ON THE DENSITY OF SUPERCUSPIDAL POINTS OF FIXED REGULAR WEIGHT IN LOCAL DEFORMATION RINGS AND GLOBAL HECKE ALGEBRAS

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Abstract. We study the Zariski closure in the deformation space of a local Galois representation of the closed points corresponding to potentially semi-stable representations with prescribed p-adic Hodge-theoretic properties. We show in favourable cases that the closure contains a union of irreducible components of the deformation space. We also study an analogous question for global Hecke algebras.

1. Introduction

In this paper we study the Zariski closure in the deformation space of a local Galois representation of the closed points corresponding to potentially semi-stable representations with prescribed properties. We show in a number of cases that the closure contains a union of irreducible components of the deformation space. Our results are most novel in the case when the family consists of potentially semi-stable representations with fixed Hodge–Tate weights and their Weil–Deligne representations correspond to supercuspidal representations via the classical local Langlands correspondence; we call such points supercuspidal. We also obtain similar results in the setting of global Hecke algebras, and thus, in certain circumstances, namely those to which the results of [25] apply, in global Galois deformation rings as well.

Let us explain our results in more detail. Let F be a finite extension of $\mathbb{Q}_p$ and let G_F be its absolute Galois group. Let L be a further finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}$, a uniformizer ϖ and residue field k. Let $\bar{\rho} : G_F \to \mathrm{GL}_n(k)$ be a continuous representation and let $\rho^{\square} : G_F \to \mathrm{GL}_n(R^{\square})$ be the universal framed deformation of $\bar{\rho}$. In particular, $R^{\square}$ is a complete local noetherian $\mathcal{O}$-algebra with residue field k. If x is a maximal ideal of $R^{\square}[1/p]$ then its residue field $\kappa(x)$ is a finite extension of L; we denote its ring of integers by $\mathcal{O}_{\kappa(x)}$. By specializing $\rho^{\square}$ at x we obtain a representation $\rho_x : G_F \to \mathrm{GL}_n(\kappa(x))$. Moreover, its image is contained in $\mathrm{GL}_n(\mathcal{O}_{\kappa(x)})$ and by reducing the matrix entries modulo the maximal ideal of $\mathcal{O}_{\kappa(x)}$ we obtain $\bar{\rho}$. If ρ_x is potentially semi-stable then we can associate to it p-adic Hodge theoretic data: a multiset of integers $\mathrm{wt}(\rho_x)$ called the Hodge–Tate weights and Weil–Deligne representation $\mathrm{WD}(\rho_x)$, which is a representation of W_F (the Weil group of F) together with a nilpotent operator satisfying certain compatibilities. The following theorem is a representative example of our results.

Theorem 1.1. *Assume that p does not divide $2n$ and $\bar{\rho}$ admits a potentially crystalline lift of regular weight, which is potentially diagonalisable. Let E be an extension of F of degree n, which is either unramified or totally ramified, and fix regular*

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Hodge–Tate weights $\underline{k}$. Let Σ be the subset of maximal ideals of $R^{\square}[1/p]$ consisting of $x \in$ m-Spec $R^{\square}[1/p]$, such that ρ_x is potentially semi-stable, $\mathrm{wt}(\rho_x) = \underline{k}$, $\mathrm{WD}(\rho_x)$ is irreducible as a representation of W_F and is isomorphic to an induction from W_E of a 1-dimensional representation. Then the closure of Σ in $\operatorname{Spec} R^{\square}$ contains a non-empty union of irreducible components of $\operatorname{Spec} R^{\square}$.

Remark 1.2. If $\mathrm{Hom}_{G_F}(\bar{\rho}, \bar{\rho}(1)) = 0$ then the deformation problem of $\bar{\rho}$ is unobstructed and thus $R^{\square}$ is formally smooth over $\mathcal{O}$, so the theorem implies the density of Σ in the whole of $\operatorname{Spec} R^{\square}$.

Remark 1.3. The proof of the theorem uses a global input: the patched module M_∞ constructed in [10]. In the process we identify G_F with a decomposition group at a prime $\mathfrak{p}$ of an absolute Galois group of a totally real field K, and $\bar{\rho}$ as a restriction to G_F of a global automorphic representation of G_K. In the general case, we obtain precisely those components of $\operatorname{Spec} R^{\square}$ which contain a closed point corresponding to the restriction to G_F of a global automorphic representation of the kind considered in [10, §2.4] which is crystalline at $\mathfrak{p}$.

Remark 1.4. The assumption $p \nmid 2n$ comes from our use of the patched module M_∞ constructed in [10] and the assumptions made there. We also have a version of Theorem 1.1, when the field extension E/F is wildly ramified. In that case, the condition on $\mathrm{WD}(\rho_x)$ is more complicated to describe; it can be phrased in terms of Bushnell–Kutzko classification of supercuspidal representations of $\mathrm{GL}_n(F)$ in terms of types. If we identify $\mathrm{GL}_n(F) = \mathrm{Aut}_F(E)$, then $\mathrm{WD}(\rho_x)$ is required to correspond to a supercuspidal representation π_x under the classical local Langlands correspondence, such that π_x contains a simple stratum $[\mathfrak{A}, n', 0, \alpha]$ with $E = F[\alpha]$. Moreover, it is enough to consider those supercuspidals where α is minimal over F.

Remark 1.5. The assumption that $\bar{\rho}$ admits a potentially crystalline lift of regular weight, which is potentially diagonalisable, also comes from our use of the patched module M_∞. It has been conjectured in [20] and it will be shown in the forthcoming joint work of Toby Gee and ME that such lift always exists. We refer the reader to [2, §1.4] for the notion of potential diagonalisability.

Remark 1.6. The first result of this flavour for arbitrary F and n was proved by Hellmann and Schraen in [26], for the family of crystabelline representations with fixed Hodge–Tate weights (i.e. those representations, which become crystalline after a restriction to the Galois group of a finite abelian extension of F.) We also handle this case; in fact our arguments allow us to mix both cases.

1.1. **An idea of the proof.** The proof of Theorem 1.1 uses in a crucial way the patched module M_∞ constructed in [10]. We will recall some properties of M_∞: it is an $R_\infty[G]$-module, where R_∞ is a complete local noetherian $R^{\square}$-algebra with residue field k, obtained as part of the patching construction, and $G := \mathrm{GL}_n(F)$. We let $K = \mathrm{GL}_n(\mathcal{O}_F)$. As part of the construction of M_∞ we know that the action of the group ring $R_\infty[K] \subset R_\infty[G]$ on M_∞ extends to the action of the completed group ring $R_\infty[\![K]\!]$, which makes M_∞ into a finitely generated $R_\infty[\![K]\!]$-module. This allows us to equip M_∞ with a canonical topology with respect to which the action of $R_\infty[\![K]\!]$ is continuous. Moreover, M_∞ is a projective $\mathcal{O}[\![K]\!]$-module in the category of linearly compact $\mathcal{O}[\![K]\!]$-modules.

The proof then breaks up into two parts. In the first part we work in an abstract setting, by axiomatising the properties of M_∞ mentioned above, and develop a theory which we call *capture*. This abstract setting applies both to M_∞ and to the zero-th completed homology of a symmetric space, when the group is compact at ∞, see section 5.1. In fact the setting applies to the completed homology of more general symmetric spaces associated to a reductive group $\mathbb{G}$ defined over $\mathbb{Q}$, whenever $\bar{\rho}$ occurs only in one homological degree. Conjecturally this holds whenever $\bar{\rho}$ is irreducible and $l_0 = 0$, see [19, Conj. 3.3]; the invariant l_0 vanishes if for example $\mathbb{G}$ is semi-simple and has discrete series.

The output of the theory says that for certain well chosen families $\{V_i\}_{i \in I}$ of finite dimensional representations of K on L-vector spaces the R_∞-annihilator of the family of R_∞-modules $M_\infty \otimes_{\mathcal{O}[\![K]\!]} V_i$, for $i \in I$, annihilates M_∞. This abstract theory is then applied with the family $\{V_i\}_{i \in I}$ consisting of Bushnell–Kutzko types for supercuspidal representations, tensored with a fixed irreducible algebraic representation W of $\operatorname{Res}^F_{\mathbb{Q}_p} \mathrm{GL}_n / F$ evaluated at L and restricted to K. Properties of M_∞ proved in [10] allow us to convert the automorphic information given by the Bushnell–Kutzko types and the algebraic representation W into the Galois theoretic information defining the set Σ.

If $f, g : \mathbb{N} \to \mathbb{N}$ are functions, which tend to infinity as n goes to infinity, then we write $f \sim g$, if $\log f(n) = \log g(n) + O(1)$ as n tends to infinity. The following proposition is a key tool in the abstract setting considered above; see section 2.4 for the definition of the δ-dimension.

Proposition 1.7. *Let K be a uniformly powerful pro-p group. Let M be a finitely generated $k[\![K]\!]$-module, let $\dim M$ be the δ-dimension of M as $k[\![K]\!]$-module, and let $\dim K$ be the dimension of K as a p-adic manifold. For each $n \geq 1$ let K_n be the closed subgroup of K generated by the p^n-th powers of the elements of K. Then*

$$\dim_k M_{K_n} \sim (K : K_n)^{\frac{\dim M}{\dim K}},$$

where M_{K_n} denotes the space of K_n-coinvariants.

A weaker version of this Proposition has been proved by Harris in [23], see also [24].

In the second part of the proof carried out in section 6 we show:

Theorem 1.8. *The support of M_∞ in $\operatorname{Spec} R_\infty$ is equal to a union of irreducible components of $\operatorname{Spec} R_\infty$.*

We follow the approach of Chenevier [11] and Nakamura [30], who proved that the crystalline points are Zariski dense in the rigid analytic generic fibre $(\operatorname{Spf} R^\square)^{\mathrm{rig}}$ of $\operatorname{Spf} R^\square$. Indeed, we prove the following result, which implies the preceding theorem.

Theorem 1.9. *The Zariski closure in $(\operatorname{Spf} R_\infty)^{\mathrm{rig}}$ of the set of crystalline points lying in the support of M_∞ is equal to a union of irreducible components of $(\operatorname{Spf} R_\infty)^{\mathrm{rig}}$.*

The argument of Chenevier and Nakamura uses "infinite fern"-style arguments. In order to modify these arguments so as to be sensitive to the support of M_∞, we use the locally analytic Jacquet module of [17], which provides a representation-theoretic approach to the study of finite slope families. Breuil, Hellmann, and Schraen [3] have made a careful study of the theory of the Jacquet module as it applies to M_∞, and we use their results.

The argument outlined above proves the analog of Theorem 1.1 with $R^\square$ replaced by R_∞. The statement for $R^\square$ is deduced from this by a commutative algebra argument using that $R_\infty \cong R^\square \widehat{\otimes}_{\mathcal{O}} A$, where A is a complete local noetherian $\mathcal{O}$-algebra, which is $\mathcal{O}$-flat.

1.2. **Historical remarks.** Since we have been talking about the results of this paper since the workshop on the p-adic Langlands correspondence in the Fields Institute in 2012, some historical comments seem to be in order. At the beginning our focus was to prove such density theorem for the global deformation ring of an absolutely irreducible representation $\bar{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$, which we assume to be modular. The strategy was to apply representation theoretical techniques in the spirit of Proposition 1.7 to the completed homology of modular curves localised at the ideal of the Hecke algebra corresponding to $\bar{\rho}$. So the first part of the proof was essentially the same as it is now. For the second part of the proof one needed to show that the completed homology of modular curves localised at the ideal of the Hecke algebra corresponding to $\bar{\rho}$ is supported on a union of irreducible components of the global deformation ring of $\bar{\rho}$, which amounted to producing a lower bound on the dimension of the Hecke algebra, see the discussion of [19, §3.1.1]. The bound was obtained by showing that the Krull dimension of the Hecke algebra is at least the δ-dimension of the completed homology, which is equal to 5, minus the δ-dimension of the fibre, which we knew to have dimension 1: the fibre is determined by results of [18] (see also [36, Lem. 5.8] for a different argument based on the results of [35]), and its δ-dimension is equal to 1; our original argument is explained in the proof of [36, Lem. 5.14].

Such arguments have since appeared in the paper of Gee–Newton [21] in a more general context; in particular they also prove a kind of converse theorem called 'miracle flatness', which we did not know at the time, by working systematically with Cohen–Macaulay modules.

We also proved density results for the local deformation rings of $\bar{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(k)$ by using results of [35] and the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Then a paper of Hellmann–Schraen appeared [26], proving the density result discussed in Remark 1.6. While studying their paper, we realised that the second part of our proof can be obtained using the 'infinite fern' arguments by constructing what is now known as the patched eigenvariety. The construction of this patched eigenvariety appeared in the paper by Breuil–Hellmann–Schraen [3], although both ME and Hellmann explained their ideas to VP independently at around the same time.

Finally a paper of Scholze [37] has appeared, in which he has a much simpler proof of one of the consequences of our theory, discussed in [19, 3.3.2]. We adapt and explain his argument in section 4. And although our abstract theory is more general, the two key arithmetically interesting examples are covered by this simpler argument due to Scholze, which avoids counting dimensions, required by our criterion for capture.

In recent preprints [39], [40] Shen-Ning Tung has used our result on the support of M_∞ to obtain a new proof of the Fontaine–Mazur conjecture for $\mathrm{GL}_2/\mathbb{Q}$, obtaining new cases of the conjecture when $p = 3$ and $p = 2$. This was one of the motivations to finally finish this paper.

1.3. **The structure of the paper.** In section 2 we develop the theory of capture in an abstract setting. Some of the results of section 2.1 have appeared in [12, §2.4], but we recall the proofs for the convenience of the reader. The main result is Proposition 2.23, which establishes a numerical criterion for a family $\{V_i\}_{i\in I}$ of smooth irreducible representations on finite dimensional L-vector spaces of a compact p-adic analytic group K, which does not contain any torsion, to *capture* every projective linearly compact $\mathcal{O}[\![K]\!]$-module in the sense of Definition 2.1. The key Proposition which establishes the link between the growth of coinvariants and dimension is proved in section 2.4.

In section 3 we exhibit two examples where our general theory applies, by taking K to be a sufficiently small compact open subgroup of $\mathrm{GL}_n(F)$ and, in the supercuspidal case in section 3.4, the family $\{V_i\}_{i\in I}$ inductions of simple characters in the sense of Bushnell–Kutzko theory. The main work is then to verify that our numerical criterion holds, which amounts to counting types and their dimensions. We also handle the principal series case in section 3.3.

In section 4 we explain that both of these examples can be dealt with in an easier way by adopting an argument of Scholze in [37]. This argument still uses Bushnell–Kutzko theory, but avoids the counting arguments.

In section 5 we apply the theory developed above to the zero-th completed homology of a symmetric space, when the group is compact at infinity.

In section 6 we show that the support of M_∞ is equal to a union of irreducible components of R_∞.

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2. Generalities

2.1. **Capture.** Let K be a pro-finite group with an open pro-p subgroup. Let $\mathcal{O}[\![K]\!]$ be the completed group algebra, and let $\mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$ be the category of compact linear-topological $\mathcal{O}[\![K]\!]$-modules. Let $\{V_i\}_{i\in I}$ be a family of continuous representations of K on finite dimensional L-vector spaces, and let $M \in \mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$. We will denote by V_i^* the L-linear dual of V_i.

Definition 2.1. We say that $\{V_i\}_{i\in I}$ *captures* M if the smallest quotient $M \twoheadrightarrow Q$ such that $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(Q, V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$, for all $i \in I$, is equal to M.

The following equivalent characterizations of capture can be easily verified, see [12, §2.4] for details.

Lemma 2.2. *Let*
$$N = \bigcap_{\phi} \mathrm{Ker}\,\phi,$$
where the intersection is taken over all $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$, *for all* $i \in I$. *Then* $\{V_i\}_{i\in I}$ *captures* M *if and only if* $N = 0$.

Lemma 2.3. *Let* $M \in \mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$ *be* $\mathcal{O}$*-torsion free, let* $\Pi(M) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L)$ *be the associated* L*-Banach space equipped with its supremum norm. Then* $\{V_i\}_{i\in I}$ *captures* M *if and only if the image of the evaluation map*
$$\oplus_i \mathrm{Hom}_K\big(V_i, \Pi(M)\big) \otimes_L V_i \longrightarrow \Pi(M)$$
is a dense subspace.

Our next lemmas describe various simple aspects of the notion of capture.

Lemma 2.4. *If $\{V_i\}_{i\in I}$ captures M, then M is $\mathcal{O}$-torsion free.*

Proof. Any $\mathcal{O}$-torsion element is in the kernel of any morphism $M \to V_i^*$ for any $i \in I$; the lemma then follows from Lemma 2.2. $\square$

Lemma 2.5. *If $\{M_j\}_{j\in J}$ is a collection of objects of $\mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$, then the family $\{V_i\}_{i\in I}$ captures the product $\prod_{j\in J} M_j$ if and only if it captures each of the M_j.*

Proof. Noting that $\prod_j M_j$ is $\mathcal{O}$-torsion free if and only if each M_j is $\mathcal{O}$-torsion free, we see from Lemma 2.4 that it is no loss of generality, when proving the lemma, to assume further that each M_j is $\mathcal{O}$-torsion free. We note also that

$$\Pi(\prod_j M_j) := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\prod_j M_j, L) = \widehat{\oplus}_j \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M_j, L) =: \widehat{\oplus}_j \Pi(M_j),$$

where the completed direct sum is taken with respect to the sup norm on each summand, and where we are using the notation from the statement of Lemma 2.3. Thus the evaluation map $\oplus_i \mathrm{Hom}_K\big(V_i, \Pi(\prod_j M_j)\big) \otimes_L V_i \to \Pi(\prod_j M_j)$ may be rewritten as the map $\oplus_i \widehat{\oplus}_j \mathrm{Hom}_K\big(V_i, \Pi(M_j)\big) \to \widehat{\oplus}_j \Pi(M_j)$. This has dense image if and only if each of the evaluation maps $\oplus_i \mathrm{Hom}_K\big(V_i, \Pi(M_j)\big) \to \Pi(M_j)$ has dense image. The present lemma thus follows from Lemma 2.3. $\square$

Lemma 2.6. *Let W be a continuous representation of K on a finite dimensional L-vector space, and let Θ be a K-invariant $\mathcal{O}$-lattice in W. If $\{V_i\}_{i\in I}$ captures $M \otimes_{\mathcal{O}} \Theta$ then $\{V_i \otimes_L W\}_{i\in I}$ captures M.*

Proof. For each $i \in I$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M \otimes_{\mathcal{O}} \Theta, V_i^*) \cong \mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M, (V_i \otimes_L W)^*). \tag{1}$$

Thus if Q is a quotient of M, such that

$$\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M, (V_i \otimes_L W)^*) = \mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(Q, (V_i \otimes_L W)^*),$$

for all $i \in I$, then we deduce from (1) that

$$\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M \otimes_{\mathcal{O}} \Theta, V_i^*) = \mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(Q \otimes_{\mathcal{O}} \Theta, V_i^*),$$

for all $i \in I$. This implies that $M \otimes_{\mathcal{O}} \Theta = Q \otimes_{\mathcal{O}} \Theta$, as $\{V_i\}_{i\in I}$ captures $M \otimes_{\mathcal{O}} \Theta$ by assumption. Since Θ is a free $\mathcal{O}$-module we deduce that $M = Q$. $\square$

Lemma 2.7. *Assume that $\{V_i\}_{i\in I}$ captures M and let $\phi \in \mathrm{End}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M)$. If ϕ kills $\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M, V_i^*)$ for all $i \in I$ then $\phi = 0$.*

Proof. The assumption on ϕ implies that

$$\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(\mathrm{Coker}\,\phi, V_i^*) \cong \mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M, V_i^*), \quad \forall i \in I.$$

Since $\{V_i\}_{i\in I}$ captures M, we deduce that $M = \mathrm{Coker}\,\phi$ and thus $\phi = 0$. $\square$

Lemma 2.8. *The following assertions are equivalent:*

(i) *$\{V_i\}_{i\in I}$ captures every indecomposable projective in $\mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$;*
(ii) *$\{V_i\}_{i\in I}$ captures every projective in $\mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$;*
(iii) *$\{V_i\}_{i\in I}$ captures $\mathcal{O}[\![K]\!]$.*

Proof. This follows from Lemma 2.5 and the fact that every projective object in $\mathrm{Mod}_K^{\mathrm{pro}}(\mathcal{O})$ is isomorphic to a product of indecomposable projective objects, which in turn are precisely the indecomposable direct summand of $\mathcal{O}[\![K]\!]$. A version of this argument appears in [12, Lem. 2.11]. $\square$

Lemma 2.9. *Let W be a continuous representation of K on a finite dimensional L-vector space. If $\{V_i\}_{i\in I}$ captures every projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$ then the same holds for $\{V_i\otimes_L W\}_{i\in I}$.*

Proof. Let Θ be a K-invariant $\mathcal{O}$-lattice in W. If K_n is an open normal subgroup of K, which acts trivially on Θ/ϖ^n, then the map $g\otimes v\mapsto g\otimes g^{-1}v$, where $g\in K/K_n$ and $v\in\Theta/\varpi^n$ induces an isomorphism of K-representations:

$$\mathcal{O}/\varpi^n[K/K_n]\otimes\Theta/\varpi^n\cong\mathcal{O}/\varpi^n[K/K_n]\otimes\Theta/\varpi^n,$$

where K acts diagonally on the left hand side and only on the first factor on the right hand side of the isomorphism. By passing to a limit we deduce that

$$\mathcal{O}[\![K]\!]\otimes_{\mathcal{O}}\Theta\cong\mathcal{O}[\![K]\!]^{\oplus d}$$

as $\mathcal{O}[\![K]\!]$-module, where $d=\dim_L W$. In particular, $\mathcal{O}[\![K]\!]\otimes_{\mathcal{O}}\Theta$ is projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$, and hence is captured by $\{V_i\}_{i\in I}$ by assumption. Lemma 2.6 implies that $\{V_i\otimes_L W\}_{i\in I}$ captures $\mathcal{O}[\![K]\!]$ and the assertion follows from Lemma 2.8. $\square$

2.2. **Density.** Let K and M and $\{V_i\}_{i\in I}$ be as above, but we additionally assume that K is p-adic analytic. In the applications K will be an open subgroup of $\mathrm{GL}_n(F)$, where F is a finite extension of $\mathbb{Q}_p$. Let R be a complete local noetherian $\mathcal{O}$-algebra with residue field k. We assume that we are given a faithful, continuous $\mathcal{O}$-linear action of R on M, which commutes with the action of K. In other words, we assume that M is a compact $R[\![K]\!]$-module, such that the action of R is faithful. For each $i\in I$ let $\mathfrak{a}_i$ be the R-annihilator of $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M,V_i^*)$.

Lemma 2.10. *If $\{V_i\}_{i\in I}$ captures M then $\bigcap_{i\in I}\mathfrak{a}_i=0$.*

Proof. This follows from Lemma 2.7. $\square$

Let $\text{m-Spec}\, R[1/p]$ be the set of maximal ideals of $R[1/p]$. If $x\in\text{m-Spec}\, R[1/p]$ then its residue field $\kappa(x)$ is a finite extension of L, and we denote by $\mathcal{O}_{\kappa(x)}$ the ring of integers of $\kappa(x)$. Then $\mathcal{O}_{\kappa(x)}\otimes_R M$ is a compact $\mathcal{O}[\![K]\!]$-module and we define

$$\Pi_x:=\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\mathcal{O}_{\kappa(x)}\otimes_R M,L).$$

Then Π_x equipped with the supremum norm is a unitary L-Banach space representation of K.

Proposition 2.11. *Assume that M is a finitely generated $R[\![K]\!]$-module, that $\{V_i\}_{i\in I}$ captures M, and that there exists an integer e such that $(\sqrt{\mathfrak{a}_i})^e\subset\mathfrak{a}_i$ for all $i\in I$. Then the set*

$$\Sigma:=\{x\in\text{m-Spec}\, R[1/p]:\mathrm{Hom}_K(V_i,\Pi_x)\neq 0\ \textit{for some}\ i\in I\}$$

is dense in $\mathrm{Spec}\, R$.

Proof. For each $i\in I$ we choose a K-invariant bounded $\mathcal{O}$-lattice Θ_i in V_i. It follows from [34, Prop. 2.15] that $M(\Theta_i):=(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M,\Theta_i^d))^d$, where $(*)^d:=\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(*,\mathcal{O})$, is a finitely generated R-module. Moreover, it follows from [34, Eqn. (11) and Rem. 2.18] that for a fixed $i\in I$ the annihilator of $M(\Theta_i)$ in R is equal to $\mathfrak{a}_i$ and it follows from [34, Prop. 2.22] that the set

$$\Sigma_i:=\{x\in\text{m-Spec}\, R[1/p]:\mathrm{Hom}_K(V_i,\Pi_x)\neq 0\}$$

is precisely $\text{m-Spec}(R/\mathfrak{a}_i)[1/p]$. Since $M(\Theta_i)$ is $\mathcal{O}$-torsion free, $R/\mathfrak{a}_i$ is $\mathcal{O}$-torsion free, and thus the intersection of all maximal ideals in Σ_i with R is equal to $\sqrt{\mathfrak{a}_i}$. Thus the intersection of all maximal ideals in Σ with R is equal to $\cap_{i\in I}\sqrt{\mathfrak{a}_i}$. Note that R

is $\mathcal{O}$-torsion free since by assumption it acts faithfully on an $\mathcal{O}$-torsion free module M. Our assumption $(\sqrt{\mathfrak{a}_i})^e \subset \mathfrak{a}_i$ for all $i \in I$ implies that the e-th power of this ideal is contained in $\cap_{i\in I}\mathfrak{a}_i$, which is zero by Lemma 2.10. Thus Σ is Zariski dense in $\operatorname{Spec} R$. □

Remark 2.12. If M is a finitely generated $\mathcal{O}[\![K]\!]$-module then $\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$ is a finite dimensional L-vector space. If for each $i \in I$ the action of $R[1/p]$ on $\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$ is semi-simple then we can conclude that $R/\mathfrak{a}_i$ is reduced, and hence $\sqrt{\mathfrak{a}_i} = \mathfrak{a}_i$, so that the conditions of Proposition 2.11 are satisfied.

Remark 2.13. The set-up of Proposition 2.11 can be relaxed to allow non-noetherian rings as follows. Let A be a projective limit of local artinian R-algebras with residue field k with transition maps homomorphisms of local R-algebras. Then A is a local R-algebra with residue field k, which need not be noetherian. Assume that the action of $R[\![K]\!]$ on M extends to a continuous action of $A[\![K]\!]$ for the projective limit topology on A. For each $i \in I$ let A_i be a quotient of A, which acts faithfully on $M(\Theta_i)$. Since R is noetherian and $M(\Theta_i)$ is a finitely generated R-module, we deduce that the rings A_i are noetherian for all $i \in I$. Thus the proof of Proposition 2.11 shows that if A_i are reduced for all $i \in I$ and A acts faithfully on M then $\bigcup_{i\in I} \text{m-Spec}\, A_i[1/p]$ is dense in $\operatorname{Spec} A$.

2.3. **Rationality.** From the representation theory point of view it is convenient to work over $\overline{L}$, the algebraic closure of L. However this causes trouble from the p-adic functional analysis point of view, see for example the proof of Lemma 2.5. The purpose of this section is to deal with this problem.

Lemma 2.14. *If V is a finite dimensional $\overline{L}$-vector space with a continuous K-action then there is a finite extension L' of L contained in $\overline{L}$ and a finite dimensional K-invariant L'-subspace W of V, such that the inclusion $W \hookrightarrow V$ induces an isomorphism $W \otimes_{L'} \overline{L} \xrightarrow{\cong} V$.*

Proof. We may rephrase the problem as follows: if $\rho : K \to \mathrm{GL}_n(\overline{L})$ is a continuous group homomorphism then its image is contained in $\mathrm{GL}_n(L')$ for some finite extension L' of L contained in $\overline{L}$. Such statement is proved in Lemma 2.2.1.1 in [4], with K equal to the absolute Galois group of $\mathbb{Q}_p$. However, only compactness of the group is used in the proof, so their argument applies in our setting. □

Lemma 2.15. *Let L' be a finite extension of L contained in $\overline{L}$. Let W be a finite dimensional L'-vector space with a continuous K-action. Then we have a natural isomorphism*

$$\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W) \otimes_{L'} \overline{L} \xrightarrow{\cong} \operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W \otimes_{L'} \overline{L}). \tag{2}$$

Proof. Since $\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, *)$ commutes with finite direct sums, the isomorphism in (2) holds if we replace $\overline{L}$ by a finite extension L'' of L' contained in $\overline{L}$. We thus obtain

$$\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W) \otimes_{L'} L'' \overset{\cong}{\to} \operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W \otimes_{L'} L'') \hookrightarrow \operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W \otimes_{L'} \overline{L}).$$

By passing to a direct limit we obtain an injection

$$\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W) \otimes_{L'} \overline{L} \hookrightarrow \operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, W \otimes_{L'} \overline{L}).$$

In order to prove surjectivity we have to show that if $\varphi : M \to W \otimes_{L'} \overline{L}$ is a continuous map of $\mathcal{O}$-modules then its image is contained in $W \otimes_{L'} L''$ for some finite extension L'' of L'. This can be proved in the same way as Lemma 2.2.1.1 in [4] using the compactness of M. □

Let $\{V_i\}_{i\in I}$ be a family of continuous representations of K on finite dimensional $\overline{L}$-vector spaces, and let $M \in \mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$ as in the previous section. We make the analogous definition to Definition 2.1.

Definition 2.16. We say that $\{V_i\}_{i\in I}$ *captures* M if the smallest quotient $M \twoheadrightarrow Q$, such that $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(Q, V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$ for all $i \in I$ is equal to M.

Lemma 2.17. *Let $\{V_i\}_{i\in I}$ be a family of continuous representations of K on finite dimensional $\overline{L}$-vector spaces. For each $i \in I$ let L_i be a finite extension of L contained in $\overline{L}$ and let W_i be a K-invariant L_i-subspace of V_i given by Lemma 2.14, so that $W_i \otimes_{L_i} \overline{L} \cong V_i$ is an isomorphism of K-representations. Then $\{V_i\}_{i\in I}$ captures M in the sense of Definition 2.16 if and only if $\{W_i\}_{i\in I}$ captures M in the sense of Definition 2.1.*

Proof. The assertion follows from Lemma 2.15. □

2.4. **Growth of coinvariants.** In this subsection we let K be a uniformly powerful pro-p group; such groups are discussed in great detail in [13, §4]. We note that uniformly powerful pro-p groups are p-adic analytic and every compact p-analytic group contains an open uniformly powerful subgroup, so the assumption is harmless. We denote by $\dim K$ the dimension of K as a p-adic manifold. Finitely generated modules over $k[\![K]\!]$ have a good dimension theory, generalizing the Krull dimension, see [1], [41]. The purpose of this section is to relate this dimension to the growth of coinvariants by certain open subgroups of K.

Let M be a finitely generated $k[\![K]\!]$-module. We let

$$\mathrm{codim}_{k[\![K]\!]}(M) := \min\{i : \mathrm{Ext}^i_{k[\![K]\!]}(M, k[\![K]\!]) \neq 0\},$$
$$\dim_{k[\![K]\!]}(M) := \dim K - \mathrm{codim}_{k[\![K]\!]}(M).$$

We will refer to $\dim_{k[\![K]\!]}(M)$ as the δ-dimension of M. For $n \geq 0$ let $K_n := K^{p^n}$ be the closed subgroup generated by the p^n-th powers of elements of K. Since K is uniformly powerful, K_n is open in K and we have

$$(K : K_n) = p^{n(\dim K)}, \quad \forall n \geq 0. \tag{3}$$

If $f, g : \mathbb{N} \to \mathbb{N}$ are functions, which tend to infinity as n goes to infinity, then we write $f \sim g$, if $\log f(n) = \log g(n) + O(1)$ as n tends to infinity.

For example, if F is a finite extension of $\mathbb{Q}_p$, U_F is the group of units of its ring of integers, and U_F^n is the n-th congruence subgroup of U_F then $(\mathrm{U}_F : \mathrm{U}_F^n) \sim q_F^n$, where q_F is the number of elements in the residue field of F.

Proposition 2.18. *Let $d = \dim_{k[\![K]\!]} M$. Then there are real numbers $a \geq b \geq \frac{1}{d!}$ such that*

$$bp^{dn} + O(p^{(d-1)n}) \leq \dim_k M_{K_n} \leq ap^{dn} + O(p^{(d-1)n}).$$

In particular, $\dim_k M_{K_n} \sim (K : K_n)^{\frac{\dim(M)}{\dim(K)}}$.

Proof. Let $g_1, \dots, g_m$ be a minimal set of topological generators of K, let $\mathfrak{m}$ be the maximal ideal of $k[\![K]\!]$. Then there is an isomorphism of graded rings

$$\mathrm{gr}^\bullet_{\mathfrak{m}}(k[\![K]\!]) \cong k[x_1, \dots, x_m],$$

which sends $g_i - 1 + \mathfrak{m}^2$ to x_i, see [1, Thm. 3.4]. Since K is uniformly powerful $g_1^{p^n}, \dots, g_m^{p^n}$ is a minimal set of generators of K^{p^n} [13, Thm. 3.6 (iii)]. Thus if we let I_n be the closed two-sided ideal of $k[\![K]\!]$ generated by $g_1^{p^n} - 1, \dots, g_m^{p^n} - 1$, then

$$M_{K_n} \cong M/I_n M \twoheadrightarrow M/\mathfrak{m}^{p^n} M. \tag{4}$$

Let A and $\widetilde{M}$ be the localizations of $\mathrm{gr}^{\bullet}_{\mathfrak{m}}(k[\![K]\!])$ and $\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)$ at the ideal $\mathrm{gr}^{>0}_{\mathfrak{m}}(k[\![K]\!])$ respectively, and let $\mathfrak{n}$ be the maximal ideal of A. Then

$$\dim_k M/\mathfrak{m}^n M = \sum_{i=0}^{n-1} \dim_k(\mathfrak{m}^i M/\mathfrak{m}^{i+1}M) = \ell(\widetilde{M}/\mathfrak{n}^n\widetilde{M}). \tag{5}$$

Proposition 5.4 of [1] implies that the Krull dimension of $\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)$, and hence of $\widetilde{M}$, is equal to d. We know that for large n, $\ell(\widetilde{M}/\mathfrak{n}^n\widetilde{M})$ is a polynomial in n of degree d and leading coefficient $e/d!$, where e is the Hilbert–Samuel multiplicity of $\widetilde{M}$, see [28, §14]. In particular,

$$\ell(\widetilde{M}/\mathfrak{n}^{p^n}\widetilde{M}) = \frac{e}{d!}p^{dn} + O(p^{(d-1)n}). \tag{6}$$

The surjection $M \twoheadrightarrow M_{K_n}$ induces a surjection of graded modules

$$\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M) \twoheadrightarrow \mathrm{gr}^{\bullet}_{\mathfrak{m}}(M_{K_n}).$$

Since $g_i^{p^n}$ act trivially on M_{K_n}, the surjection factors through

$$\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)/(x_1^{p^n},\dots,x_m^{p^n})\,\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M) \twoheadrightarrow \mathrm{gr}^{\bullet}_{\mathfrak{m}}(M_{K_n}). \tag{7}$$

Let $\mathfrak{n}^{[p^n]}$ be the ideal of A generated by x^{p^n} for all $x \in \mathfrak{n}$. If $a_1,\dots,a_m$ is a system of parameters of A then $\mathfrak{n}^{[p^n]} = (a_1^{p^n},\dots,a_m^{p^n})$ as A is a regular ring. Hence, the localization of $\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)/(x_1^{p^n},\dots,x_m^{p^n})\,\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)$ at $\mathrm{gr}^{>0}_{\mathfrak{m}}(k[\![K]\!])$ is equal to $\widetilde{M}/\mathfrak{n}^{[p^n]}\widetilde{M}$ and we obtain

$$\dim_k(\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)/(x_1^{p^n},\dots,x_m^{p^n})\,\mathrm{gr}^{\bullet}_{\mathfrak{m}}(M)) = \ell(\widetilde{M}/\mathfrak{n}^{[p^n]}\widetilde{M}). \tag{8}$$

The main result of [29] says that there is a real number e', called the Hilbert–Kunz multiplicity, such that

$$\ell(\widetilde{M}/\mathfrak{n}^{[p^n]}\widetilde{M}) = e'p^{dn} + O(p^{(d-1)n}). \tag{9}$$

Actually, as G. Dospinescu pointed out to us, we don't require the full strength of [29]. Indeed, since $\mathfrak{m}$ is generated by m generators, we have $\mathfrak{n}^{mp^n} \subseteq \mathfrak{n}^{[p^n]}$, and so (just using the theory of the Hilbert–Samuel multiplicity), we find that

$$\ell(\widetilde{M}/\mathfrak{n}^{[p^n]}\widetilde{M}) \le \ell(\widetilde{M}/\mathfrak{n}^{mp^n}\widetilde{M}) = \frac{e}{d!}m^d p^{dn} + O(p^{(d-1)n}), \tag{10}$$

which suffices for our purposes.

In any event, we conclude from (4), (5), (7), and (8) that

$$\ell(\widetilde{M}/\mathfrak{n}^{p^n}\widetilde{M}) \le \dim_k M_{K_n} \le \ell(\widetilde{M}/\mathfrak{n}^{[p^n]}\widetilde{M}).$$

The assertion then follows from (6) and (9) (or its variant (10)). □

Remark 2.19. Proposition 2.18 asserts that

$$\dim_k M_{K_n} \sim (K:K_n)^{\frac{\dim(M)}{\dim(K)}}. \tag{11}$$

However, we caution the reader that the analogous statement might not hold for an arbitrary filtration of K by open normal subgroups. For example, let $K = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and let $M = k[\![\mathbb{Z}_p]\!]$, on which K acts via the projection onto the second factor, let $K_n = p^n\mathbb{Z}_p \oplus p^{2^n}\mathbb{Z}_p$. Then $\dim_k M_{K_n} = p^{2^n}$ and $(K:K_n) = p^{n+2^n}$, $\dim M = 1$ and $\dim K = 2$, so that (11) does not hold. One should modify the statement of [9, Thm. 1.18] accordingly. The authors of [9] claim that a stronger version of Proposition 2.18 is proved in [1, §5], however we have not been able to verify this.

Remark 2.20. A weaker version of this statement follows from Theorem 1.10 in [23], see also [24].

2.5. **Numerical criterion for capture.** From now on we assume that K is a p-adic analytic group, which doesn't have any torsion. This allows for more general groups K than considered in the previous section. As already noted every such K contains an open subgroup K', which is a uniformly powerful pro-p group, see [13, Thm. 8.23]. For a finitely generated $\mathcal{O}[\![K]\!]$-module M we let $\dim_{\mathcal{O}[\![K]\!]}(M) := \dim_{\mathcal{O}[\![K']\!]}(M)$. This definition does not depend on the choice of K', see [1, §5.4]. Since the identity in K has a basis of open neighbourhoods consisting of open uniformly powerful pro-p subgroups, this implies that $\dim_{\mathcal{O}[\![K]\!]}(M) = \dim_{\mathcal{O}[\![K'']\!]}(M)$ for any open subgroup K'' of K.

Our assumption on K implies that K is pro-p and the completed group algebras $k[\![K]\!]$ and $\mathcal{O}[\![K]\!]$ are integral domains, see [1, Thm. 4.3].

Lemma 2.21. *Let M be a cyclic $k[\![K]\!]$-module. If $\dim_{k[\![K]\!]}(M) = \dim K$ then the surjection $k[\![K]\!] \twoheadrightarrow M$ is an isomorphism.*

Proof. Let $\Lambda = k[\![K]\!]$. Then $\dim_\Lambda M = \dim K$ if and only if $\mathrm{Hom}_\Lambda(M, \Lambda) \neq 0$. Since M is cyclic, it is enough to prove that $\mathrm{Hom}_\Lambda(\Lambda/\Lambda y, \Lambda) = 0$ for all non-zero y in the maximal ideal of Λ. This follows by applying the functor $\mathrm{Hom}_\Lambda(*, \Lambda)$ to the exact sequence $\Lambda \xrightarrow{\cdot y} \Lambda \to \Lambda/\Lambda y \to 0$, and using the fact that our assumptions on K imply that Λ is an integral domain, so that multiplication with y induces an injective map from $\mathrm{Hom}_\Lambda(\Lambda, \Lambda)$ to itself. □

Let $\{K_n\}_{n\geq 1}$ be a basis of open neighbourhoods of 1 in K consisting of normal subgroups. We assume that there is $n_0 \geq 0$, such that K_{n_0} is a uniform pro-p group, and for each $m \geq 1$ there is an $n(m)$, such that $K_{n_0}^{p^m} = K_{n(m)}$. Let $\mathcal{C}(K, L)$ be the space of continuous L-valued functions on K. Since K is compact the supremum norm defines a Banach space structure on $\mathcal{C}(K, L)$; it is an admissible unitary L-Banach space representation of K for the K-action given by right translations.

Lemma 2.22. *Let Π be a closed K-invariant subspace of $\mathcal{C}(K, L)$. If $\dim_L \Pi^{K_n} \sim (K : K_n)$ then $\Pi = \mathcal{C}(K, L)$.*

Proof. Let Θ be the unit ball in Π with respect to the supremum norm on $\mathcal{C}(K, L)$. The map $K \to \mathcal{O}[\![K]\!]$, $g \mapsto g$ induces an isomorphism $\mathcal{C}(K, L) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\mathcal{O}[\![K]\!], L)$ [38, Cor.2.2]. Thus, if we let $\Theta^d := \mathrm{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$ then Schikhof's duality induces a surjection of $\mathcal{O}[\![K]\!]$-modules, $\mathcal{O}[\![K]\!] \twoheadrightarrow \Theta^d$ [38, Thm. 3.5]. We claim that the surjection is an isomorphism. By dualizing back, the claim implies the Lemma. Since Θ^d is $\mathcal{O}$-torsion free, it is enough to show that $k[\![K]\!] \twoheadrightarrow \Theta^d/\varpi$ is an isomorphism. Lemma 2.21 implies that it is enough to show that $\dim_{k[\![K]\!]}(\Theta^d/\varpi) = \dim K$. Since the dimension does not change if we pass to an open subgroup, it is enough to show that $\dim_{k[\![K_{n_0}]\!]}(\Theta^d/\varpi) = \dim K_{n_0}$. Proposition 2.18 implies that it is enough to show that $\dim_k((\Theta^d/\varpi)_{K_n}) \sim (K_{n_0} : K_n)$. Since $\Theta^d/\varpi \cong (\Theta/\varpi)^\vee$, we get that $(\Theta^d/\varpi)_{K_n} \cong ((\Theta/\varpi)^{K_n})^\vee$. Thus

$$\dim_k((\Theta^d/\varpi)_{K_n}) = \dim_k((\Theta/\varpi)^{K_n}) \geq \mathrm{rank}_{\mathcal{O}} \Theta^{K_n} = \dim_L \Pi^{K_n} \sim (K : K_n).$$

On the other hand

$$\dim_k((\Theta^d/\varpi)_{K_n}) \leq \dim_k k[\![K]\!]_{K_n} = (K : K_n).$$

Since $(K : K_n) \sim (K_{n_0} : K_n)$ we obtain the claim. □

Let $\{V_i\}_{i\in I}$ be a family of pairwise non-isomorphic, smooth irreducible $\overline{L}$-representations of K. Let

$$d(n) := \sum_{i\in I} (\dim_{\overline{L}} V_i^{K_n})^2.$$

We will show in the course of the proof of the next proposition that only finitely many terms in the sum are non-zero, so that $d(n)$ is finite.

Proposition 2.23. *If $d(n) \sim (K : K_n)$ then $\{V_i\}_{i\in I}$ captures $\mathcal{O}[\![K]\!]$ and hence every projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$.*

Proof. Let $\mathcal{O}[\![K]\!] \twoheadrightarrow M$ be the smallest quotient such that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathcal{O}[\![K]\!], V_i^*) = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*), \quad \forall i \geq 0. \tag{12}$$

Then M is $\mathcal{O}$-torsion free, and $\Pi := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L)$ is a closed K-invariant subspace of $\mathcal{C}(K, L)$. Lemma 2.22 and Schikhof duality imply that it is enough to show that $\dim_L \Pi^{K_n} \sim (K : K_n)$. Let $I(n) := \{i \in I : V_i^{K_n} \neq 0\}$. Since V_i are absolutely irreducible and K_n are normal in K, $V_i^{K_n} = V_i$ for all $i \in I(n)$. In particular, such V_i are representations of a finite group K/K_n. Thus, $I(n)$ is a finite set and $d(n)$ is finite. Thus for a fixed n there is a finite extension L_n of L, such that all V_i, $i \in I(n)$ are defined over L_n. Alternatively, we could use Lemma 2.14.

Since $(\Pi_{L_n})^{K_n} \cong (\Pi^{K_n})_{L_n}$, after extending the scalars to L_n we may assume that $L_n = L$. Schikhof duality [38, Thm.3.5], together with (12), implies that

$$\mathrm{Hom}_K(V_i, \Pi) \cong \mathrm{Hom}_K(V_i, \mathcal{C}(K, L)) \cong V_i^*, \quad \forall i \in I(n) \tag{13}$$

(the second isomorphism holding by Frobenius reciprocity). This implies that

$$\oplus_{i\in I(n)} V_i \otimes V_i^*$$

occurs as a subspace of Π^{K_n}, and so $\dim_L \Pi^{K_n} \geq d(n)$. Since Π^{K_n} is a subspace of $\mathcal{C}(K, L)^{K_n}$, $\dim_L \Pi^{K_n} \leq (K : K_n)$. Since $d(n) \sim (K : K_n)$ by assumption, we deduce that $\dim_L \Pi^{K_n} \sim (K : K_n)$. It follows from Lemma 2.22 and Lemma 2.3 that $\{V_i\}_{i\in I}$ captures $\mathcal{O}[\![K]\!]$. The last assertion follows from Lemma 2.8. □

3. Examples of capture

3.1. **Locally algebraic representations.** Let G be an affine group scheme of finite type over $\mathbb{Z}_p$ such that G_L is a split connected reductive group over L. Let $\mathrm{Alg}(G)$ be the set isomorphism classes of irreducible rational representations of G_L, which we view as representations of $G(\mathbb{Z}_p)$ via the inclusion $G(\mathbb{Z}_p) \subset G(L)$.

Proposition 3.1. $\mathrm{Alg}(G)$ *captures every projective object in* $\mathrm{Mod}^{\mathrm{pro}}_{G(\mathbb{Z}_p)}(\mathcal{O})$.

Proof. [12, Prop. 2.12]. □

Corollary 3.2. *Let W be any finite dimensional L-representation of $G(\mathbb{Z}_p)$, such that the action factors through a finite quotient. Then $\{V \otimes_L W\}_{V\in \mathrm{Alg}(G)}$ captures every projective object in $\mathrm{Mod}^{\mathrm{pro}}_{G(\mathbb{Z}_p)}(\mathcal{O})$.*

Proof. This follows from Proposition 3.1 and Lemma 2.9. □

3.2. **Subgroups and filtrations.** Let F be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_F$, the maximal ideal $\mathfrak{p}_F$ and a uniformizer ϖ_F. Let V be an N-dimensional F-vector space. Let $A = \mathrm{End}_F(V)$ and let $G = \mathrm{Aut}_F(V)$. A lattice chain $\mathcal{L}$ is a set of $\mathcal{O}_F$-lattices in V, $\{L_i : i \in \mathbb{Z}\}$, such that $L_{i+1} \subset L_i$ for all $i \in \mathbb{Z}$ and there is an integer $e = e(\mathcal{L})$, called the period of $\mathcal{L}$, such that $\varpi_F L_i = L_{i+e}$, for all $i \in \mathbb{Z}$. To a lattice chain we associate a hereditary order, $\mathfrak{A} := \{a \in A : aL_i \subset L_i, \forall i \in \mathbb{Z}\}$. If $n \in \mathbb{Z}$ then let $\mathfrak{P}^n := \{a \in A : aL_i \subset L_{i+n}, \forall i \in \mathbb{Z}\}$. Then $\mathfrak{P} := \mathfrak{P}^1$ is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{P}^m\mathfrak{P}^n = \mathfrak{P}^{n+m}$, for all $n, m \in \mathbb{Z}$. Let

$$\mathrm{U}(\mathfrak{A}) := \mathrm{U}^0(\mathfrak{A}) := \mathfrak{A}^\times, \quad \mathrm{U}^n(\mathfrak{A}) := 1 + \mathfrak{P}^n, \quad \forall n \geq 1.$$

Then $\mathrm{U}(\mathfrak{A})$ is a compact open subgroup of G and $\{\mathrm{U}^n(\mathfrak{A})\}_{n\geq 0}$ is a basis of open neighborhoods of 1 by normal subgroups of $\mathrm{U}(\mathfrak{A})$.

Example 3.3. If $\mathcal{L} = \{\varpi_F^i L_0 : \forall i \in \mathbb{Z}\}$, then $\mathrm{U}(\mathfrak{A}) = \mathrm{Aut}_{\mathcal{O}_F}(L_0)$ and $\mathrm{U}^n(\mathfrak{A})$ is the kernel of the natural map $\mathrm{Aut}_{\mathcal{O}_F}(L_0) \twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_F}(L_0/\varpi_F^n L_0)$.

Lemma 3.4. *Let $e(\mathcal{L}|\mathbb{Q}_p) := e(\mathcal{L})e(F|\mathbb{Q}_p)$ and if $p > 2$ then let $c = e(\mathcal{L}|\mathbb{Q}_p)$, if $p = 2$ then let $c = 2e(\mathcal{L}|\mathbb{Q}_p)$. Then $\mathrm{U}^n(\mathfrak{A})^p = \mathrm{U}^{n+e(\mathcal{L}|\mathbb{Q}_p)}(\mathfrak{A})$ for all $n \geq c$.*

Proof. If $x \in \mathfrak{P}^n$ then $(1+x)^p \in 1 + p\mathfrak{P}^n = \mathrm{U}^{n+e(\mathcal{L}|\mathbb{Q}_p)}(\mathfrak{A})$. This yields one inclusion. For the other inclusion, we observe that the series defining exp converges on $p\mathfrak{A} = \mathfrak{P}^c$, if $p > 2$, and on $4\mathfrak{A} = \mathfrak{P}^c$, if $p = 2$, to give a homeomorhism $\exp : \mathfrak{P}^c \to \mathrm{U}^c(\mathfrak{A})$, with the inverse map given by the series for log. If $y \in \mathfrak{P}^{n+e(\mathcal{L}|\mathbb{Q}_p)}$ and $n \geq c$, then $\log(1+y)/p \in \mathfrak{P}^n$, and hence $\exp(\log(1+y)/p) \in \mathrm{U}^n(\mathfrak{A})$ is a p-th root of $1+y$. □

Proposition 3.5. *Let c be the constant defined in Lemma 3.4. If $n \geq c$ then $\mathrm{U}^n(\mathfrak{A})$ is a uniform pro-p group.*

Proof. If $2n \geq m$ then the map $g \mapsto g - 1$ induces an isomorphism of groups

$$\mathrm{U}^n(\mathfrak{A})/\mathrm{U}^m(\mathfrak{A}) \xrightarrow{\cong} \mathfrak{P}^n/\mathfrak{P}^m, \tag{14}$$

where the group operation on the right-hand-side is addition, and so both groups are abelian. Lemma 3.4 implies that the derived subgroup $[\mathrm{U}^n(\mathfrak{A}), \mathrm{U}^n(\mathfrak{A})]$ is contained in $\mathrm{U}^n(\mathfrak{A})^p$, if $p > 2$ and in $\mathrm{U}^n(\mathfrak{A})^4$ if $p = 2$. Hence, $\mathrm{U}^n(\mathfrak{A})$ is powerful. It follows from (14) and Lemma 3.4 that $\mathrm{U}^n(\mathfrak{A})/\mathrm{U}^n(\mathfrak{A})^p$ is finite. Since $\mathrm{U}^n(\mathfrak{A})$ is a pro-p group, this implies that $\mathrm{U}^n(\mathfrak{A})$ is finitely generated. Since $\exp(ma) = \exp(a)^m$, for $m \in \mathbb{Z}$, and $\mathfrak{P}^n$ is a torsion free $\mathbb{Z}_p$-module, the homeomorphism $\exp : \mathfrak{P}^n \cong \mathrm{U}^n(\mathfrak{A})$ implies that $\mathrm{U}^n(\mathfrak{A})$ is torsion free. Theorem 4.5 of [13] implies that $\mathrm{U}^n(\mathfrak{A})$ is uniformly powerful. □

3.3. **Principal series types for** $\mathrm{GL}(N, F)$**.** We keep the notation of the previous subsection, but fix a basis of V and identify $G = \mathrm{GL}(N, F)$. Let $\mathrm{U}_F := \mathrm{U}_F^0 := \mathcal{O}_F^\times$ and $\mathrm{U}_F^i = 1 + \mathfrak{p}_F^i$ for $i \geq 1$. Let T the subgroup of diagonal matrices in G and let T_i be the subgroup of T with all diagonal entries in U_F^i.

For $n \geq 1$ let I^n be the inverse image in $\mathrm{GL}(N, \mathcal{O}_F)$ of unipotent upper-triangular matrices in $\mathrm{GL}(N, \mathcal{O}_F/\mathfrak{p}_F^n)$ under the reduction map. Then I^n is a compact p-adic analytic group.

Lemma 3.6. *There is n_0 such that I^n is torsion free for all $n \geq n_0$.*

Proof. Let t be a diagonal matrix with diagonal entries $t_{ii} = \varpi_F^{(N-i)m}$, $1 \leq i \leq N$, with $n \geq Nm$. A matrix in $g \in tI^nt^{-1}$ is congruent to 1 modulo ϖ_F^m. It follows from the last part of the proof of Proposition 3.5 that if m is large enough then g cannot be a torsion element unless it is equal to the identity. □

We choose n_0, such that I^{n_0} is torsion free and $n_0 > N$. We say that a smooth character $\chi : \mathrm{U}_F^m \to \overline{L}^\times$ has conductor i, if χ is trivial on U_F^{i+1} and nontrivial on U_F^i. For $n \geq n_0$ let $X(n)$ be the set of ordered N-tuples $(\chi_1, \ldots, \chi_N)$ of smooth characters of $\mathrm{U}_F^{n_0}$, such that χ_i has conductor $n-i+1$. Then for $n \geq n_0 + N$ we have

$$|X(n)| = \prod_{i=1}^{N} (|\mathrm{U}_F^{n_0}/\mathrm{U}_F^{n-i+1}| - |\mathrm{U}_F^{n_0}/\mathrm{U}_F^{n-i}|) \sim q_F^{nN}. \tag{15}$$

Let

$$\mathfrak{J}^n := \begin{pmatrix} \mathfrak{p}_F^n & \mathcal{O}_F & \ldots & & \mathcal{O}_F \\ \mathfrak{p}_F^n & \mathfrak{p}_F^{n-1} & \mathcal{O}_F & \ldots & \mathcal{O}_F \\ \vdots & \vdots & & \ddots & \vdots \\ \mathfrak{p}_F^n & \mathfrak{p}_F^{n-1} & \ldots & & \mathcal{O}_F \\ \mathfrak{p}_F^n & \mathfrak{p}_F^{n-1} & \ldots & & \mathfrak{p}_F^{n-N+1} \end{pmatrix}.$$

If $x, y \in \mathfrak{J}^n$ then $x+y$ and xy lie in $\mathfrak{J}^n$. Hence,

$$J^n := 1 + \mathfrak{J}^n$$

is an open subgroup of G normalized by T^0. We may interpret an element of $X(n)$ as a character of T^{n_0}, which maps a diagonal matrix $(d_1, \ldots, d_N)$ to $\chi_1(d_1) \ldots \chi_N(d_N)$. Given such $\chi \in X(n)$ we may extend it to a character of $T^{n_0} J^{n+1}$ by letting $\chi(tk) = \chi(t)$ for all $t \in T^{n_0}$ and $k \in J^{n+1}$. Let

$$V_\chi := \mathrm{Ind}_{T^{n_0} J^{n+1}}^{I^{n_0}} \chi.$$

Proposition 3.7. *If π is a smooth irreducible $\overline{L}$-representation of G then*

$$\mathrm{Hom}_{I^{n_0}}(V_\chi, \pi) \neq 0 \iff \pi \cong \mathrm{Ind}_B^G \psi$$

with $\psi|_{T^{n_0}} = \chi_1 \otimes \ldots \otimes \chi_N$.

Proof. Let $\tilde{\chi} : T^0 \to \overline{L}^\times$ be any character of T^0, such that $\tilde{\chi}|_{T^{n_0}} = \chi$ and let $W_{\tilde{\chi}} := \mathrm{Ind}_{T^0 J^{n+1}}^{K} \tilde{\chi}$, where $K = \mathrm{GL}_N(\mathcal{O}_F)$. It follows from [8], see [26, Prop. 3.10], that $W_{\tilde{\chi}}$ is an irreducible representation of K and is a type for the inertial Bernstein component $[T^0, \tilde{\chi}]$. This implies that $\mathrm{Hom}_K(W_{\tilde{\chi}}, \pi) \neq 0$ if and only if $\pi \cong \mathrm{Ind}_B^G \psi$ with $\psi|_{T^0} = \tilde{\chi}$. Since

$$\mathrm{Ind}_{I^{n_0}}^K V_\chi \cong \mathrm{Ind}_{T^{n_0} J^{n+1}}^K \chi \cong \mathrm{Ind}_{T^0 J^{n+1}}^K (\mathrm{Ind}_{T^{n_0} J^{n+1}}^{T^0 J^{n+1}} \chi) \cong \bigoplus_{\tilde{\chi}} W_{\tilde{\chi}}, \tag{16}$$

where the sum is taken over all $\tilde{\chi} : T^0 \to \overline{L}^\times$, such that $\tilde{\chi}|_{T^{n_0}} = \chi$, the assertion follows from Frobenius reciprocity. □

Definition 3.8. $X := \cup_{n \geq 1} X(n)$.

Lemma 3.9. *The representation V_χ is irreducible. If $\chi_1, \chi_2 \in X$ are distinct then $V_{\chi_1} \not\cong V_{\chi_2}$.*

Proof. Let $\chi_1 \in X(n_1)$ and $\chi_2 \in X(n_2)$ and let $\tilde{\chi}_1$ and $\tilde{\chi}_2$ be characters of T^0 extending χ_1, and χ_2, respectively. We use the notation introduced in the proof of Proposition 3.7. If $\mathrm{Hom}_K(W_{\tilde{\chi}_1}, W_{\tilde{\chi}_2}) \neq 0$ then $W_{\tilde{\chi}_1} \cong W_{\tilde{\chi}_2}$. Since these representations are types for the components $[T^0, \tilde{\chi}_1]$ and $[T^0, \tilde{\chi}_2]$ respectively, we conclude that $\tilde{\chi}_1 = \tilde{\chi}_2$. It follows from (16) and Frobenius reciprocity, that if $\chi_1 \neq \chi_2$ then $\mathrm{Hom}_{I^{n_0}}(V_{\chi_1}, V_{\chi_2}) = 0$, and if $\chi_1 = \chi_2 = \chi$ then

$$\dim_{\overline{L}} \mathrm{Hom}_K(\mathrm{Ind}_{T^{n_0} J^{n+1}}^K \chi, \mathrm{Ind}_{T^{n_0} J^{n+1}}^K \chi) = \dim_{\overline{L}} \mathrm{Hom}_K(\mathrm{Ind}_{I^{n_0}}^K V_\chi, \mathrm{Ind}_{I^{n_0}}^K V_\chi)$$

is equal to the number of characters $\tilde{\chi} : T^0 \to \overline{L}^\times$, such that $\tilde{\chi}|_{T^{n_0}} = \chi$, which is equal to

$$|T^0/T^{n_0}| = |T^{n_0}J^{n+1}\backslash T^0 J^{n+1}/T^{n_0}J^{n+1}|.$$

Since $T^0 J^{n+1}$ is contained in the K-intertwining of χ, we deduce that the K-intertwining of χ is equal to $T^0 J^{n+1}$. Since $I^{n_0} \cap T^0 J^{n+1} = T^{n_0} J^{n+1}$, we deduce that the I^{n_0}-intertwining of χ is equal to $T^{n_0} J^{n+1}$ and so V_χ is irreducible. $\square$

For $n \geq 1$ let K_n be the n-th congruence subgroup in $\mathrm{GL}(N, \mathcal{O}_F)$.

Lemma 3.10.

$$\sum_{\chi \in X} (\dim V_\chi^{K_{n+1}})^2 \sim (I_{n_0} : K_{n+1}).$$

Proof. If $\chi \in X(n)$ then K_{n+1} acts trivially on V_χ and K_n does not act trivially on V_χ, and, since V_χ is irreducible, $V_\chi^{K_n} = 0$. Hence, it is enough to check that

$$\sum_{\chi \in X(n)} (\dim V_\chi)^2 \sim (I^{n_0} : K_{n+1}). \tag{17}$$

Let U^- be the subgroup of unipotent lower triangular matrices in G. Then

$$\begin{aligned} \dim V_\chi &= (I^{n_0} : T^{n_0} J^{n+1}) = (I^{n_0} \cap U^- : J^{n+1} \cap U^-) \\ &= \prod_{i=1}^{N-1} q_F^{(N-i)(n-n_0-i)} \sim q_F^{nN(N-1)/2} \end{aligned} \tag{18}$$

Since $|X(n)| \sim q_F^{nN}$ by (15) the right-hand-side of (17) grows as $q_F^{nN^2}$. Since

$$(I^{n_0} : K_{n+1}) = q_F^{(n+1-n_0)N(N-1)/2} q_F^{(n+1-n_0)N} q_F^{(n+1)N(N-1)/2} \sim q_F^{nN^2}, \tag{19}$$

we deduce that (17) holds. $\square$

Proposition 3.11. *The set $\{V_\chi\}_{\chi \in X}$ captures every projective object in $\mathrm{Mod}^{\mathrm{pro}}_{I^{n_0}}(\mathcal{O})$.*

Proof. The assertion follows from Proposition 2.23 applied with $K = I^{n_0}$ and K_n as in Example 3.3 with $L_0 = \mathcal{O}_F^{\oplus N}$, using Lemma 3.10. $\square$

3.4. **Supercuspidal types for** $\mathrm{GL}(N, F)$**.** Let F be a finite extension of $\mathbb{Q}_p$ and let E be a field extension of F of degree N. Let $f = f(E|F)$ and $e = e(E|F)$ denote the inertial degree and the ramification index of E over F respectively.

Definition 3.12 ([7, (1.4.14)])**.** An element $\alpha \in E$ is minimal over F if e is prime to $v_E(\alpha)$ and $\varpi_F^{-v_E(\alpha)} \alpha^e + \mathfrak{p}_E$ generates the field extension k_E/k_F.

Remark 3.13. If α is minimal then $E = F[\alpha]$, since the fields have the same ramification indices and inertial degrees over F.

Remark 3.14. If α is minimal over F then $\alpha + a$ is also minimal over F for all $a \in E$ with $v_E(a) > v_E(\alpha)$.

In general, not every extension E will contain an element which is minimal over F, as the following example shows.

Example 3.15. Let f be an even integer, $e = p^{f/2} + 1$, $F = \mathbb{Q}_p$ and let E be the compositum of $\mathbb{Q}_p(p^{1/e})$ and the unramified extension of $\mathbb{Q}_p$ of degree f. If $x \in (k_E^\times)^e$ then $x^{p^{f/2}} = x$, and hence x cannot generate k_E over k_F. Any $\alpha \in E$ will be of the form $p^{n/e}\xi$ for some $\xi \in \mathcal{O}_E^\times$ and the image of $\varpi^{-n}\alpha^e$ in k_E will be equal to the image of ξ^e, and hence cannot generate k_E over k_F.

However, there are plenty extensions that do. For example, let F' be a finite unramified extension of F, and let $f(x) = x^e + \ldots + \varpi_F \xi$ be an Eisenstein polynomial in $F'[x]$, such that $\xi + \mathfrak{p}_{F'}$ generates $k_{F'}/k_F$ then $E = F'[x]/(f(x))$ is a field and $x + (f(x))$ is minimal over F. In particular, all unramified or totally ramified extensions will contain a minimal element.

Lemma 3.16. *If $\alpha \in E$ is minimal over F then α^m is also minimal over F for all m prime to $e(q_E - 1)$.*

Proof. If $x \in k_E$ then x generates k_E over k_F if and only if it is not contained in any proper subfield, which is equivalent to the order of x in the group $k_E^\times$ not dividing $q_F^d - 1$ for all divisors d of f with $d < f$. If m is prime to $q_F^f - 1$ then x and x^m have the same order, which implies the assertion. □

From now on we assume that E contains an element that is minimal over F. It follows from Lemma 3.16 that for any $n \geq 0$ there exists $\alpha \in E$ minimal over F, such that $v_E(\alpha) < -n$.

Let $A = \mathrm{End}_F(E)$ and $G = \mathrm{Aut}_F(E)$, so that $G \cong \mathrm{GL}(N, F)$, $E \subset A$ and $E^\times \subset G$. Let $\mathfrak{A}$ be the hereditary $\mathcal{O}_F$-order in A associated to the lattice chain $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ and let $\mathfrak{P}$ be its Jacobson radical. Then $\mathfrak{P}^n = \{a \in A : a\mathfrak{p}_E^i \subset \mathfrak{p}_E^{i+n}, \forall i \in \mathbb{Z}\}$ for all $n \in \mathbb{Z}$. We let $\mathrm{U}(\mathfrak{A}) = \mathrm{U}^0(\mathfrak{A}) = \mathfrak{A}^\times$ and $\mathrm{U}^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$ for $n \geq 1$. Then for all $n \geq 0$, $\mathrm{U}^n(\mathfrak{A})$ is a normal subgroup of $\mathrm{U}(\mathfrak{A})$ normalized by $E^\times$.

Let $\alpha \in E$ be minimal with respect to E/F with $v_E(\alpha) = -n < 0$. Then $[\mathfrak{A}, n, 0, \alpha]$ is a simple stratum in the sense of [7, (1.5.5)]. For $m \geq 0$, we let

$$H^m(\alpha) := \mathrm{U}_E^m \mathrm{U}^{[n/2]+1}(\mathfrak{A}), \quad J^m(\alpha) := \mathrm{U}_E^m \mathrm{U}^{[(n+1)/2]}(\mathfrak{A}).$$

Since α is minimal and $[E : F] = N$, these groups coincide with those defined in [7, (3.1.14)]. If it is clear that $v_E(\alpha)$ is fixed and is equal to $-n$ then we will write H^m and J^m.

We fix an additive character $\psi_F : F \to \overline{L}^\times$, which is trivial on $\mathfrak{p}_F$ and non-trivial on $\mathcal{O}_F$. If $b \in A$ then $\psi_b : A \to \overline{L}^\times$ is the function $a \mapsto \psi_F(\mathrm{tr}_A(b(a-1)))$. The restriction of ψ_α to $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$ defines a character, which is trivial on $\mathrm{U}^{n+1}(\mathfrak{A})$ and non-trivial on $\mathrm{U}^n(\mathfrak{A})$. Let $\mathcal{C}(m, \alpha)$ be the set of characters $\theta : H^{m+1}(\alpha) \to \overline{L}^\times$, such that the restriction of θ to $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$ is equal to ψ_α. Since α is minimal and $[E : F] = N$, $\mathcal{C}(m, \alpha)$ is the set of simple characters defined in [7, (3.2.1)]. If $\theta \in \mathcal{C}(m, \alpha)$ then the G-intertwining of θ is equal to $E^\times J^{m+1}(\alpha)$. It follows from [5, (8.3.3)], using [7, (3.4.1)], that there is a unique irreducible representation $\eta(\theta)$ of $J^{m+1}(\alpha)$, such that $\mathrm{Hom}_{H^{m+1}(\alpha)}(\theta, \eta(\theta)) \neq 0$. Thus,

$$\mathrm{Ind}_{H^{m+1}(\alpha)}^{J^{m+1}(\alpha)} \theta \cong \eta(\theta)^{\oplus a}, \tag{20}$$

where a is an integer such that $a \dim \eta(\theta) = (J^{m+1}(\alpha) : H^{m+1}(\alpha))$. It is shown in [5, (8.3.3)] that

$$(\dim \eta(\theta))^2 = (J^{m+1}(\alpha) : H^{m+1}(\alpha)) = (\mathrm{U}^{[(n+1)/2]}(\mathfrak{A}) : \mathrm{U}^{[n/2]+1}(\mathfrak{A})). \tag{21}$$

In particular, $a = \dim \eta(\theta)$ and is equal to 1 if n is odd, and is equal to $\sqrt{(\mathfrak{A} : \mathfrak{P})}$ if n is even.

Definition 3.17. $V(\alpha, \theta) := \mathrm{Ind}_{J^{m+1}(\alpha)}^{\mathrm{U}^{m+1}(\mathfrak{A})} \eta(\theta)$.

Lemma 3.18. *$V(\alpha, \theta)$ is an irreducible representation of $\mathrm{U}^{m+1}(\mathfrak{A})$.*

Proof. It follows from (20) that the G-intertwining of $\eta(\theta)$ is equal to the G-intertwining of θ. Hence, the $\mathrm{U}^{m+1}(\mathfrak{A})$-intertwining of $\eta(\theta)$ is equal to $\mathrm{U}^{m+1}(\mathfrak{A}) \cap E^\times J^{m+1}(\alpha) = J^{m+1}(\alpha)$. Hence, $V(\alpha,\theta)$ is irreducible. $\square$

Proposition 3.19. *If π is a smooth irreducible $\overline{L}$-representation of G then*

$$\mathrm{Hom}_{\mathrm{U}^{m+1}(\mathfrak{A})}(V(\alpha,\theta),\pi) \neq 0 \iff \pi \cong \text{c-Ind}_{E^\times J^0(\alpha)}^G \Lambda, \tag{22}$$

where Λ is an irreducible representation, such that $\Lambda|_{H^{m+1}(\alpha)}$ is isomorphic to a direct sum of copies of θ. In particular, π is supercuspidal.

Moreover, if E is a tame extension of F and ρ is the N-dimensional representation of W_F corresponding to π via the local Langlands correspondence, then (22) is equivalent to $\rho \cong \mathrm{Ind}_{W_E}^{W_F} \psi$, where $\psi : W_E \to \overline{L}^\times$ is a character, such that if we identify $W_E^{\mathrm{ab}} \cong E^\times$ then $\psi|_{\mathrm{U}_E^{m+1}} = \theta$.

Proof. Since α is minimal over F and $[E:F]=N$, [7, (3.2.5), (3.3.18)] imply that $\mathrm{Ind}_{H^{m+1}}^{H^1}\theta \cong \bigoplus \tilde{\theta}$, where the sum is taken over all $\tilde{\theta} \in \mathcal{C}(0,\alpha)$, such that $\tilde{\theta}|_{H^{m+1}} = \theta$. It follows from (20) that $\mathrm{Ind}_{H^1}^{J^1}\tilde{\theta} \cong \eta(\tilde{\theta})^{\oplus a}$. Theorem (5.2.2) of [7] implies that $\mathrm{Ind}_{J^1}^{J^0}\eta(\tilde{\theta}) \cong \bigoplus_\sigma \kappa\otimes\sigma$, where κ is an irreducible representation of J^0, such that $\kappa|_{J^1} \cong \eta(\tilde{\theta})$, the so called β-extension, and the sum is taken over all characters σ of $\mathrm{U}_E/\mathrm{U}_E^1$, which we inflate to J^0 via the isomorphism $J^0/J^1 \cong \mathrm{U}_E/\mathrm{U}_E^1$. Thus

$$\mathrm{Ind}_{H^{m+1}}^{J^0}\theta \cong (\bigoplus \lambda)^{\oplus a} \tag{23}$$

where the sum is taken over all simple types λ, [7, (5.5.10)], associated to a simple stratum $[\mathfrak{A},n,0,\alpha]$, such that $\lambda|_{H^{m+1}}$ is a direct sum of copies of θ. If $\mathrm{Hom}_{J^0}(\lambda,\pi)\neq 0$ then π is supercuspidal and there is a unique irreducible representation Λ of $E^\times J^0$, such that $\Lambda|_{J^0}\cong\lambda$ and $\pi \cong \text{c-Ind}_{E^\times J^0}^G\Lambda$, [7, (6.2.2)]. The first part of the Proposition now follows from (20) and Frobenius reciprocity. The second part follows from [6]. $\square$

Lemma 3.20. $\mathrm{U}^{n+1}(\mathfrak{A})$ *acts trivially on $V(\alpha,\theta)$. The space of* $\mathrm{U}^n(\mathfrak{A})$*-invariants in $V(\alpha,\theta)$ is zero.*

Proof. This follows from the fact that the restriction of $V(\alpha,\theta)$ to $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$ is isomorphic to a direct sum of copies of ψ_α. $\square$

Lemma 3.21.

$$\dim V(\alpha,\theta) = \frac{(\mathrm{U}^{m+1}(\mathfrak{A}):\mathrm{U}^{[n/2]+1}(\mathfrak{A}))}{(\mathrm{U}_E^{m+1}:\mathrm{U}_E^{[n/2]+1})\dim\eta(\theta)} \sim (\mathrm{U}^{m+1}(\mathfrak{A}):\mathrm{U}^{n+1}(\mathfrak{A}))^{1/2} q_E^{-n/2}.$$

Proof. It follows from the definition of $V(\alpha,\theta)$ as induced representation that its dimension is equal to $(\mathrm{U}^{m+1}(\mathfrak{A}):J^{m+1}(\alpha))\dim\eta(\theta)$. The Lemma follows from this using (21). $\square$

We fix an additive character $\psi_E : E \to \overline{L}^\times$, which is trivial on $\mathfrak{p}_E$ and nontrivial on $\mathcal{O}_E$. There exists a unique map $s: A\to E$ such that

$$\psi_A(ab) = \psi_E(s(a)b), \quad \forall a\in A, \quad \forall b \in E. \tag{24}$$

Moreover, s is an $E\times E$-bimodule homomorphism, such that $s(\mathfrak{P}^m) = \mathfrak{p}_E^m$ for all $m\in\mathbb{Z}$, [7, (1.3.4)]. Since E is maximal in A and E is separable over F, $\mathrm{tr}_{A/F}(E) = \mathrm{tr}_{E/F}(E) = F$. It follows from (24) that $s(1)\neq 0$. We let $\delta := v_E(s(1))$.

Lemma 3.22. *Let n, r be natural numbers with $n-\delta > r$. Let $\alpha_1, \alpha_2 \in E$ be minimal over F with $v_E(\alpha_1) = v_E(\alpha_2) = -n$. If $\alpha_1 - \alpha_2 \in \mathfrak{p}_E^{-n+\delta+1}$ and $\alpha_1 + \mathfrak{P}^{-r} = x\alpha_2 x^{-1} + \mathfrak{P}^{-r}$ for some $x \in \mathrm{U}(\mathfrak{A})$ then $\alpha_1 - \alpha_2 \in \mathfrak{p}_E^{-r-\delta}$.*

Proof. Since $\mathfrak{p}_E^{-n+\delta+1} \subset \mathfrak{P}^{-n+\delta+1}$ we have

$$\alpha_2 + \mathfrak{P}^{-n+\delta+1} = \alpha_1 + \mathfrak{P}^{-n+\delta+1} = x\alpha_2 x^{-1} + \mathfrak{P}^{-n+\delta+1}.$$

Since α_2 is minimal it follows from [27, Thm. 2.4] that $x = uy$ with $u \in E^\times$ and $y \in \mathrm{U}^{\delta+1}(\mathfrak{A})$. Then $x\alpha_2 x^{-1} = y\alpha_2 y^{-1}$ and so $\alpha_1 + \mathfrak{P}^{-r} = y\alpha_2 y^{-1} + \mathfrak{P}^{-r}$. Thus $\alpha_1 y - y\alpha_2 \in \mathfrak{P}^{-r}$. By applying s, we get $s(y)(\alpha_1 - \alpha_2) \in s(\mathfrak{P}^{-r}) = \mathfrak{p}_E^{-r}$. Since $y \in \mathrm{U}^{\delta+1}(\mathfrak{A})$, $s(y) \in s(1) + s(\mathfrak{P}^{\delta+1}) = s(1) + \mathfrak{p}_E^{\delta+1}$, hence $v_E(s(y)) = \delta$ and $\alpha_1 - \alpha_2 \in \mathfrak{p}_E^{-r-\delta}$. $\square$

Lemma 3.23. *For $i = 1, 2$ let $\alpha_i \in E$ be minimal over F with $v_E(\alpha_i) = -n$ and $n > 2\delta$, and let $\theta_i \in \mathcal{C}(m, \alpha_i)$. If $\alpha_1 - \alpha_2 \in \mathfrak{p}_E^{-n+\delta+1}$ and $\alpha_1 - \alpha_2 \notin \mathfrak{p}_E^{-[n/2]-\delta}$ then θ_1 and θ_2 do not intertwine in G.*

Proof. If $g \in G$ intertwines θ_1 and θ_2 then there exists $x \in \mathrm{U}(\mathfrak{A})$ normalizing H^{m+1} such that $\theta_1^x = \theta_2$, [7, (3.5.11)]. By considering the restrictions of θ_1 and θ_2 to $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$ we obtain that $\psi_{\alpha_1} = \psi_{\alpha_2}^x = \psi_{x\alpha_2 x^{-1}}$ as characters of $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$. Hence, $\psi_A((\alpha_1 - x\alpha_2 x^{-1})a) = 1$ for all $a \in \mathfrak{P}^{[n/2]+1}$, which implies $\alpha_1 - x\alpha_2 x^{-1} \in \mathfrak{P}^{-[n/2]}$, [7, p.22]. Lemma 3.22 implies that $\alpha_1 - \alpha_2 \in \mathfrak{p}_E^{-[n/2]-\delta}$, leading to a contradiction. $\square$

Lemma 3.24. *Let $\alpha \in E$ be minimal with $v_E(\alpha) = -n$, where $n \geq 2m \geq 0$. Then $|\mathcal{C}(m, \alpha)| = (\mathrm{U}_E^{m+1} : \mathrm{U}_E^{[n/2]+1}) \sim q_E^{n/2}$. Moreover, distinct $\theta_1, \theta_2 \in \mathcal{C}(m, \alpha)$ do not intertwine.*

Proof. The set $\mathcal{C}(m, \alpha)$ is non-empty, [7, (3.3.18)]. If $\theta \in \mathcal{C}(m, \alpha)$ then all the other characters in $\mathcal{C}(m, \alpha)$ are of the form $\theta\mu$, where μ is any character of $\mathrm{U}_E^{m+1}/\mathrm{U}_E^{[n/2]+1}$ inflated to H^{m+1}. This implies the first assertion. The second assertion follows from the proof of [33, Prop. 7.3]. Namely, if g intertwines θ and $\theta\mu$ then it will intertwine their restrictions to $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$, which are both equal to ψ_α. It follows from [7, (3.3.2)] that $g \in E^\times \mathrm{U}^{[(n+1)/2]}(\mathfrak{A})$. Any such g will normalize H^{m+1}, and so the intertwining maybe rewritten as

$$\theta\mu(h) = \theta(ghg^{-1}), \quad \forall h \in H^{m+1}. \tag{25}$$

By applying [7, (3.3.2)] again, we get that g will also intertwine θ with itself. It then follows from (25) that $\mu(h) = 1$ for all $h \in H^{m+1}$. $\square$

We fix $m \geq 0$. For $n \geq 2m$, let $I(n)$ be the set of isomorphism classes of the representations of the form $V(\alpha, \theta)$, where $\theta \in \mathcal{C}(m, \alpha)$ with $\alpha \in E$ minimal over F and $v_E(\alpha) = -n$.

Proposition 3.25. $|I(n)| \sim q_E^n$.

Proof. Let us note that it follows from Lemma 3.16 that there exists $\alpha \in E$ minimal over F with $n := -v_E(\alpha)$ arbitrarily large. It is enough to show that $g(n) \leq |I(n)| \leq f(n)$, with $f(n) \sim g(n) \sim q_E^n$. For the upper bound we observe that

$$|I(n)| \leq |\bigcup_\alpha \mathcal{C}(m, \alpha)| \leq |\mathcal{C}(m, \alpha)||\mathfrak{p}_E^{-n}/\mathfrak{p}_E^{-[n/2]}| \sim q_E^n,$$

where the union is taken over all minimal $\alpha \in E$ with $v_E(\alpha) = -n$. In the above estimates we use Lemma 3.24 and the fact that if $\alpha_1 - \alpha_2 \in \mathfrak{p}_E^{-[n/2]}$ then $\psi_{\alpha_1} = \psi_{\alpha_2}$

on $\mathrm{U}^{[n/2]+1}(\mathfrak{A})$ and hence $\mathcal{C}(m,\alpha_1)=\mathcal{C}(m,\alpha_2)$. For the lower bound we choose a set of representatives $\{a_i\}_i$ of $\mathfrak{p}_E^{-n+\delta+1}/\mathfrak{p}_E^{-[n/2]-\delta}$ and let $\alpha_i:=\alpha+a_i$. Then each α_i is minimal over F, $\alpha_i-\alpha_j\in\mathfrak{p}_E^{-n+\delta+1}$ and $\alpha_i-\alpha_j=a_i-a_j\notin\mathfrak{p}_E^{-[n/2]-\delta}$ if $i\neq j$. Lemmas 3.24 and 3.23 imply that distinct $\theta_1,\theta_2\in\bigcup_i\mathcal{C}(m,\alpha_i)$ do not intertwine. This implies that $\mathrm{Hom}_{\mathrm{U}^{m+1}(\mathfrak{A})}(\mathrm{Ind}_{H^{m+1}}^{\mathrm{U}^{m+1}(\mathfrak{A})}\theta_1,\mathrm{Ind}_{H^{m+1}}^{\mathrm{U}^{m+1}(\mathfrak{A})}\theta_2)=0$, and (20) implies that $V(\alpha,\theta_1)\ncong V(\alpha',\theta_2)$. Hence

$$|I(n)|\geq|\bigcup_i\mathcal{C}(m,\alpha_i)|=|\mathcal{C}(m,\alpha)||\mathfrak{p}_E^{-n+\delta+1}/\mathfrak{p}_E^{-[n/2]-\delta}|\sim q_E^n,$$

where we use Lemma 3.24 again. □

We let $I=\cup_{n\geq 2m}I(n)$ and for each $i\in I$ we fix a representative V_i of the isomorphism class i.

Proposition 3.26. *Let m be large enough, so that $\mathrm{U}^{m+1}(\mathfrak{A})$ is torsion free. Then the set $\{V_i\}_{i\in I}$ captures every projective object in $\mathrm{Mod}^{\mathrm{pro}}_{\mathrm{U}^{m+1}(\mathfrak{A})}(\mathcal{O})$.*

Proof. We will deduce the assertion from Proposition 2.23 with $K=\mathrm{U}^{m+1}(\mathfrak{A})$ and $K_n=\mathrm{U}^{n+1}(\mathfrak{A})$. Proposition 3.5 and Lemma 3.4 imply that $\{K_n\}_{n\geq m}$ satisfy the conditions of Proposition 2.23. Lemma 3.20 implies that $i\in I(n)$ if and only if $\mathrm{U}^{n+1}(\mathfrak{A})$ acts trivially on V_i and $\mathrm{U}^n(\mathfrak{A})$ acts nontrivially on V_i. Thus it is enough to show that

$$\sum_{i\in I(n)}(\dim V_i)^2\sim(\mathrm{U}^{m+1}(\mathfrak{A}):\mathrm{U}^{n+1}(\mathfrak{A})).$$

This follows from Lemma 3.21 and Proposition 3.25. □

4. Capture a la Scholze

The following section is motivated by the proof of Corollary 7.3 in [37]. Let K be a pro-finite group and let $\{K_n\}_{n\geq 1}$ be a basis of open neighbourhood of 1 consisting of normal subgroups of K. We assume that K_n is pro-p for all $n\geq 1$. Let $\psi_n:K_n\to\overline{L}^*$ be a smooth character for all $n\geq 1$.

Let R be a complete noetherian local $\mathcal{O}$-algebra with residue field k. Let M be a finitely generated $R[\![K]\!]$-module.

Proposition 4.1. *Assume M is projective as $\mathcal{O}[\![K]\!]$-module. If $\phi\in\mathrm{End}^{\mathrm{cont}}_{R[\![K]\!]}(M)$ kills $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,\psi_n^*)$ for all $n\geq 1$ then ϕ kills M.*

Proof. Since ψ_n is smooth, $\psi_n(K_n)$ is a finite subgroup of $\overline{L}^*$. Since K_n is pro-p, the order of $\psi_n(K_n)$ is a power of p. Hence, we may assume that ψ_n takes values in $\mathcal{O}_n^\times$, where $\mathcal{O}_n$ is the ring of integers of a finite extension L_n of L obtained by adding some p-power roots of unity to L. Let $\mathcal{O}_n(\psi_n^*)$ be a free $\mathcal{O}_n$-module of rank 1, on which K_n acts by ψ_n^*. Since $\mathcal{O}_n(\psi_n^*)$ is an $\mathcal{O}[\![K_n]\!]$-submodule of ψ_n^* we have an injection

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,\mathcal{O}_n(\psi_n^*))\hookrightarrow\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,\psi_n^*).$$

Moreover, this is an injection of $\mathrm{End}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M)$-modules. Thus ϕ annihilates

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,\mathcal{O}_n(\psi_n^*))$$

for all $n\geq 1$. Let ϖ_n be a uniformizer in $\mathcal{O}_n$. Then the residue field is equal to k and all the p-power roots of unity in $\mathcal{O}_n$ are mapped to 1 in k. We have an exact sequence of K_n-representations

$$0\to\mathcal{O}_n(\psi_n^*)\xrightarrow{\varpi_n}\mathcal{O}_n(\psi_n^*)\to k\to 0 \tag{26}$$

with the trivial action of K_n on the quotient. Since M is projective as an $\mathcal{O}[\![K]\!]$-module, it is also projective as an $\mathcal{O}[\![K_n]\!]$-module and by applying $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,*)$ to (26) we see that $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,k)$ is a quotient of $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,\mathcal{O}_n(\psi_n^*))$ and thus is killed by ϕ. This module is isomorphic to the continuous k-linear dual of K_n-coinvariants of $M/\varpi M$, which we denote by $(M/\varpi M)_{K_n}$. Using Pontryagin duality we conclude that ϕ acts trivially on $(M/\varpi M)_{K_n}$.

Let $\mathfrak{m}$ be the maximal ideal of R and let $I_{m,n} = \mathfrak{m}^m R[\![K]\!] + I(K_n)R[\![K]\!]$, where $I(K_n)$ is the augmentation ideal of $\mathcal{O}[\![K_n]\!]$. Since M is finitely generated as $R[\![K]\!]$-module, its topology coincides with the topology defined by the ideals $I_{m,n}$, see [32, (5.2.17)]. Thus

$$M/\varpi M \cong \varprojlim_{n,m} (M/\varpi M)/I_{n,m}(M/\varpi M).$$

Since $(M/\varpi M)/I(K_n)(M/\varpi M) = (M/\varpi M)_{K_n}$, by the previous part we deduce that ϕ kills all the terms in this projective limit. Thus ϕ kills $M/\varpi M$ and so $\phi(M)$ is contained in ϖM.

If ϕ is non-zero then there is a largest integer n such that $\phi(M)$ is contained in $\varpi^n M$ and not contained in $\varpi^{n+1}M$. We may replace ϕ by $\frac{1}{\varpi^n}\phi$ in the argument above to obtain a contradiction. □

Corollary 4.2. *Let M be as in Proposition 4.1 and assume that R acts faithfully on M. Let $\{V_i\}_{i\in I}$ be the family of irreducible subquotients of $\mathrm{Ind}_{K_n}^K \psi_n$ for all $n \geq 1$. For each $i \in I$ let $\mathfrak{a}_i$ be the R-annihilator of $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$. Then $\cap_{i\in I}\mathfrak{a}_i = 0$.*

Proof. This follows from Proposition 4.1 by observing that $\mathrm{Ind}_{K_n}^K \psi_n$ is a finite direct sum of some V_i's with some multiplicity and then using Frobenius reciprocity to obtain an isomorphism of R-modules

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K_n]\!]}(M,\psi_n^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M,(\mathrm{Ind}_{K_n}^K \psi_n)^*).$$

□

Remark 4.3. Although the theory outlined in section 2 is more general, all the smooth examples in section 3 are covered by Corollary 4.2, which avoids all the intricate counting arguments. For example in the supercuspidal case, considered in section 3.4, it is enough to apply Corollary 4.2 with $K = \mathrm{U}(\mathfrak{A})$ and consider families $(\mathrm{U}^{[n/2]+1}(\mathfrak{A}), \psi_\alpha)$, for all $n \geq 1$, such that $[\mathfrak{A}, n, 0, \alpha]$ is a simple stratum, $E = F[\alpha]$ and α is minimal over F.

5. Applications

5.1. **Global applications.** Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$ with the property that the maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$ is a maximal $\mathbb{R}$-split torus in $\mathbb{G}$. This means that $\mathbb{G}(\mathbb{R})$ is compact modulo its centre, and that if the level K_f is small enough, then $\mathbb{G}(\mathbb{Q})$ acts with trivial stabilizers on $\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_f$. We let

$$Y(K_f) := \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_f.$$

Here we use the usual notation that A_∞ is the group of real points of the maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$, and K_∞ denotes a choice of maximal compact subgroup of $\mathbb{G}(\mathbb{R})$; and, of course, the superscript $\circ$ denotes the connected component of the identity.

We are in the situation of [14, (3.2)]. In particular, we may consider the completed cohomology $\widetilde{H}^0(K^p)$ with $\mathcal{O}$-coefficients, for some choice of tame level K^p; recall that this is defined as

$$\widetilde{H}^0(K^p) := \varprojlim_s \varinjlim_{K_p} H^0(Y(K_pK^p), \mathcal{O}/\varpi^s),$$

where K_p runs over all compact open subgroups of $\mathbb{G}(\mathbb{Q}_p)$. We may also consider the completed *homology*, with $\mathcal{O}$-coefficients, which we denote by M; it is defined as

$$M := \varprojlim_{K_p} H_0(Y(K_pK^p), \mathcal{O}).$$

We then have that M is finitely generated and (if K^p is small enough) free over $\mathcal{O}[\![K]\!]$, and

$$\widetilde{H}^0(K^p) = \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M, \mathcal{O}), \tag{27}$$

where K is a compact open subgroup of $\mathbb{G}(\mathbb{Q}_p)$.

For each K_p, let $\mathbb{T}(K_p)$ denote the commutative algebra of endomorphisms of $H_0(Y(K_pK^p), \mathcal{O})$ generated by the spherical Hecke operators at the primes $\ell \neq p$ at which K^p is unramified (i.e. at which $\mathbb{G}$ is unramified and K^p is of the form $K^{p,\ell}K_\ell$, where K_ℓ is maximal compact hyperspecial). We then define

$$\mathbb{T} := \varprojlim_{K_p} \mathbb{T}(K_p);$$

this is an algebra of G-equivariant endomorphisms of M. We endow each $\mathbb{T}(K_p)$ with its ϖ-adic topology, and endow $\mathbb{T}$ with its projective limit topology. Then $\mathbb{T}$ is a pro-Artinian $\mathcal{O}$-algebra (indeed, since each $\mathbb{T}(K_p)$ is finitely generated as an $\mathcal{O}$-modules, we may describe $\mathbb{T}$ as the projective limit of (literally) finite $\mathcal{O}$-algebras), and is the product (as a topological ring) of finitely many local pro-Artinian $\mathcal{O}$-algebras, these local factors being in bijection with the maximal ideals of $\mathbb{T}$; we let $\mathbb{T}_{\mathfrak{m}}$ denote the factor corresponding to a given maximal ideal $\mathfrak{m}$. The local factor $\mathbb{T}_{\mathfrak{m}}$ is naturally identified with the $\mathfrak{m}$-adic completion of $\mathbb{T}$ as an abstract ring; note, though, that the topology on $\mathbb{T}_{\mathfrak{m}}$ coincides with the $\mathfrak{m}$-adic topology if and only if $\mathbb{T}_{\mathfrak{m}}$ is Noetherian. In fact, because of the anticipated relationship with Galois deformation rings, we expect that $\mathbb{T}$, and hence each $\mathbb{T}_{\mathfrak{m}}$, *is* Noetherian, but as far as we know, this is not proved in the generality that we consider here. If V is any locally algebraic representation of K, then it follows from (27) and Schikhof duality [38, Thm. 3.5] that

$$\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M, V^*) = \mathrm{Hom}_K\big(V, \widetilde{H}^0(K^p)\big),$$

and the target is isomorphic to the space of algebraic automorphic forms on $\mathbb{G}$ with values in V^* [14, Prop. 3.2.4]. In particular, the action of $\mathbb{T} \otimes_{\mathcal{O}} L$ on this space is semi-simple.

Now, let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$. We may form the localization $M_{\mathfrak{m}}$; this is a direct summand of M, and so projective as an $\mathcal{O}[\![K]\!]$-module. It has a commuting action of the local ring $\mathbb{T}_{\mathfrak{m}}$. The space $\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M_{\mathfrak{m}}, V^*)$ is a finite-dimensional L-vector space. The action of $\mathbb{T}_{\mathfrak{m}}$ on this space factors through an order in the finite-dimensional L-algebra generated by the Hecke operators.

Remark 2.12 applies, and so Proposition 2.11 together with Remark 2.13 shows that if the set $\{V_i\}_{i\in I}$ captures M, then the systems of Hecke eigenvalues arising in the spaces of algebraic automorphic forms $\mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(M, V_i^*)$ are Zariski dense in $\mathrm{Spec}\,\mathbb{T}_{\mathfrak{m}}$.

5.2. **Hecke algebras and functoriality.** Let us suppose that we have two groups $\mathbb{G}_1$ and $\mathbb{G}_2$ as in the previous subsection. We keep the notation of the previous section, except that we add the index 1 or 2 to indicate with respect to which group the objects are defined. Assume that both Hecke algebras $\mathbb{T}_1$ and $\mathbb{T}_2$ are completions of the quotients of the same universal Hecke algebra $\mathbb{T}^{\mathrm{univ}}$, which is a polynomial ring over $\mathcal{O}$ with variables corresponding to the Hecke operators at the unramified places for both groups. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{\mathrm{univ}}$ and assume that $\mathfrak{m}$ is a maximal ideal of both $\mathbb{T}_1$ and $\mathbb{T}_2$. Then both $\mathbb{T}_{1,\mathfrak{m}}$ and $\mathbb{T}_{2,\mathfrak{m}}$ are quotients of the $\mathfrak{m}$-adic completion of $\mathbb{T}^{\mathrm{univ}}$, which we denote by $\widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}}$.

Let $\{V_i\}_{i\in I}$ be a family of irreducible locally algebraic representations of K_1, which captures M_1. As explained above, a maximal ideal $x \in \text{m-Spec}\, \widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}}[1/p]$ in the support of $\mathrm{Hom}_{K_1}\big(V_i, \widetilde{H}^0(K_1^p)_{\mathfrak{m}}\big)$ corresponds to an algebraic automorphic form on $\mathbb{G}_1$. Let us further assume that functoriality between $\mathbb{G}_1$ and $\mathbb{G}_2$ is available and we may transfer such automorphic forms on $\mathbb{G}_1$ to automorphic forms on $\mathbb{G}_2$. More precisely, we assume that for such $x \in \text{m-Spec}\, \widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}}[1/p]$, the map $\widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}} \to \kappa(x)$, where $\kappa(x)$ is the residue field of x, fits into a commutative diagram:

$$\begin{tikzcd} \widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}} \arrow[r, two heads] \arrow[d, two heads] & \mathbb{T}_{1,\mathfrak{m}} \arrow[d] \\ \mathbb{T}_{2,\mathfrak{m}} \arrow[r] & \kappa(x). \end{tikzcd}$$

Let Σ be the subset of $\text{m-Spec}\, \widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}}[1/p]$ consisting of maximal ideals contained in the support of $\mathrm{Hom}_{K_1}\big(V_i, \widetilde{H}^0(K_1^p)_{\mathfrak{m}}\big)$ for some $i \in I$. Since $\{V_i\}_{i\in I}$ captures M_1 it will also capture its direct summand $M_{1,\mathfrak{m}}$. Hence, the image of $\widehat{\mathbb{T}}^{\mathrm{univ}}_{\mathfrak{m}} \to \prod_{x\in\Sigma} \kappa(x)$ is equal to $\mathbb{T}_{1,\mathfrak{m}}$. The commutativity of the above diagram implies that this map factors through $\mathbb{T}_{2,\mathfrak{m}}$. Thus we obtain a surjection $\mathbb{T}_{2,\mathfrak{m}} \twoheadrightarrow \mathbb{T}_{1,\mathfrak{m}}$ and so the Hecke algebra $\mathbb{T}_{2,\mathfrak{m}}$ acts on the completed cohomology $\widetilde{H}^0(K_1^p)$.

The meta-argument outlined above can be modified to give a different proof of Corollary 7.3 in [37], where the functoriality is given by the (classical) Jacquet–Langlands correspondence, by taking $\{V_i\}_{i\in I}$ to be a family of supercuspidal types considered in subsection 3.4 and using Proposition 3.26 together with Proposition 3.19. A similar example has been discussed in [19, §3.3.2].

5.3. **The case of definite unitary groups.** In the case when $\mathbb{G} = GU(n)$, we let $\mathbb{T}'_{\mathfrak{m}}$ denote the subring of $\mathbb{T}_{\mathfrak{m}}$ topologically generated by spherical Hecke operators at unramified primes ℓ which are furthermore split in the appropriate quadratic imaginary field. The ring $\mathbb{T}'_{\mathfrak{m}}$ is a subring of $\mathbb{T}_{\mathfrak{m}}$, and (assuming that $\mathfrak{m}$ corresponds to an irreducible $\overline{\rho}$) is a quotient of a global deformation ring $R_{\overline{\rho}}$. (See [10, §2.4] for an indication of what exact kinds of Galois representations $R_{\overline{\rho}}$ parameterizes.)

Thus we get the Zariski density in $\mathrm{Spec}\, \mathbb{T}'_{\mathfrak{m}}$ of the collection of Galois representations arising from the various spaces $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M, V_i^*)$.

We formulate this discussion into a theorem to make it more concrete. We freely use the notation of [10]. Let R be the Hecke algebra denoted by $\mathbb{T}^{S_p}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$ in [10] before Corollary 2.11 and let $M = \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$. It follows from the proof of [10, Prop. 2.10] that $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$ is a finitely generated projective $\mathcal{O}[[\mathrm{GL}_n(\mathcal{O}_F)]]$-module, where $\mathfrak{p}$ is a fixed place above p of totally real field denoted by $\widetilde{F}^+$ in [10] and F is a finite extension of $\mathbb{Q}_p$ such that F is the $\mathfrak{p}$-adic completion of $\widetilde{F}^+$.

Let Σ^{cl} be the subset of m-Spec $R[1/p]$ corresponding to the Hecke eigenvalues corresponding to the classical (algebraic) automorphic forms on the unitary group considered in [10, §2.3]. To each $x \in \Sigma^{\mathrm{cl}}$ one may associate a Galois representation, which is potentially semi-stable at $\mathfrak{p}$. For each embedding $\kappa : F \to \overline{\mathbb{Q}}_p$ we fix an n-tuple of distinct integers $k_\kappa := (k_{\kappa,1} > k_{\kappa,2} > \ldots > k_{\kappa,n})$ and let $\underline{k}$ be the multiset $\{k_\kappa\}_{\kappa:F\hookrightarrow\overline{\mathbb{Q}}_p}$. Let $\Sigma(\underline{k})$ be the subset of Σ^{cl} corresponding to the automorphic forms such that the restriction of the associated Galois representation at the decomposition group at $\mathfrak{p}$ has Hodge–Tate weights equal to $\underline{k}$.

To each classical automorphic form we may associate an automorphic representation. Let Σ_{ps} be the subset of Σ^{cl} such that the local factor of the automorphic representation at $\mathfrak{p}$ is a principal series representation of $\mathrm{GL}_n(F)$. In terms of Galois representations the subset Σ_{ps} correspond to those automorphic forms such that the restriction of the associated Galois representation at the decomposition group at $\mathfrak{p}$ becomes crystalline after a restriction to the Galois group of a finite abelian extension of F.

We fix either a totally ramified or unramified extension E of F of degree n and identify $\mathrm{GL}_n(F)$ with $\mathrm{Aut}_F(E)$. More generally we could take any extension E of F of degree n, which is generated by a minimal element in the sense of section 3.4. Let Σ_{sc} be the subset of Σ^{cl} such that the local factor of the automorphic representation at $\mathfrak{p}$ is a supercuspidal representation of $\mathrm{GL}_n(F)$ containing a minimal stratum $[\mathfrak{A}, m, 0, \alpha]$ with $E = F[\alpha]$. If E over F is tamely ramified and ρ is a Galois representation associated to an automorphic form in Σ_{sc} then the Weil–Deligne representation associated to the restriction of ρ to the decomposition group at $\mathfrak{p}$ is isomorphic to the induction from W_E to W_F of a 1-dimensional representation; the requirement for $[\mathfrak{A}, m, 0, \alpha]$ to be minimal imposes further restrictions on the 1-dimensional representation of W_E.

Theorem 5.1. *The sets $\Sigma(\underline{k}) \cap \Sigma_{ps}$ and $\Sigma(\underline{k}) \cap \Sigma_{sc}$ are both Zariski dense inside* $\mathrm{Spec}\, \mathbb{T}^{S_p}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$.

Proof. We will carry out the proof in the supercuspidal case. The proof in the principal series case is the same using the results of section 3.3. Out of the multiset $\underline{k}$ one may manufacture a highest weight and hence a corresponding irreducible algebraic representation of $\mathbb{G}' := \mathrm{Res}^F_{\mathbb{Q}_p} \mathrm{GL}_n /F$, see section [10, §1.8]. Let W be its evaluation at L and restriction to $\mathrm{GL}_n(F)$ via $\mathrm{GL}_n(F) = \mathbb{G}'(\mathbb{Q}_p) \hookrightarrow \mathbb{G}'(L)$.

We will now apply the theory developed in section 3.4 with $R = \mathbb{T}^{S_p}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$ and $M = S(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$. Let $K := \mathrm{U}^{m+1}(\mathfrak{A})$. Since M is projective as $\mathcal{O}[\![\mathrm{GL}_n(\mathcal{O}_F)]\!]$-module it is also projective as $\mathcal{O}[\![K]\!]$-module. It follows from Lemma 2.6 and Proposition 3.26 that the family $\{W \otimes V(\alpha, \theta)\}_{\alpha,\theta}$, where α runs over the elements of E, such that $E = F[\alpha]$ such that $[\mathfrak{A}, n', 0, \alpha]$ is a simple stratum with $n' \geq 2m$ and α is minimal over F and θ runs over all simple characters $\theta \in \mathcal{C}(m, \alpha)$, captures M. Since the Hecke algebra acts faithfully on the module by construction we only have to understand the R-annihilator of $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, (W \otimes V(\alpha, \theta))^*)$ and then the assertion follows from Proposition 2.11. We first note that since M is finitely generated over $\mathcal{O}[\![K]\!]$ and both W and $V(\alpha, \theta)$ are finite dimensional L-vector space $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, (W \otimes V(\alpha, \theta))^*)$ is a finite dimensional L-vector space. By Schikhof duality we get that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, (W \otimes V(\alpha, \theta))^*) \cong \mathrm{Hom}_K(W \otimes V(\alpha, \theta), \tilde{S}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L),$$

where $\tilde{S}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L)$ is an admissible unitary L-Banach space representation of $\mathrm{GL}_n(F)$. Since $W \otimes V(\alpha, \theta)$ is a locally algebraic representation of K, the image of a K-invariant homomorphism will be contained in the subspace of locally algebraic vectors $(\tilde{S}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)^{\mathrm{alg}}$. It follows from [14, Prop. 3.2.4] that the representation $(\tilde{S}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)^{\mathrm{alg}}$ can be computed in terms of classical automorphic forms. In particular, it follows from [22, Prop. 6.11] that the action of $R[1/p]$ on $(\tilde{S}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)^{\mathrm{alg}}$ is semi-simple, and (essentially by definition) the support is equal to Σ^{cl}. Moreover, if $x\in\Sigma^{\mathrm{cl}}$ and $\mathfrak{m}_x$ is the corresponding maximal ideal of $R[1/p]$ then the subspace of $(\tilde{S}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)^{\mathrm{alg}}$ annihilated by $\mathfrak{m}_x$ is isomorphic to $\pi_{x,\mathfrak{p}}\otimes W_x$, where $\pi_{x,\mathfrak{p}}$ is the local component at $\mathfrak{p}$ of the automorphic representation associated to x, and W_x is the restriction to $\mathrm{GL}_n(F)$ of the algebraic representation of $\mathbb{G}'$ encoding the Hodge–Tate weights of the Galois representation attached to x. Thus the action of $R[1/p]$ on $\mathrm{Hom}_K(W\otimes V(\alpha,\theta),(\tilde{S}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)^{\mathrm{alg}})$ is semisimple and the support is equal to a finite subset of Σ^{cl} consisting of x such that $W\cong W_x$ and $\mathrm{Hom}_K(V(\alpha,\theta),\pi_{x,\mathfrak{p}})\neq 0$. The first condition implies that $x\in\Sigma(\underline{k})$, the second condition implies that $x\in\Sigma_{sc}$ via Proposition 3.19. $\square$

Remark 5.2. In the recent paper [25], Hellman, Margerin and Schraen have proved a "big R equals big $\mathbb{T}$ theorem" in the context of definite unitary groups. We refer to their paper for the precise details, but note here that, when their result applies, we may rephrase Theorem 5.1 as a statement about the Zariski density of certain sets of points in the Spec of a global Galois deformation ring.

5.4. **Local application.** By applying the theory of capture to M_∞ of the six author paper [10], we will get density in the support of M_∞. This support will equal the full local deformation space, provided that this latter space is irreducible. The proof of this is the subject of the next section.

Let $R^{\square}$ be the framed deformation ring of $\bar{\rho}: G_F\to\mathrm{GL}_n(k)$. If R is a local noetherian $R^{\square}$-algebra with residue field k then to $x\in\text{m-Spec}\,R[1/p]$ one may associate Galois representation $\rho_x: G_F\to\mathrm{GL}_n(\kappa(x))$ by specialising the universal framed deformation along the map $R^{\square}\to R\to\kappa(x)$. We will define the local analogs of the sets Σ^{cl}, $\Sigma(\underline{k})$, Σ_{ps} and Σ_{sc} defined in the previous section.

Let Σ^{pst} be the subset of $\text{m-Spec}\,R[1/p]$ consisting of those x for which the representation ρ_x is potentially semi-stable. Let $\Sigma(\underline{k})$ be the subset of Σ^{pst} such that the Hodge–Tate weights of ρ_x are equal to $\underline{k}$. Let Σ_{ps} be the subset of Σ^{pst} such that ρ_x become crystalline after a restriction to the Galois group of an abelian extension of F. Alternatively Σ_{ps} can be described as those $x\in\Sigma^{\mathrm{pst}}$ such that the Weil–Deligne representation associated to ρ_x corresponds to a principal series representation via the classical local Langlands correspondence. Let E be either a totally ramified or an unramified extension of F of degree n as in the previous section. We let Σ_{sc} be the subset of Σ^{pst} consisting of those x such that the Weil–Deligne representation associated to ρ_x corresponds to a supercuspidal representation π_x via the classical local Langlands correspondence, such that π_x contains a simple stratum $[\mathfrak{A},n',0,\alpha]$, such that $E=F[\alpha]$ and α is minimal over F.

Recall that in [10] we have constructed a local noetherian $R^{\square}$-algebra R_∞ with residue field k and an arithmetically interesting $R_\infty[\mathrm{GL}_n(F)]$-module M_∞. We assume that p does not divide $2n$ and that $\bar{\rho}$ admits a potentially crystalline lift of regular weight, which is potentially diagonalisable, since this is assumed in [10]. As noted in Remark 1.5, this last assumption will become redundant in the future. We recall that M_∞ is projective as an $\mathcal{O}[\![K]\!]$-module. (This follows from [10, Prop. 2.10], which shows that M_∞ is projective over $S_\infty[\![K]\!]$, where S_∞ is a certain formal power series ring in a finite number of variables over $\mathcal{O}$.)

Theorem 5.3. *If $\mathfrak{a}$ denotes the R_∞-annihilator of M_∞ then $\Sigma(\underline{k}) \cap \Sigma_{ps} \cap V(\mathfrak{a})$ and $\Sigma(\underline{k}) \cap \Sigma_{sc} \cap V(\mathfrak{a})$ are both Zariski dense subsets of $V(\mathfrak{a})$.*

Proof. We will treat the supercuspidal case, since the principal series is similar (and easier). The proof is essentially the same as the proof of Theorem 5.1. We only need to verify that $R_\infty/\mathfrak{a}_{\alpha,\theta}$ is reduced and all the maximal ideals of $(R_\infty/\mathfrak{a}_{\alpha,\theta})[1/p]$ lie in $\Sigma(\underline{k}) \cap \Sigma_{sc}$, where $\mathfrak{a}_{\alpha,\theta}$ is the R_∞-annihilator of $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M_\infty, (W \otimes V(\alpha,\theta))^*)$. This is proved in the same way as part 1 of [10, Lem. 4.17] by using Proposition 3.19 together with local–global compatibility. $\square$

Lemma 5.4. *Let A and B be complete local noetherian $\mathcal{O}$-algebras with residue field k. Assume that B is a flat A-algebra. Let Σ be a subset of* m-Spec $A[1/p]$ *and let Σ' be its preimage in* m-Spec $B[1/p]$*. If the closure of Σ' in* $\operatorname{Spec} B$ *is a union of irreducible components of* $\operatorname{Spec} B$ *then the closure of Σ in* $\operatorname{Spec} A$ *is a union of irreducible components of* $\operatorname{Spec} A$.

Proof. Let B_{tf} be the maximal p-torsion free quotient of B. The map $\mathfrak{p} \mapsto \mathfrak{p} \cap B_{\mathrm{tf}}$ induces a bijection between the minimal primes of $B[1/p]$ and B_{tf} and we will identify both sets. Moreover, every minimal prime of B_{tf} is also a minimal prime of B: if $\mathfrak{p}$ is a minimal prime of B_{tf} then the quotient $B_{\mathrm{tf}}/\mathfrak{p}$ is p-torsion free, as p is regular on B_{tf}. If $\mathfrak{q}$ is a minimal prime of B then either $B/\mathfrak{q}$ is p-torsion free in which case $\mathfrak{q}$ is a minimal prime of B_{tf} or p is zero in $B/\mathfrak{q}$ in which case $B/\mathfrak{q}$ cannot have $B_{\mathrm{tf}}/\mathfrak{p}$ as a quotient. Thus it is enough to study the question after inverting p.

Let $\mathfrak{b}$ be an ideal of $B[1/p]$ such that $V(\mathfrak{b})$ is the closure of Σ' and let $\mathfrak{a} = \mathfrak{b} \cap A[1/p]$. Then $V(\mathfrak{a})$ is equal to the closure of Σ. By assumption we may write $V(\mathfrak{b}) = V(\mathfrak{q}_1) \cup \ldots \cup V(\mathfrak{q}_r)$, where $\mathfrak{q}_i$ are minimal prime ideals of $B[1/p]$. We may assume that $\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$. Since B is A-flat the going down theorem implies that $\mathfrak{p}_i := \mathfrak{q}_i \cap A[1/p]$ is a minimal prime of $A[1/p]$. Since $\mathfrak{a} = \mathfrak{b} \cap A[1/p] = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we have $V(\mathfrak{a}) = V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r)$. $\square$

Corollary 5.5. *Let $\mathfrak{a}$ be the R_∞-annihilator of M_∞. If $V(\mathfrak{a})$ is equal to a union of irreducible components of* $\operatorname{Spec} R_\infty$*, then the Zariski closure of $\Sigma(\underline{k}) \cap \Sigma_{sc}$ (resp. $\Sigma(\underline{k}) \cap \Sigma_{ps}$) in* $\operatorname{Spec} R^{\square}$ *contains a non-empty union of irreducible components of* $\operatorname{Spec} R^{\square}$.

Proof. Given Lemma 5.4 and Theorem 5.3, it is enough to verify that R_∞ is flat over $R^{\square}$. Now R_∞ is formally smooth over the ring denoted by R^{loc} in [10, §2.6], and $R^{\mathrm{loc}} = R^{\square} \widehat{\otimes}_{\mathcal{O}} A$, where A is a completed tensor product over $\mathcal{O}$ of some deformation rings at a finite number of places away from $\mathfrak{p}$. All them are $\mathcal{O}$-torsion free, see [10, §2.4], thus A is $\mathcal{O}$-torsion free and hence $\mathcal{O}$-flat, which implies that R^{loc} is flat over $R^{\square}$. $\square$

The condition on $V(\mathfrak{a})$ in Corollary 5.5 will be verified in the next section.

6. The support of M_∞

We begin with a definition.

Definition 6.1. If A is a complete Noetherian local $\mathcal{O}$-algebra, and M is a finitely generated $A[\![K]\!]$-module, then we define the *support* of M in $\operatorname{Spec} A$, denoted $\operatorname{Supp}_A M$, as follows: $\operatorname{Supp}_A M := V(\mathfrak{a})$, where $\mathfrak{a}$ is the A-annihilator of M.

Since a finitely generated $A[\![K]\!]$-module is typically not finitely generated as an A-module, it is not immediately clear that defining its support in the above manner is sensible. However, the following lemma gives us assurance that the preceding definition does in fact give a reasonable notion of support.

Lemma 6.2. *If $m_1, \ldots, m_r$ are generators of M as an $A[\![K]\!]$-module, and M' is the A-submodule of M generated by $m_1, \ldots, m_r$, then* $\operatorname{Supp} M' = \operatorname{Supp} M$.

Proof. Let $\mathfrak{a}$ be the A-annihilator of M and let $\mathfrak{a}'$ be the A-annihilator of M'. Since $M' \subset M$ we have $\mathfrak{a} \subset \mathfrak{a}'$. On the other hand, since the actions of A and K on M commute, we see that $\mathfrak{a}'$ annihilates $A[\![K]\!]M' = M$. □

The main goal of this section is to prove the following theorem.

Theorem 6.3. *The support of M_∞ in* $\operatorname{Spec} R_\infty$ *is equal to a union of irreducible components of* $\operatorname{Spec} R_\infty$.

Proof. It follows from Proposition 3.1 that, if ξ runs over all the algebraic representations of G, then the support of M_∞ is equal to the Zariski closure of the unions of the supports of the various modules $M_\infty(\xi)$. (See [10] for a discussion of these patched modules.) The support of each $M_\infty(\xi)$ is reduced [10, Lem. 4.17], and so it follows from Theorem 6.5 below that the support of $M_\infty[1/p]$ in $\operatorname{Spec} R_\infty[1/p]$ is equal to a union of its irreducible components. Since M_∞ is p-torsion free by construction, the support of M_∞ is equal to a union of irreducible components of $\operatorname{Spec} R_{\infty,\mathrm{tf}}$, where $\operatorname{Spec} R_{\infty,\mathrm{tf}}$ denotes the maximal p-torsion free quotient of R_∞. As explained in the proof of Lemma 5.4 these are also irreducible components of $\operatorname{Spec} R_\infty$. □

Remark 6.4. Typically $R^\square$ is a formally smooth $\mathcal{O}$-algebra and thus irreducible. In such cases, our theorem together with Corollary 5.5 implies that M_∞ is supported on all of $\operatorname{Spf} R^\square$. We expect that M_∞ is in fact always supported on all of $\operatorname{Spf} R^\square$, but it seems *a priori* possible that this is not the case, when $\operatorname{Spf} R^\square$ has more than one irreducible component.

Theorem 6.5. *The Zariski closure in* $(\operatorname{Spf} R_\infty)^{\mathrm{rig}}$ *of the set of crystalline points lying in the support of M_∞ is equal to a union of irreducible components of* $(\operatorname{Spf} R_\infty)^{\mathrm{rig}}$.

We follow the approach of Chenevier [11] and Nakamura [30], who proved (under various technical hypotheses) that the crystalline points are Zariski dense in the rigid analytic generic fibre $(\operatorname{Spf} R^\square)^{\mathrm{rig}}$ of $\operatorname{Spf} R^\square$.

The arguments of Chenevier and Nakamura use "infinite fern"-style techniques. In order to modify these arguments so as to be sensitive to the support of M_∞, we use the locally analytic Jacquet module of [17], which provides a representation-theoretic approach to the study of finite slope families. Breuil, Hellmann, and Schraen [3] have made a careful study of the theory of the Jacquet module as it applies to M_∞, and we will use their results.[1]

Following the notation of [3], we write $\Pi_\infty := \operatorname{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M_\infty, L)$; then Π_∞ is an R_∞-admissible Banach space representation of G. We may pass to its R_∞-locally analytic vectors, see [3, §3.1], and then form the locally analytic Jacquet module $J_B(\Pi_\infty^{R_\infty-\mathrm{an}})$. The continuous dual $J_B(\Pi_\infty^{R_\infty-\mathrm{an}})'$ of $J_B(\Pi_\infty^{R_\infty-\mathrm{an}})$ may naturally be regarded as the space of global sections of a coherent sheaf $\mathcal{M}_\infty$ on the rigid analytic space $(\operatorname{Spf} R_\infty)^{\mathrm{rig}} \times \widehat{\mathrm{T}}$. We let X_∞ denote the support of $\mathcal{M}_\infty$; then X_∞ is a Zariski closed rigid analytic subspace of $(\operatorname{Spf} R_\infty)^{\mathrm{rig}} \times \widehat{\mathrm{T}}$. If x is a closed point of $\operatorname{Spec} R_\infty[1/p]$

[1] In fact, they work with a slightly different version of M_∞ to us (which is literally the construction of [10]), but all their arguments and results apply equally well in our setting, and we will cite and apply them without further discussion of this issue.

and $\mathfrak{m}_x$ is the corresponding maximal ideal, then $\Pi(x)$ is a closed subspace of Π_∞ consisting of elements in Π_∞, killed by $\mathfrak{m}_x$.

We recall again from [3], see their Definition 2.4, that there is another important Zariski closed rigid analytic subspace

$$X_\infty^{\mathrm{tri}} \hookrightarrow (\operatorname{Spf} R_\infty)^{\mathrm{rig}} \times \widehat{\mathrm{T}},$$

which by definition is the Zariski closure of the set of points

$$(x, \delta) \in (\operatorname{Spf} R_\infty)^{\mathrm{rig}} \times \widehat{\mathrm{T}}^{\mathrm{reg}},$$

where δ is the parameter of a triangulation of the Galois representation r_x, see [3, §2.2].

We then have the following key result of [3].

Theorem 6.6 ([3, Thm. 3.21]). *There is an inclusion $X_\infty \subset X_\infty^{\mathrm{tri}}$, and X_∞ is furthermore a union of irreducible components of X_∞^{tri}.*

We also need to recall the key point of the infinite fern argument of [11] and [30], in a form that is well-suited to our argument.

Definition 6.7. If $\underline{k}$ is a system of labelled Hodge–Tate weights, then we let $R_\infty^{\underline{k}}$ denote the reduced and $\mathcal{O}$-flat quotient of R_∞ characterized by the property that a point $x \in (\operatorname{Spf} R_\infty)^{\mathrm{rig}}$ lies in $(\operatorname{Spf} R_\infty^{\underline{k}})^{\mathrm{rig}}$ if and only if r_x is crystalline with labelled Hodge–Tate weights given by $\underline{k}$.

Let $f := [F_0 : \mathbb{Q}_p]$ be the degree of the maximal unramified subextension of F.

Definition 6.8. We say that a crystalline representation r of G_F with regular Hodge–Tate weights $\underline{k}$ is *benign* if the following hold:

(i) the eigenvalues of the linearization of the crystalline Frobenius φ^f of r are pairwise distinct;
(ii) all the refinements of r are non-critical;
(iii) the ratio of any two eigenvalues in (i) is not equal to $p^{\pm f}$.

We refer the reader to [3, §2.3] for the terminology used in (i) and (ii). Our definition coincides with the definition of benign in [30, Def. 2.8]. It is more restrictive than [3, Def. 2.8], since we additionally impose the condition (iii). Let $\mathrm{WD}(r)$ be the Weil-Deligne representation associated to r. The condition (iii) implies that the smooth representation π_p of $\mathrm{GL}_n(F)$ associated to $\mathrm{WD}(r)$ via the classical local Langlands correspondence is an *irreducible* unramified principal series representation.

Proposition 6.9. *If $\underline{k}$ is any regular system of labelled Hodge–Tate weights, and C is any irreducible component of $(\operatorname{Spf} R_\infty^{\underline{k}})^{\mathrm{rig}}$, then the set of points $x \in C$ for which r_x is benign forms a non-empty Zariski open subset of $(\operatorname{Spf} R_\infty^{\underline{k}})^{\mathrm{rig}}$.*

Proof. This is a result of Chenevier [11, Lem. 4.4] in the case when $K = \mathbb{Q}_p$ and Nakamura [30, Lem. 4.2] in the general case. □

Theorem 6.10. *Let z be a benign crystalline point of $\operatorname{Spec} R_\infty$.*

(1) *If (z, δ) is a point of X_∞^{tri} which lies over z, then (z, δ) is a smooth point of X_∞^{tri}.*
(2) *The tangent space $T_{(\operatorname{Spf} R_\infty)^{\mathrm{rig}}, z}$ is spanned by the images of the tangent spaces $T_{X_\infty^{\mathrm{tri}}, (z,\delta)}$, as (z, δ) runs over all points of X_∞^{tri} lying over z.*

Proof. The first claim follows by combining Corollary 2.12 and Theorem 2.6 (iii) of [3]. The second claim follows from [11] in the case that $K = \mathbb{Q}_p$ and [31, Thm. 2.62] in general once we note that the image of $T_{X_\infty^{\mathrm{tri}},(z,\delta)}$ in $T_{(\mathrm{Spf}\, R_\infty)^{\mathrm{rig}},z}$ coincides with the closed subspace of $T_{(\mathrm{Spf}\, R_\infty)^{\mathrm{rig}},z}$ classifing trianguline deformations of (z,δ). See e.g. the discussion of this in the proof of [3, Thm. 2.6]. □

With the preceding results recalled, we are now in a position to study the support of M_∞.

If ξ is an irreducible algebraic representation of G, and if $\underline{k}$ is the system of labelled Hodge–Tate weights corresponding to ξ, then (as already recalled in the proof of Theorem 6.3), we may form the $R_\infty[1/p]$-module $M_\infty(\xi)$ as in [10], whose support is a union of some number of irreducible components of $\mathrm{Spec}\, R_\infty^{\underline{k}}[1/p]$, or, equivalently, a union of some number of irreducible components of $(\mathrm{Spf}\, R_\infty^{\underline{k}})^{\mathrm{rig}}$. In particular, if $M_\infty(\xi)$ is non-zero, then Proposition 6.9 implies that its support contains a Zariski dense open subset of points x for which r_x is benign.

Lemma 6.11. *If ξ is an algebraic representation of G, if x is a point lying in the support of $M_\infty(\xi)$ for which r_x is benign, and if (x,δ) is a point of X_∞^{tri} lying over x, then $(x,\delta) \in X_\infty$.*

Proof. We see from [3, Cor. 2.12] that the δ is necessarily a triangulation of r_x. The fibre of $\mathcal{M}_\infty$ over (x,δ) is dual to the eigenspace $J_B\big(\Pi(x)^{\mathrm{an}}\big)^{T=\delta}$ in the Jacquet module of $\Pi(x)$, where an denotes the subspace of locally analytic vectors in the Banach space $\Pi(x)$, see [3, Prop. 3.7]. Thus in order to show that (x,δ) lies in X_∞, it suffices to show that for every parameter δ describing a triangulation of r_x, the corresponding eigenspace in the Jacquet module of $\Pi(x)^{\mathrm{an}}$ is non-zero.

To see this, note that local-global compatibility shows that $\Pi(x)$ contains a locally algebraic representation $\pi_p \otimes \xi$, with π_p the parabolic induction of an unramified character, see [10, Thm. 4.35]. Note that the condition (iii) in Definition 6.8 ensures that π_p is irreducible and hence generic, so that the conditions of [10, Thm. 4.35] are satisfied. The representation $\pi_p \otimes \xi$ equipped with the finest locally convex topology is a locally analytic subrepresentation of $\Pi(x)^{\mathrm{an}}$. Since the Jacquet-functor is left exact we have an inclusion $J_B(\pi_p \otimes \xi) \subset J_B\big(\Pi(x)^{\mathrm{an}}\big)$. Moreover, [17, Prop. 4.3.6] implies that $J_B(\pi_p \otimes \xi) \cong (\pi_p)_N \otimes \xi^N$, where N is the unipotent radical of B. Hence, $J_B(\pi_p \otimes \xi)$ consists of $n!$ characters which fill out all the necessary eigenspaces. (These correspond to the $n!$ possible orderings of the Frobenius eigenvalues, which in turn match with all the possible triangulations.) □

Theorem 6.12. *The Zariski closure, in $(\mathrm{Spf}\, R_\infty)^{\mathrm{rig}}$, of the union of the supports of the modules $M_\infty(\xi)$, as ξ ranges over all the irreducible algebraic representations of G, is equal to a non-empty union of components of $(\mathrm{Spf}\, R_\infty)^{\mathrm{rig}}$.*

Proof. To ease notation, write $Y := (\mathrm{Spf}\, R_\infty)^{\mathrm{rig}}$, and let $Z \hookrightarrow Y$ the Zariski closure under consideration in the statement of the theorem. We note that $M_\infty(\xi)$ is non-zero for at least one choice of ξ, so that Z is non-empty. Since Z is furthermore a reduced rigid analytic space over a field of characteristic zero, its smooth locus Z^{sm} is a non-empty and open subset. Thus the correspondence $V \mapsto U := V \cap Z^{\mathrm{sm}}$ induces a bijection between the irreducible components V of Z and the irreducible (or, equivalently, connected) components U of Z^{sm}; one recovers V from U by taking the Zariski closure of U in Z (or, equivalently, in Y).

We next note that the points of X_∞ of the form (z,δ) with z lying in the support of one of $M_\infty(\xi)$ are actually Zariski dense in X_∞. Indeed, this follows from [3, Thm. 3.19], since any $(z,\delta) \in X_\infty$, which is strongly classical in the sense of [3, Def. 3.15], is classical in the sense of [3, Def. 3.17] and hence will have z lying in the support of $M_\infty(\xi)$, where the highest weight of ξ corresponds to the algebraic part of δ. This implies that the image of X_∞ in Y under the composition $\pi : X_\infty \subset Y \times \widehat{\mathrm{T}} \to Y$ is contained in Z, as $\pi^{-1}(Z)$ is a closed subset of X_∞ containing a dense set of points.

As noted above, if $\underline{k}$ is the system of labelled Hodge–Tate weights corresponding to an algebraic representation ξ, then the support of $M_\infty(\xi)$ is a union of some number of components of $(\mathrm{Spf}\, R_\infty^{\underline{k}})^{\mathrm{rig}}$, each of which contains a Zariski dense subset of points z for which r_z is benign. Thus Z is equal to the Zariski closure of the set B of such points, i.e. the set B of points z lying in the support of some $M_\infty(\xi)$, and for which r_z is benign. Thus if V is an irreducible component of Z, and if $U := V \cap Z^{\mathrm{sm}}$ (which, as we noted above, is Zariski dense in V), then we find that the $B \cap U$ is Zariski dense in U; in particular, it is non-empty.

Choose a point $z \in B \cap U$. If (z,δ) is any point of X_∞^{tri} lying over z, the point (z,δ) lies in the smooth locus of X_∞^{tri}, by Theorem 6.10 (1). Lemma 6.11 shows that any such point (z,δ) in fact lies in X_∞. We then conclude from Theorem 6.6 that (z,δ) is a smooth point of X_∞, and that the closed immersion $X_\infty \hookrightarrow X_\infty^{\mathrm{tri}}$ is an isomorphism at any such point (z,δ).

Combining this last conclusion with the inclusion $\pi(X_\infty) \subseteq Z$ that was proved above, we thus find that $T_{Z,z} = T_{U,z}$ contains the image of the tangent space $T_{X_\infty^{\mathrm{tri}},(z,\delta)}$, for each choice of (z,δ) lying over z. Since Theorem 6.10 (2) shows that the images of the various tangent spaces $T_{X_\infty^{\mathrm{tri}},(z,\delta)}$ span $T_{Y,z}$, we find that $T_{U,z} = T_{Y,z}$, and hence that U and Y coincide locally at z. This implies that the Zariski closure V of U, which is an irreducible component of Z, is also an irreducible component of Y, as required. □

References

[1] K. Ardakov and K. A. Brown, Ring-theoretic properties of Iwasawa algebras: a survey, *Doc. Math.* 2006, Extra Vol., 7–33.

[2] T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, Potential automorphy and change of weight, *Ann. of Math.*, (2) 179 (2014), no. 2, 501–609.

[3] C. Breuil, E. Hellmann and B. Schraen, Une interpretation modulaire de la variete trianguline, *Math. Annalen*, 367 no.3-4 (2017), pp.1587–1645.

[4] C. Breuil and A. Mézard, Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $l = p$, *Duke Math. J.* 115 (2002), 205–310.

[5] C. Bushnell and A. Fröhlich, *Gauss sums and p-adic division algebras*, Lecture Notes in Mathematics, 987. Springer-Verlag, Berlin-New York, 1983. xi+187 pp.

[6] C. Bushnell and G. Henniart, The essentially tame local Langlands correspondence. I., *J. Amer. Math. Soc.* 18 (2005), no. 3, 685–710.

[7] C. Bushnell and P. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, PUP 1993.

[8] C. Bushnell, P. Kutzko, Semisimple Types in GL_n, *Compositio Mathematica* 119 (1999), 53–97.

[9] F. Calegari and M. Emerton, Completed cohomology – a survey. *Non-abelian fundamental groups and Iwasawa theory*, 239–257, London Math. Soc. Lecture Note Ser., 393, Cambridge Univ. Press, Cambridge, 2012.

[10] A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paškūnas, S.-W. Shin, Patching and the p-adic local Langlands correspondence, *Camb. J. Math.* 4 (2016), no. 2, 197–287.

[11] G. Chenevier, Sur la densité des représentations cristallines de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, *Math. Ann.* 355 (2013), no. 4, 1469–1525.

[12] P. Colmez, G. Dospinescu and V. Paškūnas, The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, *Camb. J. Math.* 2 (2014), no. 1, 1–47.

[13] J.D. Dixon, M.P.F. du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, Second edition, CUP, 2003.

[14] M. Emerton, On the interpolation of eigenvalues attached to automorphic Hecke eigenforms, *Invent. Math.* 164 (2006), no. 1, 1–84.

[15] M. Emerton, p-adic L-functions and unitary completions of representations of p-adic reductive groups, *Duke Math. J.* 130 (2005), no. 2, 353–392.

[16] M. Emerton, Locally analytic vectors in representations of locally p-adic analytic groups. *Mem. Amer. Math. Soc.* 248 (2017), no. 1175, iv+158 pp.

[17] M. Emerton, Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties. *Ann. Sci. École Norm. Sup.* (4) 39 (2006), no. 5, 775–839.

[18] M. Emerton, Local-global compatibility in the p-adic Langlands programme for $\mathrm{GL}_2/\mathbb{Q}$, Preprint (2011), `http://math.uchicago.edu/~emerton/preprints.html`.

[19] M. Emerton, Completed cohomology and the p-adic Langlands program, In: *Proceedings of the International Congress of Mathematicians*, 2014. Volume II. 2014, pp. 319–342.

[20] M. Emerton and T. Gee, A geometric perspective on the Breuil-Mézard conjecture, *J. Inst. Math. Jussieu*, 13 (2014), no. 1, 183–223.

[21] T. Gee and J. Newton, Patching and the completed homology of locally symmetric spaces, Preprint 2017, `https://arxiv.org/abs/1609.06965`.

[22] B. Gross, Algebraic modular forms, *Israel J. Math.*, 113 (1999), 61–93.

[23] M. Harris, p-adic representations arising from descent on abelian varieties, *Compositio Math.* 39 (1979), no. 2, 177–245.

[24] M. Harris, Correction to: "p-adic representations arising from descent on abelian varieties", *Compositio Math.* 121 (2000), no. 1, 105–108.

[25] E. Hellmann, C.M. Margerin and B. Schraen, Density of automorphic points, Preprint (2018), `https://arxiv.org/abs/1811.09116`.

[26] E. Hellmann and B. Schraen, Density of potentially crystalline representations of fixed weight. *Compos. Math.* 152 (2016), no. 8, 1609–1647.

[27] P. Kutzko and D. Manderscheid, On intertwining operators for $\mathrm{GL}_N(F)$, F a nonarchimedean local field, *Duke Math. J.* Volume 57, Number 1 (1988), 275–293.

[28] H. Matsumura, *Commutative ring theory*, Second edition. Cambridge Studies in Advanced Mathematics, 8. CUP, Cambridge, 1989.

[29] P. Monsky, The Hilbert–Kunz function, *Math. Ann.*, 263, 43–49 (1983).

[30] K. Nakamura, Zariski density of crystalline representations for any p-adic field, *J. Math. Sci. Univ. Tokyo* 21 (2014), no. 1, 79–127.

[31] K. Nakamura, Deformations of trianguline B-pairs and Zariski density of two dimensional crystalline representations, *J. Math. Sci. Univ. Tokyo* 20 (2013), no. 4, 461–568.

[32] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, Sec. Ed., corr.sec. print. (2013), pp. 826, Springer Grundlehren der mathematischen Wissenschaften Bd. 323.

[33] V. Paškūnas, Unicity of types for supercuspidal representations of GL_N, *Proc. London Math. Soc.* (3) 91 (2005), no. 3, 623–654.

[34] V. Paškūnas, On the Breuil–Mézard Conjecture, *Duke Math. J.* 164 (2015), no. 2, 297–359.

[35] V. Paškūnas, The image of Colmez's Montreal functor, *Publ. Math. Inst. Hautes Études Sci.* 118 (2013), 1–191.

[36] V. Paškūnas, On some consequences of a Theorem of J. Ludwig, Preprint 2018, `https://arxiv.org/abs/1804.07567`.

[37] P. Scholze, On the p-adic cohomology of the Lubin-Tate tower, *Annales de l'ENS*, (4) 51 (2018), no. 4, 811–863.

[38] P. Schneider and J. Teitelbaum, Banach space representations and Iwasawa theory, *Israel J. Math.* 127, 359–380 (2002).

[39] S.-N. Tung, On the automorphy of 2-dimensional potentially semi-stable deformation rings of $G_{\mathbb{Q}_p}$, Preprint 2018, `https://arxiv.org/abs/1803.07451`.

[40] S.-N. Tung, On the modularity of 2-adic potentially semi-stable deformation rings, Preprint 2019, `https://arxiv.org/abs/1908.06174`.

[41] O. Venjakob, On the structure theory of the Iwasawa algebra of a p-adic Lie group, *J. Eur. Math. Soc.* (JEMS) 4 (2002), no. 3, 271–311.