinoid rigid analytic group satisfying the hypotheses of Definition* 3.6.1. *If V is a Hausdorff convex K-vector space equipped with a topological G-action, and if W is a finite dimensional $\mathbb{G}$-analytic representation of G, then there is a natural G-equivariant isomorphism $(V \otimes_K W)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{G}-\mathrm{an}} \otimes_K W$, characterized by the commutativity of the diagram*

$$\begin{tikzcd} (V \otimes_K W)_{\mathbb{G}-\mathrm{an}} \arrow[r, "\sim"] \arrow[dr, "(3.4.2)"'] & V_{\mathbb{G}-\mathrm{an}} \otimes_K W \arrow[d, "(3.4.2)\otimes\mathrm{id}"] \\ & V \otimes_K W. \end{tikzcd}$$

(*Here the various tensor products are equipped with the diagonal G-action.*)

Proof. We begin by proving the proposition in the case that $\mathbb{G}$ is and affinoid V is a K-Banach space. If $\phi \in \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V \otimes_K W)$ then we denote by $\tilde{\phi}$ the function $g \mapsto (\mathrm{id} \otimes g^{-1})\phi(g)$. (Here $\mathrm{id} \otimes g^{-1}$ is denoting the indicated automorphism of $V \otimes_K W$.) Since the G-action on W is assumed to be $\mathbb{G}$-analytic, the function $\tilde{\phi}$ again lies in $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V \otimes_K W)$, and the association of $\tilde{\phi}$ to ϕ induces an automorphism of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V \otimes_K W)$. (The inverse automorphism takes a function $\tilde{\phi}$ to the function $\phi : g \mapsto (\mathrm{id} \otimes g)\phi(g)$.) In fact, it simplifies our discussion if we use the canonical isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V \otimes_K W) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \otimes_K W$ to regard $\phi \mapsto \tilde{\phi}$ as an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, V \otimes_K W) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{G}, V) \otimes_K W$. One then checks that this isomorphism intertwines the $\Delta_{1,2}(G)$-action on the source with tensor product of the $\Delta_{1,2}(G)$-action and the trivial action on the target. Passing to $\Delta_{1,2}(G)$-invariants on the source, we obtain the desired isomorphism $(V \otimes_K W)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{G}-\mathrm{an}} \otimes_K W$.

Now suppose that $\mathbb{G}$ and V are arbitrary. If we fix a value of $n \geq 1$, then it follows from proposition 1.1.8 and its proof that as U ranges over all G_n-invariant BH-subspaces of V, the tensor products $U \otimes_K W$ are cofinal among the G_n-invariant BH-subspaces of $V \otimes_K W$. Combining this observation with the result already proved for Banach spaces and affinoid groups, and the fact that tensor product with W commutes with inductive limits, we obtain a natural isomorphism

$$\begin{aligned} (V \otimes_K W)_{\mathbb{G}_n-\mathrm{an}} &\xrightarrow{\sim} \varinjlim_U (\overline{U} \otimes_K W)_{\mathbb{G}_n-\mathrm{an}} \xrightarrow{\sim} \varinjlim_U (\overline{U}_{\mathbb{G}_n-\mathrm{an}} \otimes_K W) \\ &\xrightarrow{\sim} (\varinjlim_U \overline{U}_{\mathbb{G}_n-\mathrm{an}}) \otimes_K W = V_{\mathbb{G}_n-\mathrm{an}} \otimes_K W. \end{aligned}$$

Taking the inverse limit over n of these natural isomorphisms then yields the required isomorphism $(V \otimes_K W)_{\mathbb{G}-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{G}-\mathrm{an}} \otimes_K W$. □

Corollary 3.6.7. *Let $\mathbb{G}$ be an affinoid rigid analytic group such that $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$. If V is a Hausdorff convex K-vector space equipped with a $\mathbb{G}$-analytic representation of G, and if W is a finite dimensional $\mathbb{G}$-analytic representation of G, then the diagonal G-action on $V \otimes_K W$ makes this tensor product a $\mathbb{G}$-analytic representation of G.*

Proof. Since V is barrelled and W is finite dimensional, the tensor product $V \otimes_K W$ is certainly barrelled. The corollary is now an immediate consequence of proposition 3.6.6. $\square$

Proposition 3.6.8. *If V is a Hausdorff convex K-vector space equipped with a topological G-action, and if there is an isomorphism $\varinjlim_{i \in I} V_i \xrightarrow{\sim} V$, where $\{V_i\}_{i \in I}$ is a G-equivariant inductive system of Hausdorff K-vector spaces, each equipped with a $\mathbb{G}$-analytic action of G, then the G-action on V is again $\mathbb{G}$-analytic.*

Proof. Functoriality of the formation of $\mathbb{G}$-analytic vectors yields the commutative diagram

$$\begin{tikzcd} \varinjlim_{i \in I} (V_i)_{\mathbb{G}-\mathrm{an}} \arrow[r] \arrow[d] & V_{\mathbb{G}-\mathrm{an}} \arrow[d] \\ \varinjlim_{i \in I} V_i \arrow[r] & V. \end{tikzcd}$$

The left-hand vertical arrow and lower horizontal arrow are both continuous bijections, by assumption, and thus so is the right-hand vertical arrow. $\square$

Having introduced analytic representations, we now consider locally analytic representations.

Definition 3.6.9. Let G be a locally L-analytic group. A locally analytic representation of G is a topological action of G on a barrelled Hausdorff locally convex topological K-vector space for which the natural map $V_{\mathrm{la}} \to V$ is a bijection.

This definition is a reformulation of the one given in [23, p. 12]. (That it is equivalent with the definition given in this reference follows from theorem 3.5.7.) The requirement that V be barrelled is made so as to ensure that the G-action on V is continuous. (See the discussion following definition 3.6.1.)

Note that if $\mathbb{G}$ is an affinoid rigid analytic group such that $G := \mathbb{G}(L)$ and G is Zariski dense in $\mathbb{G}$, then a $\mathbb{G}$-analytic representation of G is in particular a locally analytic representation of G.

If V is an arbitrary Hausdorff convex K-vector space equipped with a topological action of G then V_{la} is barrelled, and proposition 3.5.5 then shows that V_{la} is equipped with a locally analytic representation of G. Thus if G is compact then, by proposition 3.5.12, $\mathcal{C}^{\mathrm{la}}(G, V)$ is a locally analytic representation of G, which, if V is either of compact type or Banach (respectively Fréchet), is an LB-space (respectively an LF-space), by propositions 2.1.4, 2.1.6, 3.5.6, and 3.5.11.

If V is a locally analytic representation of G then there is induced on V an action of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G by continuous linear operators [23, p. 13]. (The continuity of this action again depends on the fact that V is barrelled.) This action is functorial in V.

Proposition 3.6.10. *Suppose that $L = \mathbb{Q}_p$ and that K is local. Then any continuous action of a locally L-analytic group G on a finite dimensional K-vector space is locally analytic.*

Proof. Replacing G by a compact open subgroup if necessary, we may assume that G is compact. Let n denote the dimension of V over K, and choose a basis of V over K. The G-action on V then determines a continuous homomorphism of locally $\mathbb{Q}_p$-analytic groups $G \to \mathrm{GL}_n(K)$, which is necessarily locally $\mathbb{Q}_p$-analytic [28, LG 5.42, thm. 2]. This implies that the action of G on V is analytic. $\square$

Proposition 3.6.11. *If H is an open subgroup of the locally L-analytic group G, and V is a barrelled Hausdorff convex K-vector space equipped with a topological G-action, then V is a locally analytic representation of G if and only if it is a locally analytic representation of H.*

Proof. This follows from lemma 3.5.4. $\square$

Theorem 3.6.12. *If G is a locally L-analytic group and V is a locally analytic representation of G on a Hausdorff LF-space, then the natural map $V_{\mathrm{la}} \to V$ is a G-equivariant topological isomorphism.*

Proof. Proposition 3.5.6 shows that V_{la} is again an LF-space. Thus the map $V_{\mathrm{la}} \to V$ is a continuous G-equivariant bijection between LF-spaces. Theorem 1.1.17 implies that it is a topological isomorphism. $\square$

The preceding theorem is the analogue for locally analytic representations of proposition 3.2.10. Taking into account theorem 3.5.7, it shows that if V is a locally analytic representation on an LF-space then there is a *continuous* orbit map $o : V \to \mathcal{C}^{\mathrm{la}}(G, V)$.

Corollary 3.6.13. *A K-Fréchet space V equipped with a topological action of G is a locally analytic representation of G if and only if there is an analytic open subgroup H of G such that the natural map $V_{\mathbb{H}-\mathrm{an}} \to V$ is an isomorphism.*

Proof. If such an H exists then proposition 3.6.11 shows that the G-action on V gives rise to a locally analytic representation of G, since by assumption V affords an $\mathbb{H}$-analytic, and so locally analytic, representation of H. Conversely, suppose that V is endowed with a locally analytic representation of G. Let H_n be a cofinal sequence of analytic open subgroups of G. Theorem 3.6.12 shows that there is an isomorphism $\varinjlim_n V_{\mathbb{H}_n-\mathrm{an}} \xrightarrow{\sim} V$, and propositions 1.1.7 and 3.3.18 then imply that the natural map $V_{\mathbb{H}_n-\mathrm{an}} \to V$ is an isomorphism for some n, as required. $\square$

(In the case of K-Banach spaces, this result is equivalent to [12, kor. 3.1.9].)

Proposition 3.6.14. *Let V be a Hausdorff convex K-vector space equipped with a locally analytic representation of G, and let W be a closed G-invariant subspace of V. If W (respectively V/W) is barrelled, then W (respectively V/W) is also a locally analytic G-representation.*

Proof. Proposition 3.5.10 implies that the natural map $W_{\mathrm{la}} \to W$ is a bijection. On the other hand, functoriality of the formation of locally analytic vectors induces a commutative diagram

$$\begin{tikzcd} V_{\mathrm{la}} \arrow[r] \arrow[d] & (V/W)_{\mathrm{la}} \arrow[d] \\ V \arrow[r] & V/W. \end{tikzcd}$$

Since the bottom horizontal arrow is surjective, while both vertical arrows are injective, we see that if the left-hand vertical arrow is a bijection, the same is true of the right-hand vertical arrow. $\square$

Proposition 3.6.15. *If V is a Hausdorff convex K-vector space equipped with a topological G-action, and if W is a finite dimensional locally analytic representation of G, then there is a natural isomorphism G-equivariant isomorphism $(V \otimes_K W)_{\mathrm{la}} \xrightarrow{\sim} V_{\mathrm{la}} \otimes_K W$. (Here the tensor products are equipped with the diagonal G-action.)*

Proof. Since W is finite dimensional, we may find an analytic open subgroup H of G for which W is $\mathbb{H}$-analytic. Let $\{H'\}$ denote the directed set of analytic open subgroups of H; this set is then cofinal in the directed set of analytic open subgroups of G. Thus we obtain isomorphisms

$$(V \otimes_K W)_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_{H'} (V \otimes_K W)_{\mathbb{H}'-\mathrm{an}} \xrightarrow{\sim} \varinjlim_{H'} (V_{\mathbb{H}'-\mathrm{an}} \otimes_K W)$$
$$\xrightarrow{\sim} (\varinjlim_{H'} V_{\mathbb{H}'-\mathrm{an}}) \otimes_K W \xrightarrow{\sim} V_{\mathrm{la}} \otimes_K W,$$

in which the second isomorphism is provided by proposition 3.6.6, and the third isomorphism follows from the fact that tensor product with the finite dimensional space W commutes with inductive limits. $\square$

Corollary 3.6.16. *If V is a Hausdorff convex K-vector space equipped with a locally analytic representation of G, and if W is a finite dimensional locally analytic representation of G, then the diagonal G-action on $V \otimes_K W$ makes this tensor product a locally-analytic representation of G.*

Proof. Since V is barrelled and W is finite dimensional, the tensor product $V \otimes_K W$ is certainly barrelled. The corollary is now an immediate consequence of proposition 3.6.15. $\square$

Proposition 3.6.17. *If V is a Hausdorff convex K-vector space equipped with a topological G-action, and if there is an isomorphism $\varinjlim_{i \in I} V_i \xrightarrow{\sim} V$, where $\{V_i\}_{i \in I}$ is a G-equivariant inductive system of Hausdorff K-vector spaces, each equipped with a locally analytic action of G, then the G-action on V is again locally analytic.*

Proof. This is proved in the same manner as proposition 3.6.8. $\square$

Proposition 3.6.18. *If U and V are compact type spaces, each equipped with a locally analytic representation of G, then the diagonal G-action on the completed tensor product $U \hat{\otimes}_K V$ is again a locally analytic representation of G.*

Proof. This follows from proposition 3.5.15, and the fact that the natural maps $U_{\mathrm{la}} \to U$ and $V_{\mathrm{la}} \to V$ are both isomorphisms, by theorem 3.6.12. $\square$

Suppose now that E is a local subfield of L. As discussed in section 2.3, we may restrict scalars from L to E, and so regard the locally L-analytic group G as a locally E-analytic group. Thus if G acts on a convex K-vector space V, we can speak of the space of locally analytic vectors in V with respect to the action of G regarded as a locally E-analytic group. (That is, with respect to $\mathrm{Res}^L_E G$.) We will refer to this space as the space of locally E-analytic vectors of V, and denote it by $V_{E-\mathrm{la}}$. We conclude this section with some discussion about the relations between locally L-analytic G-representations and locally E-analytic representations.

If $\mathfrak{g}$ denotes the Lie algebra of G (a Lie algebra over L), then the Lie algebra of $\mathrm{Res}^L_E G$ is also equal to $\mathfrak{g}$, but now regarded as a Lie algebra over E. The Lie algebra of $(\mathrm{Res}^L_E G)_{/L}$ is thus equal to $L \otimes_E \mathfrak{g}$. The natural map $(\mathrm{Res}^L_E G)_{/L} \to G$ given by (2.3.2) induces a map on Lie algebras, which is just the natural map $L \otimes_E \mathfrak{g} \to \mathfrak{g}$ given by the L-linear structure on $\mathfrak{g}$. Let $(L \otimes_E \mathfrak{g})^0$ denote the kernel of this map.

Observe that if V is a Hausdorff convex K-vector space equipped with a locally E-analytic action of G, then the $\mathfrak{g}$-action on V extends to a $K \otimes_E \mathfrak{g}$-action, and so in particular to an $L \otimes_E \mathfrak{g}$-action.

Proposition 3.6.19. *If V is a barrelled Hausdorff convex K-vector space equipped with a locally E-analytic G-action, then this action is locally L-analytic if and only if V is annihilated by the Lie subalgebra $(L \otimes_E \mathfrak{g})^0$ of $L \otimes_E \mathfrak{g}$.*

Proof. If $\mathbb{H}$ is an analytic open subgroup of G, let $\mathbb{H}_0$ denote the restriction of scalars of $\mathbb{H}$ from L to E. Then $V_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_{W,H} \overline{W}_{\mathbb{H}-\mathrm{an}}$, where W ranges over the G-invariant BH-subspaces of V, and H ranges over the analytic open subgroups of G, while $V_{E-\mathrm{la}} \xrightarrow{\sim} \varinjlim_{W,H} \overline{W}_{\mathbb{H}_0-\mathrm{an}}$. By assumption, the map $V_{E-\mathrm{la}} \to V$ is bijective, and we must show that the map $V_{\mathrm{la}} \to V$ is also bijective if and only if V is annihilated by $(L \otimes_E \mathfrak{g})^0$. For this, it suffices to show, for each fixed W and H, that the natural map

$$\overline{W}_{\mathbb{H}-\mathrm{an}} \to \overline{W}_{\mathbb{H}_0-\mathrm{an}}$$

is an isomorphism if and only if $\overline{W}_{\mathbb{H}_0-\mathrm{an}}$ is annihilated by $(L \otimes_E \mathfrak{g})^0$. This last claim follows from the fact that $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, \overline{W})$ is equal to the subspace $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_0, \overline{W})^{(L \otimes_E \mathfrak{g})^0}$ of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_0, \overline{W})$ consisting of functions annihilated by $(L \otimes_E \mathfrak{g})^0$ (under the right regular action), together with the defining isomorphisms $\overline{W}_{\mathbb{H}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}, \overline{W})^{\Delta_{1,2}(H)}$ and $\overline{W}_{\mathbb{H}_0-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}_0, \overline{W})^{\Delta_{1,2}(H)}$. $\square$

Corollary 3.6.20. *If $V \to W$ is a continuous injection of G-representations on convex K-vector spaces, such that V is locally E-analytic and W is locally L-analytic, then V is necessarily locally L-analytic.*

Proof. Proposition 3.6.19 shows that $(L \otimes_E \mathfrak{g})^0$ acts trivially on W. Thus this Lie algebra also acts trivially on V, and by that same proposition, we are done. $\square$

Proposition 3.6.21. *Suppose that the Lie algebra of G is semi-simple. If $0 \to U \to V \to W \to 0$ is a short exact sequence of locally E-analytic representations of G on convex K-vector spaces, such both U and W are locally L-analytic, then V is locally L-analytic.*

Proof. Proposition 3.6.19 shows that $(L \otimes_E \mathfrak{g})^0$ annihilates both U and W. Since $\mathfrak{g}$ is semi-simple by assumption, the same is true of $L \otimes_E \mathfrak{g}$, and hence of its subalgebra

$(L \otimes_E \mathfrak{g})^0$. Thus the extension V of W by U is also annihilated by $(L \otimes_E \mathfrak{g})^0$, and appealing again to proposition 3.6.19, we are done. $\square$

The preceding result does not hold in general if G has non-semi-simple Lie algebra. For example, suppose that $G = \mathcal{O}_L$, and suppose that ψ is a K-valued character of $\mathcal{O}_L$ that is locally E-analytic, but note locally L-analytic. The two dimensional representation of $\mathcal{O}_L$ defined by the matrix $\begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}$ is then an extension of the trivial representation of $\mathcal{O}_L$ by itself, but is not locally L-analytic.

Chapter 4. Smooth, locally finite, and locally algebraic vectors

4.1. Smooth and locally finite vectors and representations

In this section we develop some connections between the notions of smooth and locally finite representations, and the notion of locally analytic representations. These connections have already been treated in [24]; our goal is to recall some points from this reference, and to present some minor additions to its discussion.

Proposition-Definition 4.1.1. *Let G be a topological group and V a K-vector space equipped with a G-action.*

(i) *We say that a vector $v \in V$ is smooth if there is an an open subgroup of G which fixes v.*

(ii) *The set of all smooth vectors in V forms a G-invariant K-linear subspace of V, which we denote by V_{sm}.*

(iii) *We say that V is a smooth representation of G if every vector in V is smooth; that is, if $V_{\mathrm{sm}} = V$.*

(iv) *We say that V is an admissible smooth representation of G if it is a smooth representation of V having the additional property that for any open subgroup H of G, the subspace of H-fixed vectors in V is finite dimensional over K.*

Proof. The only statement to be proved is that of (ii), stating that V_{sm} is a G-invariant K-subspace of V, and this is clear. $\square$

The formation of V_{sm} is covariantly functorial in the G-representation V.

Lemma 4.1.2. *If G is a compact topological group and V is a smooth representation of G then V is semi-simple. If furthermore V is irreducible, then V is finite dimensional.*

Proof. This is well-known, and is in any case easily proved. (It easily reduces to the fact that representations of finite groups on vector spaces over a field of characteristic zero are semi-simple.) $\square$

Proposition-Definition 4.1.3. *Let G be a locally L-analytic group, and let V be a Hausdorff convex K-vector space equipped with a locally analytic representation of G.*

(i) *A vector $v \in V$ is said to be $U(\mathfrak{g})$-trivial if it is annihilated by the action of $\mathfrak{g}$.*

(ii) *The subset of V consisting of all $U(\mathfrak{g})$-invariant vectors is a closed G-invariant K-linear subspace of V, which we denote by $V^{\mathfrak{g}}$.*

(iii) *The representation V is said to be $U(\mathfrak{g})$-trivial if every vector in it is $U(\mathfrak{g})$-trivial; that is, if $V^{\mathfrak{g}} = V$.*

Proof. All that is to be proved is that for any locally analytic representation V of G, the set of $U(\mathfrak{g})$-trivial vectors $V^{\mathfrak{g}}$ is a closed G-invariant subspace of V. That $V^{\mathfrak{g}}$ is closed follows from the fact that the $U(\mathfrak{g})$-action on V is via continuous operators. That it is G-invariant follows from the formula $Xgv = g\mathrm{Ad}_{g^{-1}}(X)v$, for $g \in G$, $X \in \mathfrak{g}$, and $v \in V$. $\square$

The following lemma and its corollaries relate definitions 4.1.1 and 4.1.3.

Lemma 4.1.4. *Suppose that G is equal to the group of L-valued points of a connected affinoid rigid analytic group $\mathbb{G}$ defined over L, and that furthermore G is Zariski dense in $\mathbb{G}$. Let V be a $\mathbb{G}$-analytic representation of G. Then a vector v in V is $U(\mathfrak{g})$-trivial if and only if v is stabilized by G.*

Proof. We may find a G-invariant BH-subspace W of V such that v lies in the image of the natural map $\overline{W}_{\mathbb{G}-\mathrm{an}} \to V$. Consider the tautological G-equivariant map $\overline{W}_{\mathbb{G}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$ (the target being equipped with the right regular G-action). If v is $U(\mathfrak{g})$-trivial, then its image lies in the $U(\mathfrak{g})$-trivial subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, \overline{W})$, which consists precisely of the constant functions. Thus it is G-invariant. Conversely, if v is G-invariant, then it is certainly $U(\mathfrak{g})$-trivial. $\square$

Corollary 4.1.5. *Suppose that G is equal to the group of L-valued points of a connected affinoid rigid analytic group $\mathbb{G}$ defined over L, and that furthermore G is Zariski dense in $\mathbb{G}$. If V is a Hausdorff convex K-vector space equipped with a topological G-action then the image of the composite $(V_{\mathbb{G}-\mathrm{an}})^{\mathfrak{g}} \subset V_{\mathbb{G}-\mathrm{an}} \to V$ is equal to the closed subspace of V consisting of G-fixed vectors.*

Proof. Since $V_{\mathbb{G}-\mathrm{an}}$ is a $\mathbb{G}$-analytic representation of G, lemma 4.1.4 shows that $(V_{\mathbb{G}-\mathrm{an}})^{\mathfrak{g}}$ is the subspace of $V_{\mathbb{G}-\mathrm{an}}$ consisting of G-fixed vectors. It remains to be shown that the continuous injection $V_{\mathbb{G}-\mathrm{an}} \to V$ induces a bijection on the corresponding subspaces of G-fixed vectors. That this is so follows from proposition 3.3.3, since a vector $v \in V$ is G-fixed if and only if the orbit map o_v is constant, and constant maps are certainly rigid analytic. $\square$

Corollary 4.1.6. *If G is a locally L-analytic group and V is a Hausdorff convex K-vector space equipped with a topological action of G then the image of the composite $(V_{\mathrm{la}})^{\mathfrak{g}} \subset V_{\mathrm{la}} \to V$ is equal to the subspace V_{sm} of V.*

Proof. If we let H run over all the analytic open subgroups of G then $V_{\mathrm{la}} = \varinjlim_H V_{\mathbb{H}-\mathrm{an}}$, and so there is a continuous bijection $\varinjlim_H (V_{\mathbb{H}-\mathrm{an}})^{\mathfrak{g}} \to (V_{\mathrm{la}})^{\mathfrak{g}}$. Since the directed set of analytic open subgroups of G maps cofinally to the directed set of all open subgroups of G, the corollary follows from corollary 4.1.5. $\square$

It is important to note that if we endow V_{sm} with the topology it inherits as a subset of V then the natural map $V_{\mathrm{la}}^{\mathfrak{g}} \to V_{\mathrm{sm}}$, which is a G-equivariant isomorphism of abstract K-vector spaces, need not be a topological isomorphism. For example, if $V = \mathcal{C}(G, K)$, then, by proposition 3.5.11, there is an isomorphism $\mathcal{C}(G, K)_{\mathrm{la}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(G, K)$. The subspace $\mathcal{C}(G, K)_{\mathrm{sm}}$ consists of the locally constant G-valued functions. This is necessarily a closed subspace of $\mathcal{C}^{\mathrm{la}}(G, K)$ (being equal to $\mathcal{C}^{\mathrm{la}}(G, K)^{\mathfrak{g}}$), but is dense as a subspace of $\mathcal{C}(G, K)$.

Corollary 4.1.7. *If G is a locally L-analytic group and V is a Hausdorff convex K-vector space equipped with a locally analytic representation of G then a vector v*

in V is $U(\mathfrak{g})$-trivial if and only if it is smooth. In particular, V_{sm} is the underlying K-vector space of the closed subspace $V^{\mathfrak{g}}$ of V.

Proof. If V is locally analytic then by definition the natural map $V_{\mathrm{la}} \to V$ is a continuous bijection, and so the induced map $(V_{\mathrm{la}})^{\mathfrak{g}} \to V^{\mathfrak{g}}$ is also a continuous bijection. The result then follows from corollary 4.1.6. □

In the remainder of this section we discuss locally finite vectors and the class of locally finite G-representations (which contains the smooth representations as a subclass).

Proposition-Definition 4.1.8. *Let G be a topological group and V a K-vector space equipped with a G-action.*

(i) *We say that a vector $v \in V$ is locally finite if there is an open subgroup H of G, and a finite dimensional H-invariant subspace W of V that contains v, such that the H-action on W is continuous (with respect to the natural Hausdorff topology on the finite dimensional K-vector space W).*

(ii) *The set of all locally finite vectors in V forms a G-invariant K-linear subspace of V, which we denote by V_{lf} (or by $V_{G-\mathrm{lf}}$, if we wish to emphasize the group G).*

(iii) *We say that V is a locally finite representation of G if every vector in V is locally finite; that is, if $V_{\mathrm{lf}} = V$.*

Proof. The only statement to be proved is that of (ii), stating that V_{lf} is a G-invariant K-subspace of V, and this is clear. □

Proposition 4.1.9. *Let G be a compact topological group, and let V be a locally finite G-representation.*

(i) *If W is any finite dimensional G-invariant subspace of V, then the G-action on W is continuous (when W is given its canonical Hausdorff topology).*

(ii) *The natural map $\varinjlim_{W} W \to V$, where W ranges over the directed set of all finite dimensional G-invariant subspaces of V, is an isomorphism.*

Proof. Suppose that W is a finite dimensional G-invariant subspace of V. If $\{w_1, \dots, w_n\}$ denotes a basis of W, then for each of the vectors w_i we may find an open subgroup H_i of G, and a finite dimensional H_i-invariant subspace W_i of V that contains w_i, and on which the H_i-action is continuous. If we let $H = \bigcap_i H_i$, then the natural map $\bigoplus_i W_i \to V$ is an H-equivariant map whose image contains W, and on whose source the H-action is continuous. It follows that the H-action on W is continuous. Since H is open in G, and since any group action on a Hausdorff finite dimensional topological K-vector space is topological, corollary 3.1.3 shows that the G-action on W is also continuous. This proves (i).

To prove (ii), we must show that each vector $v \in V$ is contained in a finite dimensional G-invariant subspace of V. By assumption, we may find an open subgroup H of G and a finite dimensional H-invariant subspace W of V containing v. Since G is compact, the index of H in G is finite, and so the G-invariant subspace of V spanned by W is again finite dimensional. This proves (ii). □

Proposition-Definition 4.1.10. *Let G be a locally L-analytic group, and let V be a Hausdorff convex K-vector space equipped with a locally analytic representation of G.*

(i) *A vector $v \in V$ is said to be $U(\mathfrak{g})$-finite if it is contained in a $U(\mathfrak{g})$-invariant finite dimensional subspace of V.*

(ii) *The set of all $U(\mathfrak{g})$-finite vectors in V forms a G-invariant K-linear subspace of V, which we denote by $V_{\mathfrak{g}-\mathrm{lf}}$.*

(iii) *The representation V is said to be $U(\mathfrak{g})$-locally finite if every vector in it is $U(\mathfrak{g})$-finite; that is, if $V_{\mathfrak{g}-\mathrm{lf}} = V$.*

(iv) *The representation V is said to $U(\mathfrak{g})$-finite if there is an ideal of cofinite dimension in $U(\mathfrak{g})$ which annihilates V.*

Proof. The only statement to be proved is that of (ii), which follows from the formula $Xgv = g\mathrm{Ad}_{g^{-1}}(X)v$. □

The notion of $U(\mathfrak{g})$-finite representation was introduced and studied in [24, §3]. The following results relate definitions 4.1.8 and 4.1.10.

Lemma 4.1.11. *If G is a locally L-analytic group, and if V is a locally analytic representation of V, then a vector $v \in V$ is $U(\mathfrak{g})$-finite if and only if it is locally finite under the action of G.*

Proof. If v lies in a finite dimensional G-invariant subspace of V, then v is certainly $U(\mathfrak{g})$-finite. Conversely, if v is $U(\mathfrak{g})$-finite, let U denote a finite dimensional $U(\mathfrak{g})$-invariant subspace of V containing v. If we choose the open subgroup H of G to be sufficiently small, then we may lift the $U(\mathfrak{g})$-action on U to an action of H. The space $\mathrm{Hom}(U, V) = \check{U} \otimes_K V$ is then a locally analytic H-representation (by corollary 3.6.16), and the given inclusion of U into V gives a $U(\mathfrak{g})$-fixed point of this space. By corollary 4.1.7, replacing H by a smaller open subgroup if necessary, we may assume that this point is in fact H-invariant, and thus that U is an H-invariant subspace of V. Thus the $U(\mathfrak{g})$-finite vector v is G-locally finite, as claimed. □

Proposition 4.1.12. *If G is a locally $\mathbb{Q}_p$-analytic group, if K is local, and if V is a Hausdorff convex K-vector space equipped with a topological action of G, then the image of the composite $(V_{\mathrm{la}})_{\mathfrak{g}-\mathrm{lf}} \subset V_{\mathrm{la}} \to V$ is equal to the subspace V_{lf} of V.*

Proof. Since V_{la} is a locally analytic representation of G, lemma 4.1.11 shows that $(V_{\mathrm{la}})_{\mathfrak{g}-\mathrm{lf}}$ is equal to the space $(V_{\mathrm{la}})_{\mathrm{lf}}$. It remains to be shown that the continuous injection $V_{\mathrm{la}} \to V$ induces a bijection on the corresponding subspaces of locally finite vectors.

If $v \in V$ is a locally finite vector, then by definition we may find a compact open subgroup H of G and a finite dimensional H-invariant subspace W of V containing v on which H acts continuously. By proposition 3.6.10, W is in particular a BH-subspace of V equipped with a locally analytic action of H, and so the inclusion of W into V factors through the natural map $V_{\mathrm{la}} \to V$. □

Just in the case of smooth vectors, it is important to note that if we endow V_{lf} (respectively $(V_{\mathrm{la}})_{\mathfrak{g}-\mathrm{lf}}$) with the topology it inherits as a subspace of V (respectively V_{la}), then the natural map $(V_{\mathrm{la}})_{\mathfrak{g}-\mathrm{lf}} \to V_{\mathrm{lf}}$, which is a G-equivariant isomorphism of abstract K-vector spaces, need not be a topological isomorphism.

The preceding result implies that if V is a K-vector space equipped with a locally finite action of a locally $\mathbb{Q}_p$-analytic group G, and if we endow V with its finest convex topology, then the natural map $V_{\mathrm{la}} \to V$ is a topological isomorphism, and in particular V becomes a $U(\mathfrak{g})$-locally finite locally analytic representation of G.

Proposition 4.1.13. *Suppose that G is a locally $\mathbb{Q}_p$-analytic group. Let $0 \to U \to V \to W \to 0$ be a short exact sequence of continuous G-representations on* *topological K-vector spaces. If U and W are locally finite, then the same is true of V.*

Proof. Let v be a vector in V, with image w in W. Replacing G by a compact open subgroup if necessary, we may assume that G is compact, and so find a finite dimensional G-subrepresentation of W that contains w. Pulling back our extension along the embedding of this subspace into W, we may thus assume that W is finite dimensional. Choose a finite dimensional subspace V' of V which contains v and projects isomorphically onto W.

If we tensor our short exact sequence through with the contragredient $\check{W}$ of W we obtain a short exact sequence

$$0 \to U \otimes_K \check{W} \to V \otimes_K \check{W} \to W \otimes_K \check{W} \to 0.$$

Our choice of V' gives rise to an element $\phi \in V \otimes_K \check{W}$, which projects onto the identity map from W to itself, regarded as an element of $W \otimes_K \check{W}$. Since this identity map is G-equivariant, the element ϕ gives rise to a continuous 1-cocycle $\sigma : G \to U \otimes_K \check{W}$. Since G is compact and $\mathbb{Q}_p$-locally analytic, it is in particular topologically finitely generated. Since U is a locally finite representation of G, we may find a finite dimensional G-invariant subrepresentation $U' \subset U$ such that σ takes values in $U' \otimes_K \check{W}$.

Now consider the short exact sequence $0 \to U/U' \to V/U' \to W \to 0$. By construction, the subspace V' is G-invariant when regarded as a subspace of V/U'. Thus the subspace $U' \bigoplus V'$ of V is a finite dimensional G-equivariant subrepresentation of V that contains v. $\square$

Proposition 4.1.14. *If G is a compact locally $\mathbb{Q}_p$-analytic group with semi-simple Lie algebra, and if K is local, then the category of locally finite G-representations is semi-simple, and any irreducible locally finite G-representation is finite dimensional.*

Proof. We begin by considering a surjection $V \to W$ of finite dimensional continuous representations of G. We must show that this surjection can be split in a G-equivariant fashion. Applying $\mathrm{Hom}(W, -)$ to the given surjection yields the surjection of finite dimensional continuous representations $\mathrm{Hom}(W, V) \to \mathrm{Hom}(W, W)$. Proposition 3.6.10 shows that both of these finite dimensional representations are in fact locally analytic, and so are equipped with a natural $\mathfrak{g}$-action. Since $\mathfrak{g}$ is semi-simple, passing to $\mathfrak{g}$-invariants yields a surjection $\mathrm{Hom}_{\mathfrak{g}}(W, V) \to \mathrm{Hom}_{\mathfrak{g}}(W, W)$. This is a surjection of smooth representations of the compact group G. Since by lemma 4.1.2 smooth representations of a compact group are semi-simple, we may find a G-equivariant map from W to V lifting the identity map from W to itself.

Now suppose that $V \to W$ is any surjection of locally finite G-representations. Taking into account proposition 4.1.9, the result of the preceding paragraph shows that we may find a G-equivariant section to this map over any finite dimensional subrepresentation of W. A simple application of Zorn's lemma now allows us to split this surjection over all of W.

Finally, since G is compact, proposition 4.1.9 implies that any irreducible locally finite representation of G is finite. $\square$

Corollary 4.1.15. *If G is a locally $\mathbb{Q}_p$-analytic group with semi-simple Lie algebra, and if K is local, then any continuous G-representation on a topological G-vector* *space which may be written as an extension of smooth G-representations is again smooth.*

Proof. Replacing G by a compact open subgroup, if necessary, we may suppose that G is compact. Since any smooth representation is locally finite, the corollary is now seen to be an immediate consequence of propositions 4.1.13 and 4.1.14. □

Corollary 4.1.16. *If G is a compact locally $\mathbb{Q}_p$-analytic group with semi-simple Lie algebra, and if K is local, then any continuous G-representation on a topological vector space which may be written as an extension of two G-representations equipped with trivial G-action is itself equipped with trivial G-action.*

Proof. Corollary 4.1.15 shows that any such extension is smooth. By lemma 4.1.2 it is semi-simple. The corollary follows. □

The preceding three results are false for more general locally analytic groups, as is illustrated by the two-dimensional representation of the group $\mathbb{Z}_p$ which sends $t \in \mathbb{Z}_p$ to the matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

4.2. Locally algebraic vectors and representations

In this section we let $\mathbb{G}$ denote a connected reductive linear algebraic group defined over L, and let G denote an open subgroup of $\mathbb{G}(L)$. In this situation, G is Zariski dense in $\mathbb{G}$ (since $\mathbb{G}$ is unirational, by [2, thm. 18.2]). We let $\mathcal{C}^{\mathrm{alg}}(\mathbb{G}, K)$ denote the affine ring of $\mathbb{G}$ over K. Since G is Zariski dense in $\mathbb{G}$, the restriction of elements of $\mathcal{C}^{\mathrm{alg}}(\mathbb{G}, K)$ to G induces an injection $\mathcal{C}^{\mathrm{alg}}(\mathbb{G}, K) \to \mathcal{C}^{\mathrm{la}}(G, K)$.

Let $\mathcal{R}$ denote the category of finite dimensional algebraic representations of $\mathbb{G}$ defined over K. Since $\mathbb{G}$ is reductive the category $\mathcal{R}$ is a semi-simple abelian category. Passing to the underlying G-representation is a fully faithful functor, since G is Zariski dense in $\mathbb{G}$.

Definition 4.2.1. Let V be a K-vector space equipped with an action of G and let W be an object of $\mathcal{R}$. We say that a vector v in V is locally W-algebraic if there exists an open subgroup H of G, a natural number n, and an H-equivariant homomorphism $W^n \to V$ whose image contains the vector v. We say that the G-representation V is locally W-algebraic if each vector of V is locally W-algebraic.

Note that W-locally algebraic vectors are in particular G-locally finite, and thus a W-locally algebraic representation is a G-locally finite representation. Thus, if we equip a W-locally algebraic V with its finest convex topology, then V becomes a locally analytic G-representation, which is $U(\mathfrak{g})$-finite. (It is annihilated by the annihilator in $U(\mathfrak{g})$ of the finite dimensional representation W.)

For the next several results, we fix an object W of $\mathcal{R}$.

Proposition-Definition 4.2.2. *If V is a K-vector space equipped with an action of G, then we let $V_{W-\mathrm{lalg}}$ denote the G-invariant subspace of W-locally algebraic vectors of V.*

Proof. We must show that $V_{W-\mathrm{lalg}}$ is a G-invariant subspace of V. It is clear that the W-locally algebraic vectors do form a subspace of V. If v lies in $V_{W-\mathrm{lalg}}$, then by assumption there is an open subgroup H of G, a natural number n, and an H-equivariant map $\phi : W^n \to V$ with v in its image. Let g be an element of G.

Then $g\phi(g^{-1}-) : W^n \to V$ is gHg^{-1}-equivariant and has gv in its image. Thus $V_{W-\mathrm{lalg}}$ is G-invariant. $\square$

The formation of $V_{W-\mathrm{lalg}}$ is covariantly functorial in V. Let A_1 denote the image of $U(\mathfrak{g})$ in $\mathrm{End}(W)$, and let A_2 denote the image of the group ring $K[G]$ in $\mathrm{End}(W)$. Since each of A_1 and A_2 act faithfully on the semi-simple module W, they are both semi-simple K-algebras. Write

$$B := \mathrm{End}_{A_1}(W) = \mathrm{End}_{\mathbb{G}}(W) = \mathrm{End}_G(W) = \mathrm{End}_{A_2}(W).$$

(The second equality holds because $\mathbb{G}$ is connected, the third equality holds because G is Zariski dense in $\mathbb{G}$, and the fourth equality holds by definition of A_2.) The ring B is a semi-simple K-algebra. The double commutant theorem implies that $A_1 = A_2 = \mathrm{End}_B(W)$, and from now on we write A in place of A_1 or A_2. The left B-action on W induces a right B-action on the dual space $\check{W}$ (the transpose action).

Proposition 4.2.3. (i) *There is a natural isomorphism* $\check{W} \otimes_B W \xrightarrow{\sim} \check{A}$.

(ii) *The is a natural injection* $\check{A} \to \mathcal{F}(G)$*, such that the composite* $\check{W} \otimes_B W \xrightarrow{\sim} \check{A} \longrightarrow \mathcal{F}(G) \xrightarrow{\mathrm{ev}_e} K$ *is the natural evaluation map* $\check{W} \otimes_B W \to K$.

Proof. As already observed, the double commutant theorem implies that the faithful action of A on W induces an isomorphism $A \to \mathrm{End}_B(W)$; dualizing this yields the isomorphism of part (i). Dualizing the surjection $K[G] \to A$ induces an injection of $\check{A}$ into the space of functions $\mathcal{F}(G)$ on G; this is the map of part (ii). The claim of part (ii) is easily checked. $\square$

Since G acts on W through an algebraic representation of $\mathbb{G}$, it is immediate that the elements of $\mathcal{F}(G)$ in the image of $\check{A}$ via the map of part (ii) of the preceding proposition are the restriction to G of elements of $\mathcal{C}^{\mathrm{alg}}(\mathbb{G})$.

If U is a smooth G-representation on a right B-module, then $U \otimes_B W$ is certainly a locally W-algebraic representation of G. Note that we may recover U via the natural isomorphism

$$\mathrm{Hom}(W, U \otimes_B W)_{\mathrm{sm}} \xrightarrow{\sim} U \otimes_B \mathrm{End}_{\mathfrak{g}}(W) = U \otimes_B B \xrightarrow{\sim} U.$$

There is an important converse to this construction. Note that if V is a K-vector space equipped with a G-action, then $\mathrm{Hom}(W, V)$ is equipped with its natural G-action, and a commuting right B-action induced by the G-equivariant left B-action on W. The subspace of smooth vectors $\mathrm{Hom}(W, V)_{\mathrm{sm}}$ is a G-invariant B-submodule, on which the G-action is smooth.

Proposition 4.2.4. *If V is a K-vector space equipped with a G-representation, then the evaluation map* $\mathrm{Hom}(W, V)_{\mathrm{sm}} \otimes_B W \to V$ *is an injection, with image equal to* $V_{W-\mathrm{lalg}}$.

Proof. By definition, the space $\mathrm{Hom}(W, V)_{\mathrm{sm}}$ is the inductive limit of the spaces $\mathrm{Hom}_H(W, V)$ as H runs over all open subgroups of G. Thus $\mathrm{Hom}(W, V)_{\mathrm{sm}} \otimes_B W$ is the inductive limit of the spaces $\mathrm{Hom}_H(W, V) \otimes_B W$.

If $\sum_{i=1}^n \phi_i \otimes w_i$ is an element of $\mathrm{Hom}_H(W, V) \otimes_B W$, then its image v in V under the evaluation map is equal to $\sum_{i=1}^n \phi_i(w_i)$. Let $\phi = (\phi_1, \ldots, \phi_n) \in \mathrm{Hom}_H(W, V)^n = \mathrm{Hom}_H(W^n, V)$, and let $w = (w_1, \ldots, w_n) \in W^n$. Then $\phi(w) = v$, and thus v lies in $V_{W-\mathrm{lalg}}$. Conversely, if $v \in V_{W-\mathrm{lalg}}$ then there exists some such H and an element $\phi = (\phi_1, \ldots, \phi_n) \in \mathrm{Hom}_H(W^n, V)$ such that v lies in the image of ϕ. If $w = (w_1, \ldots, w_n)$ is such that $\phi(w) = v$, then $\sum_{i=1}^n \phi_i \otimes w_i \in \mathrm{Hom}_H(W, V) \otimes_B W$ maps to v under the evaluation map $\mathrm{Hom}_H(W, V) \otimes_B W \to V$. This shows that $V_{W-\mathrm{lalg}}$ is equal to the image of the evaluation map.

It remains to be shown that for each open subgroup H of G the evaluation map $\mathrm{Hom}_H(W, V) \otimes_B W \to V$ is injective. Let $\mathcal{F}(H)$ denote the space of K-valued functions on H. Applying proposition 4.2.3 with H in place of G, we obtain an $H \times H$-equivariant injection $\check{W} \otimes_B W \xrightarrow{\sim} \check{A} \longrightarrow \mathcal{F}(H)$. Tensoring through by V yields an $H \times H \times H$-equivariant injection

$$\mathrm{Hom}(W, V) \otimes_B W = V \otimes \check{W} \otimes_B W \to V \otimes \mathcal{F}(H).$$

Now taking $\Delta_{1,2}(H)$-equivariant invariants yields an injection $\mathrm{Hom}_H(W, V) \otimes_B W \longrightarrow (V \otimes \mathcal{F}(H))^{\Delta_{1,2}(H)} \xrightarrow{\sim} V$, as required. (The final isomorphism is provided by the map ev_e. That this composite is the natural evaluation map follows from part (ii) of proposition 4.2.3.) $\square$

Proposition 4.2.5. *If W_1 and W_2 are two objects of $\mathcal{R}$ then the map $V_{W_1-\mathrm{lalg}} \oplus V_{W_2-\mathrm{lalg}} \to V$ induced by the inclusions $V_{W_1-\mathrm{lalg}} \to V$ and $V_{W_2-\mathrm{lalg}} \to V$ has image equal to $V_{W_1 \oplus W_2-\mathrm{lalg}}$. If furthermore $\mathrm{Hom}_{\mathbb{G}}(W_1, W_2) = 0$, then the resulting surjection $V_{W_1-\mathrm{lalg}} \oplus V_{W_2-\mathrm{lalg}} \to V_{W_1 \oplus W_2-\mathrm{lalg}}$ is in fact an isomorphism.*

Proof. Obviously both $V_{W_1-\mathrm{lalg}}$ and $V_{W_2-\mathrm{lalg}}$ are contained in $V_{W_1 \oplus W_2-\mathrm{lalg}}$. Furthermore, the definition immediately implies that any element of $V_{W_1 \oplus W_2-\mathrm{lalg}}$ may be written as the sum of an element of $V_{W_1-\mathrm{lalg}}$ and an element of $V_{W_2-\mathrm{lalg}}$. Thus the image of $V_{W_1-\mathrm{lalg}} \oplus V_{W_2-\mathrm{lalg}}$ is equal to $V_{W_1 \oplus W_2-\mathrm{lalg}}$.

Now suppose that $\mathrm{Hom}_{\mathbb{G}}(W_1, W_2) = 0$. We must show that the intersection $V_{W_1-\mathrm{lalg}} \bigcap V_{W_2-\mathrm{lalg}}$ is equal to zero. If v is a vector lying in this intersection then we may find open subgroups H_i of G and H_i-equivariant maps $\phi_i : W_i^{n_i} \to V$ (for $i = 1, 2$) that contain v in their image. Let $H = H_1 \bigcap H_2$, and let U denote the H-invariant subspace of V spanned by v. By our choice of H, U is an H-equivariant subquotient of both $W_1^{n_1}$ and $W_2^{n_2}$. Since the category $\mathcal{R}$ is semi-simple, we may find both an H-equivariant surjection $W_1^{n_1} \to U$ and an H-equivariant embedding $U \to W_2^{n_2}$. Since $\mathrm{Hom}_H(W_1, W_2) = \mathrm{Hom}_{\mathbb{G}}(W_1, W_2) = 0$ (by assumption, together with the fact that H is Zariski dense), we conclude that $U = 0$, hence that $v = 0$, and hence that $V_{W_1-\mathrm{lalg}} \bigcap V_{W_2-\mathrm{lalg}} = 0$. $\square$

Proposition-Definition 4.2.6. *Let V be a K-vector space equipped with an action of G.*

(i) *We say that a vector $v \in V$ is locally algebraic if it is W-locally algebraic for some object W of $\mathcal{R}$.*

(ii) *The set of all locally algebraic vectors of V is a G-invariant subspace of V, which we by denote V_{lalg}.*

(iii) *We say that V that is a locally algebraic representation of G if every vector of V is locally algebraic; that is, if $V_{\mathrm{lalg}} = V$.*

Proof. That V_{lalg} is a subspace of V follows from proposition 4.2.5. $\square$

The formation of V_{lalg} is covariantly functorial in V.

The class of locally algebraic representations of G is introduced in the appendix [18] to [24]. We will see below that our definition of this notion coincides with that of [18] (when $\mathbb{G}$ is reductive, as we are assuming here).

Let $\hat{\mathbb{G}}$ denote a set of isomorphism class representatives for the irreducible objects of $\mathcal{R}$.

Corollary 4.2.7. *If V is equipped with an action of G, then the natural map $\bigoplus_{W\in\hat{\mathbb{G}}} V_{W-\mathrm{lalg}} \to V_{\mathrm{lalg}}$ is an isomorphism.*

Proof. Since $\mathcal{R}$ is a semi-simple abelian category, any object W of $\mathcal{R}$ is isomorphic to a direct sum of powers of elements of $\hat{\mathbb{G}}$. Given this, the result follows immediately from proposition 4.2.5. $\square$

Proposition 4.2.8. *Let V be an irreducible locally algebraic representation of G. Then there is an element W of $\hat{G}$, and, writing $B = \mathrm{End}_{\mathbb{G}}(W)$, an irreducible smooth representation of G on a right B-module U (here irreducibility is as a representation over B), such that V is isomorphic to the tensor product $U \otimes_B W$. Conversely, given such a W and U, the tensor product $U \otimes_B W$ is an irreducible locally algebraic representation of G.*

Proof. Suppose first that V is an irreducible locally algebraic representation of G. Then corollary 4.2.7 shows that $V = V_{W-\mathrm{lalg}}$ for some element W of $\hat{\mathbb{G}}$. Proposition 4.2.4 then yields an isomorphism $\mathrm{Hom}(W, V)_{\mathrm{sm}} \otimes_B W \xrightarrow{\sim} V$. This shows that V is isomorphic to the tensor product over B of a smooth representation and W. Clearly $\mathrm{Hom}(W, V)_{\mathrm{sm}}$ must be irreducible, since V is.

Conversely, given such an element W of $\hat{\mathbb{G}}$, and an irreducible smooth representation on a B-module U, consider the tensor product $V = U \otimes_B W$. We obtain a natural isomorphism of B-modules $U \xrightarrow{\sim} \mathrm{Hom}(W, V)_{\mathrm{sm}}$, and so the irreducibility of U implies the irreducibility of V. $\square$

The following result is an analogue, for locally algebraic representations, of proposition 4.1.14.

Corollary 4.2.9. *Suppose that G is compact. If V is a locally algebraic representation of G, then V is semi-simple as a representation of G, and each irreducible summand of V is finite dimensional.*

Proof. Corollary 4.2.7 shows that V is isomorphic to the direct sum $\bigoplus_{W\in\hat{\mathbb{G}}} V_{W-\mathrm{lalg}}$, and so we may restrict our attention to V being W-algebraic, for some irreducible algebraic representation W of $\mathbb{G}$. Let $B = \mathrm{End}_{\mathbb{G}}(W)$. Proposition 4.2.4 shows that the natural map $\mathrm{Hom}(W, V)_{\mathrm{sm}} \otimes_B W \to V$ is an isomorphism. Since $\mathrm{Hom}(W, V)_{\mathrm{sm}}$ is a smooth representation of the compact group G, lemma 4.1.2 shows that it is a direct sum of finite dimensional irreducible representations of G on a B-module. Thus it suffices to show that for any finite dimensional irreducible smooth representation S of G on a B-module, the tensor product representation $S \otimes_B W$ is semi-simple. In fact, proposition 4.2.8 shows that such a representation is irreducible. $\square$

The preceding corollary shows that our definition of locally algebraic representation agrees with that of [18]. Indeed, the condition of definition 4.2.6 (iii) amounts to condition (2) of the definition of [18], and corollary 4.2.9 shows that condition (2) implies condition (1) of that definition. Our proposition 4.2.8 is thus a restatement of [18, thm. 1] (and our method of proof coincides with that of [18]). Note however that [18] does not consider the possibility that $\mathrm{End}_{\mathbb{G}}(W)$ might be larger than the ground field K.

Suppose now that V is a locally convex K-vector space of compact type equipped with a topological G-action, let W be an object of $\mathcal{R}$, and write $B = \mathrm{End}_G(W)$. The evaluation map $\mathrm{Hom}(W,V)_{\mathrm{sm}} \otimes_B W \to V$ of proposition 4.2.4 is then a continuous G-equivariant map (if we equip $\mathrm{Hom}(W,V)_{\mathrm{sm}}$ with its topology as a subspace of $\mathrm{Hom}(W,V)$), which that proposition shows to be a continuous bijection between its source and the subspace $V_{W-\mathrm{lalg}}$ of its target.

Proposition 4.2.10. *If V is a locally convex K-vector space of compact type equipped with a locally analytic representation of G, then for any $W \in \mathcal{R}$, the evaluation map of proposition* 4.2.4 *is a closed embedding, and induces a topological isomorphism between its source and $V_{W-\mathrm{lalg}}$ (which is thus a closed subspace of V).*

Proof. Since V is a locally analytic representation of G, the space $\mathrm{Hom}(W,V) \xrightarrow{\sim} V \otimes_K \check{W}$ is again a locally analytic representation of G, by corollary 3.6.16. Thus $\mathrm{Hom}(W,V)_{\mathrm{sm}} = \mathrm{Hom}(W,V)^{\mathfrak{g}}$ (the equality holding by corollary 4.1.7) is a closed subrepresentation of $\mathrm{Hom}(W,V)$, and so $\mathrm{Hom}(W,V)_{\mathrm{sm}} \otimes_B W$ is a closed subrepresentation of $V \otimes_K \check{W} \otimes_B W$. The closed embedding $\mathrm{Hom}(W,V)_{\mathrm{sm}} \otimes_B W \to V \otimes_K \check{W} \otimes_B W$ has image equal to $(V \otimes_K \check{W} \otimes_B W)^{\Delta_{1,2}(\mathfrak{g})} \xrightarrow{\sim} (V \otimes_K \check{A})^{\Delta_{1,2}(\mathfrak{g})}$ (the isomorphism being given by proposition 4.2.3 (i)). We claim that the evaluation map

$$(V \otimes_K \check{A})^{\Delta_{1,2}(\mathfrak{g})} \to V \tag{4.2.11}$$

is a closed embedding. Once this is known, the proposition follows from proposition 4.2.4.

Since V is a locally analytic representation on a space of compact type, we may write $V \xrightarrow{\sim} \varinjlim_n V_n$, where each V_n is a Banach space equipped with a $\mathbb{H}_n$-analytic representation, for some open analytic subgroup H_n of G, and the transition maps are compact and injective. Thus (by corollary 3.6.7 and lemma 4.1.4) we obtain an isomorphism

$$(V \otimes_K \check{A})^{\Delta_{1,2}(\mathfrak{g})} \xrightarrow{\sim} \varinjlim_n (V_n \otimes_K \check{A})^{\Delta_{1,2}(H_n)},$$

again with compact and injective transition maps. The elements of $\check{A}$, when regarded as functions on $\mathbb{G}$, are algebraic, and so in particular, they restrict to analytic functions on each $\mathbb{H}_n$. Thus for each n we have a closed embedding $\check{A} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_n, K)$, and hence a closed embedding

$$(V_n \otimes_K \check{A})^{\Delta_{1,2}(H_n)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_n, V_n)^{\Delta_{1,2}(H_n)} \xrightarrow{\sim} V_n,$$

the isomorphism being provided by the evaluation map (and the fact that V_n is an analytic $\mathbb{H}_n$-representation). Since the transition maps for varying n on either side of this closed embedding are compact and injective, and since

$$(V_{n+1} \otimes_K \check{A})^{\Delta_{1,2}(H_{n+1})} \bigcap V_n = (V_n \otimes_K \check{A})^{\Delta_{1,2}(H_n)}$$

(as is easily seen), Proposition 1.1.41 shows that the map obtained after passing to the locally convex inductive limit in n is again a closed embedding. Thus the evaluation map (4.2.11) is a closed embedding, as claimed. $\square$

In the situation of the preceding proposition, one finds in particular that for each $W \in \hat{\mathbb{G}}$, the space $V_{W-\mathrm{lalg}}$ is again a locally analytic representation of G on a space of compact type. Hence the direct sum $\bigoplus_{W\in\hat{\mathbb{G}}} V_{W-\mathrm{lalg}}$ (equipped with its locally convex direct sum topology) is again a locally analytic G-representation on a convex K-vector space of compact type. The isomorphism of corollary 4.2.7 thus allows us to regard V_{lalg} as a locally analytic G-representation on a convex K-vector space of compact type. By construction, the natural map $V_{\mathrm{lalg}} \to V$ is continuous.

Note that V_{lalg} is contained in $V_{\mathfrak{g}-\mathrm{lf}}$. If $\mathbb{G}$ is semi-simple and simply connected, then the proof of [24, prop. 3.2] shows that in fact there is equality $V_{\mathrm{lalg}} = V_{\mathfrak{g}-\mathrm{lf}}$. However, the example of [24, p. 120] shows that this is not true for arbitrary reductive groups $\mathbb{G}$.

Chapter 5. Rings of distributions

5.1. Frobenius reciprocity and group rings of distributions

In this section we describe the convolution structure on the space of distributions on a group (whether topological, rigid analytic, or locally analytic), and those modules over the ring of distributions that can be constructed out of an appropriate representation of the group. These module structures can be related to some simple forms of Frobenius reciprocity, which we also recall.

Let G be a group acting on a K-vector space V. If W is a second vector space then the function space $\mathcal{F}(G,W)$ is endowed with its commuting left and right regular G-actions, and the space $\mathrm{Hom}_G(V,\mathcal{F}(G,W))$ (defined by regarding $\mathcal{F}(G,W)$ as a G-representation via the right regular G-action) is equipped with a G-action (via the left regular G-action on $\mathcal{F}(G,W)$). We will repeatedly apply this observation below, and the obvious variants in which G is a topological, rigid analytic, or locally analytic group, V and W are convex spaces, $\mathcal{F}(G,W)$ is replaced by $\mathcal{C}(G,W)$, $\mathcal{C}^{\mathrm{an}}(\mathbb{G},W)$, or $\mathcal{C}^{\mathrm{la}}(G,W)$ as appropriate, and Hom is replaced by $\mathcal{L}$.

The following result gives the archetypal form of Frobenius reciprocity for continuous, analytic, and locally analytic representation.

Proposition 5.1.1. (i) *Let G be a locally compact topological group and let V be a Hausdorff convex K-vector space equipped with a continuous action of G. If W is any Hausdorff convex K-vector space then the map $\mathrm{ev}_e : \mathcal{C}(G,W) \to W$ induces a G-equivariant topological isomorphism of convex spaces $\mathcal{L}_{G,b}(V,\mathcal{C}(G,W)) \xrightarrow{\sim} \mathcal{L}_b(V,W)$.*

(ii) *Let $\mathbb{G}$ be an affinoid rigid analytic group over L such that $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$ and let V be a Hausdorff convex LF-space equipped with an analytic G-representation. If W is any Hausdorff convex space then the map $\mathrm{ev}_e : \mathcal{C}(G,W) \to W$ induces a continuous G-equivariant bijection $\mathcal{L}_{G,b}(V,\mathcal{C}^{\mathrm{an}}(\mathbb{G},W)) \to \mathcal{L}_b(V,W)$. If V and W are K-Fréchet spaces then this map is even a topological isomorphism.*

(iii) *Let G be a compact locally L-analytic group and let V be a Hausdorff convex LF-space equipped with a locally analytic G-representation. If W is any Hausdorff convex K-vector space then the map $\mathrm{ev}_e : \mathcal{C}(G,W) \to W$ induces a G-equivariant continuous bijection $\mathcal{L}_{G,b}(V,\mathcal{C}^{\mathrm{la}}(G,W)) \to \mathcal{L}_b(V,W)$. If V and W are of compact type then this map is even a topological isomorphism.*

Proof. We prove each part in turn. Thus we first suppose that G is a locally compact topological group, that V is a Hausdorff convex space equipped with a continuous G-action, and that W is an arbitrary Hausdorff convex space.

Proposition 2.1.7 shows that the natural map $\mathcal{L}_b(V,W) \to \mathcal{L}_b(\mathcal{C}(G,V),\mathcal{C}(G,W))$ is continuous. Note that its image lies in $\mathcal{L}_{G,b}(\mathcal{C}(G,V),\mathcal{C}(G,V))$. Proposition 3.2.10 shows that $o : V \to \mathcal{C}(G,V)$ is continuous and G-equivariant, and so induces a continuous map $\mathcal{L}_{G,b}(\mathcal{C}(G,V),\mathcal{C}(G,W)) \to \mathcal{L}_{G,b}(V,\mathcal{C}(G,W))$. Composing this with the preceding map yields a continuous map $\mathcal{L}_b(V,W) \to \mathcal{L}_{G,b}(V,\mathcal{C}(G,W))$. It is easily checked that this is inverse to the continuous map $\mathcal{L}_{G,b}(V,\mathcal{C}(G,W)) \to \mathcal{L}_b(V,W)$ induced by ev_e, and so part (i) of the proposition is proved.

The proof of part (ii) is similar. The functoriality of the construction of $\mathcal{C}^{\mathrm{an}}(\mathbb{G},-)$ yields a natural map of abstract K-vector spaces

$$\mathcal{L}_b(V,W) \to \mathcal{L}_{G,b}(\mathcal{C}^{\mathrm{an}}(\mathbb{G},V),\mathcal{C}^{\mathrm{an}}(\mathbb{G},W)). \tag{5.1.2}$$

Since V is an LF-space, theorem 3.6.3 shows that the orbit map $o : V \to \mathcal{C}^{\mathrm{an}}(\mathbb{G},V)$ is a continuous embedding, and combining this with (5.1.2) we obtain a K-linear map $\mathcal{L}_b(V,W) \to \mathcal{L}_{G,b}(V,\mathcal{C}^{\mathrm{an}}(\mathbb{G},W))$, which is easily checked to provide a K-linear inverse to the continuous map $\mathcal{L}_{G,b}(V,\mathcal{C}(G,W)) \to \mathcal{L}_b(V,W)$ induced by ev_e. Thus this latter map is a continuous bijection, as claimed. If V and W are Fréchet spaces then the map (5.1.2) is continuous, by proposition 2.1.24, and so in this case we even obtain a topological isomorphism. This completes the proof of part (ii).

The proof of part (iii) is similar again. One uses theorem 3.6.12 and proposition 2.1.31. $\square$

Let us put ourselves in the situation of part (i) of the preceding proposition, with $W = K$. Proposition 1.1.36 shows that passing to the transpose yields a topological embedding

$$\mathcal{L}_{G,b}(V,\mathcal{C}(G,K)) \to \mathcal{L}_{G,b}(\mathcal{D}(G,K)_b,V_b'). \tag{5.1.3}$$

Thus part (i) yields a topological embedding $V_b' \to \mathcal{L}_{G,b}(\mathcal{D}(G,K)_b,V_b')$, and hence a map

$$\mathcal{D}(G,K)_b \times V_b' \to V_b', \tag{5.1.4}$$

that is G-equivariant in the first variable, and separately continuous. If $\mu \in \mathcal{D}(G,K)$ and $v' \in V'$ we let $\mu * v'$ denote the image of the pair (μ, v'). Tracing through the definitions shows that for any element $v \in V$ we have

$$\langle \mu * v', v\rangle = \int_G \langle v', gv\rangle \, d\,\mu(g). \tag{5.1.5}$$

In particular if $g \in G$ then $\delta_g * v' = g^{-1}v'$.

If we take V to be $\mathcal{C}(G,K)$, we obtain as a special case of (5.1.4) a map

$$\mathcal{D}(G,K)_b \times \mathcal{D}(G,K)_b \to \mathcal{D}(G,K)_b,$$

which we denote by $(\mu,\nu) \mapsto \mu * \nu$, which has the property that $\delta_g \times \delta_h \mapsto \delta_{hg}$. As a special case of (5.1.5) we find that

$$\int_G f(g)\, d\,(\mu * \nu)(g) = \int_G \left(\int_G f(hg)\, d\,\mu(h)\right) d\,\nu(g). \tag{5.1.6}$$

Corollary 5.1.7. *Suppose that G is a locally compact topological group.*

(i) *The formula* (5.1.6) *defines an associative product on $\mathcal{D}(G,K)_b$, which is separately continuous in each of its variables. If G is compact then it is even jointly continuous.*

(ii) *If V is a Hausdorff convex space equipped with a continuous G-action then the map* (5.1.4) *makes V_b' a left $\mathcal{D}(G,K)_b$-module, and this map is separately continuous in each of its variables. If G is compact then it is jointly continuous.*

(iii) *If V is as in* (ii)*, then there is a natural isomorphism of K-vector spaces $\mathrm{Hom}_{\mathcal{D}(G,K)}(\mathcal{D}(G,K),V') \xrightarrow{\sim} \mathcal{L}_G(V,\mathcal{C}(G,K))$.*

Proof. We have already observed that the statements concerning separate continuity are consequences of part (i) of proposition 5.1.1. The associativity statement of part (i) can be checked directly from the formula (5.1.6), and a similarly direct calculation with the formula (5.1.5) shows that (5.1.4) makes V_b' a left $\mathcal{D}(G,K)$-module.

If G is compact then $\mathcal{D}(G,K)_b$ is a Banach space. Since the product (5.1.4) is defined by the continuous map (5.1.3), we see that given any bounded subset A of $\mathcal{D}(G,K)$, and any open subset U_1 of V_b', there is an open subset U_2 of V_b' such that $A\times U_2$ maps into U_1 under (5.1.4). Since $\mathcal{D}(G,K)$ is a Banach space, it has a neighbourhood basis of the origin consisting of bounded subsets, and thus (5.1.4) is actually jointly continuous. This completes the proof of parts (i) and (ii).

To prove part (iii), note that tautologically $\mathrm{Hom}_{\mathcal{D}(G,K)}(\mathcal{D}(G,K),V') \xrightarrow{\sim} V'$, and that by part (i) of proposition 5.1.1 there is a K-linear isomorphism $V' = \mathcal{L}(V,K) \xrightarrow{\sim} \mathcal{L}_G(V,\mathcal{C}(G,K))$. $\square$

In the case where K is local and V is a Banach space, this result almost coincides with the construction of [25,§2]. The one difference is that we work with the strong topology on our dual spaces, while in [25] the authors work with the bounded weak topology on these dual spaces.

Analogous results hold in the situations of parts (ii) and (iii) of proposition 5.1.1.

Corollary 5.1.8. *Suppose that $\mathbb{G}$ is an affinoid rigid analytic group over L such that $G := \mathbb{G}(L)$ is dense in G.*

(i) *Formula* (5.1.6) *defines an associative product on $\mathcal{D}^{\mathrm{an}}(\mathbb{G},K)_b$ which is jointly continuous as a map $\mathcal{D}^{\mathrm{an}}(\mathbb{G},K)_b \times \mathcal{D}^{\mathrm{an}}(\mathbb{G},K)_b \to \mathcal{D}^{\mathrm{an}}(\mathbb{G},K)_b$.*

(ii) *If we are given an analytic G-representation on a Hausdorff convex LF-space V, then formula* (5.1.5) *makes V_b' a left $\mathcal{D}^{\mathrm{an}}(\mathbb{G},K)$-module. The resulting map*

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G},K)_b \times V_b' \to V_b'$$

is continuous in its first variable. If V is a Fréchet space then it is jointly continuous.

(iii) *If V is as in* (ii)*, then there is a natural isomorphism of K-vector spaces $\mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(G,K)}(\mathcal{D}^{\mathrm{an}}(G,K),V') \xrightarrow{\sim} \mathcal{L}_G(V,\mathcal{C}^{\mathrm{an}}(G,K))$.*

Proof. The proof of this result follows the same lines as that of corollary 5.1.7, using part (ii) of proposition 5.1.1. In order to get the joint continuity statements of (i) and (ii), one should take into account that $\mathcal{D}^{\mathrm{an}}(\mathbb{G},K)_b$ is a Banach space, being the strong dual of the Banach space $\mathcal{C}^{\mathrm{an}}(\mathbb{G},K)$. $\square$

Corollary 5.1.9. *Suppose that G is a locally L-analytic group.*

(i) *Formula* (5.1.6) *defines an associative product on $\mathcal{D}^{\mathrm{la}}(G,K)_b$ which is separately continuous as a map $\mathcal{D}^{\mathrm{la}}(G,K)_b \times \mathcal{D}^{\mathrm{la}}(G,K)_b \to \mathcal{D}^{\mathrm{la}}(G,K)_b$. If G is compact then it is even jointly continuous.*

(ii) *If we are given a locally analytic representation of G on a Hausdorff convex LF-space V, then formula* (5.1.5) *makes V_b' a left $\mathcal{D}^{\mathrm{la}}(G,K)$-module. The resulting map*

$$\mathcal{D}^{\mathrm{la}}(G,K)_b \times V_b' \to V_b'$$

is continuous in its first variable. If V is of compact type then it is separately continuous in each variable, and if furthermore G is compact then it is jointly continuous.

(iii) *If V is as in* (ii)*, then there is a natural isomorphism of K-vector spaces* $\mathrm{Hom}_{\mathcal{D}^{\mathrm{la}}(G,K)}(\mathcal{D}^{\mathrm{la}}(G,K),V') \xrightarrow{\sim} \mathcal{L}_G(V,\mathcal{C}^{\mathrm{la}}(G,K))$.

Proof. Suppose first that G is compact. The proof of this result follows the same lines as that of corollary 5.1.7, using part (iii) of proposition 5.1.1. In order to get the joint continuity statements of (i) and (ii), one takes into account the facts that because $\mathcal{C}^{\mathrm{la}}(G,K)$ is of compact type its strong dual $\mathcal{D}^{\mathrm{la}}(G,K)_b$ is a Fréchet space, that if V is of compact type then V_b' is a Fréchet space, and that a separately continuous map on a product of Fréchet spaces is necessarily jointly continuous.

If G is not compact, one can still check that (5.1.5) and (5.1.6) define an associative product and a left module structure respectively. In order to see the required continuity properties, choose a compact open subgroup H of G, and let $G = \coprod g_i H$ be a decomposition of G into right H-cosets. Then $\mathcal{D}^{\mathrm{la}}(G)_b \xrightarrow{\sim} \bigoplus \mathcal{D}^{\mathrm{la}}(g_iH)_b = \bigoplus \mathcal{D}^{\mathrm{la}}(H)_b * \delta_{g_i}$. Now right multiplication by δ_{g_i} acts as a topological automorphism of $\mathcal{D}^{\mathrm{la}}(G)$ (induced by the automorphism of G given by left multiplication by g_i), and so we see that separate continuity of the multiplication on $\mathcal{D}^{\mathrm{la}}(G)_b$ follows from the joint continuity of the multiplication on $\mathcal{D}^{\mathrm{la}}(H)_b$. Similarly, if V is an LF-space equipped with a locally analytic G-action, then the multiplication by δ_{g_i} is an automorphism of V_b' (it is just given by the contragredient action of g_i^{-1} on V_b'), and so the continuity properties of the $\mathcal{D}^{\mathrm{la}}(G)_b$ action on V_b' can be deduced from the continuity properties of the $\mathcal{D}^{\mathrm{la}}(H)_b$-action. Thus parts (i) and (ii) are proved for arbitrary G.

The proof of part (iii) proceeds by a similar reduction. One notes that

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{la}}(G,K)}(\mathcal{D}^{\mathrm{la}}(G,K),V') \xrightarrow{\sim} V' \xrightarrow{\sim} \mathcal{L}_H(V,\mathcal{C}^{\mathrm{la}}(H,K)) \xrightarrow{\sim} \mathcal{L}_G(V,\mathcal{C}^{\mathrm{la}}(G,K)),$$

where the last isomorphism can be deduced immediately using the decomposition $G = \coprod g_i H$ and the corresponding isomorphism $\mathcal{C}^{\mathrm{la}}(G,K) \xrightarrow{\sim} \prod \mathcal{C}^{\mathrm{la}}(g_iH,K)$. □

Part (i) of the preceding result is originally due to Féaux de Lacroix [11, 4.2.1], and is recalled in [23, §2]. We have included a proof here, since the work of de Lacroix is unpublished. The case of part (ii) when V is of compact type is originally due to Schneider and Teitelbaum [23, cor. 3.3].

There are alternative approaches to obtaining a multiplicative structure on the distribution spaces $\mathcal{D}(G,K)$, $\mathcal{D}^{\mathrm{an}}(\mathbb{G},K)$ and $\mathcal{D}^{\mathrm{la}}(G,K)$. For example, if G is a compact topological group one can use the isomorphism between $\mathcal{C}(G,K)\hat{\otimes}\mathcal{C}(G,K)$ and $\mathcal{C}(G\times G,K)$ to obtain a coproduct on $\mathcal{C}(G,K)$, which dualizes to yield the product on $\mathcal{D}(G,K)$, and similar approaches are possible in the analytic or locally analytic cases. (This is the approach to part (i) of corollary 5.1.9 explained in [23, §2].)

Similarly, one can obtain the module structure on V' in other ways. For example, if V is a Hausdorff convex K-vector space equipped with a continuous action of the topological group G, then thinking of $\mathcal{D}(G,K)$ as a group ring, it is natural to attempt to extend the contragredient G-action on V' to a left $\mathcal{D}(G,K)$-module structure. This is the approach adopted in [25], and in [23, §3] in the locally analytic situation.

We also remark that the product structures we have obtained on our distribution spaces are opposite to the ones usually considered. (For example, $\delta_g * \delta_h = \delta_{hg}$, rather than δ_{gh}, and $\delta_g * v' = g^{-1}v'$ rather than gv'.) If we compose these structures with the anti-involution obtained by sending $g \mapsto g^{-1}$, then we obtain the usual product structures.

Let us recall from [23] the following converse to corollary 5.1.9 (ii).

Proposition 5.1.10. *If V is a space of compact type, and V'_b is equipped with a $\mathcal{D}^{\mathrm{la}}(G)$-module structure for which the corresponding product map $\mathcal{D}^{\mathrm{la}}(G) \times V'_b \to V'_b$ is separately continuous, then the G-action on V obtained by passing to the transpose makes V a locally analytic representation of G.*

Proof. This is [23, cor. 3.3]. We recall a proof. We may replace G by a compact open subgroup, and thus assume that G is compact. Then $\mathcal{D}^{\mathrm{la}}(G)$ and V'_b are both nuclear Fréchet spaces, and the product map $\mathcal{D}^{\mathrm{la}}(G) \times V'_b \to V'_b$ induces a surjection $\mathcal{D}^{\mathrm{la}}(G) \hat{\otimes}_K V'_b \to V'_b$. Proposition 1.1.32 shows that the source is a nuclear Fréchet space, and so this surjection is strict, by the open mapping theorem. Passing to duals, and taking into account propositions 1.1.32 and 2.1.28, we obtain a closed embedding $V \to \mathcal{C}^{\mathrm{la}}(G,V)$, which is G-equivariant, with respect to the transposed G-action on V and the right regular G-action on $\mathcal{C}^{\mathrm{la}}(G,V)$. Propositions 3.5.11 and 3.6.14 show that V is a locally analytic representation of G. □

We have the following application of Frobenius reciprocity for locally analytic representations.

Proposition 5.1.11. *If V is a K-Banach space equipped with a locally analytic action of the locally L-analytic group G then the contragredient action of G on V'_b is again locally analytic.*

Proof. We may and do assume that G is compact. Frobenius reciprocity (more precisely, proposition 5.1.1 (iii), with $W = K$) yields a G-equivariant isomorphism of abstract K-vector spaces

$$V' \xrightarrow{\sim} \mathcal{L}_G(V, \mathcal{C}^{\mathrm{la}}(G,K)). \tag{5.1.12}$$

Since $\mathcal{C}^{\mathrm{la}}(G,V)$ is reflexive, passing to the transpose yields a G-equivariant isomorphism

$$\mathcal{L}_G(V, \mathcal{C}^{\mathrm{la}}(G,K)) \xrightarrow{\sim} \mathcal{L}_G(\mathcal{D}^{\mathrm{la}}(G,K)_b, V'_b). \tag{5.1.13}$$

Proposition 2.2.10 yields a natural isomorphism $\mathcal{L}(\mathcal{D}^{\mathrm{la}}(G,K)_b, V'_b) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(G, V'_b)$, and hence a G-equivariant isomorphism (of abstract K-vector spaces)

$$\mathcal{L}_G(\mathcal{D}^{\mathrm{la}}(G,K)_b, V'_b) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(G, V'_b)^{\Delta_{1,2}(G)} \xrightarrow{\sim} (V'_b)_{\mathrm{la}}. \tag{5.1.14}$$

Working through the definitions, one checks that the composite of (5.1.12), (5.1.13), and (5.1.14) provides a K-linear inverse to the natural map $(V_b')_{\mathrm{la}} \to V_b'$, and thus that this natural map is an isomorphism of abstract K-vector spaces. Thus V_b' is equipped with a locally analytic representation of G, as claimed. $\square$

A version of this result is proved in the course of proving [22, prop. 3.8].

We remark that in the context of the preceding proposition, corollary 3.6.13 guarantees that there is an analytic open subgroup H of G such that V is $\mathbb{H}$-analytic. However, it is not the case in general that V_b' is also $\mathbb{H}$-analytic (although again by corollary 3.6.13 it will be $\mathbb{H}'$-analytic for some analytic open subgroup H' of H).

We can use proposition 5.1.11 to give an example of a non-trivial locally analytic representation on a complete LB-space that is neither of compact type nor a Banach space. For this, let G be a locally L-analytic group which is σ-compact but not compact, let H be an analytic open subgroup of G, and let $G = \coprod_i Hg_i$ be the decomposition of G into (countably many) left H cosets. The right multiplication of $g_i^{-1}Hg_i$ on Hg_i makes $\mathcal{C}^{\mathrm{an}}(\mathbb{H}g_i)$ an analytic $g_i^{-1}\mathbb{H}g_i$-representation on a Banach space, and so by proposition 5.1.11 the contragredient representation on $\mathcal{D}^{\mathrm{an}}(\mathbb{H}g_i, K)_b$ is a locally analytic $g_i^{-1}Hg_i$-representation. The right multiplication of G on itself induces a natural right G-action on the countable direct sum $\bigoplus_i \mathcal{D}^{\mathrm{an}}(\mathbb{H}g_i, K)_b$, extending the $g_i^{-1}Hg_i$-action on the i^{th} summand, and thus (by proposition 3.6.17) this direct sum is a locally analytic G-representation. Since it is a direct sum of a countable collection of Banach spaces it is a complete LB-space [5, prop. 9, p. II.32]. It is not of compact type.

Theorem 5.1.15. (i) *Let G be a compact topological group and let V be a Hausdorff LB-space equipped with a continuous G-action. If we equip V' with the $\mathcal{D}(G, K)$-action of corollary* 5.1.7, *then any surjection of left $\mathcal{D}(G, K)$-modules $\mathcal{D}(G, K)^n \to V'$ (for any natural number n) is obtained by dualizing a closed G-equivariant embedding $V \to \mathcal{C}(G, K)^n$. In particular, V' is finitely generated as a left $\mathcal{D}(G, K)$-module if and only if V admits a closed G-equivariant embedding into $\mathcal{C}(G, K)^n$ for some natural number n.*

(ii) *Let $\mathbb{G}$ be an affinoid rigid analytic group defined over L, and assume that $G := \mathbb{G}(L)$ is Zariski dense in $\mathbb{G}$. Let V be a Hausdorff LB-space equipped with a $\mathbb{G}$-analytic G-action. If we equip V' with the $\mathcal{D}^{\mathrm{an}}(\mathbb{G}, K)$-action of corollary* 5.1.8 *then any surjection of left $\mathcal{D}^{\mathrm{an}}(\mathbb{G}, K)$-modules $\mathcal{D}^{\mathrm{an}}(\mathbb{G}, K)^n \to V'$ for some natural number n arises by dualizing a closed G-equivariant embedding $V \to \mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)^n$. In particular, V' is finitely generated as a left $\mathcal{D}^{\mathrm{an}}(\mathbb{G}, K)$-module if and only if V admits a closed G-equivariant embedding into $\mathcal{C}^{\mathrm{an}}(\mathbb{G}, K)^n$ for some natural number n.*

(iii) *Let G be a compact locally L-analytic group, and let V be a complete Hausdorff LB-space equipped with a locally analytic G-action. If we equip V' with the $\mathcal{D}^{\mathrm{la}}(G, K)$-action of corollary* 5.1.9, *then any surjection $\mathcal{D}^{\mathrm{la}}(G, K)^n \to V'$ of left $\mathcal{D}^{\mathrm{la}}(G, K)$-modules, for some natural number n, arises by dualizing a closed G-equivariant embedding $V \to \mathcal{C}^{\mathrm{la}}(G, K)^n$. In particular, V' is finitely generated as a left $\mathcal{D}^{\mathrm{la}}(G, K)$-module if and only if V admits a closed G-equivariant embedding into $\mathcal{C}^{\mathrm{la}}(G, K)$ for some natural number n.*

Proof. Suppose first that G and V are as in (i), and that we are given a surjection of left $\mathcal{D}(G,K)$-modules

$$\mathcal{D}(G,K)_b^n \to V_b'. \tag{5.1.16}$$

Since the map $\mathcal{D}(G,K)_b^n \times V_b' \to V_b'$ describing V_b' as a left $\mathcal{D}(G,K)_b^n$-module is continuous in its first variable, the surjection (5.1.16) is necessarily continuous.

Write $V = \varinjlim_n V_n$ as the locally convex inductive limit of a sequence of Banach spaces. Dualizing we obtain a continuous bijection

$$V_b' \to \varprojlim_n (V_n)_b'. \tag{5.1.17}$$

Since each V_n is a Banach space, so is each $(V_n)_b'$, and so $\varprojlim_n (V_n)_b'$ is the projective limit of a sequence of Banach spaces, and hence is a Fréchet space. The composite of (5.1.16) and (5.1.17) is a surjection whose source and target are Fréchet spaces, and so by the open mapping theorem it is strict. Thus each map in this composite must itself be strict.

Dualizing the strict surjection (5.1.16) yields a closed G-equivariant embedding

$$(V_b')' \to (\mathcal{D}(G,K)_b^n)' \xrightarrow{\sim} ((\mathcal{C}(G,K)_b')_b')^n. \tag{5.1.18}$$

Let $\imath_i : \mathcal{D}(G,K)_b \to V_b'$ denote the i^{th} component of (5.1.16), where $1 \leq i \leq n$. By part (iii) of corollary 5.1.7 each $\imath_i$ is obtained by dualizing a continuous G-equivariant map $V \to \mathcal{C}(G,K)$. Taking the direct sum of these we obtain a continuous G-equivariant map

$$V \to \mathcal{C}(G,K)^n, \tag{5.1.19}$$

and our original surjection (5.1.16) is obtained by dualizing (5.1.19). We can embed the maps (5.1.18) and (5.1.19) into the following diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathcal{C}(G,K)^n \arrow[d] \\ (V_b')_b' \arrow[r] & ((\mathcal{C}(G,K)_b')_b')^n, \end{tikzcd}$$

in which the vertical arrows are double duality maps, which commutes by construction. The bottom arrow is a topological embedding, and since V and $\mathcal{C}(G,K)$ are both barrelled, so are the vertical arrows. Thus the top arrow is an embedding. It remains to prove that it has closed image. But since V is an LB-space by assumption, and normable, since it can be embedded in the Banach space $\mathcal{C}(G,K)^n$, proposition 1.1.18 implies that V is a Banach space, and so must have closed image in $\mathcal{C}(G,K)^n$.

Conversely, if we dualize a closed G-equivariant embedding $V \to \mathcal{C}(G,K)^n$, then we certainly obtain a surjection of left $\mathcal{D}(G,K)$-modules $\mathcal{D}(G,K)^n \to V'$. This proves part (i).

Parts (ii) and (iii) are proved in an identical fashion. (For part (iii), rather than appealing to proposition 1.1.18, we note that V is assumed to be both an LB-space and complete, and we use the fact that any complete — or equivalently, closed — subspace of a compact type space is again of compact type. Observe also that the last part of the proof simplifies somewhat, since $\mathcal{D}^{\mathrm{la}}(G,K)$ is reflexive.) $\square$

5.2. Completions of universal enveloping algebras

In this section we explain the relationship between the rings of analytic distributions on (a certain class of) analytic open subgroups of a locally L-analytic group, and certain completions of universal enveloping algebras.

We begin with some simple rigid analysis. For any integer d, rigid analytic affine d-space $\mathbb{A}^d$ represents the functor that attaches to any rigid analytic space X over L the set $\Gamma(X, \mathcal{O}_X)^d$ of d-tuples of globally defined rigid analytic functions on X. If r is any positive real number, then the closed ball of radius r centred at the origin (denoted by $\mathbb{B}_d(r)$) is an open subdomain of $\mathbb{A}^d$. It represents the functor that attaches to X the set $(\Gamma(X, \mathcal{O}_X)^{|\,|_{\sup} \leq r})^d$ consisting of d-tuples of globally defined rigid analytic functions on X, each of whose sup-norm is bounded above by r.

There is a coordinate-free version of these constructions. Fix a free finite rank $\mathcal{O}_L$-module M, of rank d say, and write $V = L \otimes_{\mathcal{O}_L} M$. The functor

$$X \mapsto \Gamma(X, \mathcal{O}_X) \otimes_L V$$

is represented by a rigid analytic space that we denote $\mathbb{A}(V)$. The L-valued points of $\mathbb{A}(V)$ are canonically identified with V, as is the tangent space to $\mathbb{A}(V)$ at any L-valued point of $\mathbb{A}(V)$.

If r is a positive real number, then the functor

$$X \mapsto \Gamma(X, \mathcal{O}_X)^{|\,|_{\sup} \leq r} \otimes_{\mathcal{O}_L} M$$

is represented by an open subdomain of $\mathbb{A}(V)$, that we denote by $\mathbb{B}(M, r)$. The L-valued points of $\mathbb{B}(M, 1)$ are canonically identified with M.

Let us return now to the consideration of a locally L-analytic group G. Suppose that $\mathfrak{h}$ is an $\mathcal{O}_L$-Lie sublattice of $\mathfrak{g}$, i.e. that $\mathfrak{h}$ is an $\mathcal{O}_L$-Lie subalgebra of $\mathfrak{g}$ that is free of finite rank as an $\mathcal{O}_L$-module, and such that the natural map $L \otimes_{\mathcal{O}_L} \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. If in fact $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$ for some element $a \in \mathcal{O}_L$ for which $\mathrm{ord}_L(a)$ is sufficiently large, then it is proved in [28, LG Ch. V, § 4] that we can use the Baker-Campbell-Hausdorff formula to define a rigid analytic group structure on $\mathbb{B}(\mathfrak{h}, 1)$, for which the induced Lie algebra structure on $\mathfrak{g}$ (thought of as the tangent space to the identity in $\mathbb{B}(\mathfrak{h}, 1)$) agrees with the given Lie algebra structure on $\mathfrak{g}$. We let $\mathbb{H}$ denote $\mathbb{B}(\mathfrak{h}, 1)$ equipped with this rigid analytic group structure, and as usual we let H denote the underlying locally analytic group. (Thus H and $\mathfrak{h}$ are equal as sets; $\mathfrak{h}$ denotes this set regarded as an $\mathcal{O}_L$-Lie algebra, while H denotes this set regarded as a locally L-analytic group.) For any real number $0 < r < 1$, the subdomain $\mathbb{B}(\mathfrak{h}, r)$ is in fact an open subgroup of $\mathbb{B}(\mathfrak{h}, 1)$; we denote this open subgroup by $\mathbb{H}_r$, and its underlying locally L-analytic group of L-rational points by H_r. (Here and below, let us make the convention that we only consider values of r that arise as the absolute values of elements in the algebraic closure of L. These elements are dense in the interval $(0, 1)$, and have the merit that the corresponding groups $\mathbb{H}_r$ are affinoid [3, thm. 6.1.5/4].)

If $\mathfrak{h}$ is any $\mathcal{O}_L$-Lie sublattice of $\mathfrak{g}$, then since by assumption $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, we see that if $a \in \mathcal{O}_L$, then $a\mathfrak{h}$ is again an $\mathcal{O}_L$-Lie subalgebra of $\mathfrak{g}$, and in fact $[a\mathfrak{h}, a\mathfrak{h}] \subset a(a\mathfrak{h})$. Thus choosing $a \in \mathcal{O}_L \setminus \{0\}$ with $\mathrm{ord}_L(a)$ sufficiently large, and replacing $\mathfrak{h}$ by $a\mathfrak{h}$ (more colloquially, "shrinking $\mathfrak{h}$ sufficiently"), we find that we may exponentiate $\mathfrak{h}$ to a locally L-analytic group H as in the preceding paragraph. It is proved in [28, LG 5.35, cor. 2] that if we make such a replacement, then, again provided that

$\mathrm{ord}_L(a)$ is sufficiently large, we may furthermore construct an embedding of locally L-analytic groups $\exp : H \to G$, and thus realize H as an analytic open subgroup of G. We refer to the analytic open subgroups of G constructed in this manner (i.e. by exponentiating sufficiently small $\mathcal{O}_L$-Lie sublattices of $\mathfrak{g}$) as good analytic open subgroups. The discussion of [28, LG Ch. IV, §8] shows that the set of good analytic open subgroups is cofinal in the directed set of all analytic open subgroups of G. In fact, if $\mathbb{H}$ is one good analytic open subgroup of G, corresponding to the $\mathcal{O}_L$-Lie sublattice $\mathfrak{h}$ of G, then for any $a \in \mathcal{O}_L$, we may exponentiate $a\mathfrak{h}$ to a good analytic open subgroup of G, and the resulting family of good analytic open subgroups of G is clearly cofinal.

Given such an analytic open subgroup H of G, if we fix an $\mathcal{O}_L$-basis $X_1, \ldots , X_n$ for $\mathfrak{h}$, then then we may define a rigid analytic map from $\mathbb{B}_d(1)$ to $\mathbb{H}$ via the formula

$$(t_1, \ldots , t_d) \mapsto \exp(t_1X_1) \cdots \exp(t_dX_d).$$

If $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$ with $\mathrm{ord}_L(a)$ sufficiently large, then this map will be a rigid analytic isomorphism; we then say that it equips $\mathbb{H}$ (or H) with canonical coordinates of the second kind. For any $0 < r < 1$, the canonical coordinates of the second kind restrict to an isomorphism between $\mathbb{B}_d(r)$ and $\mathbb{H}_r$.

Suppose now that $\mathbb{H}$ is a good analytic open subgroup of G equipped with canonical coordinates of the second kind $t_1, \ldots , t_d$, corresponding to an $\mathcal{O}_L$-basis $X_1, \ldots , X_n$ for $\mathfrak{h}$. In terms of these coordinates, we obtain the following description of the Banach space of K-valued rigid analytic functions on the affinoid group $\mathbb{H}_r$:

$$(5.2.1) \quad \mathcal{C}^{\mathrm{an}}(\mathbb{H}_r, K) \xrightarrow{\sim} \{ \sum_{I=(i_1,\ldots,i_d)} a_I t_1^{i_1} \cdots t_d^{i_d} \quad | \quad a_I \in K, \lim_{|I|\to\infty} |a_I| r^{|I|} = 0 \}.$$

(Here I runs over all multi-indices of length d, and $|I| = i_1 + \ldots + i_d$ for any multi-index $I = (i_1, \ldots , i_d)$.)

The right regular representation of H_r on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_r, K)$ makes this latter space an $\mathbb{H}_r$-analytic representation, and thus induces a continuous action of $U(\mathfrak{g})$ on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_r, K)$. Concretely, an element $X \in \mathfrak{g}$ acts on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_r, K)$ by the endomorphism

$$(Xf)(h) = \frac{d}{dt} f(h \exp(tX))_{|t=0}.$$

(See the discussion of [23, p. 449] — but note that we consider rigid analytic functions, rather than locally analytic functions, and consider $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_r, K)$ as an H_r-representation via the right regular action rather than the left regular action.) This in turn yields a morphism $U(\mathfrak{g}) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r, K)$, defined via

$$\langle X, f\rangle := (Xf)(e).$$

In particular, any monomial $X_1^{i_1} \cdots X_d^{i_d}$ yields an element of $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r, K)$. The elements $\dfrac{X_1^{i_1} \cdots X_d^{i_d}}{i_1! \cdots i_d!}$ are "dual" to the monomials $t_1^{i_1} \cdots t_d^{i_d}$; i.e. if $I = (i_1, \ldots , i_d)$ and $J = (j_1, \ldots , j_d)$ are two multi-indices, then

$$\langle \frac{X_1^{i_1} \cdots X_d^{i_d}}{i_1! \cdots i_d!}, t_1^{j_1} \cdots t_d^{j_d} \rangle = \begin{cases} 1 \text{ if I = J} \\ 0 \text{ otherwise,} \end{cases}$$

and thus the entire space $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r, K)$ admits the following description in terms of such monomials:

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r, K) \xrightarrow{\sim} \{ \sum_{I=(i_1,\ldots,i_d)} b_I \frac{X_1^{i_1}\cdots X_d^{i_d}}{i_1!\cdots i_d!} \quad | \quad |b_I| \leq Cr^{|I|} \text{ for some } C>0\}.$$

(Compare the discussion of [23, p. 450].) In fact this is an isomorphism not just of K-vector spaces, but of K-algebras. (Note that the K-algebra structure on $U(\mathfrak{g})$ yields a topological K-algebra structure on the space of series in the monomials $X_1^{i_1}\cdots X_d^{i_d}$ appearing on the right hand side of this isomorphism.)

Actually, it will be important for us to consider not only the affinoid groups $\mathbb{H}_r$, but also certain corresponding strictly σ-affinoid subgroups. For any value of r, we define $\mathbb{H}_r^\circ = \bigcup_{r'<r} \mathbb{H}_{r'}$. The open subspace $\mathbb{H}_r^\circ$ is an open subgroup of $\mathbb{H}$, isomorphic as a rigid analytic space to the open ball of radius r. We write $H_r^\circ := \mathbb{H}_r^\circ(L)$ to denote the corresponding locally L-analytic group of L-valued points. In the particular case when $r=1$, we write simply $\mathbb{H}^\circ$ and H° rather than $\mathbb{H}_1^\circ$ and H_1°.

The space $\mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ is defined as the projective limit $\varprojlim_{r<1} \mathcal{C}^{\mathrm{an}}(\mathbb{H}_r, K)$; it is a nuclear Fréchet space (since $\mathbb{H}^\circ$ is strictly σ-affinoid). Its strong dual space $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ is then isomorphic to the inductive limit $\varinjlim_{r<1} \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r, K)_b$. It is a space of compact type. Corollary 5.1.8 shows that each of the spaces $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r, K)_b$ is naturally a topological ring. Passing to the inductive limit, we obtain a ring structure on $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$, and we may conclude *a priori* that the multiplication map

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_K \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$$

is separately continuous. However, since $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ is of compact type, it follows from proposition 1.1.31 that this map is in fact jointly continuous.

The above explicit description of each of the Banach algebras $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$ yields the following explicit description of the inductive limit $\varprojlim_{r<1} \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$:

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K) \xrightarrow{\sim}$$
$$\{\sum_I b_I \frac{X_1^{i_1}\cdots X_d^{i_d}}{i_1!\cdots i_d!} \quad | \quad |b_I| \leq Cr^{|I|} \text{ for some } C>0 \text{ and some } r<1\}.$$

We will now give another description of the compact type algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ as an inductive limit of Banach algebras, via the technique of partial divided powers introduced in [1]. To this end, we first present some constructions related to the enveloping algebra $U(\mathfrak{g})$.

To begin with, fix an integer $m \geq 0$, and for $j = 1, \ldots, d$, and for any integer $i \geq 0$, let $q(i)$ denotes the integral part of the fraction i/p^m. We then define an $\mathcal{O}_K$-submodule $A^{(m)}$ of $U(\mathfrak{g})$ via

$$A^{(m)} := \{\sum_I b_I \frac{q(i_1)!\cdots q(i_d)!}{i_1!\cdots i_d!} X_1^{i_1}\cdots X_d^{i_d} \quad | \quad b_I \in \mathcal{O}_K \text{ and } b_I = 0 \text{ for almost all } I\}.$$

We also define

$$A^{(\infty)} := \{\sum_I b_I \frac{X_1^{i_1}\cdots X_d^{i_d}}{i_1!\cdots i_d!} \quad | \quad b_I \in \mathcal{O}_K \text{ and } b_I = 0 \text{ for almost all } I\}.$$

We let $\hat{A}^{(m)}$ (respectively $\hat{A}^{(\infty)}$) denotes the p-adic completion of $A^{(m)}$ (respectively $A^{(\infty)}$).

We will need the following elementary lemma.

Lemma 5.2.2. *If n is a natural number and m a non-negative integer, and if we use the Euclidean algorithm to write $n = p^m q + r$, with $0 \leq r < p^m$, then*

$$\frac{n!}{q!} = (p^m!)^q r! u,$$

where u is a p-adic unit.

Proof. This follows directly from the well-known formula, stating that the p-adic ordinal of $n!$ is equal to $\dfrac{n - s(n)}{p-1}$, where $s(n)$ denotes the sum of the p-adic digits of n. $\square$

We also recall that $U(\mathfrak{g})$ is a filtered K-algebra. To be precise, there is a natural quotient map $T(\mathfrak{g}) \to U(\mathfrak{g})$ from the tensor algebra of $\mathfrak{g}$ onto the universal enveloping algebra. The tensor algebra $T(\mathfrak{g})$ is naturally graded, and hence filtered. More precisely, for any integer i we write

$$T(\mathfrak{g})_{\leq i} = \oplus_{j \leq i} \mathfrak{g}^{\otimes j} \subset T(\mathfrak{g}),$$

and correspondingly write $U(\mathfrak{g})_{\leq i}$ to denote the image of $T(\mathfrak{g})_{\leq i}$ in $U(\mathfrak{g})$. The key properties of this filtration on $U(\mathfrak{g})$ are that

$$U(\mathfrak{g})_{\leq i} U(\mathfrak{g})_{\leq i'} \subset U(\mathfrak{g})_{\leq i+i'} \tag{5.2.3}$$

(i.e. it does indeed make $U(\mathfrak{g})$ a filtered algebra), and that

$$[U(\mathfrak{g})_{\leq i}, U(\mathfrak{g})_{\leq i'}] \subset U(\mathfrak{g})_{\leq i+i'-1} \tag{5.2.4}$$

(i.e. the associated graded algebra is commutative).

Lemma 5.2.5. (i) *For any $m \geq 0$, the $\mathcal{O}_K$-submodule $A^{(m)}$ of $U(\mathfrak{g})$ is in fact an $\mathcal{O}_K$-subaglebra of $U(\mathfrak{g})$. More precisely, it coincides with the $\mathcal{O}_K$-subalgebra of $U(\mathfrak{g})$ generated by the elements $\dfrac{X_j^i}{i!}$ for $0 \leq i \leq p^m$ and $1 \leq j \leq d$.*

(ii) *The $\mathcal{O}_K$-submodule $A^{(\infty)}$ of $U(\mathfrak{g})$ is in fact an $\mathcal{O}_K$-subaglebra of $U(\mathfrak{g})$.*

Proof. We begin by proving part (ii). First note that $\hat{A}^{(\infty)}$ is the unit ball in $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$ with respect to the norm on $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$ that is dual to the sup norm on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)$. Thus $\hat{A}^{(\infty)}$ is closed under multipliction in $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$, and in particular, $A^{(\infty)} = \hat{A}^{(\infty)} \bigcap U(\mathfrak{g})$ (the intersection taking place in $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$) is closed under multiplication in $U(\mathfrak{g})$, as claimed.

We now turn to proving (i). It follows from Lemma 5.2.2 that $A^{(m)}$ is contained in the $\mathcal{O}_K$-subalgebra of $U(\mathfrak{g})$ generated by the elements $\dfrac{X_j^i}{i!}$ for $0 \leq i \leq p^m$ and $1 \leq j \leq d$. To prove that this inclusion is an equality, it suffices to show that $[\dfrac{X_{j_1}^{i_1}}{i_1!}, \dfrac{X_{j_2}^{i_2}}{i_2!}] \in A^{(m)}$ for any $0 \leq i_1, i_2 \leq p^m$ and $1 \leq j_1, j_2 \leq d$. For this, it suffices in turn to note that, by part (ii) together with (5.2.4), the commutator in question lies in $A^\infty \bigcap U(\mathfrak{g})_{\leq (i_1+i_2-1)}$, and that $i_1 + i_2 - 1 < p^{m+1}$. $\square$

We now write $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} := K \otimes_{\mathcal{O}_K} \hat{A}^{(m)}$. Thus $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ is naturally a K-Banach algebra.

Proposition 5.2.6. *There is a natural isomorphism of topological K-algebras of compact type $\varinjlim_m \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \xrightarrow{\sim} \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$.*

Proof. This follows from the fact that the p-adic ordinal of $q_1! \cdots q_d!$ is asymptotic to $\dfrac{i_1 + \cdots + i_d}{(p-1)p^m}$ as $i_1 + \cdots + i_d \to \infty$. $\square$

In the remainder of this section we prove some additional technical results which will be required below. We begin with a lemma that is presumably well-known, but whose proof we include due to not knowing a reference.

Lemma 5.2.7. *If $\mathfrak{h}$ is an $\mathcal{O}_L$-Lie algebra such that $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$, for some $a \in \mathcal{O}_L$, then $[X^p, Y] \in (pa, a^p)U(\mathfrak{h})$ for any X and $Y \in \mathfrak{h}$.*

Proof. If we replace $\mathfrak{h}$ by $\mathfrak{h}/a^p\mathfrak{h}$, then we may assume furthermore that $\mathfrak{h}$ is nilpotent of class $< p$, and it suffices to show that $[X^p, Y] \in p[\mathfrak{h}, \mathfrak{h}]U(\mathfrak{h})$ for any X and $Y \in \mathfrak{h}$.

If we let $\mathfrak{g}$ be the free Lie algebra over $\mathcal{O}_L$ of nilpotentcy class $< p$ generated by two elements, then there is a homomorphism $\mathfrak{g} \to \mathfrak{h}$ taking the generators of $\mathfrak{g}$ to the elements X and Y of $\mathfrak{h}$, and so it suffices to prove the analogous statement for $\mathfrak{g}$. Since $[X^p, Y] \in [\mathfrak{g}, \mathfrak{g}]U(\mathfrak{g})$, and since $[\mathfrak{g}, \mathfrak{g}]U(\mathfrak{g})$ is a direct summand of $U(\mathfrak{g})$, in fact it suffices to prove the apparently weaker statement that $[X^p, Y] \in pU(\mathfrak{g})$ for any X and $Y \in \mathfrak{g}$. Replacing $\mathfrak{g}$ by its reduction modulo p, it suffices in turn to prove that if $\mathfrak{g}$ is any nilpotent Lie algebra over a field k of characteristic p of class $< p$, then X^p lies in the centre of $U(\mathfrak{g})$ for all $X \in \mathfrak{g}$.

It is a result due to Jacobson [14, eqn. (14a)] that in an associative algebra A of characteristic p, if we write $D_a := [a, -]$ for any $a \in A$, then $D_{a^p} = (D_a)^p$ for all $a \in A$. Applying this to $X \in U(\mathfrak{g})$, and noting that $D_X^p = 0$ because of our assumption that $\mathfrak{g}$ has class $< p$, we find that X^p is indeed central in $U(\mathfrak{g})$. $\square$

We also need a variant of the preceding result.

Proposition 5.2.8. *Let A be an associative $\mathcal{O}_L$-algebra, which is torsion free as an $\mathcal{O}_L$-module, and equipped with an increasing filtration by $\mathcal{O}_L$-submodules F_iA (for $i \geq 0$), such that $F_iA \cdot F_jA \subset F_{i+j}A$ for all $i, j \geq 0$, and such that $[F_iA, F_jA] \subset aF_{i+j-1}A$ if $i > 0$ and $j \geq 0$. If $X \in F_iA$ and $Y \in F_jA$, for some $i > 0$ and $j \geq 0$, then $[X^p, Y] \subset (pa, a^p)F_{pi+j-1}A$.*

Proof. The present proposition follows from proposition 5.2.7 via a standard argument with a Rees construction. Namely, for each $i \geq -1$ we write $\mathfrak{h}(i) := F_{i+1}A$, and we write $\mathfrak{h} := \bigoplus_{i \geq -1} \mathfrak{h}(i)$. We make $\mathfrak{h}$ a Lie algebra by equipping it with the bracket

$$[-,-] : \mathfrak{h}(i) \times \mathfrak{h}(j) \to \mathfrak{h}(\max(-1, i+j))$$

$(i, j \geq 0)$ induced by the commutator on A, which by assumption gives rise to a map

$$[-,-] : F_{i+1}A \times F_{j+1}A \to F_{\max(-1,i+j)}A.$$

We write $\mathfrak{h}_+ := \bigoplus_{i \geq 0} \mathfrak{h}(i) \subset \mathfrak{h}$, and note that $\mathfrak{h}_+$ is a graded Lie subalgebra of $\mathfrak{h}$, if we declare the direct summand $\mathfrak{h}(i)$ of $\mathfrak{h}_+$ to be in degree i, for each $i \geq 0$. Our hypotheses imply that $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$, and so proposition 5.2.7 implies that

$$[(X_{(i-1)})^p, Y_{(j-1)}] \in (p, a^p)U(\mathfrak{h}); \tag{5.2.9}$$

here we have written $X_{(i-1)}$ and $Y_{(j-1)}$ to denote X (respectively Y) viewed as an element of $\mathfrak{h}$ lying in its direct summand $\mathfrak{h}(i-1)$ (respectively $\mathfrak{h}(j-1)$).

There is a natural homomorphism of $\mathcal{O}_L$-Lie algebras $\mathfrak{h} \to A$, given by mapping the summand $\mathfrak{h}(i)$ of $\mathfrak{h}$ identically to its isomorphic image $F_{i+1}A$ in A. This induces a homomorphism of $\mathcal{O}_L$-algebras $U(\mathfrak{h}) \to A$. We wish to define an increasing filtration $F_\bullet$ on $U(\mathfrak{h})$ so that this homomorphism is compatible with this filtration on $U(\mathfrak{h})$ and the given filtration on A. To this end, we declare the summand $\mathfrak{h}(i)$ of $\mathfrak{h}$ to lie in $F_{i+1}U(\mathfrak{h})$, for $i \geq -1$. This then determines uniquely a filtration $F_\bullet U(\mathfrak{h})$, satisfying the condition that $F_iU(\mathfrak{g}) \cdot F_jU(\mathfrak{g}) \subset F_{i+j}U(\mathfrak{g})$ for all $i, j \geq 0$.

Since $\mathfrak{h} = \mathfrak{h}(-1) \oplus \mathfrak{h}_+$ decomposes $\mathfrak{h}$ as the ($\mathcal{O}_L$-module) direct sum of two $\mathcal{O}_L$-Lie subalgebras, there is an isomorphism

$$U(\mathfrak{h}(-1)) \otimes_{\mathcal{O}_L} U(\mathfrak{h}_+) \xrightarrow{\sim} U(\mathfrak{h}). \tag{5.2.10}$$

By construction, $U(\mathfrak{h}(-1)) = F_0U(\mathfrak{h})$. Since $\mathfrak{h}(-1) = F_0A$ is torsion free over $\mathcal{O}_L$, so is $F_0U(\mathfrak{h})$.

On the other hand, the grading on $\mathfrak{h}_+$ induces a grading on $U(\mathfrak{h})$, whose ith graded piece we denote by $U(\mathfrak{h}_+)(i)$. As with all enveloping algebras, it is also equipped with a canonical filtration $U(\mathfrak{h}_+)_{\leq i}$ (defined by placing $\mathfrak{h}_+$ in the first stage of the filtration). We may then equip $U(\mathfrak{h}_+)$ with the total filtration, which we denote by $F_\bullet U(\mathfrak{h}_+)$. More precisely,

$$F_iU(\mathfrak{h}_+) := \bigoplus_{0 \leq j \leq i} \Big(U(\mathfrak{h}_+)(j) \bigcap U(\mathfrak{h}_+)_{\leq i-j} \Big),$$

for $i \geq 0$.

Since A is torsion free over $\mathcal{O}_L$, the same is true for each F_iA, and hence for $\mathfrak{h}$. The associated graded of $U(\mathfrak{h})$, with respect to its filtration $U(\mathfrak{h})_{\leq i}$, is thus also $\mathcal{O}_L$-torsion free, and consequently so is the associated graded with respect to $F_\bullet U(\mathfrak{h}_+)$. Furthermore, the isomorphism (5.2.10) then induces an isomorphism

$$F_0U(\mathfrak{h}) \otimes_{\mathcal{O}_L} F_iU(\mathfrak{h}_+) \xrightarrow{\sim} F_iU(\mathfrak{h}),$$

and so we see that the associated graded of $U(\mathfrak{h})$ with respect to $F_\bullet U(\mathfrak{h}_+)$ is $\mathcal{O}_L$-torsion free.

Consequently, taking into account the filtration on $U(\mathfrak{h})$, we may improve (5.2.9) to the statement that

$$[(X_{(i-1)})^p, Y_{(j-1)}] \in (p, a^p)F_{pi+j-1}U(\mathfrak{h})$$

Applying the homomorphism $U(\mathfrak{h}) \to A$ yields the proposition. $\square$

We now return to the setting considered above, in which $\mathfrak{h}$ is an $\mathcal{O}_L$- Lie sublattice of the K-Lie algebra $\mathfrak{g}$, which exponentiates to a good analytic open subgroup H of G, admitting canonical coordinates of the second kind.

Proposition 5.2.11. *If* $[\mathfrak{h},\mathfrak{h}] \subset a\mathfrak{h}$ *for some* a *such that* $a^{p-1} \in p\mathcal{O}_L$*, then*

$$[A^{(m+1)}, A^{(m)}] \subset A^{(m)}$$

for any $m \geq 0$.

Proof. Our argument will be inductive in m, and will proceed in two steps. Namely, assuming that

$$[A^{(m)}, A^{(m)}] \subset aA^{(m)}, \tag{5.2.12}$$

we will deduce first that

$$[A^{(m+1)}, A^{(m)}] \subset aA^{(m)}, \tag{5.2.13}$$

and then go on to deduce the analogue of (5.2.12) with $m+1$ in place of m. Clearly (5.2.13) implies the statement of the proposition.

Since $A^{(0)} = U(\mathfrak{h})$, and since $[\mathfrak{h},\mathfrak{h}] \subset a\mathfrak{h}$ by assumption, the base case $m = 0$ case of (5.2.12) is clear. Suppose now that we have established (5.2.12) for some value of m. Then Proposition 5.2.7, applied to $A^{(m)}$ (regarded as a Lie algebra via the commutator), together with the assumption that p divides a^{p-1} in $\mathcal{O}_L$, implies that $[(Z^p)/p, W] \in aU(\mathfrak{h})$ for any Z and $W \in A^{(m)}$. In particular, if $X \in \mathfrak{h}$, then taking $Z := X^{p^m}/(p^m)!$, and using the formula

$$\frac{X^{p^{m+1}}}{(p^{m+1})!} = \frac{u}{p}\left(\frac{X^{p^m}}{(p^m)!}\right)^p = u\frac{Z^p}{p} \tag{5.2.14}$$

for some $u \in \mathcal{O}_L^\times$, we find that

$$[\frac{X^{p^{m+1}}}{(p^{m+1})!}, A^{(m)}] \subset aA^{(m)}. \tag{5.2.15}$$

Since X was an arbitrary element of $\mathfrak{h}$, this implies (5.2.13) for our given value of m.

If we let B denote the $\mathcal{O}_L$-Lie subalgbra B of $A^{(m+1)}$ generated by $A^{(m)}$ together with the element $X^{p^{m+1}}/(p^{m+1})!$, then we deduce from (5.2.15) that $[B,B] \subset aB$, and so we may apply Proposition 5.2.7 to B. Letting Y be any element of $\mathfrak{h}$, and taking into account (5.2.14) with Y in place of X, we deduce that

$$[\frac{X^{p^{m+1}}}{(p^{m+1})!}, \frac{Y^{p^{m+1}}}{(p^{m+1})!}] \in aB \subset aA^{(m+1)},$$

which in turn (since X and Y were arbitrary elements of $\mathfrak{h}$) implies (5.2.12) with m replaced by $m+1$. The proposition follows by induction. □

Suppose now given $r \in (0,1)$ such r is equal to absolute value of some $\alpha \in L^\times$. Note then that $\alpha\mathfrak{h}$ is the $\mathcal{O}_K$-Lie subalgebra of $\mathfrak{g}$ that exponentiates to give H_r. For any natural number m', we let $A^{(m')}(r)$ denote the $\mathcal{O}_K$-subalgebra of the enveloping algebra $U(\mathfrak{g})$ generated by the monomials $(\alpha X_j)^i/i!$ for $1 \leq i \leq p^{m'}$. (Thus $A^{(m')}(r)$ stands in the same relation to $\mathbb{H}_r$ as $A^{(m')}$ does to $\mathbb{H}$; see Lemma 5.2.5 (i).)

We also note that the filtration on $U(\mathfrak{g})$ induces a filtration on $A^{(m)}$, which we denote in the obvious way, namely $A^{(m)}_{\leq i} := A^{(m)} \bigcap U(\mathfrak{g})_{\leq i}$. Explicilty,

$$A^{(m)}_{\leq i} := \{ \sum_{|I| \leq i} b_I \frac{q(i_1)! \cdots q(i_d)!}{i_1! \cdots i_d!} X_1^{i_1} \cdots X_d^{i_d} \quad | \quad b_I \in \mathcal{O}_K \}.$$

The associated graded ring of $A^{(m)}$ with respect to this filtration embeds into that of $U(\mathfrak{g})$, and so is torsion free over $\mathcal{O}_L$. Thus

$$aA^{(m)} \bigcap A^{(m)}_{\leq i} = aA^{(m)}_{\leq i}, \tag{5.2.16}$$

for each $i \geq 0$.

Proposition 5.2.17. *If $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$ for some a such that $a^{p-1} \in p\mathcal{O}_L$, and if $m, m' \geq 0$, then $[A^{(m')}(r), A^{(m)}_{\leq i}] \subset A^{(m')}(r) A^{(m)}_{\leq i-1}$ for all $i > 0$.*

Proof. The proof is similar to that of proposition 5.2.11, though a little more straightforward. We fix m' and proceed by induction on m, proving the stronger statement that

$$[A^{(m')}(r), A^{(m)}_{\leq i}] \subset aA^{(m')}(r) A^{(m)}_{\leq i-1} \tag{5.2.18}$$

for all $i > 0$ and $m \geq 0$, beginning with the case $m = 0$.

Let $0 \leq m'' \leq m'$ and $1 \leq j \leq d$, and consider one of the monomial generators $(\alpha X_j)^{p^{m''}}/(p^{m''})!$ of $A^{(m')}(r)$. If $Y \in \mathfrak{h}$, then

$$[(\alpha X_j)^{p^{m''}}/(p^{m''})!, Y] = \alpha^{p^{m''}} [X_j^{p^{m''}}/(p^{m''})!, Y] \in \alpha^{p^{m''}} aA^{(m'')}_{\leq p^{m''}} \subset aA^{(m')}(r)$$

(taking into account (5.2.12) and (5.2.16) with m'' in place of m). This implies (5.2.18) in the case when $m = 0$.

Suppose now that we have proved (5.2.18) for some $m \geq 0$ and all $i > 0$; we will deduce the analogous statement for $m + 1$. To this end, we define $F_i B := A^{(m')}(r) A^{(m)}_{\leq i}$ for all $i \geq 0$. One deduces from (5.2.18), via induction on i, that we also have $F_i B = A^{(m)}_{\leq i} A^{(m')}(r)$. If we define $B = \bigcup_{i \geq 0} F_i B$, then it follows that B is an $\mathcal{O}_K$-subalgebra of $U(\mathfrak{g})$, and that $F_i B \cdot F_j B \subset F_{i+j} B$ for all $i, j \geq 0$. Furthermore, from (5.2.18), together with (5.2.12) and (5.2.16), we deduce that $[F_i B, F_j B] \subset aF_{i+j-1} B$ for all $i > 0$ and $j \geq 0$. Proposition 5.2.8 therefore applies to B. In particular, if $1 \leq j \leq d$, and $Y \in A^{(m')}(r) = F_0 B$, then we deduce that

$$[\left(\frac{X_j^{p^m}}{(p^m)!} \right)^p, Y] \in pF_{p^{m+1}-1} B.$$

Taking into account (5.2.14) (with $X = X_j$) and the definition of $F_{p^{m+1}-1} B$, we find that

$$[\frac{X_j^{p^{m+1}}}{(p^{m+1})!}, Y] \in A^{(m')}(r) A^{(m)}_{\leq p^{m+1}-1}. \tag{5.2.19}$$

Since $A^{(m+1)}$ is generated by $A^{(m)}$ together with the monomials $X_j^{p^{m+1}}/(p^{m+1})!$ for $1 \leq j \leq d$, we deduce from (5.2.18) and (5.2.19) that the analogue of (5.2.18) with m replaced by $m + 1$ holds. The proposition follows by induction. $\square$

5.3. Rings of locally analytic distributions are Fréchet-Stein algebras

Let G be locally L-analytic group, and let H be a compact open subgroup of G. The main goal of this section is to present a proof of the fact that $\mathcal{D}^{\mathrm{la}}(H, K)$ is a Fréchet-Stein algebra (under the assumption that K is discretely valued). This result was originally proved by Schneider and Teitelbaum [27, thm. 5.1], by methods related to those of [16]. The proof we give is different; it relies on an extension of the methods used in [1] to prove the coherence of the sheaf of rings $\mathcal{D}^{\dagger}$. Our approach also shows that for any good analytic open subgroup H of G, the algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)$ is coherent.

Proposition 5.3.1. *If H is a good analytic open subgroup of G (in the sense of section 5.2), then the K-Fréchet algebra $\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)_b$ is a weak Fréchet-Stein algebra.*

Proof. Let $\{r_n\}_{n\geq 1}$ be a strictly decreasing sequence of numbers lying in the open interval between 0 and 1, and also in $|\overline{L}^{\times}|$, that converges to 0, and form the corresponding decreasing sequence $\{\mathbb{H}^{\circ}_{r_n}\}_{n\geq 1}$ of σ-affinoid rigid analytic open subgroups of $\mathbb{H}$. (As in the preceding section, we have written $\mathbb{H}^{\circ}_{r_n} = \bigcup_{r<r_n} \mathbb{H}_r$.) Each of the subgroups $\mathbb{H}^{\circ}_{r_n}$ is normalized by H°.

For each $n \geq 1$ there are continuous injections

$$\mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_{r_n}-\mathrm{an}} \to \mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_{r_{n+1}}-\mathrm{an}} \tag{5.3.2}$$

and

$$\mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_{r_n}-\mathrm{an}} \to \mathcal{C}^{\mathrm{la}}(H^{\circ}, K) \tag{5.3.3}$$

(the latter being compatible with the former) which, upon passing to the inductive limit in n, yield an isomorphism

$$\varinjlim_{n} \mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_{r_n}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(H^{\circ}, K). \tag{5.3.4}$$

For each value of $n \geq 1$, we write $D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})$ to denote the strong dual to the space $\mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_{r_n}-\mathrm{an}}$. The restriction map

$$\mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_{r_n}-\mathrm{an}} \to \mathcal{C}(H^{\circ}_{r_n}, K)_{\mathbb{H}^{\circ}_{r_n}-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}^{\circ}_{r_n}, K)$$

(the isomorphism being provided by proposition 3.4.11) yields a closed embedding $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}_{r_n}, K) \to D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})$. The topological ring structure on $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}_{r_n}, K)$ extends naturally to a ring structure on $D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})$, such that $D(\mathbb{H}^{\circ}_{r_n}, H^{\circ}) = \bigoplus_{h\in H^{\circ}/H^{\circ}_{r_n}} \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}_{r_n}, K) * \delta_h$. (As indicated, the direct sum ranges over a set of coset representatives of $H^{\circ}_{r_n}$ in H°.) Dualizing the isomorphism (5.3.4) yields an isomorphism of topological K-algebras $\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)_b \xrightarrow{\sim} \varprojlim_{n} D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})$. We will show that this isomorphism induces a weak Fréchet-Stein structure on $\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)$.

Each of the algebras $D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})$ is of compact type as a convex K-vector space, and so satisfies condition (i) of definition 1.2.6. Since for each $n \geq 1$, the inclusion $\mathbb{H}^{\circ}_{r_{n+1}} \subset \mathbb{H}^{\circ}_{r_n}$ factors through the inclusion $\mathbb{H}_{r_{n+1}} \subset \mathbb{H}^{\circ}_{r_n}$, the map (5.3.2) is compact (as follows from proposition 2.1.16), and thus so is the dual map $D(\mathbb{H}^{\circ}_{r_{n+1}}, H^{\circ}) \to$

$D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})$. Thus by lemma 1.1.14, condition (ii) of definition 1.2.6 is also satisfied. Finally, since for each $n \geq 1$ the map (5.3.3) is a continuous injection of reflexive spaces, the dual map $\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)_b \to D(\mathbb{H}^{\circ}_{r_n}, H^{\circ})_b$ has dense image. Thus condition (iii) of definition 1.2.6 is satisfied, and the proposition is proved. $\square$

We intend to show that $\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)_b$ is in fact a Fréchet-Stein algebra, and we begin by abstracting the argument used to prove [1, thm. 3.5.3]. We suppose that A is a (not necessarily commutative) $\mathbb{Z}_p$-algebra, which is p-torsion free and p-adically separated. If $\hat{A}$ denotes the p-adic completion of A, then $\hat{A}$ is also p-torsion free, and the natural map $A \to \hat{A}$ is an injection. Tensoring with $\mathbb{Q}_p$ over $\mathbb{Z}_p$, we obtain an injection $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$. We regard A, $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$, and $\hat{A}$ as subalgebras of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$. One easily checks that A is saturated in $\hat{A}$, in the sense that the inclusion $A \to \big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A\big) \bigcap \hat{A}$ is an equality.

Multiplication in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$ induces a natural map $\hat{A} \otimes_A \big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A\big) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$, which is evidently an isomorphism. More generally, if M is a left A-submodule of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$, then multiplication in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$ induces a natural map of left $\hat{A}$-modules $\imath_M : \hat{A} \otimes_A M \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$. The following lemma concerns the image of the map $\imath_M$.

Lemma 5.3.5. *Let M be a left A-submodule of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$. The image of the map $\imath_M$ is contained in $M + \hat{A}$ (the sum taking place in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$). If $A \subset M$, then image of $\imath_M$ is in fact equal to $M + \hat{A}$.*

Proof. The A-module M is the inductive limit of finitely generated A-submodules. The first statement of the lemma is clearly compatible with passing to inductive limits, and so it suffices to prove it for M that are finitely generated. We may thus assume that $p^n M \subset A$ for some natural number n. The image of $\imath_M$ is spanned as a $\mathbb{Z}_p$-module by the product $\hat{A}M$. Since $\hat{A} = A + p^n \hat{A}$, we see that this product is contained in $M + \hat{A}$, as claimed. This proves the first claim of the lemma.

To prove the second claim, note that if $1 \in M$, then $\hat{A}M$ contains both M and $\hat{A}$. Thus in this case $M + \hat{A}$ is contained in the image of $\imath_M$, and the claimed equality follows. $\square$

Corollary 5.3.6. *If B is a subring of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ that contains A, then the image of $\imath_B : \hat{A} \otimes_A B \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$ is a subring of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$.*

Proof. Since B is in particular a left A-submodule of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ that contains A, lemma 5.3.5 implies that the image of $\imath_B$ is equal to $B + \hat{A}$. Applying the obvious analogue of lemma 5.3.5 for right A-modules, we find that the same is true of the image of the natural map $B \otimes_A \hat{A} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$. It is now clear that $B + \hat{A}$ is a subring of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$. $\square$

Lemma 5.3.7. *In the situation of lemma 5.3.5, if we write N to denote the image of $\imath_M$, and if we assume that $A \subset M$, then $M = N \bigcap \big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A\big)$.*

Proof. Lemma 5.3.5 shows that $N = M + \hat{A}$. Taking this into account, the lemma follows from the fact that $A = \big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A\big) \bigcap \hat{A}$, together with the assumption that $A \subset M$. $\square$

Lemma 5.3.8. *In the situation of corollary 5.3.6, the natural map $B \to B + \hat{A}$ of $\mathbb{Z}_p$-algebras induces an isomorphism on p-adic completions.*

Proof. Since $\hat{A} = A + p^n\hat{A} \subset B + p^n(B + \hat{A})$ for any natural number n, we see that the natural map $B/p^n \to \left(B + \hat{A}\right)/p^n$ is surjective for any natural number n. Lemma 5.3.7 implies that this map is also an injection. Passing to the projective limit in n proves the lemma. $\square$

Lemma 5.3.9. *In the situation of corollary* 5.3.6*, suppose that B is equipped with an exhaustive increasing filtration by $\mathbb{Z}_p$-submodules $F_0 \subset F_1 \subset \cdots$ satisfying the following assumptions:*

(i) *For each pair $i, j \geq 0$, we have $F_iF_j \subset F_{i+j}$.* (*That is, the $\mathbb{Z}_p$-submodules F_i filter B in the ring-theoretic sense.*)

(ii) $F_0 = A$. (*Note that when combined with* (i)*, this implies that F_i is a two-sided A-submodule of B, for each $i \geq 0$.*)

(iii) *The associated graded algebra $\mathrm{Gr}^F_\bullet B$ is finitely generated over A $(= \mathrm{Gr}^F_0 B)$ by central elements.*

If we let C denote the image of $\imath_B$ (*a subalgebra of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$, by corollary* 5.3.6)*, and if for each $i \geq 0$, we let G_i denote the image of $\imath_{F_i}$, then $G_\bullet$ is an exhaustive filtration of C satisfying properties analogous to* (i)*,* (ii) *and* (iii) *above. Namely:*

(i') *For each pair $i, j \geq 0$, we have $G_iG_j \subset G_{i+j}$.*

(ii') $G_0 = \hat{A}$.

(iii') *The associated graded algebra $\mathrm{Gr}^G_\bullet C$ is finitely generated over $\hat{A}$ $(= \mathrm{Gr}^G_0 C)$ by central elements.*

Proof. Since $A = F_0 \subset F_i$, lemma 5.3.5 shows that the image G_i of $\imath_{F_i}$ is equal to $F_i + \hat{A}$, and also that $C = B + \hat{A}$. It is now immediately checked that $G_\bullet$ is an exhaustive filtration of C, satisfying conclusions (i') and (ii') of the lemma. Lemma 5.3.7 implies that for each $i \geq 1$, the inclusion $F_{i-1} \subset \left(F_i \bigcap (F_{i-1} + \hat{A})\right)$ is an equality, and thus that the natural map $\mathrm{Gr}^F_i B \to \mathrm{Gr}^G_i C$ is an isomorphism if $i \geq 1$. Part (iii') thus follows from the corresponding assumption (iii), and the lemma is proved. $\square$

We are now ready to generalize [1, thm. 3.5.3].

Proposition 5.3.10. *Let A be a* (*not necessarily commutative*) *p-torsion free and p-adically separable left Noetherian $\mathbb{Z}_p$-algebra. If B is a $\mathbb{Z}_p$-subalgebra of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ that contains A, and that admits an exhaustive increasing filtration $F^\bullet$ by $\mathbb{Z}_p$-modules that satisfies conditions* (i)*,* (ii) *and* (iii) *of lemma* 5.3.9*, then the rings $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$ are left Noetherian $\mathbb{Q}_p$-algebras, and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$ is flat as a right $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$-module.*

Proof. We begin by noting that assumption (iii) of lemma 5.3.9 and the Hilbert basis theorem imply that $\mathrm{Gr}^F_\bullet B$ is left Noetherian, and thus that the same is true of B. Since the element p is central in A and B, it follows that the p-adic completions $\hat{A}$ and $\hat{B}$ are both left Noetherian, and thus the same is true of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$. The main point of the proposition, then, is to prove that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$ is flat as a right $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$-module.

As in the statement of lemma 5.3.9, let C denote the image of $\imath_B$, equipped with its filtration $G_\bullet$. Since $\hat{A}$ is left Noetherian, conclusion (iii') of lemma 5.3.9 and the Hilbert basis theorem imply that $\mathrm{Gr}^G_\bullet C$ is left Noetherian, and hence that C is left Noetherian. As p is central in C, the Artin-Rees theorem applies to show that $\hat{C}$ is flat as a right C-module. Tensoring with $\mathbb{Q}_p$ over $\mathbb{Z}_p$, we conclude that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{C}$ is right flat over $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} C$.

Since $A \subset B \subset \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$, and since $C = B + \hat{A}$, it is immediate that the inclusion $\hat{A} \subset C$ becomes an equality after tensoring through with $\mathbb{Q}_p$ over $\mathbb{Z}_p$. Lemma 5.3.8 implies that the inclusion $B \subset C$ induces an isomorphism $\hat{B} \xrightarrow{\sim} \hat{C}$, which of course remains an isomorphism after tensoring through with $\mathbb{Q}_p$ over $\mathbb{Z}_p$. Combining these remarks with the conclusion of the preceding paragraph, we deduce that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$ is right flat over $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A}$, as required. $\square$

We now present some applications of the preceding proposition.

Proposition 5.3.11. *Let H be a good analytic open subgroup of G (in the sense of section 5.2), obtained by exponentiating the Lie sublattice $\mathfrak{h}$ of $\mathfrak{g}$. Suppose further that $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$ for some a such that $a^{p-1} \in p\mathcal{O}_L$, and suppose also that K is discretely valued.*

(i) *For each $m \geq 0$, the ring $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ (as defined in the discussion preceding proposition 5.2.6) is Noetherian.*

(ii) *If $0 \leq m_1 \leq m_2$, then the natural map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m_1)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m_2)}$ is flat.*

Proof. Recall that the filtration on $U(\mathfrak{g})$ induces a filtration $A^{(m)}_{\leq i}$ on $A^{(m)}$ $(i \geq 0)$; we will use a similar notation with $m+1$ in place of m. The associated graded algebra of $A^{(m)}$ with respect to this filtration is a commutative $\mathcal{O}_K$-algebra, and it follows from Lemma 5.2.5 (i) that it is furthermore of finite type over $\mathcal{O}_K$. In particular, it is Noetherian, and thus the same is true of $A^{(m)}$.

Now consider the inclusions $A^{(m)} \subset A^{(m+1)} \subset \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A^{(m)} = U(\mathfrak{g})$. We equip $A^{(m+1)}$ with the following filtration:

$$F_i := A^{(m)} A^{(m+1)}_{\leq i}.$$

It follows from Proposition 5.2.11 that $A^{(m)} A^{(m+1)}_{\leq i} = A^{(m+1)}_{\leq i} A^{(m)}$, for each $i \geq 0$, and thus that $F_i F_j \subset F_{i+j}$ for any $i, j \geq 0$. Since $A^{(m+1)}$ is generated over $A^{(m)}$ by finitely many monomials, the associated graded algebra $\mathrm{Gr}^F_\bullet A^{(m+1)}$ is finitely generated over $A^{(m)}$. Furthermore, these monomials reduce to central elements in the associated graded algebra, again by Proposition 5.2.11. Altogether, we find that we are in the situation of proposition 5.3.10 (if we take the ring A of that proposition to be $A^{(m)}$ and the ring B to be $A^{(m+1)}$). That proposition then implies that each of the rings $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ and $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m+1)}$ is Noetherian, and that the natural map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m+1)}$ is flat. Both parts of the present proposition now follow. $\square$

Corollary 5.3.12. *Let H be a good analytic open subgroup of G, obtained by exponentiating the Lie sublattice $\mathfrak{h}$ of $\mathfrak{g}$. If $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$, where $a^{p-1} \in p\mathcal{O}_L$, and if K is discretely valued, then $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ is coherent.*

Proof. Propositions 5.2.6 and 5.3.11 together imply that $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ is equal to the inductive limit of a sequence of Noetherian rings with flat transition maps. This immediately implies that $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ is coherent. $\square$

Proposition 5.3.13. *Let H be a good analytic open subgroup of G, obtained by exponentiating a Lie sublattice $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}] \subset a\mathfrak{h}$, where $a^{p-1} \in p\mathcal{O}_L$, and suppose that K is discretely valued. If $0 < r < 1$ is a real number lying in $|L^\times|$, and if m is a sufficiently large natural number, then the natural map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ factors through the natural map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$, and the resulting map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ is flat.*

Proof. By assumption, r is equal to the absolute value of some $\alpha \in L^\times \subset K^\times$. Note then that $\alpha\mathfrak{h}$ is the $\mathcal{O}_K$-Lie subalgebra of $\mathfrak{g}$ that exponentiates to give H_r. As in the discussion of section 5.2, we choose an $\mathcal{O}_L$-basis $X_1, \ldots, X_j$ of $\mathfrak{h}$, and for any natural number m', we let $A^{(m')}(r)$ denote the $\mathcal{O}_K$-subalgebra of the enveloping algebra $U(\mathfrak{g})$ generated by the monomials $(\alpha X_j)^i/i!$ for $1 \leq i \leq p^{m'}$. If $\hat{A}^{(m')}(r)$ denotes the p-adic completion of this ring, then

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K) \cong \varinjlim_{m'} K \otimes_{\mathcal{O}_K} \hat{A}^{(m')}(r).$$

(This follows from proposition 5.2.6, applied to $\mathbb{H}_r$ rather than $\mathbb{H}$.) Thus, to prove the proposition, it suffices to exhibit a natural number m such that, for every $m' \geq 0$, the map

$$K \otimes_{\mathcal{O}_K} \hat{A}^{(m')}(r) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K) \tag{5.3.14}$$

factors as a composite

$$K \otimes_{\mathcal{O}_K} \hat{A}^{(m')}(r) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K), \tag{5.3.15}$$

with the first factor being flat.

As in proof of lemma 5.2.2, let $s(i)$ denote the sum of the p-adic digits of i, for any natural number i. If we fix an integer i, and write $i = p^m q + r$ (respectively $i = p^{m'} q' + r'$) with $0 \leq r < p^m$ (respectively $0 \leq r' < p^{m'}$), then

$$\mathrm{ord}_K(q!) = \frac{i - s(i)}{(p-1)p^m} - \frac{r - s(r)}{(p-1)p^m},$$

while

$$\mathrm{ord}_K(\alpha^i q'!) = \mathrm{ord}_K(\alpha) i + \frac{i - s(i)}{(p-1)p^{m'}} - \frac{r' - s(r')}{(p-1)p^{m'}}.$$

The values of r and r' are bounded independently of i. Thus, if we choose m so that $\mathrm{ord}_K(\alpha) \geq 1/(p-1)p^m$ (a condition independent of m'), then we find that

$$\mathrm{ord}_K(\alpha^i q'!) \geq \mathrm{ord}_K(q!) - C,$$

for some constant $C \geq 0$ independent of i. This inequality in turn shows that $p^k A^{(m')}(r) \subset A^{(m)}$ for some sufficiently large natural number k. Thus we have inclusions

$$p^k A^{(m')}(r) \subset A^{(m')}(r) \bigcap A^{(m)} \subset A^{(m')}(r),$$

from which we conclude that the map of p-adic completions

$$(A^{(m')}(r) \bigcap A^{(m)})\hat{} \to \hat{A}^{(m')}(r)$$

becomes an isomormphism after tensoring with K. Since there is also a natural map of p-adic completions

$$(A^{(m')}(r)\bigcap A^{(m)})\hat{} \to \hat{A}^{(m)},$$

we deduce that the map (5.3.14) does indeed factor as in (5.3.15). It remains to be shown that the first factor of (5.3.15) is flat. To this end, we will give another description of the factorization (5.3.15).

It follows from Proposition 5.2.17 that $A^{(m')}(r)A^{(m)} = A^{(m)}A^{(m')}(r)$, and hence that $A^{(m')}(r)A^{(m)}$ is an $\mathcal{O}_K$-subalgebra of $U(\mathfrak{g})$. Note that

$$A^{(m)} \subset A^{(m')}(r)A^{(m)} \subset \frac{1}{p^k}A^{(m)}.$$

Thus the first inclusion induces an isomorphism of p-adic completions, and hence the inclusion $A^{(m')}(r) \to A^{(m')}(r)A^{(m)}$ induces, upon passage to p-adic completions and tensoring with K, the first morphism in (5.3.15). We will apply proposition 5.3.10 to deduce that this morphism is flat.

To this end, we equip the ring $A^{(m')}(r)A^{(m)}$ with the following filtration:

$$F_i := A^{(m')}(r)A^{(m)}_{\leq i}.$$

It follows from proposition 5.2.17 that this filtration satisfies conditions (i), (ii), and (iii) of lemma 5.3.9 (taking the algebra A of that lemma to be $A^{(m')}(r)$, and the algebra B to be $A^{(m')}(r)A^{(m)}$). Proposition 5.3.10 now implies that the natural map

$$K \otimes_{\mathcal{O}_K} \hat{A}^{(m')}(r) \to K \otimes_{\mathcal{O}_K} (A^{(m')}(r)A^{(m)})\hat{}$$

is flat. Since this is precisely the first arrow of (5.3.15), the proof of the proposition is completed. □

In the context of the preceding proposition, the homomorphism $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}_r, K) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)$ extends to a homomorphism

$$D(\mathbb{H}^{\circ}_r, H^{\circ}) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K) \tag{5.3.16}$$

(in the notation of the proof of proposition 5.3.1; this homomorphism is dual to the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{H}^{\circ}, K) \to \mathcal{C}(H^{\circ}, K)_{\mathbb{H}^{\circ}_r-\mathrm{an}}$.) We wish to strengthen proposition 5.3.13 so that is applies to the map (5.3.16).

The following lemma provides the necessary bootstrap. It provides an analogue in a simple noncommutative situation of the following commutative algebra fact: if the composite $A \to B \to C$ is a flat morphism of commutative rings, and if B is finite étale over A, then $B \to C$ is also flat.

Lemma 5.3.17. *Let A be an associative K-algebra, and let B be an extension of A, satisfying the following hypotheses:*

(i) *There exists an element $x \in B$ such that B is generated as a ring by A together with x.*

(ii) *There is a unit $a \in A^{\times}$ such that $x^n = a$.*

(iii) *The element x of B (which is a unit of B, by (ii)) normalizes A; that is, $xAx^{-1} = A$.*

If C is an extension of B that is flat as a left (respectively right) A-module, then C is flat as a left (respectively right) B-module.

Proof. If M is a right B-module, then we define a C-linear automorphism σ of $M \otimes_A C$ as follows:

$$\sigma(m \otimes c) = mx^{-1} \otimes xc.$$

The formation of σ is evidently functorial in M. It follows from hypothesis (iii) in the statement of the lemma that σ is well-defined, and hypothesis (ii) implies that $\sigma^n = 1$. Hypothesis (i) implies that $M \otimes_B C$ is isomorphic to the coinvariants of $M \otimes_A C$ with respect to the cyclic group $\langle \sigma \rangle$ generated by σ.

Suppose that C is flat as a left A-module. Since passing to $\langle \sigma \rangle$-coinvariants is exact (as K is of characteristic zero), we see that the formation of $M \otimes_B C$ is an exact functor of M. Thus C is also flat as a left B-module.

The case when C is flat as a right A-module is proved by an analogous argument. $\square$

Proposition 5.3.18. *Let H be a good analytic open subgroup of G, obtained by exponentiating a Lie sublattice $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$, where $a^{p-1} \in p\mathcal{O}_L$, and suppose that K is discretely valued. If $0 < r < 1$ lies in $|L^\times|$, and if m is a sufficiently large natural number, then the map* (5.3.16) *factors through the natural map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$, and the resulting map $D(\mathbb{H}_r^\circ, H^\circ) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ is flat.*

Proof. The ring $D(\mathbb{H}_r^\circ, H^\circ)$ is free of finite rank as a $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$-module (on either side), and so in particular is finitely generated as a ring over $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$. Since $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$ is the inductive limit of its subrings $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$, and since, by proposition 5.3.13, the image of the restriction of (5.3.16) to $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$ lies in $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ for some sufficiently large value of m, we see that $D(\mathbb{H}_r^\circ, H^\circ)$ also maps into $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}$ for some (possibly larger) value of m.

Again by proposition 5.3.13, the composite of the sequence of maps

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K) \to D(\mathbb{H}_r^\circ, H^\circ) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)}$$

is flat. Our goal is to show that the second of these maps is flat.

Since H° is a pro-p-group containing H_r° as a normal open subgroup, we may find a filtration by normal subgroups

$$H_r^\circ = G_0 \subset G_1 \subset \cdots \subset G_n = H^\circ,$$

such that each quotient G_{i+1}/G_i is cyclic. For each $i \geq 0$, let $D(\mathbb{H}_r^\circ, G_i)$ denote the dual to the space $\mathcal{C}(G_i, K)_{\mathbb{H}_r^\circ-\mathrm{an}}$, equipped with its natural ring structure. For each value of i, the inclusion $D(\mathbb{H}_r^\circ, G_i) \subset D(\mathbb{H}_r^\circ, G_{i+1})$ then satisfies the hypothesis of the inclusion $A \subset B$ of lemma 5.3.17. Taking $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)}$ to be the ring C of that lemma, and arguing by induction, starting from the case $i = 0$ (and noting that $D(\mathbb{H}_r^\circ, G_0) = D(\mathbb{H}_r^\circ, H_r^\circ) = \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$) we find that $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)}$ is flat over $D(\mathbb{H}_r^\circ, G_n) = D(\mathbb{H}_r^\circ, H^\circ)$, as required. $\square$

We next use the preceding results to give a new proof of [27, thm. 5.1], after first introducing one more piece of notation: we put ourselves in the context of proposition 5.3.18, and for any $0 < r < 1$ lying in $|L^\times|$, and any $m \geq 1$, we write

$$D(\mathbb{H}_r^\circ, H^\circ)^{(m)} := K[H^\circ] \otimes_{K[H_r^\circ]} \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)} \xrightarrow{\sim} \bigoplus_{h \in H^\circ / H_r^\circ} \delta_h * \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)}.$$

(Thus $D(\mathbb{H}_r^\circ, H^\circ)^{(m)}$ stands in the same relation to $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)}$ as $D(\mathbb{H}_r^\circ, H^\circ)$ does to $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$.) Since H° normalizes $\mathbb{H}_r^\circ$, there is a K-algebra structure on $D(\mathbb{H}_r^\circ, H^\circ)^{(m)}$, and it thus a K-Banach algebra. (Another way to describe this K-algebra structure is to note that the continuous injection $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)^{(m)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}_r^\circ, K)$ extends to a continuous injection $D(\mathbb{H}_r^\circ, H^\circ)^{(m)} \to D(\mathbb{H}_r^\circ, H^\circ)$, whose image is a K-subalgebra of $D(\mathbb{H}_r^\circ, H^\circ)$. We may then pull back the K-algebra structure on the image to obtain a K-algebra structure on $D(\mathbb{H}_r^\circ, H^\circ)^{(m)}$.)

Corollary 5.3.19. *If K is discretely valued, and if H is a compact open subgroup of G, then the nuclear Fréchet algebra $\mathcal{D}^{\mathrm{la}}(H, K)$ is a Fréchet-Stein algebra.*

Proof. It suffices to verify this after replacing H by any open subgroup. (Compare the second half of the argument in step 1 of the proof of [27, thm. 5.1].) Let $\mathfrak{h}$ be a $\mathcal{O}_L$-Lie sublattice of $\mathfrak{g}$ that exponentiates to a good analytic open subgroup of G, whose L-valued points are contained in H. Replacing $\mathfrak{h}$ by $a\mathfrak{h}$ for an appropriate choice of $a \in \mathcal{O}_L$, we may furthermore assume (if it is not already true) that $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$, where $a^{p-1} \in p\mathcal{O}_L$. If we now let H denote the L-valued points of the good analytic open subgroup $\mathbb{H}$ obtained by exponentiating $\mathfrak{h}$, then it suffices to show that $\mathcal{D}^{\mathrm{la}}(H, K)$ is a Fréchet-Stein algebra. In fact, since H° is an open subgroup of H, it suffices to show that $\mathcal{D}^{\mathrm{la}}(H^\circ, K)$ is a Fréchet-Stein algebra, and this is what we shall do.

Choose $\alpha \in L^\times$ such that $r := |\alpha| < 1$. Consider the decreasing sequence $\{\mathbb{H}_{r^n}^\circ\}_{n \geq 0}$ of affinoid subgroups of $\mathbb{H}$. Fix a value of n, and apply proposition 5.3.18 to the inclusion $\mathbb{H}_{r^{n+1}}^\circ \to \mathbb{H}_{r^n}^\circ$. We see that we may find an integer m_n such that the map $D(\mathbb{H}_{r^{n+1}}^\circ, H_{r^n}^\circ) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r^n}^\circ, K)$ factors as

$$D(\mathbb{H}_{r^{n+1}}^\circ, H_{r^n}^\circ) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r^n}^\circ, K)^{(m_n)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r^n}^\circ, K),$$

in which both arrows are flat, with dense image. (The flatness of the first arrow is the conclusion of proposition 5.3.18; that of the second arrow follows from propositions 5.2.6 and 5.3.11. To see that each arrow has dense image, note that $U(\mathfrak{g})$ is a subalgebra of $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r^{n+1}}^\circ, K)$, and hence of $D(\mathbb{H}_{r^{n+1}}^\circ, H_{r^n}^\circ)$, and is dense in each of $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r^n}^\circ, K)^{(m_n)}$ and $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r^n}^\circ, K)$.) Tensoring with $D(\mathbb{H}_{r^{n+1}}^\circ, H^\circ)$ over $D(\mathbb{H}_{r^{n+1}}^\circ, H_{r^n}^\circ)$, we obtain the sequence of continuous homomorphisms of topological K-algebras

$$D(\mathbb{H}_{r^{n+1}}^\circ, H^\circ) \to D(\mathbb{H}_{r^n}^\circ, H^\circ)^{(m_n)} \to D(\mathbb{H}_{r^n}^\circ, H^\circ), \tag{5.3.20}$$

each being flat with dense image.

If we now allow n to vary, and take into account the isomorphism $\mathcal{D}^{\mathrm{la}}(H^\circ, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{H}_{r^n}^\circ, H^\circ)$ established in the course of proving proposition 5.3.1, as well as the sequences of flat maps provided by (5.3.20), we obtain an isomorphism

$$\mathcal{D}^{\mathrm{la}}(H^\circ, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{H}_{r^n}^\circ, H^\circ)^{(m_n)}.$$

This isomorphism describes $\mathcal{D}^{\mathrm{la}}(H^\circ, K)$ as the projective limit of Noetherian Banach algebras related by transition maps that are flat with dense image, and so establishes that it is a Fréchet-Stein algebra. $\square$

In fact, we will require a more general result than the preceding corollary.

Definition 5.3.21. Assume that K is discretely valued, and let $A \to B$ be a flat morphism of commutative Noetherian Banach algebras. A good integral model for this morphism consists of a p-adically complete Noetherian $\mathcal{O}_K$-algebra $\mathcal{A}$, and a finite type $\mathcal{A}$-algebra $\mathcal{B}$ such that the induced map $K \otimes_{\mathcal{O}_K} \mathcal{A} \to K \otimes_{\mathcal{O}_K} \mathcal{B}$ is an isomorphism, together with isomorphisms $K \otimes_{\mathcal{O}_K} \mathcal{A} \xrightarrow{\sim} A$ and $K \otimes_{\mathcal{O}_K} \hat{\mathcal{B}} \xrightarrow{\sim} B$ (where $\hat{\mathcal{B}}$ denotes the p-adic completion of B), such that the resulting diagram

$$\begin{tikzcd}
K \otimes_{\mathcal{O}_K} \mathcal{A} \arrow[r, "\sim"] \arrow[d, "\sim"] & K \otimes_{\mathcal{O}_K} \mathcal{B} \arrow[r] & K \otimes_{\mathcal{O}_K} \hat{\mathcal{B}} \arrow[d, "\sim"] \\
A \arrow[rr] & & B
\end{tikzcd}$$

commutes.

For example, if $\mathbb{X} \to \mathbb{Y}$ is an open immersion of rigid analytic spaces over K, then the induced map $\mathcal{C}^{\mathrm{an}}(\mathbb{Y}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ admits a good integral model in the sense of definition 5.3.21, as follows from [4, Lem. 5.7].[3]

Proposition 5.3.22. *Suppose that K is discretely valued. If H is a compact open subgroup of G, and if A is a commutative Fréchet-Stein algebra over K which possesses a Fréchet-Stein structure $A \xrightarrow{\sim} \varprojlim_n A_n$ with the property that each of the transition maps $A_{n+1} \to A_n$ admits a good integral model (in the sense of definition* 5.3.21*), then the completed tensor product $A \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$ is again a Fréchet-Stein algebra.*

Proof. As in the proof of corollary 5.3.19, we may assume that H is a good analytic open subgroup of G obtained by exponentiating a Lie sublattice $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset a\mathfrak{h}$, where $a^{p-1} \in p\mathcal{O}_L$. For each $n \geq 1$, let $\mathcal{A}_{n+1} \to \mathcal{B}_n$ be a good integral model for the transition map $A_{n+1} \to A_n$; i.e. $\mathcal{A}_{n+1}$ is a Noetherian p-adically complete $\mathcal{O}_K$-algebra, $\mathcal{B}_n$ is a finite type $\mathcal{A}_{n+1}$-algebra, and, letting $\hat{\mathcal{B}}_n$ denote the p-adic completion of $\mathcal{B}_n$, the base-change to K over $\mathcal{O}_K$ of the induced map $\mathcal{A}_{n+1} \to \hat{\mathcal{B}}_n$ recovers the transition map $A_{n+1} \to A_n$.

Fix an element $0 < r < 1$ of $|L^\times|$. We may apply arguments analogous to those used to prove propositions 5.3.11, 5.3.13, and 5.3.18, with $U(\mathfrak{g})$ replaced by $A_{n+1} \otimes_K U(\mathfrak{g}) \xrightarrow{\sim} \mathcal{A}_{n+1} \otimes_{\mathcal{O}_K} U(\mathfrak{g})$, and with all of the various $\mathcal{O}_K$-subalgebras of $U(\mathfrak{g})$ replaced by the corresponding $\mathcal{A}_{n+1}$-subalgebras. In this way we deduce that if m_n is sufficiently large, then for any $m' \geq m_n$, the natural map

$$A_{n+1} \hat{\otimes}_K D(\mathbb{H}^\circ_{r^{n+1}}, H^\circ)^{(m')} \to A_{n+1} \hat{\otimes}_K D(\mathbb{H}^\circ_{r^n}, H^\circ)^{(m')}$$

factors through the map

$$A_{n+1} \hat{\otimes}_K D(\mathbb{H}^\circ_{r^n}, H^\circ)^{(m_n)} \to A_{n+1} \hat{\otimes}_K D(\mathbb{H}^\circ_{r^n}, H^\circ)^{(m')},$$

[3] More precisely, this result shows that we may find affine formal models $\mathcal{X}$ and $\mathcal{Y}$ for $\mathbb{X}$ and $\mathbb{Y}$, an admissible formal blowing-up $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$, and an open immersion $\mathcal{X} \to \tilde{\mathcal{Y}}$, which induces the given open immersion $\mathbb{X} \to \mathbb{Y}$ on rigid analytic generic fibres. If we write $\mathcal{Y} = \operatorname{Spf} \mathcal{A}$, then the open formal subscheme $\mathcal{X}$ of $\tilde{\mathcal{Y}}$ is equal to $\operatorname{Spf} \hat{\mathcal{B}}$, where $\hat{\mathcal{B}}$ is the p-adic completion of some finite type $\mathcal{A}$-algebra $\mathcal{B}$ with the property that $K \otimes_{\mathcal{O}_K} \mathcal{A} \xrightarrow{\sim} K \otimes_{\mathcal{O}_K} \mathcal{B}$.

and that furthermore, the resulting map

$$A_{n+1} \hat{\otimes}_K D(\mathbb{H}^{\circ}_{r^{n+1}}, H^{\circ})^{(m')} \to A_{n+1} \hat{\otimes}_K D(\mathbb{H}^{\circ}_{r^n}, H^{\circ})^{(m_n)},$$

is flat.

If we write $\mathcal{A}_{n+1} \hat{\otimes}_{\mathcal{O}_K} A^{(m_n)}(r^n)$ to denote the p-adic completion of $\mathcal{A}_{n+1} \otimes_{\mathcal{O}_K} A^{(m_n)}(r^n)$, then the same arguments also prove that $\mathcal{A}_{n+1} \hat{\otimes}_{\mathcal{O}_K} A^{(m_n)}(r^n)$ is Noetherian. (Here $A^{(m_n)}(r^n)$ is defined in analogy to $A^{(m_n)}$, but with $\mathbb{H}_{r^n}$ in place of $\mathbb{H}$.) Since $\mathcal{B}_n$ is finite type over $\mathcal{A}_{n+1}$, we find that $\mathcal{B}_n \otimes_{\mathcal{A}_{n+1}} \mathcal{A}_{n+1} \hat{\otimes}_{\mathcal{O}_K} A^{(m_n)}(r^n)$ is Noetherian. Thus the same is true of its p-adic completion, and furthermore, the map from this $\mathcal{O}_K$-algebra to its p-adic completion is flat. Clearly this p-adic completion is isomorphic to $\hat{\mathcal{B}}_n \hat{\otimes}_{\mathcal{A}_{n+1}} A^{(m_n)}(r^n)$ (the p-adic completion of $\hat{\mathcal{B}}_n \otimes_{\mathcal{A}_{n+1}} A^{(m_n)}(r^n)$), and so we conclude that each of $\mathcal{B}_n \otimes_{\mathcal{A}_{n+1}} \mathcal{A}_{n+1} \hat{\otimes}_{\mathcal{O}_K} A^{(m_n)}(r^n)$ and $\hat{\mathcal{B}}_n \hat{\otimes}_{\mathcal{A}_{n+1}} A^{(m_n)}(r^n)$ is Noetherian, and that the latter is flat over the former. Tensoring with K over $\mathcal{O}_K$, we find that each of $A_{n+1} \hat{\otimes}_K \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}_{r^n}, K)^{(m_n)}$ and $A_n \hat{\otimes}_K \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}_{r^n}, K)^{(m_n)}$ is Noetherian, and that the latter is flat over the former. Tensoring with $K[H^{\circ}]$ over $K[H^{\circ}_{r^n}]$, we find that $A_{n+1} \hat{\otimes}_K D(\mathbb{H}^{\circ}_{r^n}, H^{\circ})^{(m_n)}$ and $A_n \hat{\otimes}_K D(\mathbb{H}^{\circ}_{r^n}, H^{\circ})^{(m_n)}$ are each Noetherian, and that the latter is flat over the former. Combining this with the result of the preceding paragraph, taking $m' = m_{n+1}$, we find that the map

$$A_{n+1} \hat{\otimes}_K D(\mathbb{H}^{\circ}_{r^{n+1}}, H^{\circ})^{(m_{n+1})} \to A_n \hat{\otimes}_K D(\mathbb{H}^{\circ}_{r^n}, H^{\circ})^{(m_n)}$$

is a continuous flat map of Noetherian K-Banach algebras. It also has dense image, since this is true of each of the maps $A_{n+1} \to A_n$ and $D(\mathbb{H}^{\circ}_{r^{n+1}}, H^{\circ})^{(m_{n+1})} \to D(\mathbb{H}^{\circ}_{r^n}, H^{\circ})^{(m_n)}$ separately. The isomorphism

$$A \hat{\otimes} \mathcal{D}^{\mathrm{la}}(H^{\circ}, K) \xrightarrow{\sim} \varprojlim_n A_n \hat{\otimes}_K D(H^{\circ}_{r^n}, H^{\circ})^{(m_n)}$$

thus exhibits a Fréchet-Stein structure on $A \hat{\otimes} \mathcal{D}^{\mathrm{la}}(H^{\circ}, K)$, proving the proposition. □

If $\mathbb{X}$ is a quasi-Stein rigid analytic space over K, then the remark following definition 5.3.21 shows that the Fréchet-Stein algebra $A := \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ satisfies the condition of proposition 5.3.22.

Chapter 6. Admissible locally analytic representations

6.1. Admissible locally analytic representations

Fix a locally L-analytic group G. In this section we introduce the notion of an admissible locally analytic G-representation.

Definition 6.1.1. Let V be a locally analytic representation of G. We say that V is admissible if it an LB-space, and if for a cofinal sequence of analytic open subgroups H of G, the space $(V_{\mathbb{H}-\mathrm{an}})'$ is finitely generated as a left $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$-module.

Note that if V is an LB-space, then corollary 3.3.21 implies that $V_{\mathbb{H}-\mathrm{an}}$ is also an LB-space, and so theorem 5.1.15 (ii) implies that $(V_{\mathbb{H}-\mathrm{an}})'$ is finitely generated as a $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$-module if and only if the space $V_{\mathbb{H}-\mathrm{an}}$ of $\mathbb{H}$-analytic vectors admits a closed embedding into $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^n$ for some natural number n.

We will show below that a locally analytic G-representation V is admissible if and only if V_b' is a coadmissible topological $\mathcal{D}^{\mathrm{la}}(H, K)$ for one (or equivalently any) analytic open subgroup H of G. This will show that our definition of an admissible locally analytic representation of G is equivalent to that of [27, def., p. 33].

Proposition 6.1.2. *If V is an admissible locally analytic representation of G then $(V_{\mathbb{H}-\mathrm{an}})'$ is finitely generated as a left $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$-module for every analytic open subgroup H of G. In particular, for every such H, the space $V_{\mathbb{H}-\mathrm{an}}$ is a Banach space.*

Proof. Let H be an analytic open subgroup of G. By assumption we may find an analytic open subgroup $H' \subset H$ such that $V'_{\mathbb{H}'-\mathrm{an}}$ is finitely generated over $\mathcal{D}^{\mathrm{an}}(\mathbb{H}', K)$, or equivalently, by the remarks following definition 6.1.1, such that there is an H'-equivariant closed embedding $V_{\mathbb{H}'-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n$ for some natural number n.

We now construct a closed embedding

$$\begin{aligned} V_{\mathbb{H}-\mathrm{an}} = \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta_{1,2}(H)} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta_{1,2}(H')} &\xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V_{\mathbb{H}'-\mathrm{an}})^{\Delta_{1,2}(H')} \\ \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}, \mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n)^{\Delta_{1,2}(H')} &\xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^n. \end{aligned}$$

(Here the isomorphisms are provided by lemmas 3.3.9 and 3.3.11 respectively.) The existence of this closed embedding in turn implies that $(V_{\mathbb{H}-\mathrm{an}})'$ is finitely generated over $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$, as required. □

Proposition 6.1.3. *If V is an admissible locally analytic representation of G then V is of compact type.*

Proof. Replacing G by a compact open subgroup of itself if necessary, we assume that G is compact. Let $\{H_m\}_{m\geq 1}$ be a cofinal sequence of normal analytic open subgroups of G, chosen so that the rigid analytic map $\mathbb{H}_{m+1} \to \mathbb{H}_m$ is relatively compact (in the sense of definition 2.1.15) for each $m \geq 1$. Fix one such value of m, and write $H_m = \prod_{i\in I} h_i H_{m+1}$. Proposition 6.1.2 yields an H_m-equivariant closed embedding of K-Banach spaces $V_{\mathbb{H}_{m+1}-\mathrm{an}} \to \prod_{i\in I} \mathcal{C}^{\mathrm{an}}(h_i\mathbb{H}_{m+1}, K)^n$ for some n. Passing to $\mathbb{H}_m$-analytic vectors and applying proposition 3.3.23 yields a Cartesian diagram

$$\begin{tikzcd} V_{\mathbb{H}_m-\mathrm{an}} \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{an}}(\mathbb{H}_m, K)^n \arrow[d] \\ V_{\mathbb{H}_{m+1}-\mathrm{an}} \arrow[r] & \prod_{i\in I} \mathcal{C}^{\mathrm{an}}(h_i\mathbb{H}_{m+1}, K)^n \end{tikzcd}$$

in which the horizontal arrows are closed embeddings. Since the right-hand vertical arrow is compact, by proposition 2.1.16, and since the horizontal arrows are closed embeddings, we see that the left-hand vertical arrow is also compact. Thus $V_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_n V_{\mathbb{H}_m-\mathrm{an}}$ is of compact type. From theorem 3.6.12 we conclude that the natural map $V_{\mathrm{la}} \to V$ is a topological isomorphism, and thus that V is of compact type. □

Proposition 6.1.4. *If V is an admissible locally analytic representation of G and W is a G-invariant closed subspace of G then W is an admissible locally analytic representation of G.*

Proof. Proposition 6.1.3 shows that V is of compact type, and thus that W is also of compact type. In particular, W is an LB-space, and thus barrelled, and so proposition 3.6.14 shows that W is locally analytic.

Let H be an analytic open subgroup of G. Proposition 6.1.2 shows that $V_{\mathbb{H}-\mathrm{an}}$ is a Banach space, and so proposition 3.3.23 shows that the natural map $W_{\mathbb{H}-\mathrm{an}} \to V_{\mathbb{H}-\mathrm{an}}$ is a closed embedding. Hence $W'_{\mathbb{H}-\mathrm{an}}$ is a quotient of $V'_{\mathbb{H}-\mathrm{an}}$, and so is finitely generated over $\mathcal{D}^{\mathrm{an}}(\mathbb{H}, K)$ (since this is true of $V'_{\mathbb{H}-\mathrm{an}}$ by assumption). $\square$

Proposition 6.1.5. *If V is an admissible locally analytic representation of G and W is a finite dimensional locally analytic representation of G then the tensor product $V \otimes_K W$ is again an admissible locally analytic representation of G.*

Proof. From corollary 3.6.16 we conclude that $V \otimes_K W$ is a locally analytic representation of G. By assumption V is an LB-space and W is finite dimensional, and so their tensor product is clearly also an LB-space.

Since W is finite dimensional we may find an analytic open subgroup H of G such that W is an $\mathbb{H}$-analytic representation of H. If H' is an analytic open subgroup of H then proposition 3.6.6 yields an isomorphism $(V \otimes_K W)_{\mathbb{H}'-\mathrm{an}} \xrightarrow{\sim} V_{\mathbb{H}'-\mathrm{an}} \otimes_K W$. Proposition 6.1.2 yields an H'-equivariant closed embedding $V_{\mathbb{H}'-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n$ for some n, and thus we obtain a closed embedding $V_{\mathbb{H}'-\mathrm{an}} \otimes_K W \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n \otimes_K W$ (both source and target being equipped with the diagonal action of G). The remark following the proof of lemma 3.6.4 yields a G-equivariant isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n \otimes_K W \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n \otimes_K W$, in which the source is equipped with the diagonal action of G, and the target is equipped with the tensor product of the right regular G-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^n$ and the trivial G-action on W. Thus, if d denotes the dimension of W, we obtain altogether a G-equivariant closed embedding $(V \otimes_K W)_{\mathbb{H}'-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}', K)^{nd}$. This shows that $V \otimes_K W$ is admissible, as required. $\square$

Lemma 6.1.6. *If V is an admissible locally analytic representation of G, then for any good analytic open subgroup H of $\mathbb{G}$ (in the sense of section 5.2), the space $V_{\mathbb{H}^\circ-\mathrm{an}}$ of $\mathbb{H}^\circ$-analytic vectors in V is a nuclear Fréchet space.*

Proof. Write $\mathbb{H}^\circ = \bigcup_{n=1}^\infty \mathbb{H}_{r_n}$, where $\{r_n\}_{n\geq 1}$ is a strictly increasing sequence of real numbers belonging to $|\overline{L}^\times|$ and converging to 1 from below. Then $\{\mathbb{H}_{r_n}\}_{n\geq 1}$ is an increasing sequence of analytic open subgroups, with the property that each of the maps $\mathbb{H}_{r_n} \to \mathbb{H}_{r_{n+1}}$ is relatively compact,

In the course of proving proposition 6.1.3, it was observed that each of the maps $V_{\mathbb{H}_{r_{n+1}}-\mathrm{an}} \to V_{\mathbb{H}_{r_n}-\mathrm{an}}$ is compact. Passing to the projective limit over n, we find that $V_{\mathbb{H}^\circ-\mathrm{an}} = \varprojlim_n V_{\mathbb{H}_{r_n}-\mathrm{an}}$ is a projective limit of a sequence of Banach spaces with compact transition maps, and so is a nuclear Fréchet space. $\square$

Proposition 6.1.7. *If V is a locally analytic representation of G on an LB-space, then the following are equivalent.*

(i) *V is an admissible locally analytic representation.*

(ii) *For every good analytic open subgroup $\mathbb{H}$ of $\mathbb{G}$, there exists a natural number n and an H°-equivariant closed embedding $V_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K)^n$. (Here we are* *using the notation related to good analytic open subgroups that was introduced in section* 5.2.)

(iii) *For a cofinal sequence of good analytic open subgroups* $\mathbb{H}$ *of* $\mathbb{G}$, *there exists a natural number* n *and an* H°*-equivariant closed embedding* $V_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K)^n$.

Proof. Suppose first that V is admissible, and that $\mathbb{H}$ is a good analytic open subgroup of G. As in the proof of lemma 6.1.6, we write $\mathbb{H}^\circ = \bigcup_{n=1}^\infty \mathbb{H}_{r_n}$, where $\{r_n\}_{n\geq 1}$ is a strictly increasing sequence of real numbers belonging to $|\overline{L}^\times|$ and converging to 1 from below. Since each $\mathbb{H}_{r_n}$ is normalized by H, it follows that if V is an H-representation, then for each value of n, the space $V_{\mathbb{H}_{r_n}-\mathrm{an}}$ is naturally an H-representation, and so in particular, an H°-representation.

From proposition 6.1.2 we deduce the existence of an H_{r_1}-equivariant closed embedding $V_{\mathbb{H}_{r_1}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_{r_1}, K)^m$ for some natural number m. Since H° normalizes $\mathbb{H}_{r_1}$, this embedding lifts to an H°-equivariant closed embedding $V_{\mathbb{H}_{r_1}-\mathrm{an}} \to \mathcal{C}(H^\circ, K)^m_{\mathbb{H}_{r_1}-\mathrm{an}}$. Passing to $\mathbb{H}^\circ$-analytic vectors, and appealing to propositions 3.4.4 and 3.4.10, we obtain an H°-equivariant closed embedding

$$(V_{\mathbb{H}_{r_1}-\mathrm{an}})_{\mathbb{H}^\circ-\mathrm{an}} \to (\mathcal{C}(H^\circ, K)^m_{\mathbb{H}_{r_1}-\mathrm{an}})_{\mathbb{H}^\circ-\mathrm{an}}. \tag{6.1.8}$$

Proposition 3.4.12 (and the fact that V is assumed to be an LB-space) implies that the natural map $(V_{\mathbb{H}_{r_1}-\mathrm{an}})_{\mathbb{H}^\circ-\mathrm{an}} \to V_{\mathbb{H}^\circ-\mathrm{an}}$ is an isomorphism. Similarly, the natural map $(\mathcal{C}(H^\circ, K)^m_{\mathbb{H}_{r_1}-\mathrm{an}})_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}(H^\circ, K)^m_{\mathbb{H}^\circ-\mathrm{an}}$ is an isomorphism. Thus we may rewrite (6.1.8) as an H°-equivariant closed embedding

$$V_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}(H^\circ, K)^m_{\mathbb{H}^\circ-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K)^m \tag{6.1.9}$$

(the isomorphism being provided by proposition 3.4.11). Since $\mathbb{H}$ was an arbitrary good analytic open subgroup of G, we find that (i) implies (ii).

It is obvious that (ii) implies (iii), and so we turn to showing that (iii) implies (i). Suppose that $\mathbb{H}$ is a good analytic open subgroup of G for which we have a H°-equivariant closed embedding $V_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K)^m$ for some natural number m. Since H acts on the source (a consequence of the fact that H normalizes $\mathbb{H}^\circ$) this lifts to an H-equivariant closed embedding $V_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}(H, K)^m_{\mathbb{H}^\circ-\mathrm{an}}$. Passing to $\mathbb{H}$-analytic vectors, and appealing to Propositions 3.3.18 and 3.3.23, we obtain a closed H-equivariant embedding

$$(V_{\mathbb{H}^\circ-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to (\mathcal{C}(H, K)_{\mathbb{H}^\circ-\mathrm{an}})^m_{\mathbb{H}-\mathrm{an}}. \tag{6.1.10}$$

A consideration of propositions 3.3.7 and 3.4.14 shows that the natural map

$$\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^m \to \mathcal{C}(H, K)^m_{\mathbb{H}^\circ-\mathrm{an}}$$

induces an isomorphism

$$\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^m \xrightarrow{\sim} (\mathcal{C}(H, K)^m_{\mathbb{H}^\circ-\mathrm{an}})_{\mathbb{H}-\mathrm{an}},$$

while proposition 3.4.14 also implies that the natural map $(V_{\mathbb{H}^\circ-\mathrm{an}})_{\mathbb{H}-\mathrm{an}} \to V_{\mathbb{H}-\mathrm{an}}$ is an isomorphism. Thus (6.1.10) can be rewritten as a closed embedding $V_{\mathbb{H}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^m$. Since H was one of a cofinal sequence of analytic open subgroups of

G, we see that V satisfies the conditions of definition 6.1.1, and so is an admissible representation of G. $\square$

Let H be a good analytic open subgroup of G. If as in the proof of lemma 6.1.6 we write $\mathbb{H}^\circ = \bigcup_{n=1}^{\infty} \mathbb{H}_{r_n}$, then $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \xrightarrow{\sim} \varinjlim_n \mathcal{D}^{\mathrm{an}}(\mathbb{H}_{r_n}, K)_b$. As observed in section 5.2, $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ is a naturally a topological K-algebra of compact type.

Suppose now that V is an admissible locally analytic representation of G. As observed in the proof of lemma 6.1.6, we see that $V_{\mathbb{H}^\circ-\mathrm{an}} = \varprojlim_n V_{\mathbb{H}_n-\mathrm{an}}$ is a projective limit of Banach spaces with compact transition maps. Passing to duals, we find that $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b = \varinjlim_n (V_{\mathbb{H}_n-\mathrm{an}})'_b$. Corollary 5.1.8 shows that for each n, the Banach space $(V_{\mathbb{H}_n-\mathrm{an}})'_b$ is a left $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_n, K)_b$-module, and that the multiplication map $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_n, K)_b \times (V_{\mathbb{H}_n-\mathrm{an}})'_b \to (V_{\mathbb{H}_n-\mathrm{an}})'_b$ is jointly continuous. Passing to the locally convex inductive limit in n, we obtain a map

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \times (V_{\mathbb{H}^\circ-\mathrm{an}})'_b \to (V_{\mathbb{H}^\circ-\mathrm{an}})'_b, \tag{6.1.12}$$

which makes $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b$ a left $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$-module, and which is *a priori* separately continuous. Since each of $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ and $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b$ is of compact type, it follows from proposition 1.1.31 that in fact this map is jointly continuous, that is, that $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b$ is a topological $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$-module.

Lemma 6.1.13. *If V is an admissible locally analytic G-representation, and if $\mathbb{H}$ is a good analytic open subgroup of G, then the topological $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$-module $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b$ is finitely generated* (*in the sense of definition* 1.2.1 (iii)).

Proof. Dualizing the H°-equivariant closed embedding $V_{\mathbb{H}^\circ-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K)^m$ of part (ii) of proposition 6.1.7 yields a surjection $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^m \to (V_{\mathbb{H}^\circ-\mathrm{an}})'_b$. Since this is a surjection of spaces of compact type, it is an open map, and so $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b$ is finitely generated in the strict sense of definition 1.2.1 (iii). $\square$

Suppose now that $J \subset H$ is an inclusion of good analytic open subgroups of G, with the property that H° normalizes $\mathbb{J}^\circ$. Lemma 6.1.13 shows that the spaces $(V_{\mathbb{H}^\circ-\mathrm{an}})'_b$ and $(V_{\mathbb{J}^\circ-\mathrm{an}})'_b$ are modules over the convex K-algebras $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ and $\mathcal{D}^{\mathrm{an}}(\mathbb{J}^\circ, K)_b$, respectively. Let us simplify our notation a little, and denote these topological rings by $D(\mathbb{H}^\circ)$ and $D(\mathbb{J}^\circ)$, respectively. The inclusion $\mathbb{J}^\circ \to \mathbb{H}^\circ$ induces a continuous ring homomorphism $D(\mathbb{J}^\circ) \to D(\mathbb{H}^\circ)$.

There is another ring that we will need to consider. We let $D(\mathbb{J}^\circ, H^\circ)$ denote the strong dual to the space $\mathcal{C}(H^\circ, K)_{\mathbb{J}^\circ-\mathrm{an}}$. The natural continuous map $\mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K) \to \mathcal{C}(H^\circ, K)_{\mathbb{J}^\circ-\mathrm{an}}$ has dense image, and so induces a continuous injection $D(\mathbb{J}^\circ, H^\circ) \to D(H^\circ)$. The image of this injection is easily checked to be a K-subalgebra of $D(H^\circ)$, and we define a K-algebra structure on $D(\mathbb{J}^\circ, H^\circ)$ by requiring that this injection be a K-algebra homomorphism. In this way $D(\mathbb{J}^\circ, H^\circ)$ becomes a topological K-algebra (as one checks). The restriction map $\mathcal{C}(H^\circ, K)_{\mathbb{J}^\circ-\mathrm{an}} \to \mathcal{C}(J^\circ, K)_{\mathbb{J}^\circ-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{J}^\circ, K)$ (the isomorphism is provided by proposition 3.4.5) yields a closed embedding of topological rings $D(\mathbb{J}^\circ) \to D(\mathbb{J}^\circ, H^\circ)$, and one immediately sees that $D(\mathbb{J}^\circ, H^\circ) = \bigoplus_{h \in H^\circ/J^\circ} D(\mathbb{J}^\circ) * \delta_h$. (As indicated, the direct sum ranges over a set of coset representatives of J° in H°.) (The notions introduced in this paragraph are similar to those introduced in the proof of proposition 5.3.1.)

Since H° normalizes $\mathbb{J}^\circ$, there is a natural action of H° on $V_{\mathbb{J}^\circ-\mathrm{an}}$, and hence the $D(\mathbb{J}^\circ)$-module structure on $(V_{\mathbb{J}^\circ-\mathrm{an}})'_b$ extends in a natural fashion to a $D(\mathbb{J}^\circ, H^\circ)$-module structure. The continuous H-equivariant injection $V_{\mathbb{H}^\circ-\mathrm{an}} \to V_{\mathbb{J}^\circ-\mathrm{an}}$ induces a continuous map

$$(V_{\mathbb{J}^\circ-\mathrm{an}})'_b \to (V_{\mathbb{H}^\circ-\mathrm{an}})'_b, \tag{6.1.14}$$

which is compatible with the natural map $D(\mathbb{J}^\circ, H^\circ) \to D(\mathbb{H}^\circ)$ and the topological module structures on its source and target. The map (6.1.14) thus induces a continuous map of $D(\mathbb{H}^\circ)$-modules

$$D(\mathbb{H}^\circ) \hat{\otimes}_{D(\mathbb{J}^\circ,H^\circ)} (V_{\mathbb{J}^\circ-\mathrm{an}})'_b \to (V_{\mathbb{H}^\circ-\mathrm{an}})'_b. \tag{6.1.15}$$

The following result deals with a more general situation.

Proposition 6.1.16. *If M is a Hausdorff convex K-vector space of compact type equipped with a finitely generated topological $D(\mathbb{J}^\circ, H^\circ)$-module structure, then there is a natural H°-equivariant isomorphism*

$$(D(\mathbb{H}^\circ) \hat{\otimes}_{D(\mathbb{J}^\circ,H^\circ)} M)'_b \xrightarrow{\sim} (M'_b)_{\mathbb{H}^\circ-\mathrm{an}}.$$

Proof. Since M is of compact type, its strong dual M'_b is a nuclear Fréchet space. Corollary 3.4.5 thus implies that there is a natural isomorphism

$$(M'_b)_{\mathbb{H}^\circ-\mathrm{an}} \xrightarrow{\sim} (\mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K) \hat{\otimes}_K M'_b)^{\Delta_{1,2}(H^\circ)}. \tag{6.1.17}$$

Proposition 1.2.5, and the fact that compact type spaces are hereditarily complete, implies that the tensor product

$$D(\mathbb{H}^\circ) \hat{\otimes}_{D(\mathbb{J}^\circ,H^\circ)} M \tag{6.1.18}$$

is the quotient of the tensor product $D(\mathbb{H}^\circ) \otimes_K M$ by the closure of its subspace spanned by expressions of the form $\mu * \nu \otimes m - \mu \otimes \nu * m$, with $\mu \in D(\mathbb{H}^\circ)$, $\nu \in D(\mathbb{J}^\circ, \mathbb{H}^\circ)$, and $m \in M$. Since the delta functions δ_h, for $h \in H^\circ$, span a strongly dense subspace of $D(\mathbb{J}^\circ, H^\circ)$ (see the discussion following definition 2.2.3), we see that (6.1.18) can also be described as the quotient of $D(\mathbb{H}^\circ) \otimes_K M$ by the closure of its subspace spanned by expressions of the form $\mu * \delta_h \otimes m - \mu \otimes \delta_h * m$, with $\mu \in D(\mathbb{H}^\circ)$, $h \in H^\circ$, and $m \in M$.

This description of (6.1.18), together with proposition 1.1.32 (ii), implies that there is a natural isomorphism

$$(D(\mathbb{H}^\circ) \hat{\otimes}_{D(\mathbb{J}^\circ,H^\circ)} M)'_b \xrightarrow{\sim} (\mathcal{C}^{\mathrm{an}}(\mathbb{H}^\circ, K) \hat{\otimes}_K M'_b)^{\Delta_{1,2}(H^\circ)}.$$

Thus (6.1.17) yields the required isomorphism. $\square$

Corollary 6.1.19. *The morphism* (6.1.15) *is a topological isomorphism.*

Proof. Proposition 6.1.16 yields an isomorphism $(D(\mathbb{H}^\circ) \hat{\otimes}_{D(\mathbb{J}^\circ,H^\circ)} (V_{\mathbb{J}^\circ-\mathrm{an}})'_b)'_b \xrightarrow{\sim} (V_{\mathbb{J}^\circ-\mathrm{an}})_{\mathbb{H}^\circ-\mathrm{an}}$. By corollary 3.4.15, the natural map $(V_{\mathbb{J}^\circ-\mathrm{an}})_{\mathbb{H}^\circ-\mathrm{an}} \to V_{\mathbb{H}^\circ-\mathrm{an}}$ is an isomorphism. It is easily checked that dualizing the composite of these two isomorphisms yields the map (6.1.15). Thus this map is an isomorphism, as claimed. $\square$

Theorem 6.1.20. *Let H be a good analytic open subgroup of G. Passage to the dual induces an anti-equivalence of categories between the category of admissible locally analytic H°-representations (with morphisms being continuous H°-equivariant maps) and the category of coadmissible locally convex topological modules over the weak Fréchet-Stein algebra $\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b$.*

Proof. Suppose first that V is equipped with an admissible locally analytic representation of H°. Let $\{r_n\}$ be a strictly decreasing sequence of real numbers lying in the open interval between 0 and 1, converging to 0 from above, and such that each $r_n \in |\overline{L}^\times|$. Then $V \xrightarrow{\sim} \varinjlim_n V_{\mathbb{H}^\circ_{r_n}-\mathrm{an}}$, and so $V'_b \xrightarrow{\sim} \varprojlim_n (V_{\mathbb{H}^\circ_{r_n}-\mathrm{an}})'_b$. In the proof of proposition 5.3.1, it is proved that the isomorphism $\mathcal{D}^{\mathrm{la}}(H^\circ, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{H}^\circ_{r_n}, H^\circ)$ exhibits $\mathcal{D}^{\mathrm{la}}(H^\circ, K)$ as a weak Fréchet-Stein algebra. We will show that V'_b is coadmissible with respect to this weak Fréchet-Stein structure.

For each $n \geq 1$, lemma 6.1.13 shows that $V_{\mathbb{H}^\circ_{r_n}-\mathrm{an}}$ is finitely generated over $D(\mathbb{H}^\circ_{r_n})$, and so in particular over $D(\mathbb{H}^\circ_{r_n}, H^\circ)$, while corollary 6.1.19 yields an isomorphism $D(\mathbb{H}^\circ_{r_n}, H^\circ) \hat{\otimes}_{D(\mathbb{H}^\circ_{r_{n+1}}, H^\circ)} (V_{\mathbb{H}^\circ_{r_{n+1}}-\mathrm{an}})'_b \xrightarrow{\sim} (V_{\mathbb{H}^\circ_{r_n}-\mathrm{an}})'_b$. Thus the projective system $\{V_{\mathbb{H}^\circ_{r_n}-\mathrm{an}}\}_{n\geq 1}$ satisfies the conditions of definition 1.2.8, showing that V'_b is a coadmissible $\mathcal{D}^{\mathrm{la}}(H^\circ, K)$-module.

Conversely, suppose that $M = \varprojlim_n M_n$ is a coadmissible $\mathcal{D}^{\mathrm{la}}(H^\circ, K)$-module, so that M_n is a Hausdorff finitely generated locally convex $D(\mathbb{H}^\circ_{r_n}, H^\circ)$-module, for each $n \geq 1$. For each such value of n, we obtain a surjection of spaces of compact type $D(\mathbb{H}^\circ_{r_n}, H^\circ)^m \to M_n$, for some natural number m. Dualizing this yields a closed embedding of nuclear Fréchet spaces $(M_n)'_b \to \mathcal{C}(H^\circ, K)^m_{\mathbb{H}^\circ_{r_n}-\mathrm{an}}$. Passing to $\mathbb{H}_{r_n}$-analytic vectors, and applying propositions 3.3.23 and 3.4.14 we obtain a closed embedding

$$((M_n)'_b)_{\mathbb{H}_{r_n}-\mathrm{an}} \to \mathcal{C}(H^\circ, K)^m_{\mathbb{H}_{r_n}-\mathrm{an}}. \tag{6.1.21}$$

Condition (ii) of definition 1.2.8, together with proposition 6.1.16, yields, for each $n \geq 1$, a natural isomorphism $(M_n)'_b \xrightarrow{\sim} ((M_{n+1})'_b)_{\mathbb{H}^\circ_{r_n}-\mathrm{an}}$. Passing to $\mathbb{H}_{r_n}$-analytic vectors, and applying proposition 3.4.14, we obtain an isomorphism

$$((M_n)'_b)_{\mathbb{H}_{r_n}-\mathrm{an}} \xrightarrow{\sim} ((M_{n+1})'_b)_{\mathbb{H}_{r_n}-\mathrm{an}}.$$

Since $M'_b \xrightarrow{\sim} \varinjlim_n (M_n)'_b$, we deduce that the natural map

$$((M_n)'_b)_{\mathbb{H}_{r_n}-\mathrm{an}} \to (M'_b)_{\mathbb{H}_{r_n}-\mathrm{an}}$$

is an isomorphism. In light of the closed embedding (6.1.21), we conclude that M'_b is an admissible locally analytic H°-representation, as claimed. $\square$

The next result shows that our definition of an admissible locally analytic representation of G agrees with that of [27, p. 33].

Corollary 6.1.22. *If G is a locally L-analytic group, and if V is a convex K-vector space of compact type equipped with a locally analytic representation of G, then V is admissible if and only if V'_b is a coadmissible module with respect to the natural*

$\mathcal{D}^{\mathrm{la}}(H,K)$-module structure on V', for some (or equivalently, every) compact open subgroup H of G.

Proof. Let J be a good analytic open subgroup of G contained in H. The algebra $\mathcal{D}^{\mathrm{la}}(H,K)$ is free of finite rank (on both sides) over $\mathcal{D}^{\mathrm{la}}(J^\circ,K)$, and so V'_b is coadmissible as a $\mathcal{D}^{\mathrm{la}}(H,K)$-module if and only if it is so as a $\mathcal{D}^{\mathrm{la}}(J^\circ,K)$-module. Similarly, V is an admissible locally analytic G-representation if and only if it is an admissible locally analytic J°-representation. The corollary thus follows from theorem 6.1.20. □

The following result is a restatement of [27, prop. 6.4].

Corollary 6.1.23. *If G is a locally L-analytic group, and if K is discretely valued, then the category of admissible locally analytic G-representations and continuous G-equivariant morphisms is closed under passing to closed subobjects and Hausdorff quotients. Furthermore, any morphism in this category is necessarily strict. Consequently, this category is abelian.*

Proof. This is a consequence of theorem 6.1.20 and corollary 5.3.19, together with the general properties of coadmissible modules over Fréchet-Stein algebras that are summarized in theorem 1.2.11 and the remarks that follow it. □

6.2. Strongly admissible locally analytic representations and admissible continuous representations

The following class of locally analytic representations was introduced by Schneider and Teitelbaum in [23].

Definition 6.2.1. A locally analytic representation V of G is called strongly admissible if V is a complete LB-space, and if its dual space V' is finitely generated as a left $\mathcal{D}^{\mathrm{la}}(H,K)$-module for one (and hence every) compact open subgroup H of G.

Note that theorem 5.1.15 (iii) implies that a locally analytic representation of G on a complete LB-space V satisfies the condition of definition 6.2.1 if and only if V admits a closed embedding into $\mathcal{C}^{\mathrm{la}}(H,K)^n$ for some natural number n, and one (or equivalently, every) compact open subgroup of G. In particular, any such V is necessarily of compact type.

We also remark that the strongly admissible locally analytic representations of G are precisely the continuous duals of those $\mathcal{D}^{\mathrm{la}}(H,K)$-modules which are analytic in the sense of [24, p. 112].

Proposition 6.2.2. *If V is a strongly admissible locally analytic representation of G, then V is admissible.*

Proof. Since V is strongly admissible, it is an LB-space by assumption. If H is an analytic open subgroup of G then by assumption we may find a surjection of left $\mathcal{D}^{\mathrm{la}}(G,K)$-modules $\mathcal{D}^{\mathrm{la}}(G,K)^n \to V'$ for some natural number n, which by theorem 3.3.1 arises by dualizing a G-equivariant closed embedding $V \to \mathcal{C}^{\mathrm{la}}(G,K)^n$. Passing to $\mathbb{H}$-analytic vectors and applying proposition 3.3.23 and corollary 3.3.26 we obtain an H-equivariant closed embedding $V_{\mathbb{H}-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H},K)^n$. It follows that $(V_{\mathbb{H}-\mathrm{an}})'$ is finitely generated as a left $\mathcal{D}^{\mathrm{an}}(\mathbb{H},K)$-module. □

A basic method for producing strongly admissible locally analytic representations of G is by passing to the locally analytic vectors of continuous representations in G

that satisfy the admissibility condition introduced in [25]. We begin by reminding the reader of that definition.

Proposition-Definition 6.2.3. *A continuous G-action on a Banach space V is said to be an admissible continuous representation of G (or an admissible Banach space representation of G) if V' is finitely generated as a left $\mathcal{D}(H,K)$-module for one (and hence every) compact open subgroup H of G. (This is equivalent, by theorem* 5.1.15 (i)*, to the existence of a closed H-equivariant embedding $V \to \mathcal{C}(H,K)^n$ for some $n \geq 0$.)*

Proof. We must check that the definition is independent of the choice of compact open subgroup H of G. This follows immediately from the fact that if $H_1 \subset H_2$ is an inclusion of compact open subgroups of H, and if we write $H_2 = \coprod h_i H_1$ as a finite disjoint union of right H_1 cosets, then $\mathcal{D}(H_2,K) = \bigoplus \mathcal{D}(H_1,K)\delta_{h_i}$, and so $\mathcal{D}(H_2,K)$ is finitely generated as a left $\mathcal{D}(H_1,K)$-module. □

Note that in [25], the authors restrict their attention to the case of local K. In this case, it follows from [25, lem. 3.4] that definition 6.2.3 agrees with the notion of admissibility introduced in that reference. (See also proposition 6.5.7 below.)

Proposition 6.2.4. *If V is a Banach space equipped with an admissible continuous representation of G then V_{la} is a strongly admissible locally analytic representation of G.*

Proof. Since V is a Banach space, proposition 3.5.6 implies that V_{la} is an LB-space. By assumption, we may find a surjection $\mathcal{D}(H,K)^n \to V'$ for some natural number n and some compact open subgroup H of G, and so a closed H-equivariant embedding $V \to \mathcal{C}(H,K)^n$. Now by proposition 3.5.11 there is an isomorphism $\mathcal{C}(H,K)_{\mathrm{la}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(H,K)$. Since $\mathcal{C}^{\mathrm{la}}(H,K)$ is of compact type, we obtain by proposition 3.5.10 a closed embedding $V_{\mathrm{la}} \to \mathcal{C}^{\mathrm{la}}(H,K)^n$. Dualizing this yields a surjection $\mathcal{D}^{\mathrm{la}}(H,K)^n \to (V_{\mathrm{la}})'$, proving that V_{la} is strongly admissible. □

When K is local, the preceding result was established by Schneider and Teitelbaum [27, thm. 7.1]. (As was noted in the remark following the proof of theorem 3.5.7, the topology with which these authors equip V_{la} is *a priori* coarser than the topology with which we equip that space. In the case of V being an admissible continuous representation, however, both topologies are of compact type, since they each underly a strongly admissible locally analytic representation of G. Thus the topologies coincide.)

The remainder of this section is devoted to establishing some fundamental properties of admissible continuous G-representations. In the local case, all these results (other than propositions 6.2.6 and 6.2.7) are contained in [25]. Using a little more functional analysis, we are able to extend these results to the case where K is discretely valued.

In fact, the first three results require no additional assumption on K at all.

Proposition 6.2.5. *If V is a K-Banach space equipped with an admissible continuous representation of G and W is a closed G-invariant K-subspace of V, then the continuous G-representation on W is also admissible.*

Proof. This follows immediately from the fact that W' is a quotient of V'. □

Proposition 6.2.6. *If V is a K-Banach space equipped with an admissible continuous representation of G and W is a finite dimensional continuous representation of G then $V \otimes_K W$, equipped with the diagonal action of G, is an admissible continuous representation of G.*

Proof. Replacing G by a compact open subgroup if necessary, we may assume that G is compact and that there is a closed G-equivariant embedding $V \to \mathcal{C}(G,K)^n$ for some $n \geq 0$. Tensoring with W over K yields a closed embedding $V \otimes_K W \to \mathcal{C}(G,K)^n \otimes_K W$. The remark following the proof of lemma 3.2.11 shows that there is a natural isomorphism $\mathcal{C}(G,K)^n \otimes_K W \xrightarrow{\sim} \mathcal{C}(G,K)^n \otimes_K W$ which intertwines the diagonal G-action on the source with the tensor product of the right regular G-action and the trivial G-action on the target. If d denotes the dimension of W, then we may thus find a G-equivariant closed embedding $V \otimes_K W \to \mathcal{C}(G,K)^{nd}$. This proves that $V \otimes_K W$ is equipped with an admissible continuous representation of G. $\square$

Proposition 6.2.7. *If $0 \to U \to V \to W \to 0$ is a G-equivariant exact sequence of K-Banach spaces equipped with continuous G-actions, and if each of U and W is an admissible continuous G-representation, then V is also an admissible continuous G-representation.*

Proof. Without loss of generality, we may assume that G is compact. Passing to strong duals, and taking into account corollary 5.1.7, we obtain the short exact sequence of $\mathcal{D}(G,K)$-modules $0 \to W'_b \to V'_b \to U'_b \to 0$. If each of the end terms is finitely generated over $\mathcal{D}(G,K)$, then the same is true of the middle term. $\square$

As in [27], the remainder of our results depend on the fact that $\mathcal{D}(G,\mathcal{O}_K)$ is Noetherian. When K is local, this is provided by [16, V.2.2.5]. In fact, it is easily extended to the case where K is discretely valued.

Theorem 6.2.8. *If G is a compact locally L-analytic group and K is discretely valued, then the K-Banach algebra $\mathcal{D}(G,K)$, and its unit ball $\mathcal{D}(G,\mathcal{O}_K)$, are both Noetherian rings.*

Proof. It follows from [16, V.2.2.4] that $\mathcal{D}(G,\mathbb{Z}_p)$ is Noetherian. Indeed, that reference shows that G contains an open subgroup H such that $\mathcal{D}(H,\mathbb{Z}_p)$ is equipped with an exhaustive filtration $F^\bullet\mathcal{D}(H,\mathbb{Z}_p)$, with respect to which it is complete, whose associated graded ring $\mathrm{Gr}^\bullet_F\mathcal{D}(H,\mathbb{Z}_p)$ is isomorphic to the enveloping algebra over the ring $\Gamma := \mathbb{F}_p[\epsilon]$ of a finite rank Lie algebra over Γ.

If we endow $\mathcal{O}_K$ with its p-adic filtration, then we may endow the completed tensor product $\mathcal{O}_K \hat{\otimes}_{\mathbb{Z}_p} \mathcal{D}(H,\mathbb{Z}_p)$ with the (completion of the) tensor product filtration. The associated graded ring is then isomorphic to the tensor product $(\mathcal{O}_K/p) \otimes_{\mathbb{F}_p} \mathrm{Gr}^\bullet_F\mathcal{D}(H,\mathbb{Z}_p)$, and thus is isomorphic to the enveloping algebra over the ring $(\mathcal{O}_K/p)[\epsilon]$ of a finite rank Lie algebra over this ring. In particular, it is Noetherian, and hence the same is true of the completed tensor product $\mathcal{O}_K \hat{\otimes}_{\mathbb{Z}_p} \mathcal{D}(H,\mathbb{Z}_p)$ itself. This ring is naturally isomorphic to $\mathcal{D}(H,\mathcal{O}_K)$, and $\mathcal{D}(G,\mathcal{O}_K)$ is free of finite rank as a module over this latter ring. Thus the theorem is proved. $\square$

Proposition 6.2.9. *If K is discretely valued, then any continuous G-equivariant map between admissible continuous G-representations is strict.*

Proof. We may replace G by a compact open subgroup, and thus suppose that G is compact. If $\phi : V \to W$ is a continuous G-equivariant map between two admissible continuous G-representations, then to show that ϕ is strict, it suffices to show that $\phi' : V_b' \to W_b'$ is strict [5, cor. 3, p. IV.30]. The proposition now follows from theorem 6.2.8 and proposition 1.2.4. $\square$

Proposition 6.2.10. *Suppose that G is compact, and that K is discretely valued. The association of V' to V induces an anti-equivalence between the category of admissible continuous G-representations* (*with morphisms being continuous G-equivariant maps*) *and the category of finitely generated $\mathcal{D}(G,K)$-modules.*

Proof. Since the transpose of a continuous linear map between Hausdorff convex K-vector spaces vanishes if and only if the map itself does, we see that the passage from V to V' is faithful. We turn to proving that this functor is full.

Suppose that V and W are admissible continuous G-representations, and that we are given a $\mathcal{D}(G,K)$-linear map

$$W' \to V'. \tag{6.2.11}$$

We must show that this map arises as the transpose of a continuous G-equivariant map $V \to W$.

Let us choose surjections of $\mathcal{D}(G,K)$-modules $\mathcal{D}(G,K)^m \to V'$ and $\mathcal{D}(G,K)^n \to W'$, for some $m, n \geq 0$. Then (6.2.11) may be lifted to yield a commutative square of $\mathcal{D}(G,K)$-linear maps

$$\begin{tikzcd} \mathcal{D}(G,K)_b^n \arrow[r] \arrow[d] & \mathcal{D}(G,K)_b^m \arrow[d] \\ W_b' \arrow[r] & V_b'. \end{tikzcd} \tag{6.2.12}$$

Part (iii) of corollary 5.1.7 shows that, if we remove the lower horizontal arrow from (6.2.12), then the resulting diagram arises by dualizing a diagram of continuous G-equivariant maps of the form

$$\begin{tikzcd} V \arrow[d] & W \arrow[d] \\ \mathcal{C}(G,K)^m \arrow[r] & \mathcal{C}(G,K)^n. \end{tikzcd} \tag{6.2.13}$$

Furthermore, part (i) of theorem 5.1.15 implies that the vertical arrows in (6.2.13) are closed embeddings.

The fact that the lower horizontal arrow of (6.2.12) can be filled in implies that the lower horizontal arrow of (6.2.13) restricts to a G-equivariant map $\phi : V \to W$. By construction, the transpose of ϕ is equal to the morphism (6.2.11) with which we began, and so the functor under consideration is full, as claimed.

In order to prove that our functor is essentially surjective, it suffices to show that any surjection of $\mathcal{D}(G,K)$-modules

$$\mathcal{D}(G,K)^n \to M \tag{6.2.14}$$

arises by dualizing a closed G-equivariant embedding $V \to \mathcal{C}(G,K)^n$. For this, it suffices in turn to show that the kernel of (6.2.14) is closed in $\mathcal{D}(G,K)_s^n$. For if we then let M_s denote M equipped with the topology obtained by regarding it as a quotient of $\mathcal{D}(G,K)_s^n$, we obtain V as the closed subspace $(M_s)'$ of $\mathcal{C}(G,K)^n \xrightarrow{\sim} (\mathcal{D}(G,K)_s^n)'$.

We must show that any $\mathcal{D}(G,K)$-submodule N of $\mathcal{D}(G,K)_s^n$ is closed. Since $\mathcal{D}(G,K)$ is the dual of the Banach space $\mathcal{C}(G,K)$, it suffices to show that the intersection of N with the unit ball of $\mathcal{D}(G,K)_b^n$ is closed in $\mathcal{D}(G,K)_s$ [5, cor. 3, p. IV.25].

As in the statement of theorem 6.2.8, let $\mathcal{D}(G,\mathcal{O}_K)$ denote the unit ball of the Banach algebra $\mathcal{D}(G,K)_b$. If $\mathcal{D}(G,\mathcal{O}_K)_s$ denotes $\mathcal{D}(G,\mathcal{O}_K)$ equipped with the topology induced from $\mathcal{D}(G,K)_s$, then $\mathcal{D}(G,\mathcal{O}_K)_s$ is c-compact [20, lem. 12.10].

Passing from $\mathcal{D}(G,K)$ to $\mathcal{D}(G,K)^n$, we find that $\mathcal{D}(G,\mathcal{O}_K)_s^n$ is the unit ball of $\mathcal{D}(G,K)_b^n$, equipped with the topology induced from $\mathcal{D}(G,K)_s^n$. The intersection $N\bigcap\mathcal{D}(G,\mathcal{O}_K)_s^n$ is a $\mathcal{D}(G,K)$-submodule of $\mathcal{D}(G,\mathcal{O}_K)_s^n$, and we must show that it is closed.

Since the ring $\mathcal{D}(G,\mathcal{O}_K)$ is Noetherian, by theorem 6.2.8, the $\mathcal{D}(G,\mathcal{O}_K)$-module $N\bigcap\mathcal{D}(G,\mathcal{O}_K)_s^n$ is finitely generated. If $n_1,\ldots,n_r$ are generators of this module, then it is the image of $\mathcal{D}(G,\mathcal{O}_K)_s^r$ under the map

$$(\mu_1,\ldots,\mu_r)\mapsto \mu_1 * n_1+\cdots+\mu_r * n_r. \tag{6.2.15}$$

Now (6.2.15) is the restriction to $\mathcal{D}(G,\mathcal{O}_K)_s^r$ of a corresponding map $\mathcal{D}(G,K)_s^r \to \mathcal{D}(G,K)^n$, and part (iii) of corollary 5.1.7 shows that this map arises as the dual of a continuous map $\mathcal{C}(G,K)^n \to \mathcal{C}(G,K)^r$, and so is continuous. Thus (6.2.15) is continuous. Its source being c-compact , it has c-compact, and hence closed, image. Thus we conclude that $N\bigcap\mathcal{D}(G,\mathcal{O}_K)_s^n$ is closed in $\mathcal{D}(G,\mathcal{O}_K)_s^n$, as required. □

Corollary 6.2.16. *If K is discretely valued, then the category of admissible continuous G-representations (with morphisms being continuous G-equivariant maps) is abelian.*

Proof. If G is compact, then this follows from proposition 6.2.10. Indeed, the ring $\mathcal{D}(G,K)$ is Noetherian by theorem 6.2.8, and so the category of finitely generated $\mathcal{D}(G,K)$-modules is abelian. The general case follows easily from this one. □

We remark that proposition 6.2.9 is also a formal consequence of this result. As already noted, in the case where K is local, proposition 6.2.10, and hence corollary 6.2.16 and proposition 6.2.9, are established in [25].

6.3. Admissible smooth and admissible locally algebraic representations

In this section we develop some connections between the notions of smooth and locally algebraic representations, and of admissible locally analytic representations.

Lemma 6.3.1. *Suppose that G is equal to the group of L-valued points of an affinoid rigid analytic group $\mathbb{G}$ defined over L, and that furthermore G is Zariski dense in $\mathbb{G}$. If $\mathcal{C}^{\mathrm{an}}(\mathbb{G},K)$ is regarded as a locally analytic representation of G via the right regular G-action then $\mathcal{C}^{\mathrm{an}}(\mathbb{G},K)^{\mathfrak{g}}$ is one-dimensional, consisting precisely of the constant functions.*

Proof. Lemma 4.1.4 shows that $\mathcal{C}^{\mathrm{an}}(\mathbb{G},K)^{\mathfrak{g}}$ consists of those rigid analytic function on $\mathbb{G}$ that are stabilized by the right regular action of G. Since G is Zariski dense in $\mathbb{G}$, these are precisely the constant functions. □

Proposition 6.3.2. *If V is an admissible smooth representation of a locally L-analytic group G, then, when equipped with its finest convex topology, V becomes an admissible locally analytic representation of G that is $U(\mathfrak{g})$-trivial. Conversely, if V is an admissible locally analytic representation of G that is $U(\mathfrak{g})$-trivial then the topology on V is its finest locally convex topology, and the G-representation on the K-vector space underlying V is an admissible smooth representation.*

Proof. First suppose that V is an admissible smooth representation of G. Let H_n be a cofinal sequence of normal analytic open subgroups of G. Then $V = \bigcup_n V^{H_n}$. Since each V^{H_n} is finite dimensional by assumption, we see that V, when equipped with its finest convex topology, is isomorphic to the locally convex inductive limit $\varinjlim_n V^{H_n}$, and so is of compact type. Corollary 4.1.7 implies that $(V_{\mathrm{la}})^{\mathfrak{g}}$ maps bijectively onto V, and thus in particular that V is locally analytic and that $V^{\mathfrak{g}} = V$. Corollary 4.1.5 then shows that $V_{\mathbb{H}_n-\mathrm{an}} = V^{H_n}$. Since by assumption this is a finite dimensional space, its dual is certainly finitely generated over $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_n, K)$.

Conversely, let V be an admissible locally analytic representation that is $U(\mathfrak{g})$-trivial. It follows from corollary 4.1.6 that the G-representation on V is smooth. We have to show that it is furthermore an admissible smooth representation, and also that the topology on V is its finest convex topology.

As above, let H_n denote a cofinal sequence of normal analytic open subgroups of G. Theorem 3.3.1 shows that for each n we may find an embedding of $V_{\mathbb{H}_n-\mathrm{an}}$ into $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^m$, for some natural number m, and hence an embedding $(V_{\mathbb{H}_n-\mathrm{an}})^{\mathfrak{g}} \to (\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^m)^{\mathfrak{g}}$. Lemma 6.3.1 shows that the target of this injection is finite dimensional, and thus that $(V_{\mathbb{H}_n-\mathrm{an}})^{\mathfrak{g}}$ is also finite dimensional.

By assumption $(V_{\mathbb{H}_n-\mathrm{an}})^{\mathfrak{g}}$ is equal to $V_{\mathbb{H}_n-\mathrm{an}}$ for each n, while corollary 4.1.5 shows that $(V_{\mathbb{H}_n-\mathrm{an}})^{\mathfrak{g}}$ maps bijectively onto the subspace of H_n-fixed vectors in V. Thus on the one hand we find that the subspace of H_n-fixed vectors in V is finite dimensional, proving that the smooth G-representation on V is admissible, while on the other hand, theorem 3.6.12 provides an isomorphism $\varinjlim_n (V_{\mathbb{H}_n-\mathrm{an}})^{\mathfrak{g}} = \varinjlim_n V_{\mathbb{H}_n-\mathrm{an}} = V_{\mathrm{la}} \xrightarrow{\sim} V$, so that V is the inductive limit of a family of finite dimensional spaces, and so is endowed with its finest convex topology. $\square$

Thus there is an equivalence of categories between the category of admissible smooth representations of G and the category of $U(\mathfrak{g})$-trivial admissible locally analytic representations of G. (This result, and the next, are due to Schneider and Teitelbaum [27, thm. 6.5].)

Corollary 6.3.3. *If V is an admissible locally analytic representation of a locally L-analytic group G, then the closed subspace $V^{\mathfrak{g}}$ of V is endowed with its finest convex topology, and the G-representation on $V^{\mathfrak{g}}$ is an admissible smooth representation of G.*

Proof. Since $V^{\mathfrak{g}}$ is a closed G-invariant subspace of V, proposition 6.1.4 implies that $V^{\mathfrak{g}}$ is again an admissible representation of G, which is $U(\mathfrak{g})$-trivial by construction. The corollary follows from proposition 6.3.2. $\square$

For the sake of completeness, we also recall the following definition and result.

Definition 6.3.4. If G is a locally L-analytic group then we say that a smooth representation V of G is a strongly admissible smooth representation if for one (or equivalently, every) compact open subgroup H of G, there is an H-equivariant embedding of V into the a finite direct sum of copies of the space of locally constant functions on H.

This is immediately checked to be equivalent to the definition of [24, p. 115].

Proposition 6.3.5. *If V is a strongly admissible locally analytic representation of a locally L-analytic group G, then the closed subspace $V^{\mathfrak{g}}$ of V is endowed with its finest convex topology, and the G-representation on $V^{\mathfrak{g}}$ is a strongly admissible smooth representation of G. Conversely, if we endow any strongly admissible smooth representation of G with its finest convex topology, we obtain a strongly admissible locally analytic representation of G.*

Proof. This is [24, prop. 2.1]. $\square$

We now suppose that $\mathbb{G}$ is a connected reductive linear algebraic group defined over L, and that G is an open subgroup of $\mathbb{G}(L)$. As in section 4.2, we let $\mathcal{R}$ denote the category of finite dimensional algebraic representations of $\mathbb{G}$ defined over K. We will also use the other definitions and notations introduced in that section.

Proposition 6.3.6. *If V is an admissible locally analytic representation of G and W is an object of $\mathcal{R}$, then $V_{W-\mathrm{lalg}}$ is a closed subspace of V, and the topology induced on $V_{W-\mathrm{lalg}}$ by the topology on V is its finest convex topology. In particular, $V_{W-\mathrm{lalg}}$ is again an admissible locally analytic representation of G.*

Proof. Proposition 4.2.10 shows that $V_{W-\mathrm{lalg}}$ is a closed subspace of V, topologically isomorphic to $\mathrm{Hom}(W,V)_{\mathrm{sm}} \otimes_B W$ (where $B = \mathrm{End}_G(W)$). Since V is admissible, proposition 6.1.5 (applied to the tensor product $V \otimes_K \check{W}$) implies that $\mathrm{Hom}(W,V)$ is also admissible. Proposition 6.3.3 shows that the space $\mathrm{Hom}(W,V)_{\mathrm{sm}}$ is equipped with its finest convex topology, and thus the same is true of the tensor product $\mathrm{Hom}(W,V)_{\mathrm{sm}} \otimes_B W$. The proposition follows. $\square$

Corollary 6.3.7. *If V is an admissible locally analytic representation of G, and if V is also locally algebraic as a representation of G, then V is equipped with its finest convex topology.*

Proof. Consider the natural map

$$\bigoplus_{W \in \hat{\mathbb{G}}} V_{W-\mathrm{lalg}} \to V. \tag{6.3.8}$$

Proposition 6.3.6 shows that each summand in the source of (6.3.8) is of compact type when endowed with its finest convex topology. Since $\hat{\mathbb{G}}$ is countable, we see that if we endow the source with its finest convex topology then it is again a space of compact type. Corollary 4.2.7 then shows that (6.3.8) is a continuous bijection between spaces of compact type, and thus that (6.3.8) is a topological isomorphism. $\square$

In light of the preceding corollary, we make the following definition.

Definition 6.3.9. If V is a locally algebraic representation of G that becomes admissible as a locally analytic representation when equipped with its finest convex topology, then we say that V (equipped with its finest convex topology) is an admissible locally algebraic representation of G. If V is furthermore locally W-algebraic, for some object W of $\mathcal{R}$, then we say that V is an admissible locally W-algebraic representation of G.

Proposition 6.3.10. *If V is an admissible locally W-algebraic representation of G, then V is isomorphic to a representation of the form $U \otimes_B W$, where B denotes the semi-simple K-algebra $\mathrm{End}_{\mathbb{G}}(W)$, and U is an admissible smooth representation of G over B, equipped with its finest convex topology. Conversely, any such tensor product is an admissible locally W-algebraic representation of G.*

Proof. The first statement of the proposition was observed in the course of proving proposition 6.3.6. The converse statement follows from the remark preceding the statement of proposition 4.2.4, together with propositions 6.3.2 and 6.1.5. $\square$

Proposition 6.3.11. *If V is a locally algebraic representation of G, and if, following proposition 4.2.4 and corollary 4.2.7, we write $V \xrightarrow{\sim} \bigoplus_n U_n \otimes_{B_n} W_n$, where W_n runs over the elements of $\hat{\mathbb{G}}$, $B_n = \mathrm{End}_{\mathbb{G}}(W_n)$, and U_n is a smooth representation of G over B_n, then V is an admissible locally algebraic representation of G if and only if the direct sum $\bigoplus_n U_n$ is an admissible smooth representation of G.*

Proof. Let H be an analytic open subgroup of G. Propositions 3.3.27 and 3.6.6 and lemma 4.1.4 yield an isomorphism

$$V_{\mathbb{H}-\mathrm{an}} \xrightarrow{\sim} \bigoplus_n (U_n)^H \otimes_{B_n} W_n$$

(where each side is equipped with its finest convex topology). If $\bigoplus_n U_n$ is admissible, then the space $\bigoplus_n (U_n)^H$ is finite dimensional, and so the same is true of $V_{\mathbb{H}-\mathrm{an}}$. Since H was arbitrary, we see that V is an admissible locally analytic representation of G.

On the other hand, if $\bigoplus_n U_n$ is not an admissible smooth representation of G, then we may find H such that $\bigoplus_n (U_n)^H$ is not finite dimensional. In other words, we may find an infinite sequence $\{n_i\}_{i\geq 1}$ such that each space $(U_{n_i})^H$ is non-zero, and hence such that $\bigoplus_i W_{n_i}$ is a subspace of $V_{\mathbb{H}-\mathrm{an}}$, necessarily closed, since $V_{\mathbb{H}-\mathrm{an}}$ is equipped with its finest convex topology. However, the direct sum $\bigoplus_i W_{n_i}$ (an infinite direct sum of algebraic representations) cannot embed as a closed subspace of a finite direct sum $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, K)^m$. Thus neither can $V_{\mathbb{H}-\mathrm{an}}$, and so V is not an admissible locally analytic representation of G. $\square$

6.4. Essentially admissible locally analytic representations

We begin this section by proving some results concerning abelian locally L-analytic groups.

Proposition 6.4.1. *If Z is an abelian locally L-analytic group, then the following conditions on Z are equivalent:*

(i) *Z is topologically finitely generated.*

(ii) *For every compact open subgroup Z_0 of Z, the quotient Z/Z_0 is a finitely generated abelian group.*

(iii) *Z contains a compact open subgroup Z_0 such that the quotient Z/Z_0 is a finitely generated abelian group.*

(iv) *Z contains a (necessarily unique) maximal compact open subgroup Z_0, and the quotient Z/Z_0 is a free abelian group of finite rank.*

(v) *Z is isomorphic to a product $Z_0 \times \Lambda$, where Z_0 is a compact locally L-analytic group and Λ is a free abelian group of finite rank (equipped with its discrete topology).*

Proof. If Z is topologically finitely generated, and if Z_0 is a compact open subgroup of Z, then the quotient Z/Z_0 is a topologically finitely generated discrete group, and hence is a finitely generated abelian group. Thus (i) implies (ii), while (ii) obviously implies (iii).

Suppose now that Z satisfies (iii), and hence contains a compact open subgroup Z_0' such that the quotient Z/Z_0' is a finitely generated abelian group. Let Z_0 be the preimage in Z of the torsion subgroup of Z/Z_0'. Since this subgroup is finite, we see that Z_0 is a compact open subgroup of Z. Since any compact open subgroup of Z has finite, and hence torsion, image in the discrete subgroup Z/Z_0, we see that Z_0 is furthermore the maximal compact open subgroup of Z. Since Z/Z_0 is isomorphic to the quotient of Z/Z_0' by its torsion subgroup, it is a free abelian group of finite rank. Thus (iii) implies (iv). (Note that if Z_0' and Z_0' are two compact open subgroups of Z, then the product $Z_0'Z_0''$ is again a compact open subgroup of Z, containing both Z_0' and Z_0''. Thus if Z contains a maximal compact open subgroup, it is necessarily unique.)

That (iv) implies (v) is obvious. Finally, suppose that Z satisfies (v). Since Z_0 contains a finite index subgroup isomorphic to $\mathcal{O}_L^d$ for some natural number d, we see that Z_0 is topologically finitely generated. The same is then true of $Z \xrightarrow{\sim} Z_0 \times \Lambda$, and so Z satisfies (i). This completes the proof of the proposition. $\square$

Definition 6.4.2. If Z is an abelian locally L-analytic group, and X is a rigid analytic space over L, then we let $\hat{Z}(X)$ denote the group of characters on Z with values in $\mathcal{C}^{\mathrm{an}}(X, L)^\times$, with the property that for any admissible open affinoid subspace X_0 of X, the induced character $Z \to \mathcal{C}^{\mathrm{an}}(X, L)^\times \to \mathcal{C}^{\mathrm{an}}(X_0, L)^\times$ (the second arrow being induced by restricting functions from X to X_0) is locally analytic when regarded as a $\mathcal{C}^{\mathrm{an}}(X_0, L)$-valued function on Z.

Lemma 6.4.3. *Let X be a rigid analytic space over L, and let $\{X_i\}_{i\in I}$ be an admissible affinoid open cover of X. If $\phi : Z \to \mathcal{C}^{\mathrm{an}}(X, L)^\times$ is a character such that each of the composites $Z \xrightarrow{\phi} \mathcal{C}^{\mathrm{an}}(X, L)^\times \longrightarrow \mathcal{C}^{\mathrm{an}}(X_i, L)^\times$ (the second arrow being induced by restriction to X_i) is a locally analytic $\mathcal{C}^{\mathrm{an}}(X_i, L)$-valued function on Z, then ϕ lies in $\hat{Z}(X)$.*

Proof. Let X_0 be an admissible affinoid open in X, and write $X_{0,i} := X_0 \bigcap X_i$, for each $i \in I$. These sets form an admissible open cover of X_0. Since X_0 is affinoid, it is quasi-compact in its Grothendieck topology, and so we may find a finite subset I' of I such that $\{X_{0,i}\}_{i\in I'}$ covers X_0. The open immersions $X_{0,i} \to X_0$ for $i \in I'$ induce a surjection $\coprod_{i\in I} X_{0,i} \to X_0$. This in turn induces a closed embedding $\mathcal{C}^{\mathrm{an}}(X_0, L) \to \bigoplus_{i\in I} \mathcal{C}^{\mathrm{an}}(X_{0,i})$. Thus to prove that the composite $Z \to \mathcal{C}^{\mathrm{an}}(X, L)^\times \to \mathcal{C}^{\mathrm{an}}(X_0, L)^\times$ is locally analytic, it suffices (by proposition 2.1.23) to show that each of the composites $Z \to \mathcal{C}^{\mathrm{an}}(X, L)^\times \to \mathcal{C}^{\mathrm{an}}(X_{0,i}, L)^\times$ is locally analytic. But each such composite factors through the composite $Z \to \mathcal{C}^{\mathrm{an}}(X, L)^\times \to \mathcal{C}^{\mathrm{an}}(X_i, L)^\times$, which is locally analytic by assumption. This proves the lemma. $\square$

The preceding lemma implies that the formation of $\hat{Z}(X)$ is local in the Grothendieck topology on the rigid analytic space X. We observe in the following corollary that it also implies that $\hat{Z}$ is a functor on the category of rigid analytic spaces (and so consequently is a sheaf, when this category is made into a site by equipping each rigid analytic space with its Grothendieck topology).

Corollary 6.4.4. *The formation of $\hat{Z}(X)$ is contravariantly functorial in the rigid L-analytic space X.*

Proof. Let $f : X \to Y$ be a rigid analytic morphism between rigid L-analytic spaces, and suppose that $\phi : Z \to \mathcal{C}^{\mathrm{an}}(Y, L)^\times$ is an element of $\hat{Z}(Y)$. Composing ϕ with the pullback morphism induced by f we obtain a character $\phi' : Z \to \mathcal{C}^{\mathrm{an}}(X, L)^\times$. We must show that this character lies in $\hat{Z}(X)$. Fix an admissible open cover $\{Y_i\}_{i\in I}$ of Y, and choose an admissible open cover $\{X_j\}_{j\in J}$ of X that refines the open cover $\{f^{-1}(Y_i)\}_{i\in I}$. Then if X_j lies in $f^{-1}(Y_i)$, the map f induces a continuous map of Banach spaces $\mathcal{C}^{\mathrm{an}}(Y_i, L) \to \mathcal{C}^{\mathrm{an}}(X_j, L)$, and consequently the composite $Z \xrightarrow{\phi'} \mathcal{C}^{\mathrm{an}}(X, L)^\times \longrightarrow \mathcal{C}^{\mathrm{an}}(X_j, L)^\times$ (which is equal to the composite $Z \xrightarrow{\phi} \mathcal{C}^{\mathrm{an}}(Y, L)^\times \longrightarrow \mathcal{C}^{\mathrm{an}}(Y_i, L)^\times \longrightarrow \mathcal{C}^{\mathrm{an}}(X_j, L)^\times$) is a locally analytic $\mathcal{C}^{\mathrm{an}}(X_j, L)$-valued function on Z. Lemma 6.4.3 now implies that ϕ' lies in $\hat{Z}(X)$. $\square$

In the cases of interest to us, the functor $\hat{Z}$ is representable, as we now show.

Proposition 6.4.5. *If Z is an abelian locally L-analytic group satisfying the equivalent conditions of proposition* 6.4.1*, then the functor $\hat{Z}$ of definition* 6.4.2 *is representable by a strictly quasi-Stein rigid analytic space over L.*

Proof. Proposition 6.4.1 (v) allows us to write $Z \xrightarrow{\sim} Z_0 \times \Lambda$, with Z_0 compact and Λ a free abelian group of finite rank. Thus there is an isomorphism of functors $\hat{Z} \xrightarrow{\sim} \hat{Z}_0 \times \hat{\Lambda}$. The functor $\hat{\Lambda}$ is clearly representable by the σ-affinoid rigid analytic space $\check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{G}_m$ (where $\check{\Lambda}$ denotes the dual of the finitely generated free abelian group Λ, and $\mathbb{G}_m$ denotes the multiplicative group over L, regarded as a rigid analytic space); the point is that since Λ is discrete, the condition of locally analyticity in the definition of $\hat{\Lambda}$ is superfluous. Thus it remains to show that $\hat{Z}_0$ is representable by a strictly quasi-Stein rigid analytic space.

Since Z_0 is compact and locally L-analytic, it contains a finite index subgroup isomorphic to $\mathcal{O}_L^d$. In [26] it is shown that $\hat{\mathcal{O}}_L$ is representable on affinoid spaces by a twisted form of the open unit disk, which is in particular strictly quasi-Stein. Lemma 6.4.3 shows that this space in fact represents the functor $\hat{\mathcal{O}}_L$ on the category of all rigid analytic spaces. Restricting characters from Z_0 to $\mathcal{O}_L^d$ then realizes $\hat{Z}_0$ as a finite faithfully flat cover of $\hat{\mathcal{O}}_L^d$, which must also be strictly quasi-Stein. $\square$

If Z satisfies the equivalent criterion of proposition 6.4.1, then we let $\hat{Z}$ denote the rigid analytic space over L constructed in proposition 6.4.5, that parameterizes the rigid analytic characters of L. (Yoneda's lemma assures us that $\hat{Z}$ is determined up to natural isomorphism.) Since $\hat{Z}$ is strictly quasi-Stein, the algebra of rigid analytic functions $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ is a nuclear Fréchet algebra.

Proposition 6.4.6. *If Z is an abelian locally L-analytic group satisfying the equivalent conditions of proposition* 6.4.1*, then there is a natural continuous injection of topological K-algebras $\mathcal{D}^{\mathrm{la}}(Z, K)_b \to \mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$. This map has dense image, and if Z is compact, it is even an isomorphism.*

Proof. If $\mu \in \mathcal{D}^{\mathrm{la}}(Z, K)_b$, then integrating against μ induces a map $\hat{Z}(K) \to K$. We will show that this is obtained by evaluating an element of $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, and that the resulting map $\mathcal{D}^{\mathrm{la}}(Z, K)_b \to \mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ is continuous, with dense image. If we write $Z = Z_0 \times \Lambda$, as in proposition 6.4.1 (v), then $\mathcal{D}^{\mathrm{la}}(Z, K)_b \xrightarrow{\sim} \mathcal{D}^{\mathrm{la}}(Z_0, K)_b \otimes_K K[\Lambda]$, while $\mathcal{C}^{\text{an}}(\hat{Z}, K) \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\hat{Z}_0, K)_b \hat{\otimes}_K \mathcal{C}^{\text{an}}(\hat{\Lambda}, K)_b$, and so it suffices to prove the proposition with Z replaced by Z_0 and Λ in turn.

Since Z_0 is compact, it contains a finite index subgroup isomorphic to $\mathcal{O}_L^d$, and so our claim is easily reduced to the case where $Z_0 = \mathcal{O}_L^d$. In this case, it follows from [26, thm. 2.3] that integrating characters against distributions yields a topological isomorphism $\mathcal{D}^{\text{la}}(Z_0, K)_b \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\hat{Z}_0, K)$.

Choosing a basis $z_1, \ldots, z_r$ for Λ yields an isomorphism $\hat{Z} \xrightarrow{\sim} \mathbb{G}_m^r$. Thus $\mathcal{D}^{\text{la}}(\Lambda, K) \xrightarrow{\sim} K[\Lambda] = K[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}]$, the target being equipped with its finest complex topology (the variable z_i in the target of this isomorphism corresponds to the delta function δ_{z_i} in its source), while $\mathcal{C}^{\text{an}}(\hat{Z}, K) \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\mathbb{G}_m^r, K) \xrightarrow{\sim} K\{\{z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}\}\}$ (the variable z_i in the target of this isomorphism corresponds to the rigid analytic function that takes a character on $\hat{Z}$ to its value at z_i; that is, to the delta function δ_{z_i}). Thus we see that integrating characters against distributions yields a map $\mathcal{D}^{\text{la}}(\Lambda, K) \to \mathcal{C}^{\text{an}}(\hat{\Lambda}, K)$, which corresponds, with respect to the preceding isomorphisms, to the usual inclusion $K[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}] \subset K\{\{z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}\}\}$. In particular, this map has dense image. $\square$

The proof of the preceding proposition shows that if Z is not compact, then the map $\mathcal{D}^{\text{la}}(Z, K)_b \to \mathcal{C}^{\text{an}}(\hat{Z}, K)$ is not an isomorphism. Rather, $\mathcal{C}^{\text{an}}(\hat{Z}, K)$ is a completion of $\mathcal{D}^{\text{la}}(Z, K)_b$ with respect to a certain continuous (and locally convex) metric structure on $\mathcal{D}^{\text{la}}(Z, K)_b$.

If V is a convex K-vector space of compact type, equipped with a locally analytic action of Z, then proposition 5.1.9 (ii) shows that V' is a module over the ring $\mathcal{D}^{\text{la}}(Z, K)_b$, and that the multiplication map $\mathcal{D}^{\text{la}}(Z, K)_b \times V_b' \to V_b'$ is separately continuous.

Proposition 6.4.7. *Let V be a convex K-vector space of compact type, equipped with a topological action of the topologically finitely generated abelian locally L-analytic group Z. The following conditions are equivalent:*

(i) *The Z-action on V extends to a $\mathcal{C}^{\text{an}}(\hat{Z}, K)$-module structure for which the multiplication map $\mathcal{C}^{\text{an}}(\hat{Z}, K) \times V \to V$ is separately continuous. (Such an extension is unique, if it exists.)*

(ii) *The contragredient Z-action on V_b' extends to a topological $\mathcal{C}^{\text{an}}(\hat{Z}, K)$-module structure on V_b'. (Such an extension is unique, if it exists.)*

(iii) *The Z-action on V is locally analytic, and we may write V as a union of an increasing sequence of BH-subspaces, each invariant under Z.*

Proof. If we write $Z = Z_0 \times \Lambda$, as in proposition 6.4.1 (v), then $\mathcal{C}^{\text{an}}(\hat{Z}, K) \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\hat{Z}_0, K) \hat{\otimes}_K \mathcal{C}^{\text{an}}(\hat{\Lambda}, K)$, and it suffices to prove the above proposition separately for the two cases $Z = Z_0$ and $Z = \Lambda$.

If $Z = Z_0$, then proposition 6.4.6 yields an isomorphism of nuclear Fréchet algebras $\mathcal{D}^{\text{la}}(Z_0, K)_b \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\hat{Z}_0, K)$. The equivalence of conditions (i) and (ii) follows from the equivalence of (i) and (ii) of proposition 1.2.14. These conditions are also equivalent to the Z_0-action on V being locally analytic, as follows from corollary 5.1.9 and proposition 5.1.10. Finally, since Z_0 is compact, any compact type space V equipped with a locally analytic Z_0-representation is the inductive limit of a sequence of Z_0-invariant BH-subspaces (by proposition 3.2.15).

Now consider the case $Z = \Lambda$. Let $z_1, \dots, z_r$ be a basis for Λ. Then

$$\mathcal{C}^{\mathrm{an}}(\hat{\Lambda}, K) \xrightarrow{\sim} K\{\{z_1, z_1^{-1}, \dots, z_r, z_r^{-1}\}\} \\ \xrightarrow{\sim} \varprojlim_n K\langle\langle p^n z_1, p^n z^{-1}, \dots, p^n z_r, p^n z_r^{-1}\rangle\rangle.$$

(The second isomorphism arises from writing the rigid analytic space $\check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{G}_m \xrightarrow{\sim} \mathbb{G}_m^r$ as the union of the affinoid subdomains $|p|^n \leq |z_i| \leq |p|^{-n}$ $(i = 1, \dots, r)$ as n tends to infinity. For each $n \geq 0$, we write $K\langle\langle p^n z_1, p^n z^{-1}, \dots, p^n z_r, p^n z_r^{-1}\rangle\rangle$ to denote the Tate algebra of rigid analytic functions on the corresponding subdomain.) It follows from proposition 1.2.14 that conditions (i) and (ii) of the proposition are equivalent, and that each in turn is equivalent to the following condition:

(iii') There is an isomorphism $V \xrightarrow{\sim} \varinjlim_n V_n$, where each V_n is a Banach module over the Tate algebra $K\langle\langle p^n z_1, p^n z^{-1}, \dots, p^n z_r, p^n z_r^{-1}\rangle\rangle$, and the maps $V_n \to V$ are Z-equivariant. (Note that Z embeds as a group of units in the Tate algebra $K\langle\langle p^n z_1, p^n z^{-1}, \dots, p^n z_r, p^n z_r^{-1}\rangle\rangle$.)

Clearly (iii') implies (iii). We must show that the converse holds. For this, it suffices to show that if W is a Banach space equipped with a topological action of Λ, then for some sufficiently large value of n, the Banach space W admits a (necessarily unique) structure of topological $K\langle\langle p^n z_1, p^n z^{-1}, \dots, p^n z_r, p^n z_r^{-1}\rangle\rangle$-module extending its Λ-module structure. If we fix a norm on W, then we may certainly find some n such that $||p^n z_i^{\pm 1}|| \leq 1$ for $i = 1, \dots, r$. (Here we are considering the norm of $p^n z_i^{\pm 1}$ as an operator on the normed space W.) It is then clear that W does indeed admit the required structure of topological $K\langle\langle p^n z_1, p^n z^{-1}, \dots, p^n z_r, p^n z_r^{-1}\rangle\rangle$-module. $\square$

Lemma 6.4.8. *If V is a convex K-vector space of compact type, equipped with a locally analytic action of Z that satisfies the equivalent conditions of proposition* 6.4.7, *then the same is true of any Z-invariant closed subspace of V.*

Proof. It is clear that if V satisfies condition (i) of proposition 6.4.7, then the separately continuous $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-module structure on V restricts to a separately continuous $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-module structure on any Z-invariant closed subspace of W (since $K[Z]$ is dense in $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$). $\square$

We now suppose that G is a locally L-analytic group, whose centre Z satisfies the equivalent conditions of proposition 6.4.1.

If V is a convex K-vector space of compact type, equipped with a locally analytic G-representation, then by proposition 5.1.9 (ii), V_b' is equipped with a topological $\mathcal{D}^{\mathrm{la}}(G, K)_b$-module structure. If V furthermore satisfies the hypothesis of proposition 6.4.7, then V_b' is also equipped with a topological $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-module structure.

Since Z is central in G, the $\mathcal{D}^{\mathrm{la}}(G, K)_b$-action and $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-action on V_b' commute with one another, and so together they induce the structure of a topological $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes} \mathcal{D}^{\mathrm{la}}(G, K)_b$-module on V_b'. If H is any compact open subgroup of G, then we may restrict this to a $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)_b$-module structure on V_b'. This completed tensor product is a nuclear Fréchet algebra, by lemma 1.2.13, and hence in particular is a weak Fréchet-Stein algebra, as the discussion preceding the statement of that lemma shows. If K is discretely valued, then by proposition 5.3.22 (and the remarks following it and definition 5.3.21), it is in fact a Fréchet-Stein algebra.

Definition 6.4.9. Let V be a convex K-vector space of compact type, equipped with a locally analytic action of G. We say that V is an essentially admissible locally analytic representation of G if V satisfies the conditions of proposition 6.4.7, and if for one (equivalently, every) compact open subgroup H of G, the dual V'_b is a coadmissible module when endowed with its natural module structure over the nuclear Fréchet algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)_b$.

Note that in the situation of definition 6.4.9, we have embeddings of $\mathcal{D}^{\mathrm{la}}(Z, K)$ into each factor of the tensor product $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(G, K)$, and the action of $\mathcal{D}^{\mathrm{la}}(Z, K)$ on V' obtained by regarding $\mathcal{D}^{\mathrm{la}}(Z, K)$ as a subalgebra of either of these factors coincides. (By construction, either of these actions is the natural $\mathcal{D}^{\mathrm{la}}(Z, K)$-action on V' induced by the locally analytic Z-representation on V.) Thus the $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(G, K)$-action on G factors through the quotient algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(Z,K)} \mathcal{D}^{\mathrm{la}}(G, K)$.

Similarly, if H is a compact open subgroup of G, and if we write $Z_0 = H \bigcap Z$, then the $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(G, K)$-action on V' factors through the quotient algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(Z_0,K)} \mathcal{D}^{\mathrm{la}}(H, K)$. Thus V is essentially admissible if and only if it is coadmissible for the nuclear Fréchet algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(Z_0,K)} \mathcal{D}^{\mathrm{la}}(H, K)$. In particular, if the centre Z of G is compact, then (taking into account proposition 6.4.6) V is essentially admissible if and only if it is admissible.

In general, we have the following proposition.

Proposition 6.4.10. *Any admissible locally analytic representation of G is an essentially admissible locally analytic representation of G.*

Proof. Let $\{H_n\}_{n \geq 1}$ be a cofinal sequence of analytic open subgroups of G. If V is an admissible locally analytic G-representation, then let V_n denote the BH-subspace of V obtained as the image of the natural map $V_{\mathbb{H}_n-\mathrm{an}} \to V$. Clearly $V = \bigcup_{n=1}^{\infty} V_n$, and each V_n is invariant under Z, since multiplication by elements of Z commutes with the H_n action on V. Thus V satisfies condition (iii) of proposition 6.4.7, and so is naturally a $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module. It is certainly coadmissible with respect to this nuclear Fréchet algebra, since it is coadmissible even with respect to $\mathcal{D}^{\mathrm{la}}(H, K)$. $\Box$

If Z is not compact, then the converse to proposition 6.4.10 is false. For example, if $G = Z$ is a free abelian group of rank r, then a locally analytic G-representation is admissible if and only if it is finite dimensional. On the other hand, the essentially admissible locally analytic G-representations correspond (by passing to the dual) to coherent sheaves on the rigid analytic space $\mathbb{G}_m^r$. (Among all such sheaves, the admissible representations correspond to those skyscraper sheaves whose support is finite.)

More generally, if $G = Z$ is abelian, then the essentially admissible locally analytic Z-representations correspond (by passing to the dual) to coherent sheaves on the rigid analytic space $\hat{Z}$. Such a sheaf corresponds to an admissible locally analytic Z-representation if and only if its pushforward to $\hat{Z}_0$ (where Z_0 denotes the maximal compact subgroup of Z) is again coherent.

Proposition 6.4.11. *If G is a locally L-analytic group whose centre Z satisfies the equivalent conditions of proposition 6.4.1, and if K is discretely valued, then the category of essentially admissible locally analytic G-representations and continuous*

G-equivariant morphisms is closed under passing to closed subobjects and Hausdorff quotients. Furthermore, any morphism in this category is necessarily strict. Consequently, this category is abelian.

Proof. This is a consequence of the general properties of coadmissible modules over Fréchet-Stein algebras, summarized in theorem 1.2.11 and the remarks that follow it. $\square$

Definition 6.4.12. If V is a Hausdorff convex K-vector space satisfying the equivalent conditions of proposition 6.4.7, and if J denotes an ideal in the ring $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, then we let V^J denote the closed G-invariant subspace of V that is annihilated by the ideal J.

Proposition 6.4.13. *If V is an essentially admissible locally analytic representation of G, and if J is a finite-codimension ideal in $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, then V^J is an admissible locally analytic representation of G.*

Proof. If $\overline{J}$ denotes the closure of J in $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, then $V^J = V^{\overline{J}}$, and so, replacing J by $\overline{J}$, we may assume that J is closed.

There is a natural isomorphism $(V^J)'_b \xrightarrow{\sim} (\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)/J) \hat{\otimes}_{\mathcal{C}^{\mathrm{an}}(\hat{Z},K)} V'_b$. If we fix a compact open subgroup H of G then we see that since V'_b is coadmissible with respect to $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$, its quotient $(V^J)'_b$ is coadmissible with respect to $(\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)/J) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$. Since $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)/J$ is finite dimensional, V'_b is in fact coadmissible with respect to $\mathcal{D}^{\mathrm{la}}(H, K)$. Thus V^J is an admissible locally analytic representation of G. $\square$

If χ is an element of $\hat{Z}(K)$, then we let J_χ denote the maximal ideal of $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ consisting of functions that vanish at χ, and we write V^χ in place of V^{J_χ}. Note that V^χ is the closed G-invariant subspace of V consisting of vectors that transform via χ under the action of Z.

Corollary 6.4.14. *If V is an essentially admissible locally analytic representation of G, and if χ is an element of $\hat{Z}(K)$, then V^χ is an admissible locally analytic representation of G.*

Proof. Apply the previous proposition to the ideal J_χ. $\square$

Suppose now that G is an open subgroup of $\mathbb{G}(L)$, for some connected reductive linear algebraic group $\mathbb{G}$ over L.

In conjunction with proposition 6.4.13, the following result allows one in certain situations to reduce questions regarding morphisms from admissible locally algebraic representations into essentially admissible locally analytic representations to the case in which the target is assumed admissible.

Proposition 6.4.15. *If V is an admissible locally algebraic representation of G, and if we let J run through the directed system of ideals of finite codimension in $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, then the natural map $\varinjlim_J V^J \to V$ is an isomorphism.*

Proof. If we use proposition 4.2.4 and corollary 4.2.7 to write $V \xrightarrow{\sim} \bigoplus_n U_n \otimes_{B_n} W_n$, where W_n runs over the elements of $\hat{\mathbb{G}}$, $B_n = \mathrm{End}_{\mathbb{G}}(W_n)$, and U_n is a smooth representation of G over B_n, then we see that it suffices to prove the proposition for each of the direct summands $U_n \otimes_{B_n} W_n$. Since $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ acts on W_n through a finite dimensional quotient, we further reduce ourselves to proving the proposition for an admissible smooth representation U_n. We may write $U_n \xrightarrow{\sim} \varinjlim_H U_n^H$, where H runs over the directed set of compact open subgroups of G. Each space U_n^H is finite dimensional and Z-invariant. Thus $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ acts on it through a finite dimensional quotient, and the proposition is proved. □

6.5. Invariant lattices

We conclude this chapter with a discussion of invariant lattices in convex K-vector spaces equipped with a G-action. We suppose throughout this section that K is discretely valued.

Recall that if V is a K-vector space, then a lattice in V is an $\mathcal{O}_K$-submodule of V that spans V over K [20, def., p. 7]. We say that two lattices M_1 and M_2 in V are commensurable if there exists an element $\alpha \in K^\times$ such that $\alpha M_2 \subset M_1 \subset \alpha^{-1} M_2$. Commensurability is obviously an equivalence relation.

We will also consider seminorms on V. (All seminorms and norms on V are understood to be non-archimedean.) We say that two seminorms q_1 and q_2 are commensurable if there exists a positive real number c such that $c q_1(v) \leq q_2(v) \leq c^{-1} q_1(v)$ for all $v \in V$.

There is a natural bijection between the set of commensurability classes of lattices in V and commensurability classes of seminorms on V, obtained by passing from a lattice to its corresponding gauge, and from a seminorm to its corresponding unit ball. (See [20, lem. 2.2].)

Let π be a uniformizer of $\mathcal{O}_K$. (Recall that we are assuming that K is discretely valued.) We say that an $\mathcal{O}_K$-module M (and so in particular, a lattice M in V) is separated if $\bigcap_{n \geq 1} \pi^n M = 0$. The property of being separated is an invariant of a commensurability class of lattices. Also, the property of being a norm is an invariant of a commensurability class of seminorms. Under the above bijection, the commensurability classes of separated lattices correspond to the commensurability classes of norms.

If M is a lattice in V, then we let V_M denote V, endowed with the locally convex topology induced by the gauge of M, and let $\hat{V}_M$ denote the Hausdorff completion of V_M. Note that $\hat{V}_M$ is naturally a K-Banach space, and depends up to canonical isomorphism only on the commensurability class of M. There is a natural K-linear map $V \to \hat{V}_M$, which is an injection if and only if M is separated (so that its gauge is a norm).

If G acts on V, and if a lattice in V is G-invariant, then the same is true of its gauge, while if a seminorm is G-invariant, then the same is true of its unit ball. Thus a commensurability class of lattices in V contains a G-invariant lattice if and only if the corresponding commensurability class of seminorms contains a G-invariant seminorm.

Lemma 6.5.1. *If V is a K-vector space equipped with a G-action, and M is a lattice in V, then G preserves the commensurability class of M (that is, the lattice gM is commensurable with M, for each $g \in G$) if and only if the G-action on V_M is topological.*

Proof. Given that the topology on V_M is defined via the gauge of M, it is clear that if G preservers the commensurability class of M, then each element of G acts as a continuous automorphism of V_M. Conversely, if each element of G acts as a continuous automorphism of V_M, then for any $g \in G$, there is an inclusion $gM \subset \alpha M$ for some $\alpha \in K^\times$. Combining this inclusion with the analogous inclusion for g^{-1} shows that gM is commensurable with M, for each $g \in G$. $\square$

Lemma 6.5.2. *If V is a K-vector space equipped with a G-action, and M is a lattice in V, then the following are equivalent:*

(i) *There exists a G-invariant lattice that is commensurable with M.*
(ii) *There exists a G-invariant seminorm that defines the topology of V_M.*
(iii) *G acts as a group of equicontinuous operators on V_M.*

Proof. The gauge of a G-invariant lattice is a G-invariant seminorm, and so (i) implies (ii), while (ii) clearly implies (iii). If (iii) holds, then there is $\alpha \in K^\times$ such that $gM \subset \alpha M$ for all $g \in G$. If we let M' be the lattice generated by the lattices gM for all $g \in G$, then $M \subset M' \subset \alpha M$, and so M' is commensurable with M. Since M' is G-invariant by construction, we see that (iii) implies (i). $\square$

If V is furthermore a topological K-vector space, then the property of being an open subset of V is an invariant of a commensurability class of lattices, while the property of being continuous is an invariant of a commensurability class of seminorms. Under the above bijection, the commensurability classes of open lattices correspond to the commensurability classes of continuous seminorms. The open lattices are thus precisely those for which the natural map $V \to V_M$ is continuous.

Lemma 6.5.3. *If V is a topological K-vector space equipped with a continuous action of G, and if M is G-invariant open lattice in V, then G acts continuously on V_M. If G is compact, and if, conversely, G acts continuously on V_M, then there exists a G-invariant lattice that is commensurable with M.*

Proof. If there exists a G-invariant lattice that is commensurable with M, then G acts as a group of equicontinuous operators on V_M. Thus conditions (ii) and (iii) of lemma 3.1.1 hold with respect to the G-action on V_M. Since M is open in V, the natural map $V \to V_M$ is continuous. Since the G-action on V is continuous (and since V and V_M are equal as abstract K-vector spaces), we see that condition (i) of lemma 3.1.1 is also satisfied, and so the G-action on V_M is continuous.

Conversely, suppose that G acts continuously on V_M, and furthermore that G is compact. Lemma 3.1.4 then shows that G acts as an equicontinuous set of operators. From lemma 6.5.2, we conclude that there exists a G-invariant lattice commensurable with M. $\square$

Lemma 6.5.4. *If V is a topological K-vector space on which G acts as an equicontinuous group of operators (for example, if G is compact and acts continuously on V), then any open lattice in V contains a G-invariant open sublattice, and there is a unique maximal such G-invariant open sublattice.*

Proof. If M is a given open lattice in V, then by assumption there exists an open sublattice M' of M such $GM' \subset M$. The sublattice of M spanned by GM' is then a G-invariant open sublattice of M. If we consider the sublattice of M spanned by all invariant open sublattices, then it is obviously the unique maximal invariant open sublattice. $\square$

Lemma 6.5.5. *If V is a K-Banach space on which G acts as an equicontinuous group of operators (for example, if G is compact and acts continuously on V), then any separated open lattice in V is contained in a G-invariant separated open* *lattice of V, and there is a unique minimal G-invariant separated open lattice of V containing it.*

Proof. Let M denote the given separated open sublattice of V. We may choose the norm on the Banach space V so that M is the unit ball of V. Since G acts equicontinuously on V, there is an $\alpha \in K^\times$ such that $G\alpha M \subset M$. Then $GM \subset \alpha^{-1}M$, and so we see that the lattice spanned by GM is a G-invariant separated open lattice in V that contains M. Taking the intersection of all such lattices yields the minimal G-invariant separated open lattice containing M. □

Let $k = \mathcal{O}_K/\pi\mathcal{O}_K$ denote the residue field of $\mathcal{O}_K$.

Lemma 6.5.6. *If V is a convex K-vector space equipped with a continuous G-action, and M is an open G-invariant lattice in V, then the induced G-action on the k-vector space $M/\pi M$ is smooth.*

Proof. Lemma 6.5.3 implies that G acts continuously on V_M, and thus that we may replace V by V_M in our considerations. If we equip $M/\pi M$ with the topology induced by regarding it as a subquotient of V_M, then the G-action on $M/\pi M$ is continuous. On the hand, the topology obtained on $M/\pi M$ in this fashion is the discrete topology. Thus the G-action on $M/\pi M$ is smooth, as claimed. □

The following result slightly extends [25, lem. 3] (which treats the case when K is local).

Proposition 6.5.7. *If V is a convex K-vector space equipped with a continuous G-action, and M is an open G-invariant lattice in V, then the following are equivalent:*

(i) *The Banach space $\hat{V}_M$, when equipped with its natural G-action, becomes an admissible continuous representation of G.*

(ii) *The smooth representation of G on $M/\pi M$ is admissible.*

Proof. Replacing V with $\hat{V}_M$, and M with the closure of its image in $\hat{V}_M$ (which does not disturb the isomorphism class of the G-representation $M/\pi M$), we may assume that V is a Banach space. Also, replacing G by an appropriate pro-p open subgroup if necessary, we may assume that G is a pro-p group.

As in theorem 6.2.8, let $\mathcal{D}(G, \mathcal{O}_K)$ denote the unit ball of $\mathcal{D}(G, K)$ (that is, the polar of the unit ball of $\mathcal{C}(G, \mathcal{O}_K)$ with respect to the sup norm on this latter space). Corollary 5.1.7 shows that V'_b is naturally a topological $\mathcal{D}(G, K)$-module, and so also a $\mathcal{D}(G, \mathcal{O}_K)$-module. If we equip V with the norm obtained as the gauge of M, then the polar M' of M is a bounded open $\mathcal{O}_K$-sublattice of $(V')_b$, which is also a $\mathcal{D}(G, \mathcal{O}_K)$-submodule of V'.

Let $\mathfrak{a}$ denote the ideal of $\mathcal{D}(G, \mathcal{O}_K)$ generated by the uniformizer π of $\mathcal{O}_K$, together with the augmentation ideal I_G of $\mathcal{D}(G, \mathcal{O}_K)$. In the case where $\mathcal{O}_K = \mathbb{Z}_p$, the $\mathfrak{a}$-adic filtration of $\mathcal{D}(G, \mathbb{Z}_p)$ is cofinal with the filtration of $\mathcal{D}(G, \mathbb{Z}_p)$ considered in [16], and hence $\mathcal{D}(G, \mathbb{Z}_p)$ is $\mathfrak{a}$-adically complete. Writing $\mathcal{D}(G, \mathcal{O}_K) = \mathcal{O}_K \hat{\otimes}_{\mathbb{Z}_p} \mathcal{D}(G, \mathbb{Z}_p)$, we find that $\mathcal{D}(G, \mathcal{O}_K)$ is $\mathfrak{a}$-adically complete.

The quotient $M'/\mathfrak{a}M'$ is isomorphic to the $\mathcal{O}_K/\pi\mathcal{O}_K$-linear dual of the space $(M/\pi M)^G$. Thus $M'/\mathfrak{a}M'$ is finite dimensional over $\mathcal{O}_K/\pi\mathcal{O}_K$ if and only if $(M/\pi M)^G$ is finite dimensional over $\mathcal{O}_K/\pi\mathcal{O}_K$. On the other hand, since $\mathcal{D}(G, \mathcal{O}_K)$ is $\mathfrak{a}$-adically complete, the quotient $M'/\mathfrak{a}M'$ is finite dimensional over $\mathcal{O}_K/\pi\mathcal{O}_K$ if and only if M' is finitely generated over $\mathcal{D}(G, \mathcal{O}_K)$.

Thus if $M/\pi M$ is an admissible smooth representation of G, then we see that M' is finitely generated over $\mathcal{D}(G, \mathcal{O}_K)$, and hence that V' is finitely generated over

$\mathcal{D}(G, K)$; that is, V is an admissible continuous representation of G. Conversely, if V is an admissible continuous representation of G, then replacing G by an open subgroup H, it follows that V' is finitely generated over $\mathcal{D}(H, K)$. Hence M' is finitely generated over $\mathcal{D}(H, \mathcal{O}_K)$, and so $(M/\pi M)^H$ is finite dimensional over $\mathcal{O}_K/\pi\mathcal{O}_K$. It follows that $M/\pi M$ is an admissible smooth representation of G. $\square$

Definition 6.5.8. If V is a convex K-vector space equipped with a continuous G-action, we say that a G-invariant open lattice in V is admissible if it satisfies the equivalent conditions of proposition 6.5.7.

The following result provides something of a converse to proposition 6.2.4.

Proposition 6.5.9. *If V is an essentially admissible locally analytic representation of G, and if G is compact, then the following are equivalent:*

(i) *There is an admissible G-invariant separated open lattice in V.*

(ii) *There is a continuous G-equivariant injection $V \to W$, with W a K-Banach space equipped with an admissible continuous representation of G.*

(iii) *V is a strongly admissible locally analytic G-representation.*

Proof. Suppose that V contains an admissible G-invariant separated open lattice M. Proposition 6.5.7 shows that the G-action on $\hat{V}_M$ makes this space an admissible continuous G-representation. Since M is separated and open, the natural map $V \to \hat{V}_M$ is continuous and injective, and it is certainly G-equivariant. Thus (i) implies (ii).

Suppose that (ii) holds, so that we are given a continuous G-equivariant injection $V \to W$ with W a K-Banach space on which G acts via an admissible continuous representation. Lemma 6.5.3 shows that we may choose a G-invariant norm on W. The unit ball of this norm will be an admissible G-invariant separated open lattice in W, and hence its preimage in V will be an admissible G-invariant separated open lattice in V. Thus (ii) implies (i).

If we continue to assume given a G-equivariant injection $V \to W$ as in (ii), then passing to locally analytic vectors yields a continuous injection $V \to W_{\mathrm{la}}$. Proposition 6.2.4 shows that W_{la} is a strongly admissible locally analytic representation of G. Proposition 6.4.11 (together with propositions 6.2.2 and 6.4.10) shows that V embeds as a closed subspace of W_{la}, and so V_{la} is itself strongly admissible. Thus (ii) also implies (iii).

Finally suppose that (iii) holds, so that V is strongly admissible. The remark following definition 6.2.1 shows that we may find a closed embedding $V \to \mathcal{C}^{\mathrm{la}}(G, K)^n$ for some integer n. Composing this map with the continuous injection $\mathcal{C}^{\mathrm{la}}(G, K)^n \to \mathcal{C}(G, K)^n$ yields a continuous injection $V \to \mathcal{C}(G, K)^n$. Thus (iii) implies (ii). $\square$

Chapter 7. Representations of certain product groups

7.1. Strictly smooth representations

Throughout this section, we assume that Γ is a locally compact topological group, and that Γ admits a countable neighbourhood basis of the identity consisting of open subgroups.

If V is a K-vector space equipped with a representation of Γ, and if H is a subgroup of Γ, then we let V^H denote the subspace of V consisting of H-fixed vectors. As usual, we say that V is a smooth representation of Γ if and only if each vector of V is fixed by some compact open subgroup of V; equivalently, V is a smooth representation if and only if the natural map $\varinjlim_{H} V^H \to V$ is an isomorphism (the inductive limit being taken over all the compact open subgroups of Γ).

If V is a smooth representation of Γ, and if H is a compact open subgroup of Γ, then we let $\pi_H : V \to V^H$ denote the operator given by "averaging over H": if $v \in V$ is fixed by the subgroup H' of H, then $\pi_H(v) := [H : H']^{-1} \sum_{h \in H/H'} hv$.

We will consider a strengthening of the notion of a smooth representation, when V is assumed to be a Hausdorff locally convex topological K-vector space, and the Γ-action on V is topological. In this context, we let V^H denote the space of H-fixed vectors, equipped with the subspace topology. Thus each V^H is a closed subspace of V.

Definition 7.1.1. If V is a Hausdorff convex K-vector space equipped with a topological Γ-action, then we define $V_{\mathrm{st.sm}}$ to be the locally convex inductive limit

$$V_{\mathrm{st.sm}} := \varinjlim_{H} V^H,$$

where H runs over the compact open subgroups of Γ.

Our assumption on Γ shows that the inductive limit in Definition 7.1.1 can be replaced by a countable inductive limit. Also, the transition maps are closed embeddings. Thus the vector space $V_{\mathrm{st.sm}}$ is a strict inductive limit, in the sense of [5, p. II.33]. In particular, it is Hausdorff, and is complete if V is (since each V^H is then complete). There is a natural continuous injection $V_{\mathrm{st.sm}} \to V$, and the Γ-action on V is smooth if and only if this map is a bijection.

The map $V \mapsto V_{\mathrm{st.sm}}$ yields a covariant functor from the category of Hausdorff convex K-vector spaces equipped with topological Γ-actions (with morphisms being continuous and Γ-equivariant) to itself.

Definition 7.1.2. If V is a Hausdorff locally convex K-vector space equipped with a topological action of Γ, then we say that V is a strictly smooth representation of Γ if the natural map $V_{\mathrm{st.sm}} \to V$ is a topological isomorphism.

In particular, a strictly smooth Γ-representation on a convex K-vector space V is certainly a topological smooth Γ-action. In general, the converse need not be true.

Proposition 7.1.3. *Let V be a Hausdorff locally convex K-vector space equipped with a topological action of Γ. If either V is of compact type, or if for a cofinal sequence of compact open subgroups H of Γ, the closed subspaces V^H of Γ are of compact type, then $V_{\mathrm{st.sm}}$ is of compact type.*

Proof. If V is of compact type, then so is its closed subspace V^H, for each compact open subgroup H of Γ. Thus the first hypothesis implies the second. Let $\{H_n\}_{n \geq 1}$ be a cofinal sequence of compact open subgroups of Γ for which V^{H_n} is of compact type. The isomorphism $\varinjlim_{n} V^{H_n} \xrightarrow{\sim} V_{\mathrm{st.sm}}$ shows that V is the locally convex inductive limit of a sequence of spaces of compact type, with injective transition maps, and so is itself of compact type. $\square$

Corollary 7.1.4. *Let V be a Hausdorff locally convex K-vector space equipped with a smooth topological action of Γ. If V is of compact type, then V is a strictly smooth Γ-representation. Conversely, if the Γ-action on V is strictly smooth, and if for a cofinal collection of compact open subgroups H of Γ the subspaces V^H of V are of compact type, then V is of compact type.*

Proof. By assumption, the Γ-action on V is smooth, and hence the natural map

$$V_{\mathrm{st.sm}} \to V \tag{7.1.5}$$

is a continuous bijection. Also, either hypothesis on V implies, by lemma 7.1.3, that $V_{\mathrm{st.sm}}$ is of compact type.

If V is of compact type, then (7.1.5) is a continuous bijection between spaces of compact type, and hence is a topological isomorphism. Thus V is a strictly smooth Γ-representation. Conversely, if the Γ-action on V is strictly smooth, then (7.1.5) is a topological isomorphism, and so we conclude that V is of compact type. $\square$

Proposition 7.1.6. *If V is a Hausdorff convex K-vector space equipped with a strictly smooth action of Γ, then for each compact open subgroup H of Γ, the operator $\pi_H : V \to V^H$ is continuous.*

Proof. If H' is an open subgroup of H, then the formula given above for the restriction of π_H to $V^{H'}$, and the assumption that the Γ-action on V is topological, shows that π_H induces a continuous map $V^{H'} \to V^H$. Passing to the inductive limit over H', and using the fact that the Γ-action on V is strictly smooth, we see that the proposition follows. $\square$

Corollary 7.1.7. *If V is a Hausdorff convex K-vector space equipped with a strictly smooth action of Γ, then for each compact open subgroup H of Γ, we obtain a topological direct sum decomposition $V^H \bigoplus \ker \pi_H \xrightarrow{\sim} V$.*

Proof. This follows from the fact that the operator π_H on V is idempotent, with image equal to V^H, and is continuous, by proposition 7.1.6. $\square$

Corollary 7.1.8. *If V is a Hausdorff convex K-vector space equipped with a strictly smooth action of Γ, then V is barrelled if and only if V^H is barrelled for each compact open subgroup H of Γ.*

Proof. If each V^H is barrelled, then the isomorphism $\varinjlim_H V^H \xrightarrow{\sim} V$ shows that V is barrelled. Conversely, if V is barrelled, and if H is a compact open subgroup of Γ, then corollary 7.1.7 shows that V^H is a topological direct summand of V. It follows that V^H is barrelled [5, cor. 2, p. III.25]. $\square$

We now explain how the countable inductive limit that implicitly appears in Definition 7.1.1 can be replaced by a countable direct sum. Let us fix a cofinal decreasing sequence $\{H_n\}_{n\geq 1}$ of compact open subgroups of the identity of Γ. If V is a Hausdorff convex K-vector space, equipped with a topological action of Γ, then for each value of $n > 1$, the averaging map $\pi_{H_{n-1}}$, when restricted to V^{H^n}, induces a continuous map $\pi_n : V^{H_n} \to V^{H_{n-1}}$.

Set $V_1 = V^{H_1}$, and $V_n = \ker \pi_n$ for $n > 1$. Each of these spaces is a closed subspace of V. For each $n \geq 1$, the natural map $\bigoplus_{m=1}^n V_m \to V^{H_n}$ is a topological isomorphism, and passing to the inductive limit in n, we obtain a topological isomorphism

$$\bigoplus_{n=1}^{\infty} V_n \xrightarrow{\sim} V_{\mathrm{st.sm}}. \tag{7.1.9}$$

Thus $V_{\mathrm{st.sm}}$ can be written not only as a strict inductive limit of closed subspaces of V, but in fact as a countable direct sum of closed subspaces of V.

If W is a second Hausdorff convex K-vector space equipped with a topological Γ-action, and if $f : V \to W$ is a continuous Γ-equivariant map, then f induces continuous maps $f_n : V_n \to W_n$ for each $n \geq 1$, and the map $V_{\mathrm{st.sm}} \to W_{\mathrm{st.sm}}$ induced by f can be recovered as the direct sum of the f_n.

We now state some consequences of this analysis.

Lemma 7.1.10. *If V is a Hausdorff convex K-vector space equipped with a strictly smooth action of Γ, then the Γ-action on V is continuous.*

Proof. We apply lemma 3.1.1. Condition (i) of that result holds for any smooth action of Γ, and by definition a strictly smooth Γ-action on V is a topological action, so that condition (ii) holds. We must verify that condition (iii) holds. Let H be a compact open subgroup of Γ, and choose the cofinal sequence $\{H_n\}_{n\geq 1}$ of compact open subgroups appearing in the preceding discussion to be normal open subgroups of H. Each V_n (as defined in that discussion) is then invariant under the action of H, and the H-action on V_n factors through the finite quotient H/H_n of H. In particular, the action of H on V_n is continuous, for each $n \geq 1$. It follows that the action of H on $V \xleftarrow{\sim} \bigoplus_{n\geq 1} V_n$ is continuous, and so in particular the required condition (iii) holds. □

Lemma 7.1.11. *If V is a Hausdorff convex K-vector space equipped with a topological action of Γ, then the natural map $(V_{\mathrm{st.sm}})_{\mathrm{st.sm}} \to V_{\mathrm{st.sm}}$ is an isomorphism. Equivalently, the Γ-representation $V_{\mathrm{st.sm}}$ is strictly smooth.*

Proof. This follows from the isomorphism (7.1.9), together with the evident isomorphism $\bigoplus_{m=1}^{n} V_m \xrightarrow{\sim} (\bigoplus_{m=1}^{\infty} V_m)^{H_n}$. □

Lemma 7.1.12. *If V is a Hausdorff convex K-vector space equipped with a strictly smooth action of Γ, and if W is a Γ-invariant subspace of V, then W is closed in V if and only if W^H is closed in V^H for each compact open subgroup H of Γ. Furthermore, if these equivalent conditions hold, then the Γ-actions on each of W and V/W are again strictly smooth, and for each compact open subgroup H of Γ, the natural map $V^H/W^H \to (V/W)^H$ is a topological isomorphism.*

Proof. Certainly, if W is a closed subspace of V, then $W^H = V^H \bigcap W$ is a closed subspace of V^H, for each compact open subgroup H of Γ. Conversely, suppose that the latter is true. Then for each $n \geq 1$ we can define V_n and W_n as in the discussion preceding lemma 7.1.10, and we see that W_n is a closed subspace of V_n for each value of n. Thus $W \xleftarrow{\sim} \bigoplus_{n=1}^{\infty} W_n \subset \bigoplus_{n=1}^{\infty} V_n \xrightarrow{\sim} V$ is a closed subspace of V. There is also an isomorphism $V/W \xleftarrow{\sim} \bigoplus_{n=1}^{\infty} V_n/W_n$. (See proposition 8 and corollary 1 of [5, p. II.31].) These direct sum decompositions show that each of W and V/W is again a strictly smooth representation of Γ. Finally, if we fix a compact open subgroup H of Γ, and choose the sequence $\{H_n\}_{n\geq 1}$ so that $H_1 = H$, then the same direct sum decompositions show that $V^H/W^H \xrightarrow{\sim} (V/W)^H$, as required. □

Lemma 7.1.13. *Suppose that V and W are a pair of Hausdorff convex K-vector spaces, each equipped with a strictly smooth action of Γ, and that we are given a continuous Γ-equivariant map $V \to W$. This map is strict if and only if the same is true of the induced maps $V^H \to W^H$, for each compact open subgroup H of Γ.*

Proof. This result follows by an argument analogous to the one used to prove lemma 7.1.12. □

The following lemma provides an analysis of the "strictly smooth dual" of a strictly smooth representation of Γ.

Lemma 7.1.14. *If V is a Hausdorff convex K-vector space equipped with a strictly smooth action of Γ, let V'_b be equipped with the contragredient representation of Γ.*

(i) *For each compact open subgroup H of Γ, there is a natural isomorphism $(V'_b)^H \xrightarrow{\sim} (V^H)'_b$.*

(ii) *There is a natural Γ-equivariant isomorphism $(V'_b)_{\mathrm{st.sm}} \xrightarrow{\sim} \varinjlim_H (V^H)'_b$.*

(iii) *The natural map $V \to (V'_b)'_b$ induces a natural Γ-equivariant injective map of strictly smooth Γ-representations $V \to (((V'_b)_{\mathrm{st.sm}})'_b)_{\mathrm{st.sm}}$, which is a topological embedding if the same is true of the first map.*

(iv) *If V is of compact type, or is a nuclear Fréchet space, then the map $V \to (((V'_b)_{\mathrm{st.sm}})'_b)_{\mathrm{st.sm}}$ of* (iv) *is an isomorphism.*

Proof. Let H be a compact open subgroup of Γ. Lemma 7.1.6 provides a direct sum decomposition $V^H \bigoplus \ker \pi_H \xrightarrow{\sim} V$. Taking strong duals yields an isomorphism $V'_b \xrightarrow{\sim} (V^H)'_b \bigoplus (\ker \pi_H)'_b$. Thus to prove (i), it suffices to show that $((\ker \pi_H)')^H = 0$. If v' is an H-invariant linear functional defined on $\ker \pi_H$, then $v' \circ \pi_H = v'$, and so $v' = 0$, as required. Thus (i) is proved. Note that (ii) is an immediate consequence of (i).

To prove (iii), we first construct the required map. Note that the natural map $(V'_b)_{\mathrm{st.sm}} \to V'_b$ induces a continuous map $(V'_b)'_b \to ((V'_b)_{\mathrm{st.sm}})'_b$. Composing with the double duality map $V \to (V'_b)'_b$ yields a map $V \to ((V'_b)_{\mathrm{st.sm}})'_b$, and hence (since $V_{\mathrm{st.sm}} \xrightarrow{\sim} V$) a map

$$V \to (((V'_b)_{\mathrm{st.sm}})'_b)_{\mathrm{st.sm}}, \tag{7.1.15}$$

as required.

Parts (i) and (ii), applied to first to V and then to $(V'_b)_{\mathrm{st.sm}}$, yield an isomorphism $\varinjlim_H ((V^H)'_b)'_b \xrightarrow{\sim} (((V'_b)_{\mathrm{st.sm}})'_b)_{\mathrm{st.sm}}$. The map (7.1.15) admits a more concrete description, in terms of this isomorphism and the isomorphism $\varinjlim_H V^H \xrightarrow{\sim} V$: it is the locally convex inductive limit over the compact open subgroups H of Γ of the double duality maps

$$V^H \to ((V^H)'_b)'_b. \tag{7.1.16}$$

If the double duality map $V \to (V'_b)'_b$ is an embedding, then corollary 7.1.7 implies that the same is true for each of these maps. Since the strict inductive limit of embeddings is an embedding, this completes the proof of part (iii).

To prove part (iv), note that if V is of compact type (respectively, is a nuclear K-Fréchet space), then the same is true of each of its closed subspaces V^H.

In particular, each of these spaces is reflexive, and so each of the double duality maps (7.1.16) is a topological isomorphism. Passing to the inductive limit in H, we see that the same is true of the map (7.1.15). $\square$

Note that if V is an admissible smooth representation of Γ, equipped with its finest convex topology, then V becomes a compact type convex K-vector space, equipped with a smooth topological action of Γ, necessarily strictly smooth, by corollary 7.1.4. The preceding lemma thus implies that the strictly smooth dual $(V_b')_{\mathrm{st.sm}}$ to V is then isomorphic to the smooth dual to V, equipped with its finest convex topology.

We close this section by introducing a notion that we will require below.

Definition 7.1.17. Let A be a Fréchet-Stein K-algebra, in the sense of definition 1.2.10. A coadmissible (A, Γ)-module consists of a topological left A-module M, equipped with a topological action of Γ that commutes with the A-action on M, such that:

(i) The Γ-action on A is strictly smooth.

(ii) For each compact open subgroup H of Γ, the closed A-submodule M^H of M is a coadmissible A-module in the sense of definition 1.2.8.

In particular, since any coadmissible A-module is a K-Fréchet space, we see that $M \xrightarrow{\sim} \varinjlim_{H} M^H$ is a strict inductive limit of Fréchet spaces. (In fact, the discussion preceding lemma 7.1.10 shows that it is even a countable direct sum of Fréchet spaces.)

Lemma 7.1.18. *If A is a locally convex topological K-algebra, and if M is a topological left A-module equipped with a topological action of Γ that commutes with the A-action on M, then $M_{\mathrm{st.sm}}$ is naturally a topological left A-module.*

Proof. It is clear by functoriality of the formation of $M_{\mathrm{st.sm}}$ that the A-module structure on M induces an A-module structure on $M_{\mathrm{st.sm}}$. We must show that the multiplication map $A \times M_{\mathrm{st.sm}} \to M_{\mathrm{st.sm}}$ is continuous.

If we adopt the notation introduced preceding lemma 7.1.10, then the isomorphism (7.1.9) shows that it suffices to prove that the multiplication map $A \times M_n \to M_n$ is jointly continuous for each $n \geq 1$. Since M_n is an A-invariant subspace of M, this follows from the assumption that the multiplication map $A \times M \to M$ is jointly continuous (that is, that M is a topological left A-module). $\square$

7.2. Extensions of notions of admissibility for representations of certain product groups

As in the preceding section, in this section we suppose that Γ is a locally compact topological group, admitting a countable neighbourhood basis of the identity consisting of open subgroups. We also suppose that G is a locally L-analytic group. In this section we introduce the notions of admissible continuous, admissible locally analytic, and essentially admissible locally analytic representations of the topological group $G \times \Gamma$.

Definition 7.2.1. Let V be a locally convex Hausdorff K-vector space equipped with a topological action of $G \times \Gamma$. We say that V is an admissible continuous representation of $G \times \Gamma$ if:

(i) For each compact open subgroup H of Γ the closed subspace V^H of H-invariant vectors in V is a Banach space, and the G-action on V^H is an admissible continuous representation of G.

(ii) The Γ-action on V is strictly smooth, in the sense of Definition 7.1.2. (That is, the natural map $\varinjlim_H V^H \to V$ is a topological isomorphism, the locally convex inductive limit being taken over all compact open subgroups of Γ.)

It follows from proposition 6.2.5 that it suffices to check condition (i) of this definition for a cofinal collection of compact open subgroups H of Γ. The discussion preceding lemma 7.1.10 shows that any V satisfying the conditions of Definition 7.2.1 is in fact isomorphic to a countable direct sum of Banach spaces.

Proposition 7.2.2. (i) *If V is an admissible continuous representation of $G \times \Gamma$, then the $G \times \Gamma$-action on V is continuous.*

(ii) *If V is an admissible continuous representation of $G\times\Gamma$, and if W is a closed $G \times \Gamma$-invariant subspace of V, then the $G \times \Gamma$-action on W is also an admissible continuous representation.*

(iii) *If K is discretely valued, then any $G\times\Gamma$-equivariant morphism of admissible continuous representations of $G \times \Gamma$ is strict, with closed image.*

(iv) *If K is discretely valued, then the category of admissible continuous representations of $G \times \Gamma$ is abelian.*

(v) *If H is a compact open subgroup of Γ, then the functor that takes V to the closed subspace of H-invariants V^H is an exact functor from the category of admissible continuous representations of $G \times \Gamma$ to the category of admissible continuous G-representations.*

Proof. If V is an admissible continuous representation of $G \times \Gamma$, then V is in particular a direct sum of Banach spaces, and so is barrelled. Since V is isomorphic to a locally convex inductive limit of admissible continuous representations of G, the G-action on V is separately continuous, and hence continuous. On the other hand, lemma 7.1.10 shows that the Γ-action on V is continuous. Thus (i) is proved.

Now suppose that we are given a closed $G \times \Gamma$-invariant subspace W of V. If H is a compact open subgroup of Γ, then by assumption V^H is an admissible continuous representation of G. Since $W^H = V^H \bigcap W$, we see that W^H is a closed G-invariant subspace of the admissible continuous G-representation V^H, and so by proposition 6.2.5 is again an admissible continuous G-representation. We are also assuming that V is a strictly smooth Γ-representation, and so it follows from lemma 7.1.12 that the same is true of W. This proves (ii).

Part (iii) follows from lemmas 7.1.12 and 7.1.13 and the fact that, when K is discretely valued, G-equivariant morphisms of continuous admissible G-representations are strict with closed image, by proposition 6.2.9.

In light of what we have already proved, and the fact that the direct sum of two admissible continuous representations of $G \times \Gamma$ is again such a representation, to prove part (iv) we need only show that if W is a closed subrepresentation of an admissible continuous representation V of $G \times \Gamma$, then the quotient V/W is again an admissible continuous representation of $G \times \Gamma$. Lemma 7.1.12 shows that the representation of Γ on this quotient is strictly smooth. Furthermore, the isomorphism $V^H/W^H \xrightarrow{\sim} (V/W)^H$, and the fact that, by corollary 6.2.16, the category of admissible continuous representations of G is abelian when K is discretely valued, shows that $(V/W)^H$ is an admissible representation of G, for each compact open subgroup H of Γ. Thus V/W is an admissible continuous representation of $G \times \Gamma$, as required.

Part (v) follows from the fact that passing to H-invariants yields an exact functor on the category of smooth Γ-representations. $\square$

If V is a Hausdorff convex K-vector space equipped with a topological action of $G \times \Gamma$, then we will let V_{la} denote the space of G-locally analytic vectors attached to V, as in definition 3.5.3. The functoriality of the construction of locally analytic vectors shows that the Γ-action on V lifts to a Γ-action on V_{la}, and thus that V_{la} is equipped with a topological $G \times \Gamma$-action, uniquely determined by the requirement that the natural continuous injection $V_{\mathrm{la}} \to V$ be $G \times \Gamma$-equivariant.

Definition 7.2.3. We say that a topological action of $G \times \Gamma$ on a Hausdorff convex K-vector space V is a locally analytic $G \times \Gamma$-representation if V is barrelled, if the natural map $V_{\mathrm{la}} \to V$ is a bijection, and if the Γ-action on V is strictly smooth.

If V is equipped with a locally analytic $G \times \Gamma$-representation, then the fact that the G-action on V is locally analytic, together with lemma 7.1.10, implies that the $G \times \Gamma$-action on V is continuous. Note that if V is of compact type, then corollary 7.1.4 implies that in the preceding definition, we may replace the requirement that Γ act via a strictly smooth representation on V with the apparently weaker condition that the action of Γ on V be smooth.

Lemma 7.2.4. *If V is a Hausdorff convex K-vector space equipped with a topological action of $G \times \Gamma$, such that the Γ-action is strictly smooth, then the following are equivalent:*

(i) *The $G \times \Gamma$-action on V is locally analytic.*

(ii) *The G-action on V is locally analytic in the sense of definition* 3.6.9.

(iii) *For each compact open subgroup H of G, the G-action on V^H is locally analytic.*

Proof. Note that since the Γ-action on V is assumed to be strictly smooth, conditions (i) and (ii) are equivalent by definition.

Corollary 7.1.8 shows that if V is equipped with a strictly smooth Γ-action, then V is barrelled if and only if V^H is barrelled for each compact open subgroup H of G. It follows from proposition 3.6.14 that (ii) implies (iii).

Conversely, if (iii) holds, then $(V^H)_{\mathrm{la}} \to V^H$ is a continuous bijection, for each compact open subgroup H of Γ. The sequence of isomorphisms

$$V_{\mathrm{la}} \xleftarrow{\sim} (\varinjlim_H V^H)_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_H (V^H)_{\mathrm{la}}$$

(the first following from the assumption that the Γ-action on V is strictly smooth, and the second from proposition 3.5.14) then shows that the natural map $V_{\mathrm{la}} \to V$ is a continuous bijection. Again taking into account corollary 7.1.8, we find that (iii) implies (ii). $\square$

Proposition 7.2.5. *If V is a Hausdorff convex K-vector space, equipped with a topological $G \times \Gamma$-action, such that the Γ-action is strictly smooth, then for any compact open subgroup H of Γ there is a natural isomorphism $(V^H)_{\mathrm{la}} \xrightarrow{\sim} (V_{\mathrm{la}})^H$.*

Proof. Corollary 7.1.7 shows that V^H is a topological direct summand of V. Thus $(V^H)_{\mathrm{la}}$ is naturally identified with a topological direct summand of V_{la}, which clearly must equal $(V_{\mathrm{la}})^H$. $\square$

Corollary 7.2.6. *If V is a Hausdorff convex K-vector space, equipped with a topological $G\times\Gamma$-action, such that the Γ-action is strictly smooth, then V_{la} is a locally analytic $G\times\Gamma$-representation.*

Proof. Propositions 3.5.5 and 7.2.5 show that for each compact open subgroup H of Γ, the space of H-invariants $(V_{\mathrm{la}})^H$ is a locally analytic G-representation. By lemma 7.2.4, the corollary will be proved if we show that the Γ-action on V_{la} is strictly smooth.

Let $\{H_n\}_{n\geq 1}$ be a cofinal sequence of compact open subgroups of G. By assumption there is a natural isomorphism $\varinjlim_{n\geq 1} V^{H_n} \xrightarrow{\sim} V$. Passing to locally analytic vectors, and again appealing to proposition 7.2.5, we obtain isomorphisms

$$(V_{\mathrm{la}})_{\mathrm{st.sm}} \xrightarrow{\sim} \varinjlim_{n\geq 1} (V_{\mathrm{la}})^{H_n} \xleftarrow{\sim} \varinjlim_{n\geq 1} (V^{H_n})_{\mathrm{la}} \xrightarrow{\sim} (\varinjlim_{n\geq 1} V^{H_n})_{\mathrm{la}} \xrightarrow{\sim} V_{\mathrm{la}}$$

(the second-to-last isomorphism following from proposition 3.5.14, and the last from the fact that V is assumed to be a strictly smooth Γ-representation), and so the Γ-action on V_{la} is strictly smooth, as claimed. $\square$

We now extend the notions of admissible and essentially admissible representations of G to the context of representations of $G\times\Gamma$. Whenever we speak of essentially admissible locally analytic representations of G or of $G\times\Gamma$, we assume that the centre of G is topologically finitely generated, so that the notion of an essentially admissible locally analytic representation of G is in fact defined.

Definition 7.2.7. Let V be a locally convex Hausdorff K-vector space equipped with a locally analytic representation of $G\times\Gamma$. We say that V is an (essentially) admissible locally analytic representation of $G\times\Gamma$ if for each compact open subgroup H of Γ the closed subspace V^H of H-invariant vectors in V is an (essentially) admissible locally analytic representation of G.

Lemma 7.1.4 shows that if V is equipped with an (essentially) admissible locally analytic representation of $G\times\Gamma$, then V is of compact type.

Proposition 7.2.8. *Suppose that K is discretely valued.*

(i) *If V is an* (*essentially*) *admissible locally analytic representation of $G\times\Gamma$, then the $G\times\Gamma$-action on V is continuous.*

(ii) *If V is an* (*essentially*) *admissible locally analytic representation of $G\times\Gamma$, and if W is a closed $G\times\Gamma$-invariant subspace of V, then the $G\times\Gamma$-action on W is also an* (*essentially*) *admissible locally analytic representation.*

(iii) *Any $G\times\Gamma$-equivariant morphism of* (*essentially*) *admissible locally analytic representations of $G\times\Gamma$ is strict, with closed image.*

(iv) *The category of* (*essentially*) *admissible locally analytic representations of $G\times\Gamma$ is abelian.*

(v) *If H is a compact open subgroup of Γ, then the functor that takes V to the closed subspace of H-invariants V^H is an exact functor from the category of* (*essentially*) *admissible locally analytic representations of $G\times\Gamma$ to the category of* (*essentially*) *admissible locally analytic representations of G.*

Proof. This is proved in an analogous manner to proposition 7.2.2, once one recalls corollary 6.1.23 in the admissible case (respectively proposition 6.4.11 in the essentially admissible case). $\square$

We can give an alternative characterization of (essentially) admissible locally analytic representations of $G \times \Gamma$ in terms of their strictly smooth duals.

Suppose that V is a convex K-vector space of compact type, equipped with a locally analytic representation of $G \times \Gamma$. The strictly smooth dual $(V_b')_{\mathrm{st.sm}}$ of V is then a topological left $\mathcal{D}^{\mathrm{la}}(H, K)$-module (by corollary 5.1.9 (ii) together with lemma 7.1.18), for any compact open subgroup H of G, equipped with a commuting strictly smooth action of Γ (the contragredient action).

Lemma 7.2.9. *Suppose that the centre Z of G is topologically finitely generated. If V is as in the preceding discussion, then the Z-representation on V satisfies the equivalent conditions of proposition* 6.4.7 *if and only if the same is true for the closed Z-subrepresentation V^H of V, for each compact open subgroup H of G.*

Proof. If V is the union of an increasing sequence of Z-invariant BH-subspaces, then so is any closed subspace of V, and so in particular, so is each V^H. Conversely, if each V^H is the union of an increasing sequence of Z-invariant BH-subspaces, then letting H run through a cofinal sequence $\{H_n\}_{n \geq 1}$ of compact open subgroups of G, and writing $V = \bigcup V^{H_n}$, we find that the same is true of V. □

If the centre Z of G is topologically finitely generated, and if the locally analytic $G \times \Gamma$-representation V satisfies the conditions of proposition 6.4.7, then lemmas 7.1.18 and 7.2.9, together with that proposition, show that the strictly smooth dual $(V_b')_{\mathrm{st.sm}}$ of V is a topological left $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module, for any compact open subgroup H of G, equipped with a commuting strictly smooth action of Γ.

We now state the criterion on $(V_b')_{\mathrm{st.sm}}$ for V to be an (essentially) admissible representation of $G \times \Gamma$.

Proposition 7.2.10. *If V is a convex K-vector space of compact type, equipped with a locally analytic $G \times \Gamma$-action* (*respectively, a locally analytic $G \times \Gamma$-action, whose restriction to the centre Z of G satisfies the conditions of proposition* 6.4.7), *then V is an admissible* (*respectively, essentially admissible*) *locally analytic $G \times \Gamma$-representation if and only if $(V_b')_{\mathrm{st.sm}}$ is a coadmissible* $(\mathcal{D}^{\mathrm{la}}(H, K), \Gamma)$*-module* (*respectively, a coadmissible* $(\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K), \Gamma)$*-module*), *for one, or equivalently every, compact open subgroup H of G, in the sense of definition* 7.1.17.

Proof. Let H be a compact open subgroup of Γ. Lemma 7.1.14 shows that the space of invariants $((V_b')_{\mathrm{st.sm}})^H$ $(= (V_b')^H)$ is naturally isomorphic to $(V^H)_b'$. Thus, by the definition of admissibility (respectively, essential admissibility) of a locally analytic G-representation, the space $((V_b')_{\mathrm{st.sm}})^H$ is a coadmissible $\mathcal{D}^{\mathrm{la}}(H, K)$-module (respectively, a coadmissible $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module), equipped with its canonical topology, if and only if V^H is an admissible (respectively, essentially admissible) locally analytic representation of G. Taking into account the definition of coadmissibility of a topological $(\mathcal{D}^{\mathrm{la}}(H, K), \Gamma)$-module (respectively, $(\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K), \Gamma)$-module), the proposition follows. □

Proposition 7.2.11. *If V is an admissible continuous representation of $G \times \Gamma$, then V_{la} is an admissible locally analytic representation of $G \times \Gamma$.*

Proof. It follows from corollary 7.2.6 that V_{la} is a locally analytic representation of $G \times \Gamma$. It remains to show that for each compact open subgroup H of G, the space $(V_{\mathrm{la}})^H$ is an admissible locally analytic representation of G. Proposition 7.2.5

shows that this space is naturally isomorphic to $(V^H)_{\mathrm{la}}$, which proposition 6.2.4 shows is an admissible locally analytic representation of G. $\square$

Proposition 7.2.12. *If V is an admissible locally analytic representation of G, and if W is a finite dimensional locally analytic representation of G, then the tensor product $V \otimes_K W$, regarded as a $G \times \Gamma$-representation via the diagonal action of G and the action of Γ on the left factor, is an admissible locally analytic representation of $G \times \Gamma$.*

Proof. As a Γ-representation, the tensor product $V \otimes_K W$ is isomorphic to a finite direct sum of copies of V, and so is strictly smooth, since V is assumed to be so. If H is a compact open subgroup of Γ, then the natural map $V^H \otimes_K W \to (V \otimes_K W)^H$ is evidently a G-equivariant topological isomorphism. Since V^H is assumed to be an admissible locally analytic representation of G, we see by proposition 6.1.5 that the same is true of $V^H \otimes_K W$. Thus $V \otimes_K W$ satisfies the conditions of Definition 7.2.7. $\square$

Now suppose that $\mathbb{G}$ is a linear reductive algebraic group defined over L, and that G is an open subgroup of $\mathbb{G}(L)$. We can extend the notion of a locally algebraic representation of G, discussed in section 4.2, to representations of $G \times \Gamma$. As in that section we let $\mathcal{R}$ denote the semi-simple abelian category of finite dimensional representations of $\mathbb{G}$. The functor from $\mathcal{R}$ to the category of G-representations is fully faithful.

Fix an object W of $\mathcal{R}$. If V is a representation of $G \times \Gamma$ on an abstract K-vector space, then we may in particular regard V is a representation of G, and so (following proposition-definition 4.2.2) construct the space $V_{W-\mathrm{lalg}}$ of locally W-algebraic vectors of V. Since the formation of locally W-algebraic vectors is functorial in V, we see that $V_{W-\mathrm{lalg}}$ is a $G \times \Gamma$-invariant subspace of V.

Similarly, we may define the space V_{lalg} of all local algebraic vectors of V, as in proposition-definition 4.2.6. This is again a $G \times \Gamma$-invariant subspace of V. If $\hat{G}$ denotes a set of isomorphism class representatives of irreducible objects of $\mathcal{R}$, then corollary 4.2.7 implies that the natural map $\bigoplus_{W \in \hat{\mathbb{G}}} V_{W-\mathrm{lalg}} \to V_{\mathrm{lalg}}$ is an isomorphism.

Proposition 7.2.13. *If V is an admissible locally analytic representation of $G \times \Gamma$, and if W is an object of $\mathcal{R}$, then $V_{W-\mathrm{lalg}}$ is a closed $G \times \Gamma$-invariant subspace of V.*

Proof. It is evident that the natural map $(V^H)_{W-\mathrm{lalg}} \to (V_{W-\mathrm{lalg}})^H$ is an isomorphism. Thus proposition 4.2.10 shows that $(V_{W-\mathrm{lalg}})^H$ is a closed subspace of V^H for each compact open subgroup H of Γ. The proposition follows from lemma 7.1.12. $\square$

Proposition 7.2.14. *If V is an admissible locally analytic representation of $G \times \Gamma$ that is also locally algebraic as a representation of G, then V is equipped with its finest convex topology.*

Proof. If H is a compact open subgroup of Γ then V^H is an admissible locally analytic representation of G that is also locally algebraic as a representation of G. It follows from corollary 6.3.7 that V^H is equipped with its finest convex topology. By assumption V is isomorphic to the locally convex inductive limit $\varinjlim_H V^H$, as H ranges over all compact open subgroups of Γ. Thus V is also equipped with its finest convex topology. $\square$

Definition 7.2.15. If V is a representation of $G \times \Gamma$ that is locally algebraic as a representation of G, and that becomes admissible as a locally analytic representation when equipped with its finest convex topology, then we say that V (equipped with its finest convex topology) is an admissible locally algebraic representation of $G \times \Gamma$. If V is furthermore locally W-algebraic, for some object W of $\mathcal{R}$, then we say that V is an admissible locally W-algebraic representation.

Proposition 7.2.16. *If V is an admissible locally W-algebraic representation of $G \times \Gamma$, for some object W of $\mathcal{R}$, then V is isomorphic to a representation of the form $U \otimes_B W$, where $B = \mathrm{End}_{\mathbb{G}}(W)$ (a finite rank semi-simple K-algebra), U is an admissible smooth representation of $G \times \Gamma$ on a B-module (equipped with its finest convex topology), and the $G \times \Gamma$-action on the tensor product is defined by the diagonal action of G, and the action of Γ on the first factor. Conversely, any such tensor product is an admissible locally W-algebraic representation of $G \times \Gamma$.*

Proof. Proposition 4.2.10 yields a $G \times \Gamma$-equivariant topological isomorphism $U \otimes_B W \xrightarrow{\sim} V$. Let $\check{W}$ denote the contragredient representation to W. Proposition 7.2.12 shows that $\mathrm{Hom}(W, V)$ (which is naturally isomorphic to the tensor product $V \otimes_K \check{W}$) is an admissible locally analytic representation of $G \times \Gamma$, and hence (by proposition 7.2.8 (ii)) the same is true of its closed subspace $U = \mathrm{Hom}_{\mathfrak{g}}(W, V)$, equipped with its natural $G \times \Gamma$ action. Thus there is a topological isomorphism $\varinjlim_H U^H \xrightarrow{\sim} U$ (the locally convex inductive limit being taken over all compact open subgroups H of Γ). By corollary 6.3.3, each of the spaces U^H is an admissible smooth representation of G, equipped with its finest convex topology. Thus U is an admissible smooth representation of $G \times \Gamma$, and is equipped with its finest convex topology, as claimed.

Now suppose given an object W of $\mathcal{R}$, with algebra of endomorphisms B, and a B-linear admissible smooth representation U of $G \times \Gamma$ (equipped with its finest convex topology). The remark preceding the statement of proposition 4.2.4 shows that $U \otimes_B W$ is a locally W-algebraic representation of $G \times \Gamma$. If H is any compact open subgroup of Γ, then the natural map $U^H \otimes_B W \to (U \otimes_B W)^H$ is obviously an isomorphism. The source of this map is the tensor product of an admissible smooth representation of G and an object of $\mathcal{R}$, and hence by proposition 6.3.9 is an admissible locally analytic representation of G. Since the natural map $\varinjlim_H U^H \otimes_B W \to U \otimes_B W$ is obviously an isomorphism, we see that $U \otimes_B W$ is an admissible locally analytic isomorphism of $G \times \Gamma$, as claimed. $\square$

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