A p-ADIC VARIATIONAL HODGE CONJECTURE AND MODULAR FORMS WITH COMPLEX MULTIPLICATION

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If p is a prime number, N a positive integer prime to p, and k an integer, then we let $S_k^\dagger(N)$ denote the space of overconvergent p-adic cuspforms with coefficients in $\mathbb{C}_p$ (of some radius of overconvergence) of weight k on $\Gamma_1(N)$ (see [10, chap. I] for the definition of overconvergent modular forms). In [4, prop. 4.3], Coleman proves that if $k \geq 2$, then the differential operator $\theta^{k-1} = \left(q\frac{d}{dq}\right)^{k-1}$ induces a transformation from $S^\dagger_{-(k-2)}$ to $S_k^\dagger$. If f is a weight k normalized Hecke eigenform lying in the image of θ^{k-1}, then one sees that the U_p eigenvalue of f is divisible by p^{k-1}, and so f has slope at least $k-1$ ([4, lem. 6.3]).

Suppose that f is actually a normalized classical eigenform on $\Gamma_1(N) \cap \Gamma_0(p)$. It follows by Atkin-Lehner theory [1] and the fact that f has slope at least $k-1$ that f has slope exactly $k-1$, and that f is one of the twin oldforms attached to an ordinary classical form g on $\Gamma_1(N)$.

Conversely, suppose that g is a classical normalized Hecke eigenform of weight k on $\Gamma_1(N)$, defined over $\overline{\mathbb{Q}}_p$, which is ordinary at p. Then g gives rise to two oldforms (a pair of twins) on $\Gamma_1(N)\cap\Gamma_0(p)$, one of slope zero and one of slope $k-1$, which we may think of as overconvergent Hecke eigenforms on $\Gamma_1(N)$. Denote the slope $k-1$ twin by f. In [4, §7], Coleman discusses the question as to whether or not f is in the image of θ^{k-1}. He shows that if g is a CM modular form, then f *is* in the image of θ^{k-1}. In fact, if g is attached to the $(k-1)^{st}$ power of a Grössencharacter ψ, then f is obtained by applying θ^{k-1} to a slope zero p-adic modular form of weight $-(k-2)$ attached to the $(1-k)^{th}$ power of ψ [4, prop. 7.1]. Coleman also asks whether there are any non-CM forms in the image of θ^{k-1} [4, remark 2, p. 232].

Continue to suppose that g is a a normalized classical eigenform. The Fourier coefficients span a finite extension E of $\mathbb{Q}$. Let $\wp$ range over the primes of E lying over p. Each prime $\wp$ gives an embedding of E into $\overline{\mathbb{Q}}_p$ (well-defined up to conjugation in $\overline{\mathbb{Q}}_p$). Let $g_\wp$ denote g regarded as a modular form over $\overline{\mathbb{Q}}_p$ (well-defined up to algebraic conjugacy in $\overline{\mathbb{Q}}_p$) in this way. If $g_\wp$ is ordinary, then as above we may form its slope $k-1$ twin, which we denote $f_\wp$.

The aim of this article is to state a certain analogue in p-adic Hodge theory of the variational Hodge conjecture [12], and to explain how it implies the following conjecture.

Conjecture (0.1). *Let g be a classical Hecke eigenform on $\Gamma_1(N)$, let E be the finite extension of $\mathbb{Q}$ generated by the Fourier coefficients of g, and let $\wp$ run over* *all the prime ideals of E lying over p. Suppose that each $g_\wp$ is ordinary, and that $f_\wp$ lies in the image of θ^{k-1} for every prime $\wp$. Then g is a CM eigenform.*

Whether or not it is reasonable to weaken conjecture (0.1) by asking only that $g_\wp$ be ordinary and $f_\wp$ be in the image of θ^{k-1} for *one* prime $\wp$ is something on which the author is undecided for the moment.

In section 1, after making some preliminary observations valid for all weights, we *prove* conjecture (0.1) in the case that the weight k equals two. In this case, we appeal directly to the Serre-Tate theory of deformations of abelian varieties in characteristic p [15, 19] and Tate's theorem on morphisms of p-divisible groups in mixed characteristic [24, thm. 4].

In section 2, we state our p-adic variational Hodge conjecture (conjecture (2.2)) and elaborate on some aspects of it. In particular, proposition (2.7) gives a useful alternative formulation of the conjecture, in terms of motives with good reduction modulo p.

In section 3 we explain how conjecture (2.2) implies conjecture (0.1) for arbitrary weights $k \geq 2$. For this we will depend heavily on the existence of the motive attached to a Hecke eigenform [21].

That conjecture (0.1) would be implied by an appropriate conjecture on algebraic cycles was noticed independently by B. Mazur. The author first observed the relation between conjectures (0.1) and (2.2) some time in 1997. He would like to thank M. Kisin for encouraging him to write up this account of his observation, as well as B. Conrad and S. Bloch for helpful conversations.

1. Preliminaries, and the case of weight two eigenforms

Let us fix a choice of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and let $\mathbb{C}_p$ denote the p-adic completion of $\overline{\mathbb{Q}}_p$. If E is any finite extension of $\mathbb{Q}$ then $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on the set of embeddings of E into $\overline{\mathbb{Q}}_p$, and the orbits correspond to primes $\wp$ of E lying over p.

If g is a classical normalized Hecke eigenform of weight at least two on $\Gamma_1(N)$ and if E is the finite extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ generated by the q-expansion coefficients of g, then there is attached to g a representation $\rho_g : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ [6]. Write $E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \prod_{\wp|p} E_\wp$, where the product ranges over the primes $\wp$ of E lying over p, and $E_\wp$ denotes the $\wp$-adic completion of E. There is a corresponding factorization $\rho_g \xrightarrow{\sim} \prod_{\wp|p} \rho_{g,\wp}$. The representation $\rho_{g,\wp}$ reflects the properties of g regarded as a p-adic modular form via the embedding $E \to \overline{\mathbb{Q}}_p$ corresponding to $\wp$. As in the introduction, we let $g_\wp$ denote g regarded as a p-adic modular form in this way.

If f is any p-adic normalized Hecke eigenform of some weight on $\Gamma_1(N)$ with coefficients in $\mathbb{C}_p$, we may attach to f a Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C}_p)$. (See [10, §III.5] and [14, thm. II] for the construction of these representations. The first reference treats the case $p > 7$, and the second, the case $p > 2$ (in the more general context of p-adic Hilbert modular forms). We will only need the existence of these representations in the ordinary case, for p-adic modular forms on $\mathrm{GL}_{2/\mathbb{Q}}$, and in this case a modification of the methods of [13] can also be used to construct the necessary representations when $p = 2$. One can also modify the set-up of the argument of [14, §1] so that it treats the case $p = 2$. If the reader prefers, they may assume that p is odd for the remainder of the paper.)

Now suppose as above that g is a classical eigenform on $\Gamma_1(N)$, that $\wp$ is a prime lying over p in the coefficient field E of g, and that $g_\wp$ is ordinary. As in the introduction let $f_\wp$ denote the slope $k-1$ oldform on $\Gamma_0(p)\cap\Gamma_1(N)$ attached to $g_\wp$. Then $f_\wp$ is a p-adic Hecke eigenform, and we have an equality of Galois representations

$$\rho_{f_\wp} = \rho_{g,\wp}.$$

Suppose that $f_\wp$ lies in the image of θ^{k-1}. Let $F_\wp$ be the weight $-(k-2)$ overconvergent form satisfying $\theta^{k-1}F_\wp = f_\wp$. Then since $f_\wp$ has slope $k-1$, we see that $F_\wp$ must be ordinary.

The Galois representations $\rho_{f,\wp}$ and $\rho_{F,\wp}$ are related by the equation

$$\rho_{f,\wp} = \chi^{k-1}\rho_{F,\wp} \tag{1.1}$$

(where χ denotes the p-adic cyclotomic character).

The following proposition was also observed independently by Gouvea and Kisin. It is the characteristic zero analogue of Serre's observation concerning mod p Galois representations and companion forms [11, thm. 13.8].

Proposition (1.2). *Suppose that the classical eigenform g on $\Gamma_1(N)$ satisfies the hypothesis of conjecture (0.1). Then the Galois representation ρ_g, when restricted to an inertia group at p in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, splits as the direct sum of the trivial character and the character χ^{k-1}.*

Proof. It suffices to prove this for each factor $\rho_{g,\wp} = \rho_{f_\wp}$ of ρ_g. Since $\rho_{f_\wp}$ is the Galois representation attached to an ordinary p-adic modular form of weight k, when restricted to inertia at p it becomes a reducible representation, which is an extension of the trivial character by the character χ^{k-1} (as was proved for classical forms by Deligne in an unpublished letter to Serre; see [14, prop. 2.3] for a published proof, in the more general context of p-adic Hilbert modular forms). Similarly, $\rho_{F_\wp}$ becomes an extension of the trivial character by $\chi^{-(k-1)}$, when restricted to an inertia group at p. Equation (1.1) shows that $\rho_{f_\wp}$ thus also restricts to an extension of the character χ^{k-1} by the trivial character. Since $k \neq 1$, so that χ^{k-1} is distinct from the trivial character, these two descriptions of $\rho_{f_\wp}$ as an extension prove the proposition. □

We now prove conjecture (0.1) in the case of weight two.

Theorem (1.3). *Let N be a positive integer, p a prime not dividing N, g a classical normalized Hecke eigenform of weight two on $\Gamma_1(N)$, and E the finite extension of $\mathbb{Q}$ generated by the coefficients of g. Suppose that for every prime $\wp$ of E lying over p, the form $g_\wp$ is ordinary, and the slope one form $f_\wp$ lies in the image of $\theta : S_0^\dagger(N) \to S_2^\dagger(N)$. Then g is a CM form on $\Gamma_1(N)$.*

Proof. Since we are in the case $k=2$, the modular form g determines an abelian variety factor A of the Jacobian of $X_1(N)$ defined over $\mathbb{Q}$, of dimension equal to the degree of E over $\mathbb{Q}$, and equipped with an embedding $E \to \mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$.

Let $T_p(A)$ denote the p-adic Tate module of A, and write $V_p(A) = T_p(A)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Then $V_p(A)$ is a naturally a two-dimensional E-vector space, and as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, is isomorphic to ρ_g. Proposition (1.2) implies that we have the isomorphism

$$V_p(A) \xrightarrow{\sim} E\otimes_{\mathbb{Q}}\mathbb{Q}_p \oplus E\otimes_{\mathbb{Q}}\mathbb{Q}_p(1) \tag{1.4}$$

after restricting to an inertia group at p. Replacing A by an isogenous abelian variety if necessary, we may assume that the ring of integers $\mathcal{O}_E$ of E acts on A as a ring of endomorphisms, and that the isomorphism (1.4) arises from an isomorphism

$$T_p(A) \xrightarrow{\sim} \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p \oplus \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p(1) \tag{1.5}$$

after restricting to an inertia group at p.

Let $A[p^\infty]$ denote the p-divisible group of A. Then the isomorphism (1.5) yields a splitting

$$A[p^\infty] \xrightarrow{\sim} \mathcal{O}_E \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p) \oplus \mathcal{O}_E \otimes_{\mathbb{Z}} \hat{\mathbb{G}}_m \tag{1.6}$$

after restricting the p-divisible group $A[p^\infty]$ to the maximal unramified extension of $\mathbb{Q}_p$.

The abelian variety A has good reduction at p, since p is prime to N. Let $\mathcal{A}$ denote the Néron model of A over $W(\overline{\mathbb{F}}_p)$. Tate's theorem [24, thm. 4] implies that the factorization (1.6) induces an analogous splitting of the the p-divisible group $\mathcal{A}[p^\infty]$ of $\mathcal{A}$:

$$\mathcal{A}[p^\infty] \xrightarrow{\sim} \mathcal{O}_E \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p) \oplus \mathcal{O}_E \otimes_{\mathbb{Z}} \hat{\mathbb{G}}_m.$$

Thus $\mathcal{A}$ is the canonical lift of its special fibre A_0 [19, proof of thm. V.3.3], and is endowed with an $\mathcal{O}_E$-linear endomorphism F lifting the Frobenius endomorphism of the special fibre.

The morphism F is *not* given by an element of $\mathcal{O}_E$, since its kernel is entirely contained in (indeed, is equal to the p-torsion subgroup of) $\hat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathcal{O}_E$. Thus $\mathcal{O}_E[F]$ is a commutative $\mathbb{Z}$-algebra of endomorphisms of A of rank equal to twice the dimension of A.

If $\mathcal{O}_E[F]$ is an integral domain, then A is endowed with endomorphisms by an order in a field of twice its dimension, and so is a CM abelian variety. Consequently [23, prop. 1.6] implies that g is a CM modular form.

If $\mathcal{O}_E[F]$ is not an integral domain, then we may use it to cut out a proper abelian subvariety B of A which is closed under the action of $\mathcal{O}_E$. Then this abelian subvariety has CM (since $[E:\mathbb{Q}] = \dim(A) > \dim(B)$), and again we conclude that g is a CM form [23, prop. 1.6]. This completes the proof of the theorem. □

2. The p-adic variational Hodge conjecture

In order to state our conjecture in p-adic Hodge theory, we introduce some notation. Let k be a perfect field in characteristic p, let W denote the Witt ring of k, and let K denote the fraction field of W.

Let $\mathcal{X}$ be a smooth proper W-scheme, and let Z_0 be a codimension n-cycle in the special fibre $\mathcal{X}_0$ of $\mathcal{X}$. (Let us remark that we will always take our cycles to have coefficients in $\mathbb{Q}$, thus avoiding any problems with torsion.) Corresponding to Z_0 is a cycle class c_0 in the crystalline cohomology group $H^{2n}_{crys}(\mathcal{X}_0, K) := H^{2n}_{crys}(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0/W}) \otimes_W K$. Let X denote the generic fibre of $\mathcal{X}$, which is a smooth and proper K-scheme. The comparison between crystalline and de Rham cohomology [2] yields an isomorphism

$$H^{2n}_{crys}(\mathcal{X}_0, K) \xrightarrow{\sim} H^{2n}_{dR}(X, K). \tag{2.1}$$

Suppose that Z_0 is the specialization of a cycle Z on X. The cycle class c of Z lies in the n^{th} filtered piece of $H^{2n}_{dR}(X,K)$ (where the filtration on this vector space is of course the Hodge filtration), and under the isomorphism (2.1), c_0 is identified with c. (See [9] for the construction of cycle classes in crystalline cohomology, and a proof of this last claim.) We conjecture that the converse is true.

Conjecture (2.2). *Suppose that Z_0 is a codimension n cycle on the special fibre $\mathcal{X}_0$ of the smooth proper W-scheme $\mathcal{X}$, such that under the isomorphism (2.1) the class c_0 lies in the n^{th} filtered piece of $H^{2n}_{dR}(X,K)$. Then there is a cycle Z on X which specializes to Z_0.*

This conjecture is an obvious p-adic analogue of the variational Hodge conjecture discussed in [12].

Remark (2.3) (a) Conjecture (2.2) is known in the case when $n=1$ [3, thm. 3.8]. The method of proof is analogous to that used by Kodaira and Spencer to prove the classical Lefschetz $(1,1)$-theorem [17].

(b) Conjecture (2.2) is also known in the special case when $\mathcal{X}$ is the product of abelian varieties $\mathcal{A}$ and $\mathcal{B}$, and Z_0 is the graph of a morphism between the special fibres $\mathcal{A}_0$ and $\mathcal{B}_0$ [3, thm. 3.15]. (As the proof of this result makes clear, this is a crystalline incarnation of the Serre-Tate deformation theory of morphisms of abelian varieties.)

Remark (2.4) We will elaborate on conjecture (2.2) in the particular case when Z_0 is a correspondence between the special fibres $\mathcal{X}_0$ and $\mathcal{Y}_0$ of two smooth proper W-schemes $\mathcal{X}$ and $\mathcal{Y}$. That is, Z_0 is a cycle on the product $\mathcal{X}_0\times\mathcal{Y}_0$, of codimension $n+dim(\mathcal{X}_0)$, say.

The Künneth formula and Poincaré duality together show that

$$\begin{aligned} &H^{2(dim\mathcal{X}_0+n)}_{crys}(\mathcal{X}_0\times\mathcal{Y}_0,K)\\ &\qquad=\oplus_i H^{2(dim\mathcal{X}_0+n)-i}_{crys}(\mathcal{X}_0,K)\otimes H^i_{crys}(\mathcal{Y}_0,K)\\ &\qquad\qquad=\oplus_i \operatorname{Hom}(H^{i-2n}_{crys}(\mathcal{X}_0,K),H^i_{crys}(\mathcal{Y}_0,K)),\end{aligned}$$

and the cycle class $c(Z_0)$ decomposes as a direct sum $c(Z_0)=\oplus_i c_i(Z_0)$, where each

$$c_i(Z_0)\in\operatorname{Hom}(H^{i-2n}_{crys}(\mathcal{X}_0,K),H^i_{crys}(\mathcal{Y}_0,K))$$

is the map on cohomology induced by the correspondence Z_0. In particular, if Z_0 is the graph Γ_f of a morphism $f:\mathcal{Y}_0\to\mathcal{X}_0$ of k-schemes (so that $n=0$), then each $c_i(\Gamma_f)$ is the map on degree i cohomology induced by f.

Let X and Y denote the generic fibres of X and Y respectively. The Künneth formula and Poincaré duality are equally valid for the de Rham cohomology of $X\times Y$ (regarded as filtered K-vector spaces), and are respected by the isomorphism (2.1). Hence we see that $c(Z_0)$ lies in the $(dimX+n)^{th}$ piece of the Hodge filtration on $H^{2(dimX+n)}_{dR}(X\times Y,K)$ if and only if each $c_i(Z_0)$ is of degree n with respect to the Hodge filtration on its source and target. In particular, if $Z_0=\Gamma_f$, we see that the hypothesis of conjecture (2.2) is satisfied for $c(\Gamma_f)$ if and only if the maps induced by f on cohomology respect the Hodge filtration on their source and target.

It is useful (indeed, crucial for the applications to conjecture (0.1)) to observe that conjecture (2.2) has a reformulation as a statement generalizing remark (2.4) to the context of motives (statement (ii) of proposition (2.7) below). We now explain this.

We work in the setting of homological equivalence motives (also called Grothendieck motives) over K admitting good reduction over the ring W. It is not our intention to attempt to expound a complete theory of such objects. We recall as much as we need of this theory in order to explain the statement for motives that we need in order to deduce conjecture (0.1) from conjecture (0.2). We refer to [16, 18, 22] for a more detailed discussions of motives, and to [9, rem. B.3.8] for a discussion of motives over K with good reduction over W.

A homological equivalence motive M over K is a triple (X, π, n) consisting of a smooth proper K-scheme X, a correspondence π from X to itself, and an integer n. The correspondence π is required to induce an idempotent endomorphism of the total cohomology of X. (As to what cohomology theory we are using, see the following remark.) The motive $M = (X, \pi, n)$ is to be thought of as the n^{th} Tate twist of the image of π acting on the total cohomology of n. In particular, we define the total cohomology of M to be equal to be the graded vector space obtained as the n^{th} Tate twist of the kernel of the total cohomology of X. We say that M has good reduction over W if X admits an extension to a smooth proper W-scheme $\mathcal{X}$,

Remark (2.5) We can use any of a number of cohomology theories and obtain the same notion of motive: for example, $\mathbb{Q}_\ell$-adic étale cohomology of X for any prime ℓ, or the de Rham cohomology of X. To see that these do give the same notion of motive, one uses the comparison isomorphisms afforded by the existence of topological singular cohomology on X (together with the Lefschetz principle). Note that these comparison isomorphisms are compatible with the action of correspondences.

We can also use cohomology theories attached to the special fibre of $\mathcal{X}$; for example $\mathbb{Q}_\ell$-adic étale cohomology (for $\ell \neq p$), or crystalline cohomology. To see that these give the same notion of motive, one uses the proper base change and local acyclicity of smooth morphisms to compare $\mathbb{Q}_\ell$-adic cohomology on the generic and special fibres of $\mathcal{X}$ (assuming $\ell \neq p$), and the comparison isomorphism (2.1) between the de Rham cohomology of the general fibre and the crystalline cohomology of the special fibre of $\mathcal{X}$. Although it is redundant for the purposes of this remark, one also has the p-adic comparison isomorphism between p-adic étale cohomology of the geometric generic fibre and the crystalline cohomology of the special fibre [7, thm. 5.6].

In the case that the cohomology theory that we use is computed on the special fibre of X, we should also note that the correspondence π can be specialized to this special fibre, and thus induces an endomorphism on cohomology in this case also. Furthermore, the comparison isomorphisms between cohomology on the generic and special fibres of $\mathcal{X}$ are compatible with respect to specialization and the action of correspondences.

If $M = (X, \pi, n)$ is a homological equivalence motive over K, admitting good reduction over W, then we define the special fibre M_0 of M to be the homological equivalence motive over k given by the triple $(\mathcal{X}_0, \pi_0, n)$ (here π_0 is the specialization of the correspondence π to the special fibre $\mathcal{X}_0$ of the smooth proper $\mathcal{X}$ which extends X). Although $\mathcal{X}$ may not be uniquely determined, the motive M_0 is well-defined up to isomorphism in the category of homological equivalence motives over k. (If $\mathcal{X}'$ is another smooth proper W-scheme extending X, then the graph of the identity in the generic fibre of $\mathcal{X} \times \mathcal{X}'$ specializes to a correspondence from $\mathcal{X}_0$ to $\mathcal{X}_0'$ which induces an isomorphism on cohomology.)

If M is a homological equivalence motive over K, admitting good reduction over W, we define $H^*_{w.a.}(M, K)$ to be the crystalline cohomology of the special fibre $H^*_{crys}(M_0, K)$, equipped with the filtration induced by the Hodge filtration on $H^*_{dR}(M, K)$ and the isomorphism (2.1). (The subscript *w.a.* is for *weakly admissible*, and is justified by the fact that $H^*_{w.a.}(M, K)$, equipped with its Frobenius endomorphism and filtration, is a weakly admissible filtered ϕ-module in the sense of [8, §4.4]; this follows from the comparison theorem of [7, thm. 5.6], together with [8, prop. 5.4.2 (i)].)

Now suppose given a pair of homological equivalence motives M and M' over K, both having good reduction over W. Let Z_0 be a morphism from the special fibre M_0 of M to the special fibre M_0' of the motive M'. Then Z_0 induces a morphism

$$c_0^*(Z_0) : H^*_{crys}(M_0) \to H^*_{crys}(M_0'),$$

and hence a morphism

$$c_0^*(Z_0) : H^*_{w.a.}(M_0) \to H^*_{w.a.}(M_0'), \tag{2.6}$$

which respects the Frobenius endomorphisms of these filtered ϕ-modules, but which may not respect their filtrations.

If Z_0 is the specialization of a morphism $Z : M \to M'$, then (2.6) certainly will respect the filtrations. The following proposition discusses the converse of this statement.

Proposition (2.7). *Conjecture (2.2) is equivalent to the following statement:*

If $Z_0 : M_0 \to M_0'$ is a morphism between the special fibres of two homological equivalence motives M and M' over K, both of which admit good reduction over W, for which the morphism (2.6) respects the filtrations on its source and target, then there is a morphism of motives over K, $Z : M \to M'$, which specializes to Z_0.

Proof. Let us suppose first that the given statement holds, and see that conjecture (2.2) follows. For this, it suffices to note that if $\mathcal{X}$ is a smooth proper variety over W, then giving a codimension n cycle on the special (respectively, the generic) fibre of $\mathcal{X}$ is the same as giving a morphism of the special (respectively, the generic) fibre of the motive $\mathbb{Q}(-n)$ to the motive of $\mathcal{X}_0$ (respectively, the motive of X). (Recall that $\mathbb{Q}(-n)$ denotes the motive given by the triple $(\operatorname{Spec} K, \mathrm{id}, -n)$.)

Now suppose that conjecture (2.2) holds. Let $Z_0 : M_0 \to M_0'$ be a morphism of the special fibres of the motives M and M', and suppose that the morphism (2.6) preserves the filtrations on its source and target. Suppose that $M = (X, p, m)$ and $M' = (Y, q, n)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be extensions of X and Y respectively to smooth proper schemes over W. By definition, there is a correspondence U_0 on $\mathcal{X}_0 \times \mathcal{Y}_0$, of codimension $n - m + \dim(X_0)$, such that $q_0 \circ U_0$ and $U_0 \circ p_0$ are both homologically equivalent to U_0, and such that Z_0 is induced by the correspondence U_0. The assumption that (2.6) preserves the filtrations on its source and target is equivalent (as the discussion of remark (2.4) shows) to the assumption that $c(U_0)$ lies in the $(n - m + \dim(X_0))^{th}$ piece of the Hodge filtration of $H_{dR}^{2(n-m+\dim(X_0))}(X \times Y, K)$. Conjecture (2.2) implies that there is a cycle U on $X \times Y$ specializing to U_0. Now take $Z : M \to M'$ to be the morphism induced by the correspondence U. $\square$

Let us fix a motive M over K, admitting good reduction over W. Let σ denote the canonical Frobenius automorphism of K. Then we may pull-back M along σ to obtain the motive M^{σ} over K, which again admits good reduction over W (since σ restricts to an automorphism of W). There is a natural morphism from the special fibre of M^{σ} to the special fibre of M, given by the relative Frobenius morphism.

Lemma 2.8. *Let M be a homological equivalence motive over K, which admits good reduction over W. The morphism $H^i_{w.a.}(M^{\sigma}, K) \to H^i_{w.a.}(M, K)$ induced by the relative Frobenius map $M_0^{\sigma} \to M_0$ respects the filtrations on its source and target if and only if the same is true of the canonical Frobenius endomorphism of $H^i_{w.a.}(X, K)$.*

Proof. This follows by the usual comparison between relative and absolute Frobenius. For notational simplicity we treat the case when M is the motive corresponding to the generic fibre of a smooth proper W scheme $\mathcal{X}$. The general case follows from this. The Frobenius endomorphism on $H^i_{w.a.}(X, K)$ is determined by the isomorphism $H^i_{w.a.}(X, K) \cong H^i_{crys}(\mathcal{X}_0, K)$, and the endomorphism of the latter cohomology group induced by the absolute Frobenius endomorphism of $\mathcal{X}_0$. This absolute Frobenius endomorphism can be factored as the composite

$$\mathcal{X}_0 \xrightarrow{F_{\mathcal{X}_0/k}} \mathcal{X}_0^{(p)} \xrightarrow{\sim} \mathcal{X}_0,$$

where the second arrow is the base-change of the isomorphism $k \xrightarrow{\sim} k$ given by absolute Frobenius. Thus we can think of the Frobenius endomorphism ϕ of $H^i_{w.a.}(X, K)$ as being the composite of the map (of K-vector spaces; it will generally *not* respect the filtrations on its source and target) $H^i_{w.a.}(X^{(\sigma)}, K) \to H^i_{w.a.}(X, K)$ induced by the relative Frobenius $F_{\mathcal{X}_0/k}$ and the isomorphism $H^i_{w.a.}(X, K) \to H^i_{w.a.}(X^{(\sigma)}, K)$ determined by the isomorphism $X^{(\sigma)} \xrightarrow{\sim} X$ obtained by base-changing the automorphism σ of K, which is σ-linear and which does preserve the filtrations. It follows that ϕ preserves Hodge filtrations if and only if the same is true of the K-linear endomorphism induced by the relative Frobenius of $\mathcal{X}_0$. The lemma follows from this discussion. □

3. HIGHER WEIGHT FORMS.

In this section we explain how conjecture (2.2) implies conjecture (0.1) for modular forms of arbitrary weight $k \geq 2$.

Theorem (3.1). *Let N be a positive integer, p a prime not dividing N, g a classical normalized Hecke eigenform of weight $k \geq 2$ on $\Gamma_1(N)$, and E the finite extension of $\mathbb{Q}$ generated by the coefficients of g. Suppose that for every prime $\wp$ of E lying over p, the form $g_{\wp}$ is ordinary, and the slope $k-1$ form $f_{\wp}$ lies in the image of $\theta : S^{\dagger}_{-(k-2)}(N) \to S^{\dagger}_k(N)$. Then conjecture (2.2) implies that g is a CM form on $\Gamma_1(N)$.*

Proof. Scholl [21] explains how to attach to g a homological equivalence motive M defined over $\mathbb{Q}$ equipped with an action of E which makes it of rank two over E, having good reduction modulo p (when regarded as a motive over the fraction field $\mathbb{Q}_p$ of $\mathbb{Z}_p = W(\mathbb{F}_p)$), and whose associated p-adic Galois representation is exactly ρ_g.

The crystalline comparison theorem [7, thm. 5.6] shows that the restriction of ρ_g to a decomposition group at p is a crystalline Galois representation (in the sense of [8, 5.1.4]), and that $H^*_{w.a.}(M)$ is the crystalline Dieudonné-module of ρ_g. The graded pieces of the Hodge filtration of $H^*_{w.a.}(M)$ are supported in degrees zero and $k-1$, and each is free of rank one over $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Proposition (1.2) shows that the restriction of ρ_g to an inertia group at p splits as a direct sum of two distinct characters (the trivial character and χ^{k-1}). It follows that $H^*_{w.a.}(M)$ also splits as a direct sum

$$H^*_{w.a.} = H_0 \oplus H_{k-1},$$

with each of H_0, H_{k-1} free of rank one over $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$, such that Hodge filtration of H_0 (respectively, of H_{k-1}) is supported in degree zero (respectively, degree $k-1$).

As a corollary, we see that the Frobenius endomorphism of $H^*_{w.a}(M)$ must respect the Hodge filtration of $H^*_{w.a.}(M)$. Lemma (2.8) then implies that the graph of the Frobenius endomorphism of M_0 preserves the Hodge filtration on $H^*_{w.a.}(M)$. Assuming that conjecture (2.2) holds, we find (by appealing to the equivalent form of it given by proposition (2.7)) that there is a morphism Z of M which lifts the Frobenius endomorphism of M_0. This morphism (that is, the underlying cycle that gives rise to it) is defined *a priori* only over $\mathbb{Q}_p$, but can in fact be assumed to be defined over some finite extension L of $\mathbb{Q}$, since M is defined over $\mathbb{Q}$, and since specializing cycles preserves their homological equivalence class.

To complete the argument, it is convenient to work with a field of coefficients (rather than the ring of coefficients $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$). Corresponding to the factorization $E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \prod_{\wp | p} E_{\wp}$, we have factorizations $\rho_g \xrightarrow{\sim} \prod_{\wp|p} \rho_{g,\wp}$ and $H^*_{w.a.}(M) \xrightarrow{\sim} \prod_{\wp|p} H^*_{w.a.}(M)_{\wp}$. For each $\wp$, the weakly admissible module $H^*_{w.a.}(M)_{\wp}$ corresponds via the crystalline comparison theorem to the restriction of $\rho_{g,\wp}$ to the decomposition group at p, and for each $\wp$ we also have an isomorphism $H^*_{w.a.}(M)_{\wp} \xrightarrow{\sim} H_{0,\wp} \oplus H_{k-1,\wp}$.

Now the morphism Z of M over L induces an endomorphism of each $\rho_{g,\wp}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$. We claim that this endomorphism is non-scalar. For if it were scalar, then (again by applying the crystalline comparison theorem) we would see that the Frobenius endomorphism of $H^*_{w.a.}(M)_{\wp}$ would be scalar. But this is not possible, because the Frobenius endomorphism must act on $H_{0,\wp}$ with slope zero and on $H_{k-1,\wp}$ with slope $k-1$ (since each of these filtered ϕ-modules is weakly admissible). (This is a particularly transparent case of [5, thm 3.1].)

We conclude that each $\rho_{g,\wp}$ is *not* absolutely irreducible when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$, since the endomorphism induced by Z will have at least one *one*-dimensional eigenspace defined over $\overline{E}_{\wp}$ (since it is non-scalar) and this will be a $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-closed subspace of $\rho_{g,\wp}$. It follows from [20, prop. 4.4, thm. 4.5] that g is a CM modular form. (An examination of the arguments shows that [20, prop. 4.4] remains true if one replaces "irreducible" with "absolutely irreducible" in its statement.) This shows that conjecture (2.2) implies conjecture (0.1) for arbitrary weights $k \geq 2$. □

Note that the proof of theorem (1.3) is really just a specialization of this argument to the case of $k = 2$, taking advantage of remark (2.3) (b) to know that the relevant case of conjecture (2.2) is satisfied, and using the structure theory of the p-divisible groups of abelian varieties and [24, thm. 4] to replace the general comparison theorem.

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