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ON THE INTERPOLATION OF SYSTEMS OF EIGENVALUES ATTACHED TO AUTOMORPHIC HECKE EIGENFORMS

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To Barry Mazur, for his 65th birthday

Contents

The goal of this paper is to illustrate how the techniques of locally analytic p-adic representation theory (as developed in [28, 29, 30, 31] and [13, 14, 17]; see also [16] for a short summary of some of these results) may be applied to study arithmetic properties of automorphic representations. More specifically, we consider the problem of p-adically interpolating the systems of eigenvalues attached to automorphic Hecke eigenforms (as well as the corresponding Galois representations, in situations where these appear in the étale cohomology of Shimura varieties). We can summarize our approach to the problem as follows: rather than attempting to directly interpolate the systems of eigenvalues attached to eigenforms, we instead attempt to interpolate the automorphic representations that these eigenforms give rise to.

To be more precise, we fix a connected reductive linear algebraic group $\mathbb{G}$ defined over a number field F, and a finite prime $\mathfrak{p}$ of F. We let $F_{\mathfrak{p}}$ denote the completion of F at $\mathfrak{p}$, let E be a finite extension of $F_{\mathfrak{p}}$ over which the group $\mathbb{G}$ splits, let $\mathbb{A}$ denote the ring of adèles of F, and let $\mathbb{A}_f$ denote the ring of finite adèles of F. The representations that we construct are admissible locally analytic representations of the group $\mathbb{G}(\mathbb{A}_f)$ on certain locally convex topological E-vector spaces. These representations are typically not irreducible; rather, they contain as closed subrepresentations many locally algebraic representations of $\mathbb{G}(\mathbb{A}_f)$ which are closely related to automorphic representations of $\mathbb{G}(\mathbb{A})$ of cohomological type. (It is for this reason that we regard the representations that we construct as forming an "interpolation" of those automorphic representations.)

Once we have our locally analytic representations of $\mathbb{G}(\mathbb{A}_f)$ in hand, we may apply to them the Jacquet module functors of [14]. In this way we obtain $\mathfrak{p}$-adic analytic families of systems of Hecke eigenvalues, which (under a suitable hypothesis, for which see the statement of Theorem 0.7 below) $\mathfrak{p}$-adically interpolate (in the

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traditional sense!) the systems of Hecke eigenvalues attached to automorphic representations of cohomological type. More precisely, the resulting families interpolate the systems of Hecke eigenvalues attached to those automorphic representations which contribute to the cohomology of the arithmetic quotients associated to $\mathbb{G}$, and whose local factor at $\mathfrak{p}$ embeds in a principal series representation. (This last property is a representation theoretic interpretation of the finite slope condition appearing in [11].)

Most of our efforts in this paper are devoted to constructing the representations of $\mathbb{G}(\mathbb{A}_f)$ alluded to above, and to establishing their basic properties; considerably less space is devoted to the detailed analysis of the $\mathfrak{p}$-adic families that we obtain by applying the Jacquet module functors to these representations. However, we do consider the case of $\mathrm{GL}_{2/\mathbb{Q}}$ in enough detail to explain how our results, in that particular case, lead to a new construction of the eigencurve of [11], valid for all primes and arbitrary tame level. (In [11] the authors restrict themselves to odd primes and trivial tame level.) We also indicate how the construction of this paper, when combined with the results of the forthcoming paper [17], may be used to construct a two-variable p-adic L-function parameterized by the eigencurve.

The arrangement of the paper is as follows: In Section 1 we extend some of the definitions and results of [25, V.2.4] (which compares the group cohomology and Lie algebra cohomology of finite dimensional representations of locally p-adic analytic groups) to the context of infinite dimensional representations. Section 2 presents our main constructions and results. In Section 3 we describe how to produce locally analytic $\mathfrak{p}$-adic representations out of certain classical automorphic representations, thus providing a bridge between the classical theory of automorphic representations and the locally analytic representation theory that underlies the methods of this paper. Related to this, we interpret our results in the case of a group that is compact at infinity in terms of the notion of "locally analytic $\mathfrak{p}$-adic automorphic forms". Section 4 is devoted to the example of $\mathrm{GL}_{2/\mathbb{Q}}$. In the remainder of the introduction, we briefly sketch the constructions and results of Section 2, and also comment on the relation between our work and other recent work related to the p-adic interpolation of automorphic forms.

Cohomology of arithmetic quotients. Let K_∞ denote a fixed maximal compact subgroup of the real group $\mathbb{G}(\mathbb{R}\otimes_{\mathbb{Q}} F)$, and let K_∞° be its connected component of the identity. Let π_0 denote the quotient K_∞/K_∞°. For any compact open subgroup K_f of $\mathbb{G}(\mathbb{A}_f)$ we write

$$Y(K_f) := \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K_\infty^\circ K_f.$$

If K_f is sufficiently small, and if K_f' is a normal open subgroup of K_f, then the natural map $Y(K_f') \to Y(K_f)$ is a Galois covering map, with group of deck transformations isomorphic to the quotient K_f/K_f'.

Let $\mathbb{A}_f^{\mathfrak{p}}$ denote the prime-to-$\mathfrak{p}$ part of the ring of finite adèles of F, so that $\mathbb{A}_f = F_{\mathfrak{p}} \times \mathbb{A}_f^{\mathfrak{p}}$. We are particularly interested in K_f of the form $K_f = K_{\mathfrak{p}}K^{\mathfrak{p}}$, where $K_{\mathfrak{p}}$ is a variable compact open subgroup in $\mathbb{G}(F_{\mathfrak{p}})$, and $K^{\mathfrak{p}}$ is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$, fixed for the moment. (We refer to such a subgroup $K^{\mathfrak{p}}$ as a "tame level".) If we allow $K_{\mathfrak{p}}$ to shrink to the identity, while keeping the tame level fixed, the spaces $Y(K_f)$ form a projective system, equipped with an action of $\pi_0 \times \mathbb{G}(F_{\mathfrak{p}})$. Passing to cohomology with coefficients in a ring A yields an inductive system of A-modules, equipped with an A-linear action of $\pi_0 \times \mathbb{G}(F_{\mathfrak{p}})$.

Taking A to be $\mathcal{O}_E/p^s$ for some natural number $s > 0$ (here $\mathcal{O}_E$ denotes the ring of integers of E) we define $H^n(K^{\mathfrak{p}}, \mathcal{O}_E/p^s) := \varinjlim_{K_{\mathfrak{p}}} H^n(Y(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{O}_E/p^s)$. We then obtain an E-Banach space $\tilde{H}^n(K^{\mathfrak{p}})$, equipped with a continuous action of $\pi_0 \times \mathbb{G}(F_{\mathfrak{p}})$, by passing to the projective limit in s, and tensoring with E:

$$\tilde{H}^n(K^{\mathfrak{p}}) := E \otimes_{\mathcal{O}_E} \varprojlim_s H^n(K^{\mathfrak{p}}, \mathcal{O}_E/p^s).$$

Let $K^{\mathfrak{p}\prime} \subset K^{\mathfrak{p}}$ be an inclusion of tame levels. We obtain a corresponding $\pi_0 \times \mathbb{G}(F_{\mathfrak{p}})$-equivariant closed embedding of E-Banach spaces $\tilde{H}^n(K^{\mathfrak{p}}) \to \tilde{H}^n(K^{\mathfrak{p}\prime})$. Passing to the locally convex inductive limit over all tame levels yields a locally convex E-vector space $\tilde{H}^n$ equipped in a natural way with an action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$.

In the following results, we will apply some terminology introduced in [13, §7] to the group $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$. When applying this terminology, we will regard $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ as being factored as the product of the locally $F_{\mathfrak{p}}$-analytic group $\mathbb{G}(F_{\mathfrak{p}})$ and the locally compact group $\pi_0 \times \mathbb{G}(\mathbb{A}_f^p)$.

The following result (which follows from Theorem 2.2.16 below) describes the basic properties of the $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation $\tilde{H}^n$.

Theorem 0.1. (*i*) *The representation $\tilde{H}^n$ is an admissible continuous representation of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ (in the sense of* [13, Def. 7.2.1]).

(*ii*) *For any tame level $K^{\mathfrak{p}}$, the Banach space $\tilde{H}^n(K^{\mathfrak{p}})$ is recovered as the subspace of $K^{\mathfrak{p}}$-invariants in $\tilde{H}^n$.*

Our next result describes the result of passing to the locally L-analytic vectors of the representation $\tilde{H}^n$, for any local field L intermediate between $F_{\mathfrak{p}}$ and $\mathbb{Q}_p$. It is a consequence of Theorem 0.1 and the theory of [13, §7].

Theorem 0.2. (*i*) *The representation $\tilde{H}^n_{L-\mathrm{la}}$ is an admissible locally L-analytic representation of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ (in the sense of* [13, Def. 7.2.7]).

(*ii*) *For any tame level $K^{\mathfrak{p}}$, the natural map $\tilde{H}^n(K^{\mathfrak{p}})_{L-\mathrm{la}} \to (\tilde{H}^n_{L-\mathrm{la}})^{K^{\mathfrak{p}}}$ from the space of locally L-analytic vectors of the $\mathbb{G}(F_{\mathfrak{p}})$-representation $\tilde{H}^n(K^{\mathfrak{p}})$ to the space of $K^{\mathfrak{p}}$-invariants in $\tilde{H}^n_{L-\mathrm{la}}$ is an isomorphism.*

If W is a finite dimensional representation of $\mathbb{G}$ over E, then there is a local system of E-vector spaces $\mathcal{V}_W$ defined on each of the spaces $Y(K_f)$ (which is the trivial local system when W is the trivial representation), and we may form the E-vector space $H^n(\mathcal{V}_W) := \varinjlim_{K_f} H^n(Y(K_f), \mathcal{V}_W)$, which is equipped with a smooth action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$. It is in fact an admissible smooth representation of this group (as is well known); indeed the space of K_f-invariants in $H^n(\mathcal{V}_W)$ is precisely $H^n(Y(K_f), \mathcal{V}_W)$, and so is finite dimensional.

Let $\check{W}$ denote the contragredient representation to W. We equip the tensor product $H^n(\mathcal{V}_W) \otimes_E \check{W}$ with an action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ induced by the tensor product action of $\mathbb{G}(F_{\mathfrak{p}})$, and the action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ on the first factor. This tensor product is then an admissible locally algebraic representation of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ (by [13, Prop. 7.2.16]), and we show that there is a natural $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant map

$$H^n(\mathcal{V}_W) \otimes_E \check{W} \to \tilde{H}^n. \tag{0.3}$$

Since the $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-action on $H^n(\mathcal{V}_W)$ is smooth, giving such a map is equivalent to giving a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant map

$$H^n(\mathcal{V}_W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n_{L-\mathrm{la}}) \tag{0.4}$$

(where $\mathfrak{g}$ denotes the Lie algebra of $\mathbb{G}$). When the map (0.4) is an isomorphism, it yields an intrinsic representation theoretic description of the smooth $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation $H^n(\mathcal{V}_W)$, in terms of the locally analytic representation $\tilde{H}^n_{L-\mathrm{la}}$ and the given finite dimensional representation W. Furthermore, the map (0.3) then identifies $H^n(\mathcal{V}_W) \otimes_E \check{W}$ with the subspace of locally $\check{W}$-algebraic vectors in $\tilde{H}^n$. We are able to give some examples for which (0.4) is an isomorphism (see the discussion below); in general, we establish the following result.

Theorem 0.5. *If W is a finite dimensional representation of $\mathbb{G}$ defined over E, with contragredient $\check{W}$, then the map (0.4) (with $L = \mathbb{Q}_p$) is the edge map of a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathfrak{g}}(\check{W}, \tilde{H}^j_{\mathbb{Q}_p-\mathrm{la}}) \implies H^{i+j}(\mathcal{V}_W).$$

This result is a restatement of Corollary 2.2.18.

In this paper we consider two examples for which the higher Ext terms appearing in the spectral sequence of Theorem 0.5 can be shown to vanish, and thus for which (0.4) is an isomorphism (for any choice of L as above). One is the case when (the restriction of scalars to $\mathbb{Q}$ of) $\mathbb{G}$ satisfies the conditions of [19, Prop. 1.4] (the spaces $Y(K_f)$ are then the union of finitely many contractible connected components, and so we naturally take $n = 0$); the other is the case of $\mathrm{GL}_{2/\mathbb{Q}}$ (for which we take $n = 1$). In such situations, we may regard the locally analytic representation $\tilde{H}^n_{F_{\mathfrak{p}}-\mathrm{la}}$ as providing a $\mathfrak{p}$-adic interpolation of the representations $H^n(\mathcal{V}_W)$ – or more precisely, of the locally algebraic representations $H^n(\mathcal{V}_W) \otimes_E \check{W}$, since it is these representations that are then identified by (0.3) with locally algebraic subrepresentations of $\tilde{H}^n_{F_{\mathfrak{p}}-\mathrm{la}}$.

Let us remark that we also establish analogues of Theorems 0.1, 0.2 and 0.5 with cohomology replaced by compactly supported cohomology.

Eigenvarieties. Suppose now that $\mathbb{G}$ is quasi-split over $F_{\mathfrak{p}}$. We fix a Levi factor $\mathbb{T}$ in a Borel subgroup $\mathbb{B}$ of $\mathbb{G}_{/F_{\mathfrak{p}}}$ (so that $\mathbb{T}$ is a torus), and let $\hat{T}$ denote the rigid analytic space over E that parameterizes the locally $F_{\mathfrak{p}}$-analytic characters of the abelian locally $F_{\mathfrak{p}}$-analytic group $T := \mathbb{T}(F_{\mathfrak{p}})$. Fix a cohomological degree n, as well as a tame level $K^{\mathfrak{p}}$. Write $\mathcal{H}(K^{\mathfrak{p}}) := \mathcal{H}(\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})//K^{\mathfrak{p}})$ to denote the prime-to-$\mathfrak{p}$ Hecke algebra of level $K^{\mathfrak{p}}$ (over E), and as in Subsection 2.3 below, factor $\mathcal{H}(K^{\mathfrak{p}})$ as $\mathcal{H}(K^{\mathfrak{p}}) := \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}} \otimes_E \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$. Note that $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$, the spherical Hecke algebra of tame level $K^{\mathfrak{p}}$, is a central subalgebra of $\mathcal{H}(K^{\mathfrak{p}})$.

Fix an embedding of E into $\overline{\mathbb{Q}}_p$. Suppose that π_f is an irreducible $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation appearing as a subquotient of $\overline{\mathbb{Q}}_p \otimes_E H^n(\mathcal{V}_W)$, for some irreducible $\mathbb{G}$-representation W (necessarily definable over E, since $\mathbb{G}$ is assumed to split over E), with the property that $\pi_{\mathfrak{p}}$ (the local factor at $\mathfrak{p}$ of π_f) embeds into $\mathrm{Ind}_{\mathbb{B}(F_{\mathfrak{p}})}^{\mathbb{G}(F_{\mathfrak{p}})} \theta$ for some smooth $\overline{\mathbb{Q}}_p$-valued character θ of T, and for which $\pi_f^{K^{\mathfrak{p}}} \neq 0$. Since π_f is irreducible, the spherical Hecke algebra $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ acts on $\pi_f^{K^{\mathfrak{p}}}$ through a $\overline{\mathbb{Q}}_p$-valued character λ, which determines a $\overline{\mathbb{Q}}_p$-valued point of $\mathrm{Spec}\,\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$. Let ψ

denote the highest weight character of $\check{W}$ (with respect to the chosen Borel $\mathbb{B}$), regarded as a character of T, and write $\chi := \theta\psi$. Let $E(n, K^{\mathfrak{p}})_{\mathrm{cl}}$ denote the set of points $(\chi, \lambda) \in \left(\hat{T} \times \operatorname{Spec} \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}\right)(\overline{\mathbb{Q}}_p)$ constructed in this manner from such irreducible subquotients π_f.

Definition 0.6. We define the degree n cohomological eigenvariety of $\mathbb{G}$ of tame level $K^{\mathfrak{p}}$ to be the rigid analytic Zariski closure of $E(n, K^{\mathfrak{p}})_{\mathrm{cl}}$ in $\hat{T} \times \operatorname{Spec} \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$, and denote it by $E(n, K^{\mathfrak{p}})$.

The following theorem is a consequence of the results proved in Subsection 2.3 below.

Theorem 0.7. *If the map (0.4) is an isomorphism for each irreducible representation W of $\mathbb{G}$ (and for our fixed choice of cohomological degree n), then the following are true:*

(i) Projection onto the first factor in the product $\hat{T} \times \operatorname{Spec} \mathcal{H}(K^{\mathfrak{p}})$ induces a finite map $E(n, K^{\mathfrak{p}}) \to \hat{T}$.

(ii) The composite of the finite map $E(n, K^{\mathfrak{p}}) \to \hat{T}$ of (i), and the map $\hat{T} \to \check{\mathfrak{t}}$ given by differentiating characters, has discrete fibres. (Here $\check{\mathfrak{t}}$ denotes the dual to the Lie algebra $\mathfrak{t}$ of T.) In particular, the dimension of $E(n, K^{\mathfrak{p}})$ is at most equal to the dimension of T.

(iii) If (χ, λ) is a point of $E(n, K^{\mathfrak{p}})$ in which the first factor χ is locally algebraic and of non-critical slope (in the sense of [14, Def. 4.4.3]*), then (χ, λ) is in fact a point of $E(n, K^{\mathfrak{p}})_{\mathrm{cl}}$.*

(iv) There is a coherent sheaf $\mathcal{M}$ of right $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}[\pi_0]$-modules over $E(n, K^{\mathfrak{p}})$, whose fibre over any point (χ, λ) of $E(n, K^{\mathfrak{p}})_{\mathrm{cl}}$ of non-critical slope is naturally isomorphic (as an $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}[\pi_0]$-module) to the E-linear dual of the (θ, λ)-eigenspace of the Jacquet module (taken with respect to $\mathbb{B}(F_{\mathfrak{p}})$) of the smooth representation $H^n(K^{\mathfrak{p}}, \mathcal{V}_{\check{W}})$. (Here we have factored $\chi = \theta\psi$ as a product of a smooth and an algebraic character, and written W to denote the irreducible representation of $\mathbb{G}$ whose contragredient has highest weight ψ with respect to $\mathbb{B}$. Also, we have written $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}[\pi_0]$ to denote the group ring of π_0 with coefficients in the algebra $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}$.)

Theorem 0.7 applies in particular to the two examples discussed above. In the case when $\mathbb{G}$ satisfies the conditions of [19, Prop. 1.4], so that the only non-vanishing cohomology is in degree zero, we naturally take $n = 0$. This class of examples is already extremely interesting; for example, taking $\mathbb{G}$ to be a totally definite quaternion algebra over a totally real field, one obtains a construction of eigenvarieties parameterizing systems of Hecke eigenvalues attached to Hilbert modular forms over F. In this case we are able to improve on Theorem 0.7 (ii), to show that in fact $E(n, K^{\mathfrak{p}})$ is equidimensional of dimension equal to the dimension of $\mathbb{T}$.

In the case of $\mathrm{GL}_{2/\mathbb{Q}}$, the important cohomological degree is $n = 1$. We recover the eigencurve of [11] (or more precisely, the surface obtained from it by allowing twists by wild characters at p) as the eigenvariety of trivial tame level. Note that our method imposes no constraints on the tame level, and so in this case we may equally well construct an eigenvariety for arbitrary tame level. (We are also able to include the case $p = 2$, which is omitted in [11]. A construction of the eigencurve for all primes and arbitrary tame level has been given independently by Buzzard [6], using a direct extension of the method of [11].) By working with compactly supported cohomology, and applying the results on Jacquet functors to appear in [17], we can use this point of view on the eigencurve to construct a two-variable p-adic L-function over the eigencurve. This L-function will be constructed as a certain section of (a slight modification of) the coherent sheaf $\mathcal{M}$ of Theorem 0.7 (iv).

We close our summary with an observation that allows us in some instances to interpolate Galois representations as well as systems of Hecke eigenvalues. Suppose that $\mathbb{G}$ is a group for which the arithmetic quotients $Y(K_f)$ (possibly modified in the manner discussed in Subsection 2.4 below) admit the structure of Shimura varieties, with canonical models defined over some number field F'. Then throughout the preceding discussion, we may take "cohomology" to mean "étale cohomology", and thus endow the space $\tilde{H}^n$ with a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$, commuting with the $\mathbb{G}(\mathbb{A}_f)$-action. In the cases when Theorem 0.7 applies, this Galois action will be inherited by the coherent sheaf $\mathcal{M}$, and so $\mathcal{M}$ will be a coherent sheaf of Galois representations over $E(n, K^{\mathfrak{p}})$. For example, in the case of GL_2 over $\mathbb{Q}$, for each tame level we obtain a coherent sheaf of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Other approaches. There are at least two other approaches to the study of eigenvarieties that appear in the literature. Coleman and Mazur's original construction of the eigencurve [11], via the geometric theory of modular forms, has been extended to the contexts of Shimura curves by Kassaei [22] and to the context of Hilbert modular varieties by Kisin and Lai [23].

There is also an approach, the basic idea of which is due to Stevens (generalizing the approach of Hida [20] in the ordinary case), that has been developed by Ash and Stevens (for the group GL_n over $\mathbb{Q}$ [2]), Buzzard (for tori and for definite quaternion algebras over $\mathbb{Q}$ [5], and for totally definite quaternion algebras over totally real fields [6, §§8,9]) and Chenevier (for groups $\mathbb{G}$ over $\mathbb{Q}$ that are compact over $\mathbb{R}$ and isomorphic to GL_n over $\mathbb{Q}_p$ [8]). Stevens' method has in common with the method of this paper that it uses cohomology of arithmetic quotients as a means of achieving the desired p-adic interpolation. (In the case of [5], [6, §§8,9] and [8], the arithmetic quotients that are considered are zero-dimensional. This is the situation that we consider in Subsection 3.2 below.) Nevertheless, it is quite different to the approach taken here.

Here is a sketch of Stevens' method: Take an Iwahori subgroup I of $\mathbb{G}(\mathbb{Q}_p)$, with image Ω in the flag variety of $\mathbb{G}(\mathbb{Q}_p)$, and consider the space of locally analytic distributions on Ω. Out of this space construct a quasi-coherent sheaf $\mathcal{D}$ on "weight space" – that is, the space of locally analytic characters of the intersection $I \bigcap T$. Let Γ be an arithmetic subgroup of $\mathbb{G}(\mathbb{R} \otimes_{\mathbb{Q}} F)$ which embeds into I; then Γ acts on $\mathcal{D}$, and passing to Γ cohomology (in some chosen degree) yields a sheaf over weight space on which analogues of the U_p-operators act. Passing to the finite slope part of this sheaf yields the desired eigenvariety, and a coherent sheaf over it (similar to the coherent sheaf $\mathcal{M}$ appearing in Theorem 0.7 (iv)). In practice, it can be difficult to carry out this process sheaf theoretically over all of weight space, and one can instead restrict to a sufficiently small affine open neighbourhood of a classical point of weight space and work directly over the corresponding Tate algebra. (This is done in [2], for example.)

The basic distinction between our approach and either of the approaches mentioned above is our emphasis on the point of view and methods of representation theory. One advantage of this is that our approach is not specifically tailored to the study of finite-slope automorphic forms. Spaces of overconvergent automorphic forms (which form the basic objects in the geometric approach) and the arithmetic cohomology of the sheaf $\mathcal{D}$ (which forms the basic object in Stevens' approach) both seem to be hopelessly large as Hecke modules until one passes to the finite slope parts. By contrast, the representations $\tilde{H}^n$ that we construct, and their associated spaces of analytic vectors, satisfy reasonable finiteness conditions – namely they are admissible in the appropriate sense. (This is the pay-off for working with the entire group, and not just the Hecke algebra.) Thus it is reasonable to hope that our approach will have applications to the construction of families of forms in the infinite slope (i.e. non-principal series at $\mathfrak{p}$) case also.

The representations $\tilde{H}^n$ that we construct are similar to certain representations constructed by Shimura. In the paper [32] Shimura considers an inductive limit of cohomology of arithmetic quotients of Shimura curves attached to a quaternion algebra over a totally real field F that is split at exactly one infinite place, and at all the primes lying over a fixed prime p (denoted ℓ in [32]), equipped with the natural action of the group $\mathbb{G}(\mathbb{Q}_p)$. (Here $\mathbb{G}$ denotes the restriction of scalars of the quaternion algebra from F to $\mathbb{Q}$.) Using a base-change argument, he is able to establish a strong control theorem, which in the particular case when $\mathbb{G} = \mathrm{GL}_2$, is analogous to the control theorem established by isomorphism (4.3.4) below. One difference between the results of [32] and ours is that Shimura considers cohomology with coefficients in $\mathbb{Q}_p/\mathbb{Z}_p$, whereas we consider cohomology with $\mathbb{Q}_p$-coefficients, completed with respect to the norm induced by the cohomology with $\mathbb{Z}_p$-coefficients. More importantly, in [32] there is no analogue of the passage to locally analytic vectors that plays such an important role in this paper. (I would like to thank Haruzo Hida for drawing my attention to the details of [32].)

Let us close this discussion of alternative methods by pointing out that Stevens' approach lends itself in a natural way to the construction of two-variable p-adic L-functions along the eigencurve (or at least, in the neighbourhood of classical points on the eigencurve) [33]. As we explain in Section 4, our point of view also gives a construction of such two-variable p-adic L-functions. In that section, we will recall Stevens' result more precisely, and compare it with the construction yielded by our representation theoretic approach.

Notation and conventions. We follow closely the notational and terminological conventions introduced in [13] and [14]. In particular, we refer to those papers for the meaning of any technical terms and pieces of notation that are used without explanation in the present text.

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1. Continuous cohomology and Lie algebra cohomology

(1.1) In [13, §4.1] we studied some basic properties of the functor "pass to smooth vectors". In applications, it is necessary to consider not only this functor, but also its derived functors. The object of this subsection is to introduce these derived functors, and to establish those properties of them that will be required for the applications that we have in mind.

We begin by supposing that G is a compact locally $\mathbb{Q}_p$-analytic group. (Equivalently, a compact p-adic analytic group, in the terminology of [25].) We fix a finite extension E of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_E$. Recall that the category of admissible continuous representations of G over E (as defined and studied in [30] and [13, §6.2]; we employ the notation of the latter reference) is abelian.

Lemma 1.1.1. *If G is a compact locally $\mathbb{Q}_p$-analytic group then the abelian category of admissible continuous representations of G over E has enough injectives.*

Proof. Any admissible continuous representation V admits a closed G-equivariant embedding into $\mathcal{C}(G,E)^n$ for some $n \geq 0$, and so it suffices to show that the admissible continuous representation $\mathcal{C}(G,E)^n$ is injective. This easily follows from the fact that its dual space is equal to $\mathcal{D}(G,E)^n$, which is a free module over $\mathcal{D}(G,E)$. $\square$

Since the inclusion $E \to \mathcal{C}(G,E)$ induces an isomorphism $E \xrightarrow{\sim} \mathcal{C}(G,E)^G$ (where the superscript G denotes "G-invariants"), we see that for any admissible continuous G-representation V, the space V^G is finite dimensional.

Definition 1.1.2. If G is a compact locally $\mathbb{Q}_p$-analytic group then we let $H^\bullet(G,-)$ denote the derived functors of the functor "pass to G-invariants" from the category of admissible continuous representations of G over E to the category of finite dimensional E-vector spaces.

That these derived functors exist follows from Lemma 1.1.1. The following proposition guarantees that they agree with the usual continuous cohomology of G.

Proposition 1.1.3. *The derived functors of Definition 1.1.2 agree with the usual cohomology of G computed with continuous cochains.*

Proof. We remark that cohomology with continuous cochains yields a δ-functor on the category of admissible continuous representations of G. (To see that one obtains the required long exact sequences, it is necessary to know that any surjection between objects in our category can be split continuously as a map of topological spaces. However, such a surjection can even be split as a map of topological vector spaces [27, Prop. 10.5].)

Since the derived functor of Definition 1.1.2 and continuous cohomology yield the same H^0 term, to obtain the required isomorphism, it suffices to show that continuous cohomology is effaceable. The argument of Lemma 1.1.1 then shows that it suffices to prove that $\mathcal{C}(G,E)$ has trivial continuous cohomology. This is no doubt standard; in any case, we recall the proof.

By definition, the continuous cohomology of $\mathcal{C}(G,E)$ is computed as the cohomology of G-invariants of the complex

$$0 \to \mathcal{C}(G,\mathcal{C}(G,E)) \to \mathcal{C}(G^2,\mathcal{C}(G,E)) \to \cdots, \tag{1.1.4}$$

with the usual boundary maps, on which the G-action is defined via the left regular action of G on the products G^n, together with the right-regular action of G on $\mathcal{C}(G,E)$. The example of [27, pp. 111–112] yields natural isomorphisms $\mathcal{C}(G^n,\mathcal{C}(G,E)) \xrightarrow{\sim} \mathcal{C}(G^n \times G,E) \xrightarrow{\sim} \mathcal{C}(G,\mathcal{C}(G^n,E))$, and [13, Lem. 3.2.11] yields a G-equivariant isomorphism of $\mathcal{C}(G,\mathcal{C}(G^n,E))$ with itself, but regarded as a G-representation via the right regular action of G on itself, and the trivial G-action on $\mathcal{C}(G^n, E)$. Thus the space of G-invariants of the complex (1.1.4) is naturally isomorphic to the complex

$$0 \to \mathcal{C}(G, E) \to \mathcal{C}(G^2, E) \to \cdots$$

with its usual boundary maps, which has no higher cohomology. □

We now turn to defining the derived functors of the functor "pass to smooth vectors".

Definition 1.1.5. If G is a compact locally $\mathbb{Q}_p$-analytic group, then we denote by $H^\bullet_{\mathrm{st}}(G, -)$ the derived functors of the functor "pass to smooth vectors" from the category of admissible continuous representations of G over E to the category of abstract E-vector spaces equipped with a G-action.

The subscript "st " stands for stable, and is explained by the following proposition. (This choice of notation is taken from [25, V.2.4.10].)

Proposition 1.1.6. *If G is a compact locally $\mathbb{Q}_p$-analytic group and V is an admissible continuous representation of G, then for any $i \geq 0$ there is a natural isomorphism of E-vector spaces $\varinjlim_H H^i(H, V) \xrightarrow{\sim} H^i_{\mathrm{st}}(G, V)$, where the inductive limit is taken over all the open subgroups H of G.*

Proof. For any admissible continuous representation V of G, by definition there is a natural isomorphism $\varinjlim_H V^H \to V_{\mathrm{sm}}$. Passing to derived functors (and noting that any V that is injective as a G-representation is also injective as an H-representation, since the forgetful functor from admissible continuous G-representations to admissible continuous H-representations is right adjoint to the exact functor $U \mapsto E[G] \otimes_{E[H]} U$), the proposition follows. □

Corollary 1.1.7. *If H is an open subgroup of the compact locally $\mathbb{Q}_p$-analytic group G, and if V is an admissible continuous representation of G, then for each $i \geq 0$, the natural map $H^i_{\mathrm{st}}(G, V) \to H^i_{\mathrm{st}}(H, V)$ is an isomorphism.*

Proof. This follows immediately from Proposition 1.1.6. □

Working directly with continuous cochains, we could use the inductive limit formula of Proposition 1.1.6 to define the functors $H^\bullet_{\mathrm{st}}(G, -)$ on the category of continuous G-representations for any topological group G. The reason that we restrict our attention to admissible continuous representations of locally $\mathbb{Q}_p$-analytic groups, and define $H^\bullet_{\mathrm{st}}(G, -)$ as a derived functor rather than directly via continuous cochains, is because we then have the standard homological machinery available with no expenditure of effort.

We now describe a construction that we will require in our subsequent applications, and which depends on this homological machinery. We suppose that G is a compact locally $\mathbb{Q}_p$-analytic group, and we let $V^\bullet$ denote a bounded-below complex of admissible continuous G-representations, each of which is isomorphic to $\mathcal{C}(G, E)^n$ for some natural number n, and in which the boundary maps are continuous and G-equivariant. The cohomology spaces $H^\bullet(V^\bullet)$ are again admissible continuous representations of G.

If we form the subcomplex $V^\bullet_{\mathrm{sm}}$ of $V^\bullet$ then [13, Prop. 6.2.4] and [29, prop. 2.1] show that each term of $V^\bullet_{\mathrm{sm}}$ is a strongly admissible smooth representation of G.

If we regard $V^\bullet_{\rm sm}$ as a subcomplex of $V^\bullet$, and if we choose a norm that induces the Banach space structure on each of the terms of $V^\bullet$, then these norms induce a norm on each term of $V^\bullet_{\rm sm}$, and the G-action on the terms of $V^\bullet_{\rm sm}$ is continuous with respect to this norm. We thus regard $V^\bullet_{\rm sm}$ as a complex of normed spaces each equipped with a continuous strongly admissible smooth action of G, in which the boundary maps are continuous and G-equivariant.

For each integer n the cohomology space $H^n(V^\bullet_{\rm sm})$ is thus naturally a semi-normed space, equipped with a continuous strongly admissible smooth representation of G. If $\hat{H}^n(V^\bullet_{\rm sm})$ denotes the completion of $H^n(V^\bullet_{\rm sm})$ with respect to its semi-norm then the continuous G-action on $H^n(V^\bullet_{\rm sm})$ induces a continuous G-action on $\hat{H}^n(V^\bullet_{\rm sm})$. There is a natural G-equivariant map $H^n(V^\bullet_{\rm sm}) \to \hat{H}^n(V^\bullet_{\rm sm})$, which will be injective precisely when the semi-norm on $H^n(V^\bullet_{\rm sm})$ is in fact a norm.

The inclusion of $V^\bullet_{\rm sm}$ into $V^\bullet$ induces for each integer n a G-equivariant map

$$H^n(V^\bullet_{\rm sm}) \to H^n(V^\bullet)_{\rm sm}. \tag{1.1.8}$$

Since $H^n(V^\bullet)$ is complete with respect to its natural topology, the map (1.1.8) extends to a continuous G-equivariant map

$$\hat{H}^n(V^\bullet_{\rm sm}) \to H^n(V^\bullet). \tag{1.1.9}$$

Proposition 1.1.10. (*i*) *For each integer n the natural map (1.1.9) is a closed embedding.*

(*ii*) *The map (1.1.8) is the edge map of a G-equivariant spectral sequence*

$$E_2^{i,j} = H^i_{\rm st}(G, H^j(V^\bullet)) \implies H^{i+j}(V^\bullet_{\rm sm}).$$

Proof. Fix an integer n, let $B^n(V^\bullet)$ denote the image of the boundary map from V^{n-1} to V^n, and let $Z^n(V^\bullet)$ denote the kernel of the boundary map from V^n to V^{n+1}. Both $B^n(V^\bullet)$ and $Z^n(V^\bullet)$ are closed subspaces of V^n (the former by [13, Prop. 6.2.9], and the latter obviously, since the boundary maps in the complex $V^\bullet$ are continuous by assumption). Similarly, let $B^n(V^\bullet_{\rm sm})$ denote the image of the boundary map from $V^{n-1}_{\rm sm}$ to $V^n_{\rm sm}$, and let $Z^n(V^\bullet_{\rm sm})$ denote the kernel of the boundary map from $V^n_{\rm sm}$ to $V^{n+1}_{\rm sm}$.

The cohomology group $H^n(V^\bullet)$ is equal to the quotient of $Z^n(V^\bullet)$ by $B^n(V^\bullet)$, while the cohomology group $H^n(V^\bullet_{\rm sm})$ is equal to the quotient of $Z^n(V^\bullet_{\rm sm})$ by $B^n(V^\bullet_{\rm sm})$. If $\hat{B}^n(V^\bullet_{\rm sm})$ denotes the closure of $B^n(V^\bullet_{\rm sm})$ in V^n, and $\hat{Z}^n(V^\bullet_{\rm sm})$ denotes the closure of $Z^n(V^\bullet_{\rm sm})$ in V^n, then the completion $\hat{H}^n(V^\bullet_{\rm sm})$ is isomorphic to the quotient of $\hat{Z}^n(V^\bullet_{\rm sm})$ by $\hat{B}^n(V^\bullet_{\rm sm})$.

Since by assumption V^{n-1} is isomorphic to $\mathcal{C}(G,E)^m$ for some integer m, we see that $V^{n-1}_{\rm sm}$ is dense in V^{n-1}, and so $B^n(V^\bullet_{\rm sm})$ is dense in $B^n(V^\bullet)$. Thus $\hat{B}^n(V^\bullet_{\rm sm}) = B^n(V^\bullet)$, $\hat{Z}^n(V^\bullet_{\rm sm})$ contains $B^n(V^\bullet)$, and $\hat{H}^n(V^\bullet)$ is equal to the quotient of $\hat{Z}^n(V^\bullet_{\rm sm})$ by $B^n(V^\bullet)$. This certainly embeds as a closed subspace of $H^n(V^\bullet)$, and so (i) is proved.

By assumption each term of $V^\bullet$ is acyclic for $H^\bullet_{\rm st}(G,-)$, and so the existence of the spectral sequence of part (ii) is a consequence of standard homological algebra. $\square$

We remark that in general $\hat{Z}^n(V^\bullet_{\mathrm{sm}})$ is a proper closed subspace of $Z^n(V^\bullet)$, and so the map (1.1.9) is not a topological isomorphism. (We will see examples of this in Subsection 4.2 below.)

We now extend Definition 1.1.5 to the case where G is a (not necessarily compact) locally $\mathbb{Q}_p$-analytic group. In this more general situation we cannot show the existence of enough injectives, and so we resort to a more *ad hoc* definition.

Definition 1.1.11. If G is a locally $\mathbb{Q}_p$-analytic group, and if V is an E-Banach space equipped with an admissible continuous representation of G, then for each $i \geq 0$ we define $H^i_{\mathrm{st}}(G, V) = \varinjlim_H H^i(H, V)$, where H runs over the directed set of all compact open subgroups of G.

Proposition 1.1.12. *The construction of Definition 1.1.11 satisfies the following properties:*

(i) For any admissible continuous representation V of G, there is a natural isomorphism $H^0_{\mathrm{st}}(G, V) \xrightarrow{\sim} V_{\mathrm{sm}}$.

(ii) For each $i \geq 0$ and any admissible continuous representation V of G, the E-vector space $H^i_{\mathrm{st}}(G, V)$ is equipped with a natural admissible smooth G-action.

(iii) If H is a compact open subgroup of G, and V an admissible continuous representation of G, then $H^i(H, V)$ maps isomorphically onto the subspace of H-invariants in $H^i_{\mathrm{st}}(G, V)$.

(iv) The collection $H^\bullet_{\mathrm{st}}(G, -)$ forms a covariant δ-functor.

(v) Let V be an admissible continuous representation of G. If there exists a compact open subgroup H of G and an H-equivariant isomorphism $V \xrightarrow{\sim} \mathcal{C}(H, E)^n$ for some $n \geq 0$, then $H^i_{\mathrm{st}}(G, V) = 0$ for all $i > 0$.

Proof. Part (i) is immediate.

If g is an element of G, and if H is an open subgroup of G, then conjugation by g induces an isomorphism $\phi_g : H \xrightarrow{\sim} gHg^{-1}$. The action of g on V induces an automorphism of V such that $\phi_g(h)gv = ghv$, for $h \in H$ and so we obtain an induced isomorphism $H^i(H, V) \xrightarrow{\sim} H^i(gHg^{-1}, V)$. Passing to the inductive limit as H shrinks to the identity, we obtain an automorphism σ_g of $H^i_{\mathrm{st}}(G, V)$. It is immediately verified that the association of σ_g to g defines a G-action on $H^i_{\mathrm{st}}(G, V)$. It is a standard fact that inner automorphisms act trivially on cohomology, implying that $H^i(H, V)$ is fixed by H. Thus the G-action on $H^i_{\mathrm{st}}(G, V)$ is smooth. To complete the proof of (ii), we must also show that this action is admissible. This follows from part (iii) (which we prove next).

If H is a compact open subgroup of G, then the set of normal open subgroups of H is cofinal in the directed set of open subgroups of G. Thus there is a natural isomorphism $\varinjlim_{H_1} H^i(H_1, V) \xrightarrow{\sim} H_{\mathrm{st}}(G, V)$, where H_1 runs over all normal open subgroups of H. To prove part (iii), it thus suffices to show that for each choice of such a subgroup H_1, the natural map $H^i(H, V) \to H^i(H_1, V)$ identifies the source with the H-invariants in the target.

The quotient H/H_1 is finite, and so has trivial higher cohomology with coefficients in an E-vector space (E being of characteristic zero). The Hochschild-Serre spectral sequence thus implies that restricting cohomology classes maps $H^i(H, V)$ isomorphically onto the H-invariants of $H^i(H_1, V)$. This proves (iii).

Note also that since V is an admissible continuous G-representation (by assumption), it is also an admissible continuous H-representation (since H is compact open in G). Thus $H^i(H,V)$ is finite dimensional. This completes the proof of (ii).

Part (iv) is an immediate consequence of the fact that the functors $H^\bullet_{st}(G,-)$ are defined as the inductive limit of the δ-functors $H^i(H,-)$.

If H is a compact open subgroup of G then the proof of Lemma 1.1.1 shows that $\mathcal{C}(H,E)$ is an injective object in the category of admissible continuous representations of G, and thus that $H^i(H,\mathcal{C}(H,E))$ vanishes for each $i>0$. This proves (v). $\square$

We now observe that the functor $V \mapsto V_{\mathrm{sm}}$ on the category of admissible continuous representations of G admits an alternative description as the functor $V \mapsto (V_{\mathrm{la}})^{\mathfrak{g}}$, where the subscript "la" denotes the space of locally analytic vectors in V (see [13, Def. 3.5.3]; the local field L is equal to $\mathbb{Q}_p$ in the situation we are considering), and $\mathfrak{g}$ denotes the Lie algebra of G. This suggests that there should be a relation between the functors $H^\bullet_{\mathrm{st}}(G,V)$ and the Lie algebra cohomology $H^\bullet(\mathfrak{g},V_{\mathrm{la}})$. Indeed, there is such a relation, given by the following proposition. The key ingredient is [31, Thm. 7.1], which shows that the functor $V \mapsto V_{\mathrm{la}}$ is exact.

Theorem 1.1.13. *If G is a locally $\mathbb{Q}_p$-analytic group, then there is a natural G-equivariant isomorphism*

$$H^\bullet_{\mathrm{st}}(G,-) \xrightarrow{\sim} H^\bullet(\mathfrak{g},(-)_{\mathrm{la}})$$

of δ-functors on the category of admissible continuous G-representations.

Proof. As already noted, the functor $V \mapsto V_{\mathrm{la}}$ is exact, by [31, Thm. 7.1]. As is implicit in the proof of Lemma 1.1.1, any admissible continuous representation of G has a resolution by a complex each of whose terms is a finite direct sum of copies of the injective object $\mathcal{C}(G,E)$. Thus to prove the proposition, it suffices to show that $\mathcal{C}(G,E)_{\mathrm{la}}$ has vanishing higher $\mathfrak{g}$-cohomology.

By [13, 3.5.11], there is a natural isomorphism $\mathcal{C}^{\mathrm{la}}(G,E) \xrightarrow{\sim} \mathcal{C}(G,E)_{\mathrm{la}}$. Thus we must show that $\mathcal{C}^{\mathrm{la}}(G,E)$ has trivial $\mathfrak{g}$-cohomology. We use the standard complex $\wedge^\bullet \mathfrak{g} \otimes_E \mathcal{C}^{\mathrm{la}}(G,E)$ to compute this cohomology, and observe that this complex is naturally isomorphic to the locally analytic de Rham complex on the locally $\mathbb{Q}_p$-analytic manifold G. A simple locally analytic "Poincaré lemma" shows that this complex has vanishing higher cohomology, and we are done. $\square$

The preceding result is proved for finite dimensional G-representations in [25, V.2.4.10 (i)].

(1.2) Let G be a compact locally $\mathbb{Q}_p$-analytic group. Up to this point we have restricted our attention to to E-Banach spaces with continuous G-action. However, we will also require some more refined results for certain $\mathcal{O}_E$-modules with G-action.

If V is an E-Banach space, then any bounded open $\mathcal{O}_E$-lattice in V is a p-adically separated and complete torsion free $\mathcal{O}_E$-module. Conversely, if M is a p-adically separated and complete torsion free $\mathcal{O}_E$-module, then the tensor product $E \otimes_{\mathcal{O}_E} M$ is naturally an E-Banach space. Indeed, the embedding $M \to E \otimes_{\mathcal{O}_E} M$ identifies M with an $\mathcal{O}_E$-lattice of $E \otimes_{\mathcal{O}_E} M$, and the gauge of M is a complete norm on $E \otimes_{\mathcal{O}_E} M$. (More generally, if M is a p-adically complete and separated $\mathcal{O}_E$-module, then the image of M in $E \otimes_{\mathcal{O}_E} M$ under the natural map becomes an $\mathcal{O}_E$-lattice, whose gauge is a complete norm on $E \otimes_{\mathcal{O}_E} M$.)

If M is any $\mathcal{O}_E$-submodule, we will let M_{tor} denote the torsion subgroup of M, and let M_{tf} denote the quotient M/M_{tor}. Thus M_{tor} is the kernel of the natural map $M \to E \otimes_{\mathcal{O}_E} M$, while M_{tf} is naturally isomorphic to the image of this map.

We now introduce an analogue for $\mathcal{O}_E[G]$-modules of the notion of an admissible continuous G-representation on an E-Banach space.

Definition 1.2.1. If M is an $\mathcal{O}_E[G]$-module, p-adically complete and separated as an $\mathcal{O}_E$-module, then we say that M is admissible if it satisfies the following two conditions:

(i) The torsion subgroup M_{tor} of M has bounded exponent.

(ii) The induced G-action on $E \otimes_{\mathcal{O}_E} M$ makes this latter space an admissible continuous representation of G (when equipped with its natural E-Banach space structure).

Our goal in this section is to establish some basic homological properties of admissible $\mathcal{O}_E[G]$-modules.

We begin with some lemmas. Before stating the first one, note that if M is an $\mathcal{O}_E[G]$-module, then M_{tor} is an $\mathcal{O}_E[G]$-submodule of M, and so M_{tf} is also naturally an $\mathcal{O}_E[G]$-module.

Lemma 1.2.2. *If M is an $\mathcal{O}_E$-module (respectively an $\mathcal{O}_E[G]$-module) such that M_{tor} is of bounded exponent, then M is p-adically separated and complete (respectively admissible) if and only if M_{tf} is p-adically separated and complete (respectively admissible).*

Proof. Both claims are immediate. (For the parenthetical claim, one should use the fact that the natural map $E \otimes_{\mathcal{O}_E} M \to E \otimes_{\mathcal{O}_E} M_{\mathrm{tf}}$ is an isomorphism.) $\square$

Lemma 1.2.3. *Let M be a p-adically separated and complete $\mathcal{O}_E$-module, and suppose that M_{tor} is of bounded exponent. If $N \subset M$ is an inclusion of $\mathcal{O}_E$-modules, then the p-adic topology on M induces the p-adic topology on N if and only if the p-adic topology on M_{tf} induces the p-adic topology on N_{tf}.*

Proof. This is easily proved, and we leave the verification to the reader. $\square$

Proposition 1.2.4. *Let $\phi : M \to N$ be an $\mathcal{O}_E$-linear and G-equivariant morphism between admissible $\mathcal{O}_E[G]$-modules.*

(*i*) *The $\mathcal{O}_E[G]$-module* $\ker\phi$ *is admissible, and the natural map* $\ker\phi \to M$ *is a closed embedding, when each of the source and target are endowed with their p-adic topologies.*

(*ii*) *The $\mathcal{O}_E[G]$-module* $\operatorname{im}\phi$ *is admissible, and the natural map* $\operatorname{im}\phi \to N$ *is a closed embedding, when each of the source and target are endowed with their p-adic topologies.*

(*iii*) *The $\mathcal{O}_E[G]$-module* $\operatorname{coker}\phi$ *is again an admissible $\mathcal{O}_E[G]$-module.*

Proof. Taking into account Lemmas 1.2.2 and 1.2.3, we see that we may replace M and N by M_{tf} and N_{tf} in the proof of the lemma. Thus we assume that M and N are both torsion free.

Tensoring with E, the map ϕ induces a continuous G-equivariant map $\mathrm{id} \otimes \phi : E \otimes_{\mathcal{O}_E} M \to E \otimes_{\mathcal{O}_E} N$ of Banach spaces equipped with admissible continuous G-representations. The kernel of $\mathrm{id} \otimes \phi$ is a closed G-invariant Banach subspace of $E \otimes_{\mathcal{O}_E} M$, on which the resulting G-action is admissible continuous. Thus $\ker\phi$ is closed in M when equipped with its p-adic topology, and forms an admissible $\mathcal{O}_E[G]$-module. This proves (i).

Any continuous G-equivariant map between admissible continuous representations of G is necessarily strict, with closed image [13, Prop. 6.2.9]. Thus $\operatorname{im}\phi$ is closed in N, when endowed with its p-adic topology, and is again an admissible $\mathcal{O}_E[G]$-module. This proves (ii).

If $\tilde{M}$ denotes the saturation of $\operatorname{im}\phi$ in N, then $\tilde{M}$ is a bounded subset of the image of $\operatorname{id}\otimes\phi$ (being contained in N), and so we see that $p^r\tilde{M}\subset\operatorname{im}\phi$ for some sufficiently large integer r. This proves that the torsion subgroup of the cokernel of ϕ has bounded exponent. The cokernel of $\operatorname{id}\otimes\phi$ is again an E-Banach space, equipped with an admissible continuous representation of G [13, Cor. 6.2.16]. It follows that the torsion free quotient of the cokernel of ϕ is an admissible $\mathcal{O}_E[G]$-module. Thus $\operatorname{coker}\phi$ is an extension of a torsion free admissible $\mathcal{O}_E[G]$-module by a torsion subgroup of bounded exponent. By Lemma 1.2.2, the proof of (iii) is complete. □

Lemma 1.2.5. *Let $0\to M\to N\to P\to 0$ be a short exact sequence of $\mathcal{O}_E$-modules. If M and P are p-adically separated and complete, and if M_{tor} and P_{tor} are both of bounded exponent, then N is also p-adically separated and complete, and N_{tor} is of bounded exponent. Furthermore, the p-adic topology on M coincides with the topology obtained by regarding M as a subspace of N, equipped with its p-adic topology.*

Proof. The sequence $0\to M_{\mathrm{tor}}\to N_{\mathrm{tor}}\to P_{\mathrm{tor}}$ is exact, and thus N_{tor} has bounded exponent, since this is true of M_{tor} and P_{tor}, by assumption.

Consider now the exact sequence of projective systems

$$\{P[p^s]\}_{s\geq 1}\to\{M/p^s\}_{s\geq 1}\to\{N/p^s\}_{s\geq 1}\to\{P/p^s\}_{s\geq 1}\to 0.$$

Since P_{tor} is of bounded exponent, we see that the transition maps in the projective system $\{P[p^s]\}_{s\geq 1}$ eventually vanish, and so $\varprojlim_s P[p^s]$ and $R^1\varprojlim_s P[p^s]$ are both trivial. Since the transition maps in the projective systems $\{M/p^s\}_{s\geq 1}$ are surjective, we also find that $R^1\varprojlim_s M/p^s=0$. Passing to the projective limit, we thus obtain a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_s M/p^s \arrow[r]\arrow[d] & \varprojlim_s N/p^s \arrow[r]\arrow[d] & \varprojlim_s P/p^s \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0.
\end{tikzcd}$$

The outer two vertical arrows are isomorphisms, by assumption, and thus so is the middle vertical arrow. Thus N is p-adically separated and complete. The claim about topologies also follows from the fact that the transition maps in the projective system $\{P[p^s]\}_{s\geq 1}$ are eventually trivial. □

Lemma 1.2.6. *Any extension (in the category of $\mathcal{O}_E[G]$-modules) of admissible $\mathcal{O}_E[G]$-modules is again an admissible $\mathcal{O}_E[G]$-modules.*

Proof. Let $0\to M\to N\to P\to 0$ be an extension of $\mathcal{O}_E[G]$-modules, and suppose that M and P are admissible. Lemma 1.2.5 shows that N is p-adically separated and complete, and that N_{tor} has bounded exponent. Tensoring this short exact sequence with E over $\mathcal{O}_E$, we obtain a short exact sequence of Banach spaces equipped with G-action

$$0\to E\otimes_{\mathcal{O}_E}M\to E\otimes_{\mathcal{O}_E}N\to E\otimes_{\mathcal{O}_E}P\to 0.$$

(Note that Lemma 1.2.5 shows that $E \otimes_{\mathcal{O}_E} M \to E \otimes_{\mathcal{O}_E} N$ is a closed embedding.) Since any extension of admissible continuous G-representations is again an admissible continuous G-representation [13, Prop. 6.2.7], the lemma follows. $\square$

Lemma 1.2.7. *If M is a torsion free admissible $\mathcal{O}_E[G]$-module, then for some natural number n we may find an embedding of $\mathcal{O}_E[G]$-modules $M \to \mathcal{C}(G, \mathcal{O}_E)^n$. Furthermore, if N denotes the cokernel of this embedding, then we may find an $\mathcal{O}_E$-linear map $\phi : N \to \mathcal{C}(G, \mathcal{O}_E)^n$ such that the composite of ϕ with the projection $\mathcal{C}(G, \mathcal{O}_E)^n \to N$ is equal to multiplication by p^s on N, for some natural number s.*

Proof. Since M is torsion free, the natural map $M \to E \otimes_{\mathcal{O}_E} M$ is an embedding. Since by assumption $E \otimes_{\mathcal{O}_E} M$ is an admissible continuous G-representation, it admits a closed embedding $E \otimes_{\mathcal{O}_E} M \to \mathcal{C}(G, E)^n$ for some natural number n. If we rescale sufficiently, then the restriction of this embedding yields a closed embedding $M \to \mathcal{C}(G, \mathcal{O}_E)^n$.

If N denotes the cokernel of this embedding, then we obtain a short exact sequence of E-Banach spaces

$$0 \to E \otimes_{\mathcal{O}_E} M \to \mathcal{C}(G, E)^n \to E \otimes_{\mathcal{O}_E} N \to 0.$$

By [27, Prop. 10.5] we may split this short exact sequence, say by a map $\sigma : E \otimes_{\mathcal{O}_E} N \to \mathcal{C}(G, E)^n$. If we take a suitably large power $p^{s'}$ of p, then $p^{s'}\sigma$ carries N_{tf} (identified with the image of N in $E \otimes_{\mathcal{O}_E} N$) into $\mathcal{C}(G, \mathcal{O}_E)^n$. Since N_{tor} has bounded exponent, multiplication by another suitably large power $p^{s''}$ of p on N factors through the natural map $N \to N_{\mathrm{tf}}$. Let $s = s' + s''$, and define ϕ to be the composite

$$N \longrightarrow N_{\mathrm{tf}} \xrightarrow{p^s \sigma} \mathcal{C}(G, \mathcal{O}_E)^n.$$

By construction, the composite of ϕ with the projection $\mathcal{C}(G, \mathcal{O}_E) \to N$ induces multiplication by p^s on N. $\square$

It will be useful to have an alternative description of the category of admissible $\mathcal{O}_E[G]$-modules, in terms of projective systems. We begin by introducing some definitions and notation.

We let $\mathcal{A}'_G$ denote the abelian category of $\mathcal{O}_E[G]$-modules, and let $\mathcal{A}_G$ denote the full subcategory of $\mathcal{A}'_G$ consisting of admissible $\mathcal{O}_E[G]$-modules.

We let $\mathcal{B}''_G$ denote the abelian category of projective systems $\{M_s\}_{s \geq 1}$ of $\mathcal{O}_E[G]$-modules, for which M_s is annihilated by p^s for each $s \geq 1$.

Definition 1.2.8. We say that a projective systems $\{M_s\}_{s \geq 1}$ in $\mathcal{B}''_G$ is essentially null if for any $s \geq 1$, the transition map $M_{s'} \to M_s$ vanishes if s' is sufficiently large.

The full subcategory of essentially null projective systems is immediately checked to be a Serre subcategory of $\mathcal{B}''_G$. We let $\mathcal{B}'_G$ denote the Serre quotient category.

There is a functor $\mathcal{A}'_G \to \mathcal{B}''_G$, defined by sending an $\mathcal{O}_E[G]$-module M to the projective system $\{M/p^s\}_{s \geq 1}$. We let $S : \mathcal{A}'_G \to \mathcal{B}'_G$ denote the composite of this functor with the natural quotient functor $\mathcal{B}''_G \to \mathcal{B}'_G$. Passing to the projective limit induces a functor $\mathcal{B}''_G \to \mathcal{A}'_G$ that factors through the natural quotient functor $\mathcal{B}''_G \to \mathcal{B}'_G$. We let $T : \mathcal{B}'_G \to \mathcal{A}'_G$ denote the induced functor. The functor S is left adjoint to the functor T. In particular, S is right exact and T is left exact.

Lemma 1.2.9. (*i*) *The functor $S : \mathcal{A}'_G \to \mathcal{B}'_G$ becomes exact when it is restricted to $\mathcal{A}_G$.*

(*ii*) *The natural transformation $\mathrm{id}_{\mathcal{A}_G} \to T \circ S$ induced by the adjointness of S and T becomes an isomorphism when restricted to $\mathcal{A}_G$.*

Proof. As observed above, the functor S is right exact. Thus to prove part (i), we must show that it is also left exact. For this, we must show that for any inclusion $M \subset N$ of admissible $\mathcal{O}_E[G]$-modules, the kernel of the natural map $\{M/p^s\}_{s\geq 1} \to \{M/(M \bigcap p^s N)\}_{s\geq 1}$, that is, the projective system $\{(M \bigcap p^s N)/p^s M\}_{s\geq 1}$, is essentially null. In other words, we must show that for any $s \geq 0$, there is an inclusion $M \bigcap p^{s'} N \to p^s M$ for s' sufficiently large. This follows from part (ii) of Proposition 1.2.4.

Part (ii) is just a rephrasing of the fact that objects of $\mathcal{A}_G$ are p-adically separated and complete. □

Let $\mathcal{B}_G$ denote the essential image of the functor $\mathcal{A}_G \to \mathcal{B}'_G$ obtained by restricting S. Thus the objects of $\mathcal{B}_G$ are those projective systems $\{M_s\}_{s\geq 1}$ of $\mathcal{O}_E/p^s[G]$-modules for which the projective limit $M := \varprojlim_s M_s$ is an admissible $\mathcal{O}_E[G]$-module, and for which the natural map of projective systems $\{M/p^s\}_{s\geq 1} \to \{M_s\}_{s\geq 1}$ has essentially null kernel and cokernel.

Proposition 1.2.10. *The functor S, when restricted to $\mathcal{A}_G$, yields an equivalence of categories between $\mathcal{A}_G$ and $\mathcal{B}_G$, with T providing a quasi-inverse.*

Proof. This is general categorical nonsense, given the definition of $\mathcal{B}_G$ as the essential image of S, the fact that T is right adjoint to S, and the conclusions of Lemma 1.2.9. □

Proposition 1.2.11. *The subcategory $\mathcal{B}_G$ of $\mathcal{B}'_G$ is closed under passage to kernels and cokernel of morphisms.*

Proof. Since the restriction of S to $\mathcal{A}_G$ is exact, and since $\mathcal{A}_G$ is closed under passing to kernels and cokernels in $\mathcal{A}'_G$ (by Proposition 1.2.4), we see that $\mathcal{B}_G$ is closed under passing to kernels and cokernels in $\mathcal{B}'_G$. □

Proposition 1.2.12. *If $M^\bullet$ is a cochain complex in the category $\mathcal{A}_G$, then for each integer n the natural map $\{H^n(M^\bullet)/p^s\}_{s\geq 1} \to \{H^n(M^\bullet/p^s)\}_{s\geq 1}$ is an isomorphism of objects in $\mathcal{B}_G$. Consequently, the natural map $H^n(M^\bullet) \to \varprojlim_s H^n(M^\bullet/p^s)$ is an isomorphism of objects in $\mathcal{A}_G$.*

Proof. Propositions 1.2.4 and 1.2.11 show that $\mathcal{A}_G$ and $\mathcal{B}_G$ are each closed under passing to kernels and cokernels (when regarded as subcategories of $\mathcal{A}'_G$ and $\mathcal{B}'_G$ respectively), while Proposition 1.2.10 shows that the functors S and T induce an equivalence of categories between $\mathcal{A}_G$ and $\mathcal{B}_G$. Thus for each n the cohomology module $H^n(M^\bullet)$ lies in $\mathcal{A}_G$, the projective system $H^n(S(M^\bullet))$ lies in $\mathcal{B}_G$, and there is a natural isomorphism $S(H^n(M^\bullet)) \xrightarrow{\sim} H^n(S(M^\bullet))$. This gives the required isomorphism in $\mathcal{B}'_G$. Applying the quasi-inverse T we obtain the isomorphisms

$$H^n(M^\bullet) \xrightarrow{\sim} TS(H^n(M^\bullet)) \xrightarrow{\sim} T(H^n(S(M^\bullet)))$$

in $\mathcal{A}_G$, completing the proof of the proposition. □

In the remainder of this subsection we will analyze the continuous cohomology of G with coefficients in an admissible $\mathcal{O}_E[G]$-module. The results that we obtain will be applied only in the proof of Proposition 2.4.1, and the reader willing to take that result on faith may safely pass on to Section 2.

We begin by remarking that the preceding definitions and results apply in particular when the group G is trivial. In this case we speak simply of admissible $\mathcal{O}_E$-modules, and write $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, and $\mathcal{B}'$ for the categories introduced above. Explicitly, the objects of $\mathcal{A}$ consist of those $\mathcal{O}_E$-modules M for which M_{tor} is of bounded exponent, and M_{tf} is finitely generated. (Take into account Lemma 1.2.2 and the fact that any finitely generated $\mathcal{O}_E$-module is p-adically separated and complete.) In addition to our preceding results, in this case we have the following lemmas.

Lemma 1.2.13. *The category $\mathcal{A}$ of admissible $\mathcal{O}_E$-modules is a Serre subcategory of the category $\mathcal{A}'$ of all $\mathcal{O}_E$-modules.*

Proof. Propositions 1.2.4 and 1.2.6 imply that $\mathcal{A}$ is closed under passing to extensions and cokernels in $\mathcal{A}'$. Thus it suffices to show that $\mathcal{A}$ is closed under passing to subobjects. This follows directly from the fact that $\mathcal{O}_E$ is Noetherian. $\square$

Lemma 1.2.14. *Suppose that we are given $\mathcal{O}_E$-modules M and N, and a pair of $\mathcal{O}_E$-linear morphisms $M \to N$ and $N \to M$ whose composition is equal to multiplication by p^s on M, for some natural number s. If N is an admissible $\mathcal{O}_E$-module, then the same is true of M.*

Proof. If X denotes the kernel of the map $M \to N$, then $X \subset M[p^s]$, and there is an exact sequence $0 \to X \to M_{\text{tor}} \to N_{\text{tor}}$. Thus if N_{tor} is of bounded exponent, the same is true of M_{tor}. Also, the map $M \to N$ induces an embedding $M_{\text{tf}} \to N_{\text{tf}}$, and so if N_{tf} is of finite type over $\mathcal{O}_E$, the same is true of M_{tf}. $\square$

If M is any topological $\mathcal{O}_E$-module equipped with a continuous G-action, then we let $C^\bullet_{\text{con}}(G, M)$ denote the complex of continuous cochains on G with values in M, and let $H^\bullet_{\text{con}}(G, M)$ denote the cohomology of the complex $C^\bullet_{\text{con}}(G, M)$ (the continuous cohomology of G with coefficients in M). If M is an admissible $\mathcal{O}_E[G]$-module, then when we speak of the continuous cochains or continuous cohomology of G with coefficients in M, we will always regard M as being equipped with its p-adic topology.

If $0 \to M \to N \to P \to 0$ is a sequence of continuous $\mathcal{O}_E$-linear and G-equivariant maps of topological $\mathcal{O}_E$-modules equipped with continuous G-actions, that is short exact as a sequence of $\mathcal{O}_E$-modules, then we have an exact sequence of complexes of $\mathcal{O}_E$-modules

$$0 \to C^\bullet_{\text{con}}(G, M) \to C^\bullet_{\text{con}}(G, N) \to C^\bullet_{\text{con}}(G, P), \tag{1.2.15}$$

which need not be exact on the right in general. (This is one of the complications of working with continuous cohomology.) However, we do have the following lemma.

Lemma 1.2.16. *If in the situation of the preceding paragraph, the topology on P is the discrete topology, then (1.2.15) is exact on the right.*

Proof. Since the topology on P is discrete, the maps in $C^\bullet_{\text{con}}(G, P)$ are locally constant, and so may be lifted to locally constant elements of $C^\bullet_{\text{con}}(G, N)$ (since $N \to P$ is surjective). $\square$

We begin by noting that the continuous cohomology of $\mathcal{C}(G, \mathcal{O}_E)$ has the expected values.

Lemma 1.2.17. *The natural embedding $\mathcal{O}_E \to \mathcal{C}(G, \mathcal{O}_E)$ induces an isomorphism $\mathcal{O}_E \xrightarrow{\sim} H^0_{\mathrm{con}}(G, \mathcal{C}(G, \mathcal{O}_E))$. The continuous cohomology $H^i_{\mathrm{con}}(G, \mathcal{C}(G, \mathcal{O}_E))$ vanishes if $i > 0$.*

Proof. The assertion regarding $H^0_{\mathrm{con}}(G, \mathcal{C}(G, \mathcal{O}_E))$ is evident. The assertion regarding the higher cohomology modules provides a refinement of the vanishing result discussed in the proof of Proposition 1.1.3, and is proved in the same way. $\square$

Proposition 1.2.18. *If M is an admissible $\mathcal{O}_E[G]$-module, then for each natural number i the cohomology module $H^i_{\mathrm{con}}(G, M)$ is an admissible $\mathcal{O}_E$-module.*

Proof. Let p^s be the exponent of M_{tor}. Then the endomorphism of M induced by multiplication by p^s factors as $M \to M_{\mathrm{tf}} \to M$. Thus the endomorphism of $H^i_{\mathrm{con}}(G, M)$ induced by multiplication by p^s factors as

$$H^i_{\mathrm{con}}(G, M) \to H^i_{\mathrm{con}}(G, M_{\mathrm{tf}}) \to H^i_{\mathrm{con}}(G, M).$$

By Lemma 1.2.14, it suffices to show that $H^i_{\mathrm{con}}(G, M_{\mathrm{tf}})$ is an admissible $\mathcal{O}_E$-module, and so we may restrict our attention to torsion free M.

We proceed by induction on i. Lemma 1.2.17 gives $H^0_{\mathrm{con}}(G, \mathcal{C}(G, \mathcal{O}_E)) = \mathcal{O}_E$. If M is torsion free, then by Lemma 1.2.7 we may find an embedding $M \to \mathcal{C}(G, \mathcal{O}_E)^n$ for some natural number n, and hence an embedding $H^0_{\mathrm{con}}(G, M) \to \mathcal{O}_E^n$. This establishes the case $i = 0$.

We now suppose the result is known for $i - 1$. Let M be torsion free, and again use Lemma 1.2.7 to regard M as a submodule of $\mathcal{C}(G, \mathcal{O}_E)^n$, for some natural number n. If N denotes the quotient $\mathcal{C}(G, \mathcal{O}_E)^n / M$, then by that same lemma we may find a map $\phi : N \to \mathcal{C}(G, \mathcal{O}_E)^n$ such that the composite

$$N \xrightarrow{\phi} \mathcal{C}(G, \mathcal{O}_E)^n \longrightarrow N$$

(the second arrow being the natural projection) is equal to multiplication by p^s, for some natural number s.

We consider the diagram (1.2.15) attached to the short exact sequence $0 \to M \to \mathcal{C}(G, \mathcal{O}_E)^n \to N \to 0$. If $C^\bullet$ denotes the cokernel of the natural map $C^\bullet_{\mathrm{con}}(G, M) \to C^\bullet_{\mathrm{con}}(G, \mathcal{C}(G, \mathcal{O}_E)^n)$, then there is an injection $C^\bullet \to C^\bullet_{\mathrm{con}}(G, N)$. Also, ϕ induces a map $C^\bullet_{\mathrm{con}}(G, N) \to C^\bullet$, and the composite $C^\bullet \to C^\bullet_{\mathrm{con}}(G, N) \to C^\bullet$ is equal to multiplication by p^s. Thus we obtain a sequence of maps

$$H^{i-1}(C^\bullet) \to H^{i-1}_{\mathrm{con}}(G, N) \to H^{i-1}(C^\bullet)$$

whose composite equals multiplication by p^s. By induction, we may assume that $H^{i-1}_{\mathrm{con}}(G, N)$ is an admissible $\mathcal{O}_E$-module, and Lemma 1.2.14 then implies that $H^{i-1}(C^\bullet)$ is an admissible $\mathcal{O}_E$-module.

Consider now the long exact sequence of cohomology attached to the short exact sequence of complexes

$$0 \to C^\bullet_{\mathrm{con}}(G, M) \to C^\bullet_{\mathrm{con}}(G, \mathcal{C}(G, \mathcal{O}_E)^n) \to C^\bullet \to 0.$$

From this long exact sequence we may extract the exact sequence

$$H^{i-1}(C^\bullet) \to H^i_{\mathrm{con}}(G,M) \to H^i_{\mathrm{con}}(G,\mathcal{C}(G,\mathcal{O}_E)^n).$$

Lemma 1.2.17 shows that $H^i_{\mathrm{con}}(G,\mathcal{C}(G,\mathcal{O}_E)^n)$ is trivial for $i>0$, and so we see that $H^i_{\mathrm{con}}(G,M)$ is a quotient of the admissible $\mathcal{O}_E$-module $H^{i-1}(C^\bullet)$. Lemma 1.2.13 implies that $H^i_{\mathrm{con}}(G,M)$ is itself an admissible $\mathcal{O}_E$-module. This completes the proof of the proposition. □

Proposition 1.2.19. *If M is an admissible $\mathcal{O}_E[G]$-module, then the natural map of projective systems $\{H^i_{\mathrm{con}}(G,M)/p^s\}_{s\geq 1} \to \{H^i_{\mathrm{con}}(G,M/p^s)\}_{s\geq 1}$ is an isomorphism in $\mathcal{B}'$. (That is, its kernel and cokernel are essentially null projective systems.)*

Proof. If s_0 is such that M_{tor} has exponent p^{s_0}, then multiplication by p^{s_0} factors through the natural projection $M \to M_{\mathrm{tf}}$, say as

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "p^{s_0}"] & M_{\mathrm{tf}} \arrow[r, "\phi"] & M, \end{tikzcd}$$

for some map $\phi : M_{\mathrm{tf}} \to M$. If $s \geq s_0$, then write $\phi_s = p^{s-s_0}\phi$. For each such value of s, we have a short exact sequence

$$0 \longrightarrow M_{\mathrm{tf}} \xrightarrow{\phi_s} M \longrightarrow M/p^s \longrightarrow 0.$$

Passing to continuous cochains, we obtain a short exact sequence of complexes

$$0 \longrightarrow \mathcal{C}^\bullet(G,M_{\mathrm{tf}}) \xrightarrow{\phi_s} \mathcal{C}^\bullet(G,M) \longrightarrow \mathcal{C}^\bullet(G,M/p^s) \longrightarrow 0.$$

(We have exactness on the right by Lemma 1.2.16.) Passing to cohomology, we obtain for any natural number i a short exact sequence

$$0 \to H^i_{\mathrm{con}}(G,M)/\phi_s(H^i_{\mathrm{con}}(G,M_{\mathrm{tf}})) \to H^i_{\mathrm{con}}(G,M/p^s) \to H^{i+1}_{\mathrm{con}}(G,M_{\mathrm{tf}})[\phi_s] \to 0.$$

(By abuse of notation, we have continued to denote by the same symbol the map that ϕ_s induces on cohomology.)

The composite $M_{\mathrm{tf}} \xrightarrow{\phi_s} M \longrightarrow M_{\mathrm{tf}}$ is equal to multiplication by p^s, and thus $H^{i+1}_{\mathrm{con}}(G,M_{\mathrm{tf}})[\phi_s]$ is contained in $H^{i+1}_{\mathrm{con}}(G,M_{\mathrm{tf}})[p^s]$. Since $H^{i+1}_{\mathrm{con}}(G,M_{\mathrm{tf}})$ is an admissible $\mathcal{O}_E$-module, by Proposition 1.2.18, we see that the projective system $\{H^{i+1}_{\mathrm{con}}(G,M_{\mathrm{tf}})[p^s]\}_{s\geq s_0}$ (the transition maps being given by multiplication by p) is essentially null, and hence the same is true of the projective system $\{H^{i+1}_{\mathrm{con}}(G,M_{\mathrm{tf}})[\phi_s]\}_{s\geq s_0}$. Thus the natural injection of projective systems

$$\{H^i_{\mathrm{con}}(G,M)/\phi_s(H^i_{\mathrm{con}}(G,M_{\mathrm{tf}}))\}_{s\geq s_0} \to \{H^i_{\mathrm{con}}(G,M/p^s)\}_{s\geq s_0}$$

has essentially null cokernel.

For any $s \geq s_0$, we have inclusions

$$p^s H^i_{\mathrm{con}}(G,M) \subset \phi_s(H^i_{\mathrm{con}}(G,M_{\mathrm{tf}})) = p^{s-s_0}\phi(H^i_{\mathrm{con}}(G,M_{\mathrm{tf}})) \subset p^{s-s_0}H^i_{\mathrm{con}}(G,M).$$

Thus the natural surjection of projective systems

$$\{H^i_{\mathrm{con}}(G,M)/p^s\}_{s\geq s_0} \to \{H^i_{\mathrm{con}}(G,M)/\phi_s(H^i_{\mathrm{con}}(G,M_{\mathrm{tf}}))\}_{s\geq s_0}$$

has essentially null kernel. Combining this conclusion with that of the preceding paragraph, we find that the natural map of projective systems

$$\{H^i_{\mathrm{con}}(G,M)/p^s\}_{s\geq 1} \to \{H^i_{\mathrm{con}}(G,M/p^s)\}_{s\geq 1}$$

has essentially null kernel and cokernel, as required. □

Proposition 1.2.20. *If M is an admissible $\mathcal{O}_E[G]$-module, then for any natural number i, the natural map $E \otimes_{\mathcal{O}_E} H^i_{\mathrm{con}}(G, M) \to H^i_{\mathrm{con}}(G, E \otimes_{\mathcal{O}_E} M)$ (where the continuous cohomology of $E \otimes_{\mathcal{O}_E} M$ is computed with respect to the Banach space topology on this tensor product) is an isomorphism.*

Proof. As G is compact, the natural map $E \otimes_{\mathcal{O}_E} C^\bullet_{\mathrm{con}}(G, M) \to C^\bullet_{\mathrm{con}}(G, E \otimes_{\mathcal{O}_E} M)$ is an isomorphism. Passing to cohomology (and taking into account the fact that E is flat over $\mathcal{O}_E$) yields the proposition. □

2. Cohomology of arithmetic quotients of symmetric spaces

(2.1) The goal of this subsection is to explain how one can obtain admissible continuous representations of locally $\mathbb{Q}_p$-analytic groups by taking appropriate limits of cohomology in a tower of covering spaces. The main result is Theorem 2.1.5 below.

Throughout this subsection we assume that G is a compact locally $\mathbb{Q}_p$-analytic group, and we fix a countable basis of open normal subgroups

$$G = G_0 \supset G_1 \supset \ldots \supset G_r \supset \ldots .$$

We suppose given a sequence of continuous maps of topological spaces

$$\cdots \to X_r \to \cdots \to X_1 \to X_0,$$

and assume that each of these spaces is equipped with a right action of G, such that

1. the maps in this sequence are G-equivariant;
2. the open subgroup G_r of G acts trivially on X_r;
3. if $0 \le r' \le r$ then the map $X_r \to X_{r'}$ is a Galois covering map with deck transformations provided by the natural action of $G_{r'}/G_r$ on X_r.

(If X_0 is connected with base point x_0, then it is equivalent to give a morphism $\pi_1(X_0, x_0) \to G$. For any $r \ge 0$, reduction modulo G_r yields a map $\pi_1(X_0, x_0) \to G/G_r$, and thus a (not necessarily connected) Galois covering space $X_r \to X_0$ whose group of deck transformations is isomorphic to G/G_r (acting on the right). Precisely, $X_r = \tilde{X} \times_{\pi_1(X, x_0)} G/G_r$, where $\tilde{X}$ is the universal cover of X, on which $\pi_1(X_0, x_0)$ acts on the right as deck transformations (after furthermore fixing a base-point $\tilde{x}_0$ of $\tilde{X}$ lying over x_0).)

We also suppose given a local system of free finite rank $\mathcal{O}_E$-modules $\mathcal{V}_0$ on X_0 (where $\mathcal{O}_E$ is the ring of integers in a finite extension E of $\mathbb{Q}_p$), and denote by $\mathcal{V}_r$ the pullback of $\mathcal{V}_0$ to X_r for any $r \ge 0$. The sheaf $\mathcal{V}_r$ is G/G_r-equivariant.

Let $*$ denote one of $\emptyset$ or c, so that $H^\bullet_*$ denotes either cohomology or compactly supported cohomology. Define

$$(2.1.1)\qquad \begin{aligned} H^n_*(\mathcal{V}) &:= \varinjlim_r H^n_*(X_r,\mathcal{V}_r),\\ H^n_*(\mathcal{V})_E &:= E\otimes_{\mathcal{O}_E} H^n_*(\mathcal{V}) \xrightarrow{\sim} \varinjlim_r H^n_*(X_r, E\otimes_{\mathcal{O}_E}\mathcal{V}_r),\\ \hat{H}^n_*(\mathcal{V}) &:= \varprojlim_s H^n_*(\mathcal{V})/p^s,\\ \hat{H}^n_*(\mathcal{V})_E &:= E\otimes_{\mathcal{O}_E}\hat{H}^n_*(\mathcal{V}),\\ \tilde{H}^n_*(\mathcal{V}) &:= \varprojlim_s \varinjlim_r H^n_*(X_r,\mathcal{V}_r/p^s),\\ \tilde{H}^n_*(\mathcal{V})_E &:= E\otimes_{\mathcal{O}_E}\tilde{H}^n_*(\mathcal{V}),\\ T_pH^n_*(\mathcal{V}) &:= \varprojlim_s H^n_*(\mathcal{V})[p^s],\\ V_pH^n_*(\mathcal{V}) &:= E\otimes_{\mathcal{O}_E} T_pH^n_*(\mathcal{V}). \end{aligned}$$

(Here r and s range over all non-negative integers.) Each of the $\mathcal{O}_E$-modules and E-vector spaces that we have defined is equipped with a natural G-action. (The G-action on $H^n_*(\mathcal{V})$ and $H^n_*(\mathcal{V})_E$ is smooth, but this will typically not be so for the G-action on the other objects appearing in (2.1.1).)

If we fix for the moment non-negative integers r and s then the exact sequence of sheaves $0\longrightarrow\mathcal{V}_r\xrightarrow{p^s\cdot}\mathcal{V}_r\longrightarrow\mathcal{V}_r/p^s\longrightarrow 0$ on X_r gives rise to a short exact sequence

$$0\to H^n_*(X_r,\mathcal{V}_r)/p^s\to H^n_*(X_r,\mathcal{V}_r/p^s)\to H^{n+1}_*(X_r,\mathcal{V}_r)[p^s]\to 0.$$

Passing now to the inductive limit over r yields the short exact sequence

$$0\to H^n_*(\mathcal{V})/p^s\to\varinjlim_r H^n_*(X_r,\mathcal{V}_r/p^s)\to H^{n+1}_*(\mathcal{V})[p^s]\to 0.$$

Each $\mathcal{O}_E/p^s$-module appearing in this exact sequence is equipped with a natural G-action, and the exact sequence is obviously G-equivariant. Passing to the projective limit over s, we obtain the short exact sequence of G-equivariant maps

$$(2.1.2)\qquad 0\to\hat{H}^n_*(\mathcal{V})\to\tilde{H}^n_*(\mathcal{V})\to T_pH^{n+1}_*(\mathcal{V})\to 0.$$

(To see that this sequence is short exact, note that for each value of s the transition map $H^n_*(\mathcal{V})/p^{s+1}\to H^n_*(\mathcal{V})/p^s$ is surjective.) Tensoring with E over $\mathcal{O}_E$ yields the G-equivariant short exact sequence

$$(2.1.3)\qquad 0\to\hat{H}^n_*(\mathcal{V})_E\to\tilde{H}^n_*(\mathcal{V})_E\to V_pH^{n+1}_*\to 0.$$

Lemma 2.1.4. *The short exact sequence (2.1.2) is a short exact sequence of p-adically complete and separated $\mathcal{O}_E$-modules. Furthermore, the arrow $\hat{H}^n_*(\mathcal{V})\to\tilde{H}^n_*(\mathcal{V})$ is a closed embedding, if we endow its source and target with their p-adic topologies.*

Proof. If M is any $\mathcal{O}_E$-module, then its p-adic completion $\varprojlim_s M/p^s$ is a p-adically complete and separated $\mathcal{O}_E$-module. Its p-adic Tate module $\varprojlim_s M[p^s]$ is also p-adically complete and separated, and is furthermore torsion free. Reducing the short exact sequence (2.1.2) modulo p^s thus yields, for each natural number s, a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \hat{H}^n_*(\mathcal{V}) \arrow[r] \arrow[d] & \tilde{H}^n_*(\mathcal{V}) \arrow[r] \arrow[d] & T_p H^{n+1}_*(\mathcal{V}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{H}^n_*(\mathcal{V})/p^s \arrow[r] & \tilde{H}^n_*(\mathcal{V})/p^s \arrow[r] & T_p H^{n+1}_*(\mathcal{V})/p^s \arrow[r] & 0.
\end{tikzcd}$$

Passing to the projective limit in s, we obtain a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \hat{H}^n_*(\mathcal{V}) \arrow[r] \arrow[d] & \tilde{H}^n_*(\mathcal{V}) \arrow[r] \arrow[d] & T_p H^{n+1}_*(\mathcal{V}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim_s \hat{H}^n_*(\mathcal{V})/p^s \arrow[r] & \varprojlim_s \tilde{H}^n_*(\mathcal{V})/p^s \arrow[r] & \varprojlim_s T_p H^{n+1}_*(\mathcal{V})/p^s \arrow[r] & 0.
\end{tikzcd}$$

The left-hand and right-hand vertical arrows of this diagram are isomorphisms, by the remarks of the first paragraph. Thus the middle arrow is also an isomorphism. We conclude that $\tilde{H}^n_*(\mathcal{V})$ is p-adically complete and separated, and also that the map $\hat{H}^n_*(\mathcal{V}) \to \tilde{H}^n_*(\mathcal{V})$ is a closed embedding. $\square$

The preceding lemma implies that each of the tensor products $\hat{H}^n_*(\mathcal{V})_E$, $\tilde{H}^n_*(\mathcal{V})_E$ and $V_p H^n_*(\mathcal{V})$ is naturally an E-Banach space on which G acts continuously, and that (2.1.3) is a G-equivariant short exact sequence of E-Banach spaces.

Theorem 2.1.5. *Take $* = \emptyset$, so that we are discussing cohomology. If X_0 is homotopic to a finite simplicial complex then the following results hold:*

(i) The p-adic locally analytic group G acts on each of the E-Banach spaces appearing in the exact sequence (2.1.3) through an admissible continuous representation, and the exact sequence (2.1.3) is a short exact sequence in the abelian category of admissible continuous representations of G.

(ii) The composite $H^n(\mathcal{V})_E \to \hat{H}^n(\mathcal{V})_E \to \tilde{H}^n(\mathcal{V})_E$ (the first arrow being the natural map and the second arrow being that which appears in the short exact sequence (2.1.3)) has an alternative factorization as a composite

$$H^n(\mathcal{V})_E \to (\tilde{H}^n(\mathcal{V})_E)_{\mathrm{sm}} \to \tilde{H}^n(\mathcal{V})_E,$$

and the first of these maps is the edge map of a spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{st}}(G, \tilde{H}^j(\mathcal{V})_E) \implies H^{i+j}(\mathcal{V})_E.$$

Proof. In order to prove the theorem, we introduce some notation. First note that by replacing X_0 by a finite simplicial complex homotopic to it, we may assume that X_0 is itself a finite simplicial complex. Let $T_\bullet(X_0)$ be a choice of finite triangulation of X_0 (where $T_n(X_0)$ denotes the collection of n-dimensional simplices occuring in the triangulation), and let $T_\bullet(X_r)$ denote the pullback of $T_\bullet(X_0)$ under the finite covering map $X_r \to X_0$. The set $T_\bullet(X_r)$ is equipped with a free action of G/G_r. If $\Delta \in T_n(X_0)$ for some n then we let $T_n(X_r)_{/\Delta}$ denote the set of simplices in $T_n(X_r)$

lying over Δ; the set $T_n(X_r)_{/\Delta}$ is then a principal homogeneous G/G_r-set. We let $\hat{T}_{n/\Delta}$ denote the projective limit $\varprojlim_r T_n(X_r)_{/\Delta}$; this is a profinite set which is principal homogeneous with respect to its natural G-action.

For any $r \geq 0$, the cohomology of the sheaf $\mathcal{V}_r$ is computed by a complex $S^\bullet(\mathcal{V}_r)$ whose nth term is

$$S^n(\mathcal{V}_r) := \prod_{\Delta' \in T_n(X_r)} H^0(\Delta', \mathcal{V}_r) \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \prod_{\Delta' \in T_n(X_r)_{/\Delta}} H^0(\Delta, \mathcal{V}_0).$$

The complex $S^\bullet(\mathcal{V}_r)$ is equipped with a natural action of G/G_r, and if $0 \leq r' \leq r$ then the natural (pullback) map $S^\bullet(\mathcal{V}_{r'}) \to S^\bullet(\mathcal{V}_r)$ identifies $S^\bullet(\mathcal{V}_{r'})$ with the $G_{r'}/G_r$-invariants of $S^\bullet(\mathcal{V}_r)$. Setting $S^\bullet(\mathcal{V}) := \varinjlim_r S^\bullet(\mathcal{V}_r)$, we see that $S^\bullet(\mathcal{V})$ is equipped with an action of G. The nth term of $S^\bullet(\mathcal{V})$ is given by

$$\begin{aligned} S^n(\mathcal{V}) &\xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \varinjlim_r \prod_{\Delta' \in T_n(X_r)_{/\Delta}} H^0(\Delta, \mathcal{V}_0) \\ &\xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \mathcal{C}^{\mathrm{sm}}(\hat{T}_{n/\Delta}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} H^0(\Delta, \mathcal{V}_0). \end{aligned}$$

We let $S^\bullet(\mathcal{V})_E$ denote the tensor product of the complex $S^\bullet(\mathcal{V})$ with E over $\mathcal{O}_E$. Since tensor products commute with inductive limits and finite products, the nth term is given by

$$S^n(\mathcal{V})_E \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \mathcal{C}^{\mathrm{sm}}(\hat{T}_{n/\Delta}, E) \otimes_E H^0(\Delta, \mathcal{V}_0 \otimes_{\mathcal{O}_E} E). \tag{2.1.6}$$

The nth cohomology module of the complex $S^\bullet(\mathcal{V})$ is naturally isomorphic to $H^n(\mathcal{V})$; the nth cohomology space of $S^\bullet(\mathcal{V})_E$ is naturally isomorphic to $H^n(\mathcal{V})_E$. (We are using the fact that passage to the inductive limit is exact, and commutes with tensor products.)

If s is a natural number then we can form similar constructions with $\mathcal{V}_0$ replaced by $\mathcal{V}_0/p^s$, and obtain a complex $S^\bullet(\mathcal{V}/p^s) = \varinjlim_r S^\bullet(\mathcal{V}_r/p^s)$, equipped with a G-action, whose nth term is

$$S^n(\mathcal{V}/p^s) \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \mathcal{C}^{\mathrm{sm}}(\hat{T}_{n/\Delta}, \mathcal{O}_E/p^s) \otimes_{\mathcal{O}_E/p^s} H^0(\Delta, \mathcal{V}_0/p^s).$$

The nth cohomology module of $S^\bullet(\mathcal{V}/p^s)$ is naturally isomorphic to the inductive limit $\varinjlim_r H^n(X_r, \mathcal{V}_r/p^s)$.

Taking the projective limit $\varprojlim_s S^\bullet(\mathcal{V}/p^s)$ as s grows to ∞, we obtain a complex that we denote $\tilde{S}^\bullet(\mathcal{V})$, again equipped with a G-action, whose nth term is given by

$$\begin{aligned} \tilde{S}^n(\mathcal{V}) &\xrightarrow{\sim} \varprojlim_s \prod_{\Delta \in T_n(X_0)} \mathcal{C}(\hat{T}_{n/\Delta}, \mathcal{O}_E/p^s) \otimes_{\mathcal{O}_E/p^s} H^0(\Delta, \mathcal{V}_0/p^s) \\ &\xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \mathcal{C}(\hat{T}_{n/\Delta}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} H^0(\Delta, \mathcal{V}_0). \end{aligned} \tag{2.1.7}$$

Tensoring with E over $\mathcal{O}_E$ we obtain a complex $\tilde{S}^\bullet(\mathcal{V})_E$ equipped with a G-action, whose nth term is given by

$$\tilde{S}^n(\mathcal{V})_E \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \mathcal{C}(\hat{T}_{n/\Delta}, E) \otimes_E H^0(\Delta, \mathcal{V}_0 \otimes_{\mathcal{O}_E} E). \tag{2.1.8}$$

Let Δ be a simplex in $T_n(X_0)$. As already observed, the right action of G on $\hat{T}_{n/\Delta}$ makes $\hat{T}_{n/\Delta}$ a principal homogeneous profinite G-set, and so $\mathcal{C}(\hat{T}_{n/\Delta}, \mathcal{O}_E)$ (respectively $\mathcal{C}(\hat{T}_{n/\Delta}, E)$) is isomorphic as a G-representation to $\mathcal{C}(G, \mathcal{O}_E)$ (respectively $\mathcal{C}(G, E)$) equipped with its right regular G-action. In light of this, the description of each of the terms of $\tilde{S}^\bullet(\mathcal{V})$ (respectively $\tilde{S}^\bullet(\mathcal{V})_E$) afforded by (2.1.7) (respectively (2.1.8)) shows that $\tilde{S}^\bullet(\mathcal{V})$ (respectively $\tilde{S}^\bullet(\mathcal{V})_E$) is a complex of admissible $\mathcal{O}_E[G]$-modules, in the sense of Definition 1.2.1 (respectively admissible continuous representations of G).

Passing to cohomology (and taking into account the natural isomorphism

$$\tilde{S}^\bullet(\mathcal{V})/p^s \xrightarrow{\sim} S^\bullet(\mathcal{V}/p^s)$$

for each natural number s), Proposition 1.2.12 shows that for each value of n, there is a natural isomorphism

$$H^n(\tilde{S}^\bullet(\mathcal{V})) \xrightarrow{\sim} \varprojlim_s H^n(S^\bullet(\mathcal{V}/p^s)) \xrightarrow{\sim} \varprojlim_s \varinjlim_r H^n(X_r, \mathcal{V}_r/p^s) = \tilde{H}^n(\mathcal{V}),$$

and hence, after tensoring with E, a natural isomorphism

$$H^n(\tilde{S}^\bullet(\mathcal{V})_E) \xrightarrow{\sim} \tilde{H}^n(\mathcal{V})_E. \tag{2.1.9}$$

Since $\tilde{S}^\bullet_E$ is a complex of admissible continuous representations of G, the same is true of its cohomology spaces, and thus we see that $\tilde{H}^n(\mathcal{V})_E$, when endowed with its natural Banach space structure, is an admissible continuous representation of G.

It was already observed that (2.1.3) is a short exact sequence of Banach spaces, and we have seen that its middle term is equipped with an admissible continuous representation of G. Any G-invariant closed subspace or Hausdorff quotient of an E-Banach space equipped with an admissible continuous representation of G is again an E-Banach space equipped with an admissible continuous representation of G, and so part (i) of the theorem follows.

The morphisms $\mathcal{V}_0 \to \mathcal{V}_0/p^s$, as s ranges over all natural numbers, induce a G-equivariant morphism $S^\bullet(\mathcal{V}) \to S^\bullet(\mathcal{V}/p^s)$ for each s, and hence G-equivariant morphisms $S^\bullet(\mathcal{V}) \to \tilde{S}^\bullet(\mathcal{V})$ and $S^\bullet(\mathcal{V})_E \to \tilde{S}^\bullet(\mathcal{V})_E$. A comparison of (2.1.6) and (2.1.8) shows that this latter morphism identifies $S^\bullet(\mathcal{V})_E$ with the complex of G-smooth vectors in the complex $\tilde{S}^\bullet(\mathcal{V})_E$. Part (ii) now follows from Proposition 1.1.10, together with the isomorphism (2.1.9), upon taking the complex $V^\bullet$ of that proposition to be the complex $\tilde{S}^\bullet(\mathcal{V})_E$. $\square$

Using the results on continuous cohomology proved in Subsection 1.2 we can give an alternative construction of the spectral sequence of Theorem 2.1.5 (ii). Its only application will be to the proof of Proposition 2.4.1.

For any fixed values of r and s, and any $r' \geq r$, we have the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(G_r/G_{r'}, H^j(X_{r'}, \mathcal{V}/p^s)) \implies H^{i+j}(X_r, \mathcal{V}/p^s).$$

Passing to the inductive limit in r', we obtain a spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{con}}(G_r, \varinjlim_{r'} H^j(X_{r'}, \mathcal{V}/p^s)) \implies H^{i+j}(X_r, \mathcal{V}/p^s). \tag{2.1.10}$$

Proposition 2.1.11. *If we fix a value of r, then passing to the projective limit in s of the spectral sequences (2.1.10), followed by tensoring with E over $\mathcal{O}_E$, yields a spectral sequence*

$$E_2^{i,j} = H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V})_E) \implies H^{i+j}(X_r, E \otimes_{\mathcal{O}_E} \mathcal{V}).$$

The inductive limit (with respect to r) of these spectral sequences yields a spectral sequence that is naturally isomorphic to the spectral sequence of Theorem 2.1.5 (ii).

Proof. The proof of Theorem 2.1.5 shows that $\tilde{H}^j(\mathcal{V})$ is an admissible $\mathcal{O}_E[G]$-module for each $j \geq 0$, which can be computed as the degree j cohomology of the complex of admissible $\mathcal{O}_E[G]$-modules $\tilde{S}^\bullet(\mathcal{V})$. Proposition 1.2.12 then implies that for each value of j, the natural map of projective systems

$$\{\tilde{H}^j(\mathcal{V})/p^s\}_{s\geq 1} \to \{\varinjlim_{r'} H^j(X_{r'}, \mathcal{V}/p^s)\}_{s\geq 1}$$

has essentially null kernel and cokernel. Thus the induced map of projective systems

$$\{H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V})/p^s)\}_{s\geq 1} \to \{H^i_{\mathrm{con}}(G_r, \varinjlim_{r'} H^j(X_{r'}, \mathcal{V}/p^s))\}_{s\geq 1}$$

also has essentially null kernel and cokernel. Taking into account Proposition 1.2.19, we find that the map of projective systems

$$\{H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V}))/p^s\}_{s\geq 1} \to \{H^i_{\mathrm{con}}(G_r, \varinjlim_{r'} H^j(X_{r'}, \mathcal{V}/p^s))\}_{s\geq 1}$$

has essentially null kernel and cokernel. In the terminology of Subsection 1.2, this map is an isomorphism of objects in the category $\mathcal{B}$.

Since X_r is a homotopic to a finite simplicial complex, the natural map

$$\{H^j(X_r, \mathcal{V})/p^s\}_{s\geq 1} \to \{H^j(X_r, \mathcal{V}/p^s)\}_{s\geq 1}$$

is also an isomorphism in the category $\mathcal{B}$. Thus we obtain a spectral sequence

$$E_2^{i,j} = \{H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V}))/p^s\}_{s\geq 1} \implies \{H^{i+j}(X_r, \mathcal{V})/p^s\}_{s\geq 1}$$

in the category $\mathcal{B}$. Since passing to the projective limit induces an equivalence of categories between $\mathcal{B}$ and the category $\mathcal{A}$ of admissible $\mathcal{O}_E$-modules (Proposition 1.2.10), we obtain a spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V})) \implies H^{i+j}(X_r, \mathcal{V}).$$

(Here we have used the isomorphism $H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V})) \xrightarrow{\sim} \varprojlim_s H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V}))/p^s$, which follows from the fact that $\tilde{H}^j(\mathcal{V})$ is an admissible $\mathcal{O}_E[G]$-module, together with Proposition 1.2.18, and the isomorphism $H^{i+j}(X_r, \mathcal{V}) \xrightarrow{\sim} \varprojlim_s H^{i+j}(X_r, \mathcal{V})/p^s$, which is obvious, since $H^{i+j}(X_r, \mathcal{V})$ is finitely generated as an $\mathcal{O}_E$-module.) Tensoring this spectral sequence through by E over $\mathcal{O}_E$, and taking into account the isomorphism of Proposition 1.2.20, we obtain a spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{con}}(G_r, \tilde{H}^j(\mathcal{V})_E) \implies H^{i+j}(X_r, E \otimes_{\mathcal{O}_E} \mathcal{V}),$$

as in the statement of the proposition.

Passing to the inductive limit in r, and taking into account Propositions 1.1.3 and 1.1.6, yields a spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{st}}(G, \tilde{H}^j(\mathcal{V})_E) \implies H^{i+j}(\mathcal{V})_E.$$

We leave it to the reader to chase through the various isomorphisms used in the construction of this spectral sequence, and so verify that it coincides with the spectral sequence of Theorem 2.1.5 (ii). $\square$

We now state an analogue of Theorem 2.1.5 for compactly supported cohomology. For this, we assume that X_0 is a topological manifold.

Theorem 2.1.12. *Suppose that X_0 is a topological manifold that is homotopic to a finite simplicial complex. Then the analogue of Theorem 2.1.5 holds with cohomology replaced by compactly supported cohomology.*

Proof. It suffices to prove the theorem with X_0 replaced by a connected component; thus we may assume that X_0 is equidimensional, say of dimension d. Poincaré duality yields a canonical isomorphism between compactly supported cohomology in degree n with coefficients in $\mathcal{V}$ and homology in degree $d-n$ with coefficients in $\mathcal{V}$; pullback of compactly supported cohomology classes corresponds to pullback of homology classes with respect to the finite covering maps $X_{r+1} \to X_r$. Thus we work from now on with homology rather than with compactly supported cohomology. As in the proof of Theorem 2.1.5 we replace X_0 by a finite simplicial complex, and we use the same notation $T_\bullet(X_r)$ as in that proof for the induced triangulation on each X_r.

The homology of X_r with coefficients in $\mathcal{V}_r$ is computed by a complex $S_\bullet(\mathcal{V}_r)$ whose nth term is

$$S_n(\mathcal{V}_r) := \prod_{\Delta' \in T_n(X_r)} H_0(\Delta', \mathcal{V}_r) \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \prod_{\Delta' \in T_n(X_r)_{/\Delta}} H_0(\Delta, \mathcal{V}_0).$$

The complex $S_\bullet(\mathcal{V}_r)$ is equipped with a natural action of G/G_r, and if $0 \leq r' \leq r$ then the natural (pullback) map $S_\bullet(\mathcal{V}_{r'}) \to S_\bullet(\mathcal{V}_r)$ identifies $S_\bullet(\mathcal{V}_{r'})$ with the $G_{r'}/G_r$-invariants of $S_\bullet(\mathcal{V}_r)$. Setting $S_\bullet(\mathcal{V}) := \varinjlim_r S_\bullet(\mathcal{V}_r)$, we see that $S_\bullet(\mathcal{V})$ is equipped with an action of G. The nth term of $S_\bullet(\mathcal{V})$ is given by

$$\begin{aligned} S_n(\mathcal{V}) \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \varinjlim_r \prod_{\Delta' \in T_n(X_r)_{/\Delta}} & H_0(\Delta, \mathcal{V}_0) \\ & \xrightarrow{\sim} \prod_{\Delta \in T_n(X_0)} \mathcal{C}^{\mathrm{sm}}(\hat{T}_{n/\Delta}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} H_0(\Delta, \mathcal{V}_0), \end{aligned}$$

and so is isomorphic as a topological G-module to $S^n(\mathcal{V})$ (in the notation of the proof of Theorem 2.1.5). Similarly, if we set $\tilde{S}_\bullet(\mathcal{V}) = \varprojlim_s \varinjlim_r \tilde{S}_\bullet(\mathcal{V}_r/p^s)$, then there is an isomorphism $\tilde{S}_n(\mathcal{V}) \xrightarrow{\sim} \tilde{S}^n(\mathcal{V})$ for each n.

The chain complexes $S_\bullet(\mathcal{V})$ and $\tilde{S}_\bullet(\mathcal{V})$ are thus term-by-term isomorphic to the cochain complexes $S^\bullet(\mathcal{V})$ and $\tilde{S}^\bullet(\mathcal{V})$; it is merely the boundary and coboundary maps that differ. The proof of Theorem 2.1.5 is hence seen to carry over from cohomology to homology, and so yields a proof of the present theorem. □

Let us note that in the context of the preceding theorem, the Hochschild-Serre spectral sequence for compactly supported cohomology yields a spectral sequence analogous to (2.1.10), with cohomology replaced by compactly supported cohomology. The proof of Proposition 2.1.11 then carries over in the compactly supported case, and thus yields an analogue of that proposition for compactly supported cohomology.

(2.2) We return to the situation considered in the introduction. Thus we fix a number field $F \subset \mathbb{C}$, a reductive connected linear algebraic group $\mathbb{G}$ defined over F, an isomorphism $\imath : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, and hence a prime $\mathfrak{p}$ of F lying over p. We let $F_{\mathfrak{p}}$ denote the completion $F_{\mathfrak{p}}$ of F at $\mathfrak{p}$ (equivalently, the closure of F in $\overline{\mathbb{Q}}_p$), and let G denote the locally $F_{\mathfrak{p}}$-analytic group $\mathbb{G}(F_{\mathfrak{p}})$. We also let $\mathbb{A}$ denote the ring of adèles of F, let $\mathbb{A}_f$ denote the ring of finite adèles (which we may write as $F_{\mathfrak{p}} \times \mathbb{A}_f^{\mathfrak{p}}$, where the second factor denotes the prime-to-$\mathfrak{p}$ finite adèles), let F_∞ denote $\mathbb{R} \otimes_{\mathbb{Q}} F$ (so $\mathbb{A} = F_\infty \times \mathbb{A}_f$), and write $G_\infty := \mathbb{G}(F_\infty)$. We fix once and for all a maximal compact subgroup K_∞ of G_∞.

We write H° to denote the connected component of the identity of a real Lie group H, and write $\pi_0(H)$ to denote the group of connected components of H; that is, the quotient H/H°. The structure theory of real reductive groups (in particular, the Iwasawa decomposition) shows that the natural map

$$\pi_0(K_\infty) \to \pi_0(G_\infty)$$

is an isomorphism. We thus identity these two connected component groups, and denote them simply by π_0.

If K_f is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$, then we write

$$Y(K_f) := \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K_\infty^\circ K_f. \tag{2.2.1}$$

The quotient G_∞/K_∞ is the symmetric space attached to the real Lie group G_∞, and so the quotient G_∞/K_∞° is a finite union of copies of this symmetric space, equipped with a natural action of π_0, which acts simply transitively on the set of connected components. Thus $Y(K_f)$ is a finite union of quotients of this symmetric space by congruence subgroups. If K_f is sufficiently small, then $\mathbb{G}(F)$ acts without fixed points on the quotient $\mathbb{G}(\mathbb{A})/K_\infty^\circ K_f$, and hence $Y(K_f)$ is a smooth manifold. If K_f' is a normal open subgroup of such a subgroup K_f, the natural map $Y(K_f') \to Y(K_f)$ realises $Y(K_f')$ as an unramified Galois cover of $Y(K_f)$, with Galois group isomorphic to the quotient K_f/K_f' (acting on the right). The natural continuous action of π_0 on the quotient G_∞/K_∞° induces a natural continuous action of π_0 on each of the quotients $Y(K_f)$.

Let W be a finite dimensional complex vector space equipped with a representation of the algebraic group $\mathbb{G}_{/\mathbb{C}}$. We define a π_0-equivariant local system of complex vector spaces $\mathcal{V}_W$ on the manifold $Y(K_f)$ (for K_f sufficiently small) as follows:

$$\mathcal{V}_W := \mathbb{G}(F)\backslash(W \times (\mathbb{G}(\mathbb{A})/K_\infty^\circ K_f)). \tag{2.2.2}$$

(Here $\mathbb{G}(F)$ acts on W through the given representation, and on the second factor by left multiplication.) The local system $\mathcal{V}_W$ on $Y(K_f)$ is locally isomorphic to the constant sheaf defined by W.

As the compact open subgroup K_f shrinks down to the identity, the manifolds $Y(K_f)$ form a projective system. The local systems $V_{\mathcal{W}}$ are compatible with the maps in this projective system in an obvious sense: if $K'_f \subset K_f$, then the pullback of the local system $\mathcal{V}_W$ on $Y(K_f)$ to $Y(K'_f)$ is naturally isomorphic to the local system $\mathcal{V}_W$ on $Y(K'_f)$. Since we intend to apply the results of Subsection 2.1 in the context of the arithmetic quotients $Y(K_f)$, let us recall that, although $Y(K_f)$ may not be compact, and so may not be homeomorphic to a finite simplicial complex, it does contain a deformation retract which is a finite simplicial complex.

Let us fix a degree n and a representation W, and consider the cohomology spaces $H^n_*(Y(K_f), \mathcal{V}_W)$, as K_f varies over all compact open subgroups of $\mathbb{G}(\mathbb{A}_f)$, and where as in the preceding subsection, $*$ denotes either $\emptyset$ or c (so that we are considering either cohomology or compactly supported cohomology). Passing to the inductive limit as K_f shrinks down to the identity, we obtain a smooth representation $H^n_*(\mathcal{V}_W) := \varinjlim_{K_f} H^n_*(Y(K_f), \mathcal{V}_W)$ of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$, which is well known to be admissible. In fact, passing to the K_f-fixed points exactly recovers the cohomology $H^n_*(Y(K_f), \mathcal{V}_W)$.

Our intention is to "$\mathfrak{p}$-adically complete" the vector space $H^n_*(\mathcal{V}_W)$ appropriately (using the isomorphism $\imath$ to regard W, and hence $H^n_*(\mathcal{V}_W)$, as a $\overline{\mathbb{Q}}_p$-vector space) so as to obtain an admissible continuous representation of $\mathbb{G}(\mathbb{A}_f)$.

We first recall an alternative description of the local system $\mathcal{V}_W$ (regarded as a $\overline{\mathbb{Q}}_p$-local system via $\imath$).

Definition 2.2.3. Let K_f be a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$ (chosen sufficiently small, so that $Y(K_f)$ is a manifold), and let G_0 denote the projection of K_f onto G. (Thus G_0 is a compact open subgroup of G). If M is a G_0-module then we define

$$\mathcal{V}_M := (M \times (\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K^\circ_\infty))/K_f,$$

a π_0-equivariant local system over $Y(K_f)$. (The right action of K_f on the product is defined via $m \cdot k = k_{\mathfrak{p}}^{-1} m$ on the first factor (where $k_{\mathfrak{p}} \in G_0$ is the $\mathfrak{p}$th component of an element $k \in K_f$) and via right multiplication on the second factor.)

If W is a complex representation of $\mathbb{G}$, then via $\imath$ we may regard W as a representation of $\mathbb{G}$ defined over $\overline{\mathbb{Q}}_p$, and so in particular as a representation of G, via the inclusion $G := \mathbb{G}(F_{\mathfrak{p}}) \subset \mathbb{G}(\overline{\mathbb{Q}}_p)$. Let W' denote W regarded as a representation of G in this way.

Lemma 2.2.4. *If K_f is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$ then there is a natural isomorphism of π_0-equivariant local systems over $Y(K_f)$ between $\mathcal{V}_W$ (as defined by (2.2.2), and then regarded as a $\overline{\mathbb{Q}}_p$-local system via $\imath$) and the local system $\mathcal{V}_{W'}$ defined via Definition 2.2.3.*

Proof. If $g \in \mathbb{G}(\mathbb{A})$ then let $g_{\mathfrak{p}}$ denote the $\mathfrak{p}$th component of g. The required isomorphism is provided by the automorphism $(w, g) \mapsto (g_{\mathfrak{p}}^{-1} w, g)$ of $W \times \mathbb{G}(\mathbb{A})$. $\square$

The preceding lemma justifies the duplication of notation in (2.2.2) and Definition 2.2.3. For the duration of this subsection we will consider only local systems arising from representations of compact open subgroups of G via Definition 2.2.3. It is technically easier to work with vector spaces defined over a finite extension of $\mathbb{Q}_p$, rather than over $\overline{\mathbb{Q}}_p$. Thus we fix a finite extension E of $F_{\mathfrak{p}}$ contained in $\overline{\mathbb{Q}}_p$, and from now on we let W denote a finite dimensional representation of $\mathbb{G}$

defined over E (rather than over $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$). If we choose E so that $\mathbb{G}$ is split over E, then any representation of $\mathbb{G}$ over $\overline{\mathbb{Q}}_p$ descends uniquely (up to an isomorphism) to a representation of $\mathbb{G}$ defined over E, and so we do not lose any generality by considering only W that are defined over E.

In order to make our constructions as natural as possible, it helps to consider certain directed sets. We let $\mathcal{S}$ denote the set of all compact open subgroups of G, directed downward by inclusion. If W_0 is a separated lattice in W, then we let $\mathcal{S}_{W_0}$ denote the directed subset of $\mathcal{S}$ consisting of those compact open subgroups $K_{\mathfrak{p}}$ of G that leave W_0 invariant. If G_0 is the maximal subgroup of G leaving W_0 invariant, then G_0 is a compact open subgroup of G, and $\mathcal{S}_{W_0}$ consists of all open subgroups $K_{\mathfrak{p}}$ of G_0. In particular, $\mathcal{S}_{W_0}$ is cofinal in $\mathcal{S}$, and if W_0' is a second separated lattice in W, then the intersection $\mathcal{S}_{W_0} \bigcap \mathcal{S}_{W_0'}$ is also cofinal in each of $\mathcal{S}_{W_0}$, $\mathcal{S}_{W_0'}$, and $\mathcal{S}$.

Let us temporarily fix a separated lattice W_0 in W. If K_f is any open subgroup of $\mathbb{G}(\mathbb{A}_f)$ whose image under the projection onto G is contained in $\mathcal{S}_{W_0}$, then via Definition 2.2.3 we obtain on $Y(K_f)$ a π_0-equivariant local system of $\mathcal{O}_E$-submodules $\mathcal{V}_{W_0}$ of $\mathcal{V}_W$ such that $E \otimes_{\mathcal{O}_E} \mathcal{V}_{W_0} \xrightarrow{\sim} \mathcal{V}_W$. If $K_f' \subset K_f$, then the equivariant local system so obtained on $Y(K_f')$ is naturally isomorphic to the pullback of the corresponding local system on $Y(K_f)$.

We now fix a "tame level" $K^{\mathfrak{p}}$ (that is, a compact open subgroup $K^{\mathfrak{p}}$ of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$). We form the projective system of arithmetic quotients $\{Y(K_{\mathfrak{p}}K^{\mathfrak{p}})\}_{K_{\mathfrak{p}} \in \mathcal{S}_{W_0}}$, each member of which is equipped with a local system $\mathcal{V}_{W_0}$. Let us fix for a moment a sufficiently small element $K_{\mathfrak{p}}$ of $\mathcal{S}_{W_0}$, chosen so that $Y(K_{\mathfrak{p}}K^{\mathfrak{p}})$ is a manifold. If $K_{\mathfrak{p}}'$ is a normal open subgroup of $K_{\mathfrak{p}}$, then the natural map $Y(K_{\mathfrak{p}}'K^{\mathfrak{p}}) \to Y(K_{\mathfrak{p}}K^{\mathfrak{p}})$ is unramified and Galois, with Galois group $K_{\mathfrak{p}}/K_{\mathfrak{p}}'$. Thus we are in the situation of Subsection 2.1, if we take the group G of that subsection to be $K_{\mathfrak{p}}$, and we may form the constructions introduced in that subsection. We write

$$\begin{aligned} H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}) &:= \varinjlim_{K_{\mathfrak{p}} \in \mathcal{S}_{W_0}} H^n_*(Y(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{V}_{W_0}) \\ &\xrightarrow{\sim} \varinjlim_{K_{\mathfrak{p}} \in \mathcal{S}_{W_0}} \varprojlim_s H^n_*(Y(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{V}_{W_0/p^s}), \end{aligned}$$

$$\begin{aligned} H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E &:= E \otimes_{\mathcal{O}_E} H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}), \\ \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}) &:= \varprojlim_s \varinjlim_{K_{\mathfrak{p}} \in \mathcal{S}_{W_0}} H^n_*(Y(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{V}_{W_0}/p^s), \\ \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E &:= E \otimes_{\mathcal{O}_E} \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}), \\ T_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}) &:= \varprojlim_s H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})[p^s], \\ V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}) &:= E \otimes_{\mathcal{O}_E} T_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}). \end{aligned}$$

(These correspond to the $\mathcal{O}_E$-modules and E-vector spaces that in Subsection 2.1 are denoted $H^n_*(\mathcal{V})$, $H^n_*(\mathcal{V})_E$, $\tilde{H}^n_*(\mathcal{V})$, $\tilde{H}^n_*(\mathcal{V})_E$, $T_pH^n_*(\mathcal{V})$, and $V_pH^n_*(\mathcal{V})$, respectively.)

The image of $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})$ in $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E$ is a lattice in this E-vector space, and we let $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E$ denote the E-Banach space obtained by completing $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E$ with respect to the gauge of that lattice. Also, the image of $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})$ in $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E$ is a lattice in this E-vector space, and we regard $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E$ as a semi-normed space, the semi-norm being given by the gauge of that lattice. Similarly, we use the gauge of $T_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})$ to regard $V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})$ as a semi-normed E-vector space. The results of Subsection 2.1 show that both these semi-normed spaces are in fact E-Banach spaces.

An inclusion $W_0 \subset W_0'$ of separated lattices in W induces an injection of sheaves $\mathcal{V}_{W_0} \to \mathcal{V}_{W_0'}$, and hence morphisms of $\mathcal{O}_E$-modules $H^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0}) \to H^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})$, $\tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0}) \to \tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})$, and $T_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0}) \to T_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})$. (Recall that the intersection of the directed sets $\mathcal{S}_{W_0}$ and $\mathcal{S}_{W_0'}$ is cofinal in each of them.) Such an inclusion thus induces continuous maps of E-Banach spaces

$$\hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E \to \hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})_E, \tag{2.2.5}$$

$$\tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E \to \tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})_E, \tag{2.2.6}$$

and

$$V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E \to V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})_E. \tag{2.2.7}$$

Lemma 2.2.8. *If $W_0 \subset W_0'$ is an inclusion of separated lattices in W then all three morphisms (2.2.5), (2.2.6) and (2.2.7) are topological isomorphisms.*

Proof. If the natural number r is chosen sufficiently large, then there is an inclusion $p^rW_0' \subset W_0$. Thus we may embed (2.2.5) into the sequence of maps

$$\hat{H}^n_*(K^{\mathfrak{p}},p^r\mathcal{V}_{W_0'})_E \longrightarrow \hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E \\ \stackrel{(2.2.5)}{\longrightarrow} \hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0'})_E \longrightarrow \hat{H}^n_*(K^{\mathfrak{p}},p^{-r}\mathcal{V}_{W_0})_E.$$

The composite of the first two arrows is obviously an isomorphism, as is the composite of the second two arrows. Thus the middle arrow is an isomorphism. A similar argument applies to (2.2.6) and (2.2.7). $\square$

Definition 2.2.9. We make the following definitions:

(i) $\hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_W) := \varinjlim_{W_0} \hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E$.

(ii) $\tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_W) := \varinjlim_{W_0} \tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_K$.

(iii) $V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_W) := \varinjlim_{W_0} V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})$.

(In each case the locally convex inductive limit is taken over the directed set of separated lattices W_0 in W, directed by inclusion).

Lemma 2.2.8 shows that the transition isomorphisms in each of the three inductive systems occuring in Definition 2.2.9 are topological isomorphisms. Thus for any choice of W_0, each of the natural maps $\hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E \to \hat{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_W)$, $\tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0})_E \to \tilde{H}^n_*(K^{\mathfrak{p}},\mathcal{V}_W)$ and $V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_{W_0}) \to V_pH^n_*(K^{\mathfrak{p}},\mathcal{V}_W)$ is a topological isomorphism. In particular, each of the topological E-vector spaces defined in Definition 2.2.9 is an E-Banach space.

If $g \in G$ and W_0 is a separated lattice in W then gW_0 is again a separated lattice in W. Thus we obtain an action of G on the directed set of all separated lattices in W.

Lemma 2.2.10. *The action of G on the directed set of separated lattices in W lifts to a continuous actions of G on each of the inductive systems that occurs in Definition 2.2.9. Thus each of the E-Banach spaces $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ and $V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ is equipped with a continuous G-action.*

If W is the trivial representation, then each of these E-Banach spaces admits a norm with respect to which the action of G is isometric.

Proof. If $g \in G$ and W_0 is a separated lattice in W then multiplication by g induces isomorphisms $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E \xrightarrow{\sim} \hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{g^{-1}W_0})_E$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})_E \xrightarrow{\sim} \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{g^{-1}W_0})_E$, and $V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}) \xrightarrow{\sim} V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{g^{-1}W_0})$. Passing to the inductive limit over all W_0 yields the required action of G on each of $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, and $V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$. There is a compact open subgroup G_0 of G that leaves any W_0-invariant, and this implies that the G-action on each of these E-Banach spaces is continuous.

Note that when W is the trivial representation of G, any separated lattice W_0 in W is invariant under all of G, and so in this case the G-action on $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ (respectively $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, $V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$) is isometric with respect to the norm defined as the gauge of the image of $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})$ (respectively $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})$, $V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})$). $\square$

Theorem 2.2.11. (*i*) *The group $\pi_0 \times G$ acts on each of the E-Banach spaces $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, and $V_p H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ via an admissible continuous representation.*

(*ii*) *If W is the trivial local system, then each of these Banach spaces can be topologized by a norm with respect to which $\pi_0 \times G$ acts by isometries.*

(*iii*) *If $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ denotes the space of $K^{\mathfrak{p}}$-invariants in the admissible smooth $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation $H^n_*(\mathcal{V}_W)$, then there is a $\pi_0 \times G$-equivariant natural map $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, whose image is dense.*

(*iv*) *There is a short exact sequence in the category of admissible continuous $\pi_0 \times G$-representations*

$$0 \to \hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to V_p H^{n+1}_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to 0.$$

(*v*) *The composite of the map of* (*iii*) *with the closed embedding $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ of* (*iv*) *yields a map*

$$H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)_{\mathrm{sm}},$$

which is the edge map of a $\pi_0 \times G$-equivariant spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{st}}(G, \tilde{H}^j_*(K^{\mathfrak{p}}, \mathcal{V}_W)) \implies H^{i+j}_*(K^{\mathfrak{p}}, \mathcal{V}_W).$$

Proof. Since all the local systems under consideration are π_0-equivariant, each of the $\mathcal{O}_K$-modules and E-vector spaces that we have defined is equipped in a natural way with an action of π_0, compatible with the various maps between them. The theorem then follows by applying Theorems 2.1.5 and 2.1.12 to the sheaves $\mathcal{V}_{W_0}$ for each separated lattice $W_0 \subset W$, passing to the limit with respect to the transition isomorphisms (2.2.5), (2.2.6) and (2.2.7), and taking into account the naturality of the G-action given by Lemma 2.2.10, and of the π_0-action. $\square$

Up to this point, we have kept fixed the tame level $K^{\mathfrak{p}}$. We now consider what happens when the tame level is allowed to change.

Suppose that $K^{\mathfrak{p}\prime} \subset K^{\mathfrak{p}}$ is an inclusion of compact open subgroups of $\mathbb{G}(\mathbb{A}^{\mathfrak{p}})$, such that $K^{\mathfrak{p}\prime}$ is normal in $K^{\mathfrak{p}}$. If $K_{\mathfrak{p}}$ is any compact open subgroup of G, then we obtain a surjective map $Y(K_{\mathfrak{p}}K^{\mathfrak{p}\prime}) \to Y(K_{\mathfrak{p}}K^{\mathfrak{p}})$. Consequently, we obtain a map of exact sequences in the category of admissible continuous $\pi_0 \times G$-representations

(2.2.12)
$$\begin{tikzcd}
0 \arrow[r] & \hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \arrow[r] \arrow[d] & \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \arrow[r] \arrow[d] & V_p H^{n+1}_*(K^{\mathfrak{p}}, \mathcal{V}_W) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \arrow[r] & \tilde{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \arrow[r] & V_p H^{n+1}_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \arrow[r] & 0.
\end{tikzcd}$$

Since $K^{\mathfrak{p}\prime}$ is normal in $K^{\mathfrak{p}}$, each of the E-Banach spaces in the lower short exact sequence is equipped with a natural continuous action of the quotient $K^{\mathfrak{p}}/K^{\mathfrak{p}\prime}$, and (by virtue of the naturality of these actions) the arrows in this short exact sequence are equivariant with respect to these actions.

Proposition 2.2.13. *The vertical arrows of (2.2.12) are closed embeddings, and identify the upper exact sequence with the $K^{\mathfrak{p}}/K^{\mathfrak{p}\prime}$-invariants of the lower exact sequence.*

Proof. Let a denote the order of the quotient $K^{\mathfrak{p}}/K^{\mathfrak{p}\prime}$. If U is any Hausdorff topological E-vector space equipped with a continuous action of $K^{\mathfrak{p}}/K^{\mathfrak{p}\prime}$, the averaging operator $\pi_{K^{\mathfrak{p}}} : u \mapsto a^{-1}\sum_{k \in K^{\mathfrak{p}}/K^{\mathfrak{p}\prime}} ku$ induces a continuous projection of U onto its closed subspace of $K^{\mathfrak{p}}$-invariants.

Let W_0 be a choice of separated lattice in W, and let $W_0' = a^{-1}W_0$. Then multiplication by a^{-1} is a well-defined map $H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0}) \to H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0'})$, while there is a natural trace map $\mathrm{tr} : H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0'}) \to H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0'})$ (induced by the natural maps $Y(K_{\mathfrak{p}}K^{\mathfrak{p}\prime}) \to Y(K_{\mathfrak{p}}K^{\mathfrak{p}})$ for each compact open subgroup $K_{\mathfrak{p}}$ of G). Also $\pi_{K^{\mathfrak{p}}}$ projects $H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W)$ onto $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$. These maps fit together into the commutative diagram

(2.2.14)
$$\begin{tikzcd}
H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0}) \arrow[r] \arrow[d] \arrow[ddd, bend right] & H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \arrow[d] \arrow[ddd, bend left, "\mathrm{id}"] \\
H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0}) \arrow[r] \arrow[d, "a^{-1}\cdot"] & H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \arrow[dd, "\pi_{K^{\mathfrak{p}}}"] \\
H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0'}) \arrow[d, "\mathrm{tr}"] & \\
H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0'}) \arrow[r] & H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)
\end{tikzcd}$$

(where the unlabelled arrows are the natural maps corresponding to the inclusions $W_0 \subset W_0' \subset W$, and $K^{\mathfrak{p}\prime} \subset K^{\mathfrak{p}}$).

Lemma 2.2.8 shows that the E-Banach spaces $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ and $\hat{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W))$ may be obtained as the completion of $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$ and $H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W)$ respectively with respect to the gauge of either the images of $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})$ and of $H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0})$, or the images of $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0'})$ and of $H^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_{W_0'})$. A consideration of (2.2.14) thus shows that $\pi_{K_{\mathfrak{p}}}$ extends to a continuous map $\hat{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \to \hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, splitting the natural map $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \hat{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W)$ induced by the inclusion $K^{\mathfrak{p}\prime} \subset K^{\mathfrak{p}}$. Thus the natural map $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \hat{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W)$ is a closed embedding that identifies its source with the $K^{\mathfrak{p}}$ invariants of its target.

A similar argument, in which we pass to p^s-torsion in the spaces on the left-hand side of the diagram (2.2.14), pass to the limit in s, and then tensor with E, shows that $\pi_{K^{\mathfrak{p}}}$ induces a map $V_pH^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \to V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, splitting the natural map $V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to V_pH^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W)$.

If instead of working with $\mathcal{V}_{W_0}$ and $\mathcal{V}_{W_0'}$, we work with the quotient sheaves $\mathcal{V}_{W_0}/p^s$ and $\mathcal{V}_{W_0'}/p^s$, then after passing to the limit in s and tensoring with E, we similarly conclude that $\pi_{K^{\mathfrak{p}}}$ induces a map $\tilde{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W) \to \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, splitting the natural map $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \tilde{H}^n_*(K^{\mathfrak{p}\prime}, \mathcal{V}_W)$. This completes the proof of the proposition. □

Definition 2.2.15. Write

$$\begin{aligned}\hat{H}^n_*(\mathcal{V}_W) &:= \varinjlim_{K^{\mathfrak{p}}} \hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W),\\ \tilde{H}^n_*(\mathcal{V}_W) &:= \varinjlim_{K^{\mathfrak{p}}} \tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W),\\ V_pH^n_*(\mathcal{V}_W) &:= \varinjlim_{K^{\mathfrak{p}}} V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W).\end{aligned}$$

(Here, the locally convex inductive limits are taken with respect to the set of compact open subsets of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$, directed downward.)

We will apply the terminology of [13, §7] to the group $\pi_0 \times \mathbb{G}(\mathbb{A}_f) = \pi_0 \times G \times G(\mathbb{A}_f^{\mathfrak{p}})$, taking the locally analytic group G of that reference to be the group G (in our current notation), and the auxiliary locally compact group Γ of that reference to be the product $\pi_0 \times \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$.

Theorem 2.2.16. (*i*) *The group $\pi_0 \times G(\mathbb{A}_f)$ acts on each of the locally convex topological E-vector spaces $\hat{H}^n_*(\mathcal{V}_W)$, $\tilde{H}^n_*(\mathcal{V}_W)$, and $V_pH^n_*(\mathcal{V}_W)$ via an admissible continuous representation.*

(*ii*) *For each compact open subgroup $K^{\mathfrak{p}}$ of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ there are natural isomorphisms of admissible continuous $\pi_0 \times G$-representations $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} \hat{H}^n_*(\mathcal{V}_W)^{K^{\mathfrak{p}}}$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} \tilde{H}^n_*(\mathcal{V}_W)^{K^{\mathfrak{p}}}$, and $V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} V_pH^n_*(\mathcal{V}_W)^{K^{\mathfrak{p}}}$.*

(*iii*) *There is a natural $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant map $H^n_*(\mathcal{V}_W) \to \hat{H}^n_*(\mathcal{V}_W)$, whose image is dense.*

(*iv*) *There is a short exact sequence in the category of admissible continuous $\pi_0 \times G(\mathbb{A}_f)$-representations*

$$0 \to \hat{H}^n_*(\mathcal{V}_W) \to \tilde{H}^n_*(\mathcal{V}_W) \to V_pH^{n+1}_*(\mathcal{V}_W) \to 0.$$

Proof. If $g \in \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$, then right multiplication by g on the spaces $Y(K_{\mathfrak{p}}K^{\mathfrak{p}})$ (as $K_{\mathfrak{p}}$ ranges over the directed set of compact open subgroups of G) induces isomorphisms of the directed systems of spaces $\{Y(K_{\mathfrak{p}}gK^{\mathfrak{p}}g^{-1})\}_{K_{\mathfrak{p}}\in\mathcal{S}} \xrightarrow{\sim} \{Y(K_{\mathfrak{p}}K^{\mathfrak{p}})\}_{K_{\mathfrak{p}}\in\mathcal{S}}$, and hence topological isomorphisms

$$\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} \hat{H}^n_*(gK^{\mathfrak{p}}g^{-1}, \mathcal{V}_W),$$
$$\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} \tilde{H}^n_*(gK^{\mathfrak{p}}g^{-1}, \mathcal{V}_W),$$

and

$$V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} V_pH^n_*(gK^{\mathfrak{p}}g^{-1}, \mathcal{V}_W),$$

compatible with the $\pi_0 \times G$-action on these spaces. Thus we obtain a topological $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$-action on each of the spaces $\hat{H}^n_*(\mathcal{V}_W)$, $\tilde{H}^n_*(\mathcal{V}_W)$, and $V_pH^n_*(\mathcal{V}_W)$ of Definition 2.2.15, commuting with the $\pi_0 \times G$-action on these spaces. Propositions 2.2.13 shows that each of the locally convex inductive limits appearing in Definition 2.2.15 is a strict inductive limit (in the sense of [4, p. II.33]), and the same proposition thus implies that the natural maps $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \hat{H}^n_*(\mathcal{V}_W)^{K^{\mathfrak{p}}}$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to \tilde{H}^n_*(\mathcal{V}_W)^{K^{\mathfrak{p}}}$, and $V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \to V_pH^n_*(\mathcal{V}_W)^{K^{\mathfrak{p}}}$ are topological isomorphisms. In particular, the $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$-action on each of the spaces of Definition 2.2.15 is strictly smooth, in the sense of [13, Def. 7.1.2]. Taking into account Theorem 2.2.11 (i), we see that we have proved part (i) and (ii) of the current theorem. Parts (iii) and (iv) are easy consequences of the corresponding parts of Theorem 2.2.11, and so we are done. $\square$

Note that by taking W to be the trivial representation (so that $\mathcal{V}_W$ is the trivial local system), this result yields Theorem 0.1 of the introduction.

To simplify notation, when W is the trivial representation of G on E we will write simply H^n_*, $\hat{H}^n_*$, $\tilde{H}^n_*$, and $V_pH^n_*$ in place of $H^n_*(\mathcal{V}_W)$, $\hat{H}^n_*(\mathcal{V}_W)$, $\tilde{H}^n_*(\mathcal{V}_W)$, and $V_pH^n_*(\mathcal{V}_W)$, and will write $H^n_*(K^{\mathfrak{p}})$, $\hat{H}^n_*(K^{\mathfrak{p}})$, $\tilde{H}^n_*(K^{\mathfrak{p}})$, and $V_pH^n_*(K^{\mathfrak{p}})$ in place of $H^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, $\hat{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, and $V_pH^n_*(K^{\mathfrak{p}}, \mathcal{V}_W)$, for any choice of tame level $K^{\mathfrak{p}}$.

Theorem 2.2.17. *If W is any finite dimensional algebraic representation of $\mathbb{G}$ defined over E, then there is a natural isomorphism of admissible continuous $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representations $\tilde{H}^n_*(\mathcal{V}_W) \xrightarrow{\sim} \tilde{H}^n_* \otimes_E W$. (The action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f) = \pi_0 \times G \times \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ on the target is via the diagonal action of G, and the action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ on the first factor.)*

Proof. Let us first fix a tame level $K^{\mathfrak{p}}$. If W_0 is a separated lattice in W and s is any natural number then there is a natural isomorphism

$$\varinjlim_{K_{\mathfrak{p}}} H^n_*(Y(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{V}_{W_0/p^s}) \xrightarrow{\sim} \varinjlim_{K_{\mathfrak{p}}} H^n_*(Y(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{O}_E/p^s) \otimes_{\mathcal{O}_E/p^s} \mathcal{V}_{W_0/p^s}.$$

(If $K_{\mathfrak{p}}$ is sufficiently small, it acts trivially on W_0/p^s.) Thus we obtain a natural isomorphism of inductive systems

$$\{\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_{W_0})\}_{W_0\subset W} \xrightarrow{\sim} \{\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} W_0\}_{W_0\subset W}$$

(where these inductive systems are indexed by the collection of separated lattices W_0 in W). The naturality of this isomorphism shows that it is $\pi_0 \times G$-equivariant.

Tensoring these $\mathcal{O}_E$-modules over $\mathcal{O}_E$ with E, and then passing to the inductive limit in W_0, we obtain a natural isomorphism $\tilde{H}^n_*(K^{\mathfrak{p}}, \mathcal{V}_W) \xrightarrow{\sim} \tilde{H}^n_* \otimes_E W$. Now passing to inductive limit in $K^{\mathfrak{p}}$ yields the isomorphism of the theorem. (The $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariance follows from the naturality of the isomorphism.) $\square$

In light of this result, we may rewrite the spectral sequence of Theorem 2.2.11 (v) in the following manner.

Corollary 2.2.18. *There is a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant map*

$$H^n_*(\mathcal{V}_W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n_{*,\,\mathbb{Q}_p-\mathrm{la}})$$

which is the edge map of a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathfrak{g}}(\check{W}, \tilde{H}^j_{*,\,\mathbb{Q}_p-\mathrm{la}}) \implies H^{i+j}_*(\mathcal{V}_W).$$

Proof. Theorem 1.1.13 allows us to rewrite the spectral sequence given by part (v) of Theorem 2.2.11 in the form

$$E_2^{i,j} = H^i(\mathfrak{g}, \tilde{H}^j_*(K^{\mathfrak{p}}, \mathcal{V}_W)_{\mathbb{Q}_p-\mathrm{la}}) \implies H^{i+j}_*(K^{\mathfrak{p}}, \mathcal{V}_W).$$

Passing to the inductive limit in $K^{\mathfrak{p}}$ and taking into account the fact that computing both locally analytic vectors and Lie algebra cohomology commutes with the inductive limit (the former by [13, Prop. 3.5.14]) yields a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant spectral sequence

$$E_2^{i,j} = H^i(\mathfrak{g}, \tilde{H}^j_*(\mathcal{V}_W)_{\mathbb{Q}_p-\mathrm{la}}) \implies H^{i+j}_*(\mathcal{V}_W).$$

Passing to locally $\mathbb{Q}_p$-analytic vectors in Theorem 2.2.17, and taking into account [13, Prop. 3.6.15], yields an isomorphism $\tilde{H}^j_*(\mathcal{V}_W)_{\mathbb{Q}_p-\mathrm{la}} \xrightarrow{\sim} \tilde{H}^j_{*,\,\mathbb{Q}_p-\mathrm{la}} \otimes_E W$. Combining this with the preceding spectral sequence, we obtain the spectral sequence of the corollary. $\square$

This yields Theorem 0.5 of the introduction.

Remark 2.2.19. The edge map $H^n_*(\mathcal{V}_W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n_{*,\,\mathbb{Q}_p-\mathrm{la}})$ of the preceding corollary may be rewritten as a morphism $H^n_*(\mathcal{V}_W) \to (\tilde{H}^n_* \otimes_E W)_{\mathrm{sm}}$, where the subscript sm denotes the subspace of vectors on which G acts smoothly (as follows from the degree 0 case of the isomorphism of Theorem 1.1.13). Giving this morphism is in turn equivalent to giving a morphism $H^n_*(\mathcal{V}_W) \otimes_E \check{W} \to \tilde{H}^n_{*,\,\check{W}-\mathrm{lalg}}$, by [13, Prop. 4.2.4]. Composing this with the inclusion of $\tilde{H}^n_{*,\,\check{W}-\mathrm{lalg}}$ in $\tilde{H}^n_*$ gives the map (0.3) of the introduction.

Since $\check{W}$ is an algebraic representation of $\mathbb{G}$ over $F_{\mathfrak{p}}$, the locally $\check{W}$-algebraic vectors of $\tilde{H}^n_*$ are not merely locally $\mathbb{Q}_p$-analytic, but are in fact locally $F_{\mathfrak{p}}$-analytic. Thus the edge map of the preceding corollary factors as

$$H^n_*(\mathcal{V}_W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n_{*,\,F_{\mathfrak{p}}-\mathrm{la}}) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n_{*,\,\mathbb{Q}_p-\mathrm{la}}),$$

where the second arrow is induced by the closed embedding of the locally $F_{\mathfrak{p}}$-analytic vectors into the locally $\mathbb{Q}_p$-analytic vectors.

Remark 2.2.20. Theorem 2.2.17 shows that $\tilde{H}^n_*$, equipped with the data of the spectral sequences of the preceding corollary, one for each choice of W, forms the primitive object in our construction, from which all the other objects that we have considered may be obtained. Indeed, we recover $\tilde{H}^n_*(\mathcal{V}_W)$ as the tensor product $\tilde{H}^n_* \otimes_E W$. We may then recover $\hat{H}^n_*(\mathcal{V}_W)$ as the closure of the image of the map $H^n_*(\mathcal{V}_W) \to (\tilde{H}^n_* \otimes_E W)_{\mathrm{sm}} \to \tilde{H}^n_* \otimes_E W$ (where the first arrow is the edge map of the spectral sequence of Corollary 2.2.18, reinterpreted as in Remark 2.2.19, and the second arrow is the obvious inclusion), while $V_p H^n_*(\mathcal{V}_W)$ arises as the cokernel of the embedding $\hat{H}^{n-1}_*(\mathcal{V}_W) \to \tilde{H}^{n-1}_*(\mathcal{V}_W)$.

Our constructions are compatible with the cup-product on cohomology.

Proposition 2.2.21. *Fix a pair of natural numbers m and n. Also fix $*_1, *_2 \in \{\emptyset, c\}$, and take $*$ to be $\emptyset$ if $*_1 = *_2 = \emptyset$, and to be c otherwise.*

(i) *The cup-product on cohomology induces continuous $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant morphisms $\hat{H}^m_{*_1} \hat{\otimes}_E \hat{H}^n_{*_2} \to \hat{H}^{m+n}_*$, $\tilde{H}^m_{*_1} \hat{\otimes}_E \tilde{H}^n_{*_2} \to \tilde{H}^{m+n}_*$, and $V_p H^m_{*_1} \hat{\otimes}_E V_p H^n_{*_2} \to V_p H^{m+n}_*$, compatible with the exact sequences provided by Theorem 2.2.16 (iv).*

(ii) *If W_1 and W_2 are two finite dimensional representations of $\mathbb{G}$ defined over E, then the following diagram, in which the left-hand vertical arrow arises from the cup-product, the right-hand vertical arrow is that of part (i) (restricted to the indicated spaces of locally algebraic vectors) and the horizontal arrows are given by the edge-maps of the spectral sequences of Corollary 2.2.18, as reinterpreted in Remark 2.2.19, is commutative:*

$$\begin{CD} \big(H^m_{*_1}(\mathcal{V}_{W_1}) \otimes \check{W}_1\big) \otimes \big(H^n_{*_2}(\mathcal{V}_{W_2}) \otimes \check{W}_2\big) @>>> \tilde{H}^m_{*_1,\, \check{W}_1-\mathrm{lalg}} \hat{\otimes} \tilde{H}^n_{*_2,\, \check{W}_2-\mathrm{lalg}} \\ @VVV @VVV \\ H^{m+n}_*(\mathcal{V}_{W_1 \otimes W_2}) \otimes \big(\check{W}_1 \otimes \check{W}_2\big) @>>> \tilde{H}^{m+n}_{*,\, \check{W}_1 \otimes \check{W}_2-\mathrm{lalg}}. \end{CD}$$

Proof. This is an easy consequence of the naturality of the cup-product, and of our constructions. $\square$

In particular, taking $* = \emptyset$ and $m = n = 0$, we find that $\tilde{H}^0$ is naturally a topological E-algebra, equipped with an action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ as algebra automorphisms, and that for either choice of $*$ and each $n \geq 0$, the space $\tilde{H}^n_*$ is naturally a topological $\tilde{H}^0$-module, equipped with a compatible $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-action.

We now pass from the admissible continuous G-representation $\tilde{H}_*$ to its associated space of locally analytic vectors. More precisely, if L is any local field intermediate between $F_{\mathfrak{p}}$ and $\mathbb{Q}_p$, then we can consider the associated space of locally L-analytic vectors.

Theorem 2.2.22. (i) *The space $\tilde{H}^n_{*,\, L-\mathrm{la}}$ is an admissible locally analytic representation of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ (in the sense of* [13, Def. 7.2.7]).

(ii) *For any compact open subgroup $K^{\mathfrak{p}}$ of $\mathbb{G}(\mathbb{A}^{\mathfrak{p}}_f)$, the natural map*

$$\tilde{H}^n_*(K^{\mathfrak{p}})_{L-\mathrm{la}} \to (\tilde{H}^n_{*,\, L-\mathrm{la}})^{K^{\mathfrak{p}}}$$

is a $\pi_0 \times G$-equivariant isomorphism.

(iii) *If W is any finite dimensional representation of $\mathbb{G}$ over E, then there is a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant closed embedding of $\tilde{H}^n_{*,\, W-\mathrm{lalg}}$, equipped with its finest convex topology, into $\tilde{H}^n_{*,\, L-\mathrm{la}}$.*

Proof. Part (i) (respectively part (ii)) follows from the corresponding part of Theorem 2.2.16, together with Proposition 7.2.11 (respectively Proposition 7.2.5) of [13], while part (iii) follows from part (i) together with Propositions 7.2.13 and 7.2.14 of [13]. $\square$

This yields Theorem 0.2 of the introduction.

Proposition 2.2.23. *If m and n are two natural numbers, and $*_1$, $*_2$, and $*$ are as in Proposition 2.2.21, then for any local field L intermediate between $F_{\mathfrak{p}}$ and $\mathbb{Q}_p$, the cup-product of cohomology classes induces a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant pairing $\tilde{H}^m_{*_1,\, L-\mathrm{la}} \hat{\otimes}_E \tilde{H}^n_{*_2,\, L-\mathrm{la}} \to \tilde{H}^{m+n}_{*,\, L-\mathrm{la}}$.*

Proof. This follows from [13, 3.5.15], applied to the pairing $\tilde{H}^m_{*_1} \hat{\otimes}_E \tilde{H}^n_{*_2} \to \tilde{H}^{m+n}_{*}$ of Proposition 2.2.21. □

Since π_0 is naturally identified with the group of connected components of G_∞, we may regard $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$ as being a quotient of the group $\mathbb{G}(\mathbb{A})$ in a natural way, and so regard the $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representations considered above as being $\mathbb{G}(\mathbb{A})$-representations. We have preferred to describe them explicitly as $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representations, since this emphasizes that the real Lie group G_∞ intervenes only through the group of connected components π_0.

We close this subsection with the following simple propositions and their corollaries.

Proposition 2.2.24. *If $n = 0$ or 1 then $V_p H^n$ vanishes.*

Proof. It is a general fact that $H^0(-, \mathcal{O}_E)$ and $H^1(-, \mathcal{O}_E)$ are torsion free. □

Corollary 2.2.25. *The embedding $\hat{H}^0 \to \tilde{H}^0$ is an isomorphism. Furthermore, for any representation W of $\mathbb{G}$ defined over E, the natural map $H^0(\mathcal{V}_W) \otimes_E \check{W} \to \hat{H}^0_{W-\mathrm{lalg}}$ is an isomorphism.*

Proof. The first claim of the corollary follows from the exact sequence of Theorem 2.2.16 (iv), and the case $n = 1$ of Proposition 2.2.24. The second claim follows from an examination of the spectral sequence of Corollary 2.2.18 in the case when $i = j = 0$. □

Proposition 2.2.26. (*i*) *If $n = 0$ or 1 then $V_p H^n_c$ vanishes.*

(*ii*) *If d is the dimension of G_∞ / K_∞, then $V_p H^d_c$ and $V_p H^{d+1}_c$ both vanish.*

Proof. Part (i) follows from the general fact that $H^0_c(-, \mathcal{O}_E)$ and $H^1_c(-, \mathcal{O}_E)$ are torsion free. Since the dimension of each quotient $Y(K_f)$ is equal to d, we see that $H^d_c(Y(K_f), \mathcal{O}_E)$ is the free $\mathcal{O}_E$-module spanned by the fundamental classes of the connected components of $Y(K_f)$, while $H^{d+1}_c(Y(K_f), \mathcal{O}_E)$ vanishes. Thus part (ii) holds. □

Corollary 2.2.27. *The embeddings $\hat{H}^{d-1}_c \to \tilde{H}^{d-1}_c$ and $\hat{H}^d_c \to \tilde{H}^d_c$ are both isomorphisms.*

Proof. The corollary follows from the exact sequence of Theorem 2.2.16 (iv), and Proposition 2.2.26. □

(**2.3**) This subsection presents the proof of Theorem 0.7. We maintain the notation of the preceding subsection, and suppose in addition that $\mathbb{G}$ is split over E (so that any irreducible representation of $\mathbb{G}$ over E is absolutely irreducible). We also suppose that $\mathbb{G}$ is quasi-split over $F_{\mathfrak{p}}$, and choose a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$ defined over $F_{\mathfrak{p}}$, as well as a maximal torus $\mathbb{T}$ of $\mathbb{B}$ (which then splits over E, since $\mathbb{G}$ is assumed to split over E). Write $B := \mathbb{B}(F_{\mathfrak{p}})$ and $T := \mathbb{T}(F_{\mathfrak{p}})$.

Let us fix a tame level $K^{\mathfrak{p}}$.

Definition 2.3.1. We say that $K^{\mathfrak{p}}$ is unramified at a place $v \neq \mathfrak{p}$ of F if:

(i) $\mathbb{G}$ is unramified at v; that is, if $\mathbb{G}$ is quasi-split over F_v, and splits over an unramified extension of F_v;

(ii) The compact open subgroup $K_v^{\mathfrak{p}} := K^{\mathfrak{p}} \bigcap \mathbb{G}(F_v)$ of $\mathbb{G}(F_v)$ is a hyperspecial maximal compact subgroup of $\mathbb{G}(F_v)$.

Otherwise, we say that $K^{\mathfrak{p}}$ is ramified at v.

We let S denote the (finite) set of ramified primes of $K^{\mathfrak{p}}$. We write $\mathbb{A}_f^{\mathfrak{p},S}$ to denote the prime to $S \bigcup \{\mathfrak{p}\}$ finite adèles, and also write $F_S = \prod_{v \in S} F_v$ (so that $\mathbb{A}_f^{\mathfrak{p}} = F_S \times \mathbb{A}_f^{\mathfrak{p},S}$). If we write $K_S^{\mathfrak{p}} := K^{\mathfrak{p}} \bigcap \mathbb{G}(F_S)$, and write $K^{\mathfrak{p},S} := K^{\mathfrak{p}} \bigcap \mathbb{G}(\mathbb{A}_f^{\mathfrak{p},S})$, then $K^{\mathfrak{p}} = K_S^{\mathfrak{p}} \times K^{\mathfrak{p},S}$. Also, $K^{\mathfrak{p},S} = \prod_{v \notin S \bigcup \{\mathfrak{p}\}} K_v^{\mathfrak{p}}$.

If H is a compact open subgroup of a locally compact group G, then $\mathcal{H}(G//H)$ will denote the Hecke algebra of H double cosets in G, with coefficients in E. We will abbreviate $\mathcal{H}(\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})//K^{\mathfrak{p}})$, $\mathcal{H}(\mathbb{G}(F_S)//K_S^{\mathfrak{p}})$, and $\mathcal{H}(\mathbb{G}(\mathbb{A}_f^{\mathfrak{p},S})//K^{\mathfrak{p},S})$ by $\mathcal{H}(K^{\mathfrak{p}})$, $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}$, and $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ respectively. The product decomposition of $K^{\mathfrak{p}}$ induces a corresponding tensor product decomposition of Hecke algebras:

$$\mathcal{H}(K^{\mathfrak{p}}) \xrightarrow{\sim} \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}} \otimes_E \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}.$$

Similarly, the product decomposition of $K^{\mathfrak{p},S}$ induces a decomposition of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ as a restricted tensor product: $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}} \xrightarrow{\sim} \bigotimes' \mathcal{H}(\mathbb{G}(F_v)//K_v^{\mathfrak{p}})$. The Satake isomorphism shows that each of the algebras $\mathcal{H}(\mathbb{G}(F_v)//K_v^{\mathfrak{p}})$ is a commutative algebra of finite type over E. (See [19, §16] for a clear discussion of the Satake isomorphism.) Thus $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ is commutative, and forms a central subalgebra of $\mathcal{H}(K^{\mathfrak{p}})$.

Since, by Theorem 2.2.22 (i), the space $\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}}$ is naturally identified with the space of $K^{\mathfrak{p}}$-invariants in the admissible locally $F_{\mathfrak{p}}$-analytic $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation $\tilde{H}^n_{F_{\mathfrak{p}}-\mathrm{la}}$, it is equipped with a locally $F_{\mathfrak{p}}$-analytic action of G, together with commuting actions of π_0 and $\mathcal{H}(K^{\mathfrak{p}})$ by continuous operators. The Jacquet module $J_B(\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}})$ (as defined in [14]) is then an essentially admissible locally $F_{\mathfrak{p}}$-analytic representation of the torus T, again equipped with commuting actions of π_0 and $\mathcal{H}(K^{\mathfrak{p}})$ by continuous operators.

As in the introduction, let $\hat{T}$ denote the rigid analytic variety over E that parameterizes the locally $F_{\mathfrak{p}}$-analytic characters of T. (We refer to [13, §6.4] for an explanation of the construction of $\hat{T}$.) Let $\mathcal{C}^{\mathrm{an}}(\hat{T}, E)$ denote the E-Fréchet algebra of rigid analytic functions on $\hat{T}$. If $\mathcal{F}$ is a coherent rigid analytic sheaf on $\hat{T}$, then the space of global sections $\Gamma(\hat{T}, \mathcal{F})$ is naturally a topological Fréchet module over $\mathcal{C}^{\mathrm{an}}(\hat{T}, E)$, and the theory of quasi-Stein rigid analytic spaces shows that taking global sections induces an equivalence of categories between the category of coherent rigid analytic sheaves on $\hat{T}$, and the category of coadmissible (in the sense of [31]) $\mathcal{C}^{\mathrm{an}}(\hat{T}, E)$-modules. Evaluating characters at elements of T induces a continuous injection $T \to \mathcal{C}^{\mathrm{an}}(\hat{T}, E)^{\times}$. Thus if $\mathcal{F}$ is a coherent rigid analytic sheaf on $\hat{T}$, then the space $\Gamma(\hat{T}, \mathcal{F})$ is naturally equipped with a continuous T-representation, and hence so is its strong dual.

The following proposition then amounts to little more than the definition of essentially admissible locally $F_{\mathfrak{p}}$-analytic T-representations. (See [13, §6.4].)

Proposition 2.3.2. *There is an antiequivalence of categories between the category of coherent rigid analytic sheaves on $\hat{T}$, and the category of essentially admissible* *locally $F_{\mathfrak{p}}$-analytic representations of T defined over E, given by associating to any coherent sheaf $\mathcal{F}$ on $\hat{T}$ the strong dual of its Fréchet space of global sections $\Gamma(\hat{T}, \mathcal{F})$, equipped with the T-action described above.*

We let $\mathcal{E}$ denote the coherent rigid analytic sheaf on $\hat{T}$ corresponding via the antiequivalence of categories of Proposition 2.3.2 to $J_B(\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}})$, and let $\mathrm{Exp}(\mathcal{E})$ denote the support of $\mathcal{E}$ (the "set of exponents" appearing in $\mathcal{E}$).

The action of the commutative E-algebra $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ on $J_B(\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}})$ induces an action of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ on the coherent sheaf $\mathcal{E}$, and thus gives rise to a coherent subsheaf of commutative rings $\mathcal{A}$ inside the endomorphism sheaf of $\mathcal{E}$. We form the relative Spec of $\mathcal{A}$ over $\hat{T}$; by construction it admits a Zariski closed embedding $\mathrm{Spec}\,\mathcal{A} \to \hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$. Since $\mathcal{A}$ acts as endomorphisms of $\mathcal{E}$, we may localize $\mathcal{E}$ to a coherent sheaf $\mathcal{M}$ on $\mathrm{Spec}\,\mathcal{A}$.

The action of the group ring $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}[\pi_0]$ on $J_B(\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}})$ commutes with the $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$-action, and so the sheaf $\mathcal{M}$ is naturally a coherent sheaf of right $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ram}}[\pi_0]$-modules.

Proposition 2.3.3. (*i*) *The natural projection* $\mathrm{Spec}\,\mathcal{A} \to \hat{T}$ *is a finite morphism, with set-theoretic image equal to* $\mathrm{Exp}(\mathcal{E})$.

(*ii*) *The map* $\mathrm{Spec}\,\mathcal{A} \to \check{\mathfrak{t}}$ (*where* $\check{\mathfrak{t}}$ *is the dual to the Lie algebra* $\mathfrak{t}$ *of* T) *obtained by composing the projection of* (*i*) *with differentiation of characters has discrete fibres. In particular, the dimension of* $\mathrm{Spec}\,\mathcal{A}$ *is at most equal to the dimension of* T.

(*iii*) *The fibre of* $\mathcal{M}$ *over a point* (χ, λ) *of* $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ *is dual to the* $(T = \chi,\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}} = \lambda)$*-eigenspace of* $J_B(\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}})$. *In particular, the point* (χ, λ) *lies in* $\mathrm{Spec}\,\mathcal{A}$ *if and only if this eigenspace is non-zero.*

Proof. Parts (i) and (iii) are immediate from the construction of $\mathcal{A}$. Part (ii) follows from [14, 0.11]. $\square$

We let $\tilde{\mathbb{G}}$, $\tilde{\mathbb{B}}$, $\tilde{\mathbb{T}}$ denote the restriction of scalars from $F_{\mathfrak{p}}$ to $\mathbb{Q}_p$ of $\mathbb{G}$, $\mathbb{B}$, and $\mathbb{T}$ respectively. Since E contains $F_{\mathfrak{p}}$, there are natural projections

$$\tilde{\mathbb{G}}_{/E} \to \mathbb{G}_{/E}, \tag{2.3.4}$$

and

$$\tilde{\mathbb{T}}_{/E} \to \mathbb{T}_{/E}. \tag{2.3.5}$$

Let W be an irreducible algebraic representation of $\mathbb{G}$ over E, whose highest weight with respect to $\mathbb{B}$ is the character ψ of $\mathbb{T}$. Then we may regard W as an irreducible representation of $\tilde{\mathbb{G}}$ via (2.3.4), whose highest weight with respect to $\tilde{\mathbb{B}}$ is then ψ, regarded as a character of $\tilde{\mathbb{T}}$ via (2.3.5). Also, by definition $\tilde{\mathbb{T}}(\mathbb{Q}_p) = \mathbb{T}(F_{\mathfrak{p}}) = T$. Thus if θ is a smooth character of T, the locally algebraic character $\theta\psi$ of T may equally well be regarded as a locally algebraic character of $\tilde{\mathbb{T}}(\mathbb{Q}_p)$, to which we may apply Definition 4.4.3 of [14].

Proposition 2.3.6. *If* $\chi := \theta\psi$ *is of non-critical slope* (*in the sense of* [14, Def. 4.4.3]), *then the closed embedding* $J_B(\tilde{H}^n(K^{\mathfrak{p}})_{W-\mathrm{lalg}}) \to J_B(\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\mathrm{la}})$ *induces an isomorphism on* χ*-eigenspaces.*

Proof. This follows from [14, Thm. 4.4.5] and the fact that $\tilde{H}^n(K^{\mathfrak{p}})$ admits a $\mathbb{G}(F_{\mathfrak{p}})$-invariant norm. $\square$

Suppose now that the natural map

$$H^n(\mathcal{V}_{\check{W}}) \otimes_E W \to \tilde{H}^n_{W-\mathrm{lalg}} \tag{2.3.7}$$

provided by Corollary 2.2.18 (see Remark 2.2.19) is an isomorphism for every choice of irreducible representation W of $\mathbb{G}$ over E (and consequently for any such representation, since this map is compatible with the formation of direct sums of representations). Recall that $\operatorname{Spec}\mathcal{A}$ and the eigenvariety $E(n, K^{\mathfrak{p}})$ (as defined in Definition 0.6) are both Zariski closed subspaces of $\hat{T} \times \operatorname{Spec}\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.

Proposition 2.3.8. *If (2.3.7) is an isomorphism for all W then* $\operatorname{Spec}\mathcal{A}$ *contains* $E(n, K^{\mathfrak{p}})$ *as a closed subspace.*

Proof. Since $E(n, K^{\mathfrak{p}})$ is defined to be the Zariski closure in $\hat{T} \times \operatorname{Spec}\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ of $E(n, K^{\mathfrak{p}})_{\mathrm{cl}}$, it suffices to show that the latter set of points is contained in $\operatorname{Spec}\mathcal{A}$. A typical point of $E(n, K^{\mathfrak{p}})$ is of the form $(\theta\psi, \lambda)$, where as above θ is a smooth character of T and ψ is the highest weight of an irreducible $\mathbb{G}$-representation W, such that there exists an irreducible subquotient of $H^n(K^{\mathfrak{p}}, \mathcal{V}_{\check{W}})$ which may be embedded into $\operatorname{Ind}_{\mathbb{B}(F_{\mathfrak{p}})}^{\mathbb{G}(F_{\mathfrak{p}})} \theta$, and on which $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ acts via λ. (Here and in the ensuing discussion, it is implicit that we base-change to a field of definition of λ.) Since the Jacquet functor is exact when applied to smooth representations [7, Prop. 3.2.3], we see that θ appears in the λ-eigenspace of $J_B(H^n(K^{\mathfrak{p}}, \mathcal{V}_{\check{W}}))$, and so by [14, Prop. 4.3.6] the product $\theta\psi$ appears in $J_B(H^n(K^{\mathfrak{p}}, \mathcal{V}_{\check{W}}) \otimes_E W)$. By assumption $H^n(K^{\mathfrak{p}}, \mathcal{V}_{\check{W}}) \otimes_E W$ maps isomorphically to the closed subrepresentation $\tilde{H}^n_{W-\mathrm{lalg}}$ of $\tilde{H}^n_{F_{\mathfrak{p}}-\mathrm{la}}$, and so $\theta\psi$ also appears in the λ-eigenspace of $J_B(\tilde{H}^n_{F_{\mathfrak{p}}-\mathrm{la}})$. The present proposition thus follows from Proposition 2.3.3 (iii). $\square$

Theorem 0.7 follows from Propositions 2.3.3, 2.3.6, and 2.3.8 taken together.

In light of Proposition 2.3.8, it is natural to ask whether or not $E(n, K^{\mathfrak{p}})$ is equal to $\mathcal{A}$, or equivalently, whether or not the points of $E(n, K^{\mathfrak{p}})_{\mathrm{cl}}$ are Zariski dense in $\mathcal{A}$. We do not know the answer to this question in general (even under the hypotheses of Theorem 0.7).

(2.4) The aim of this short subsection is to show that, when the arithmetic quotients attached to $\mathbb{G}$ admit the structure of Shimura varieties, the constructions of Subsection 3.3 are compatible with the comparison isomorphism between usual and étale cohomology, and hence that the various spaces considered are equipped with Galois actions, and the various short exact and spectral sequences constructed are Galois equivariant.

Let $\mathbb{G}'$ denote the restriction of scalars of $\mathbb{G}$ from F to $\mathbb{Q}$, let $\mathbb{S}$ denotes the maximal split torus in the centre of $\mathbb{G}'$, and let $\mathbb{S}'$ denote the maximal split torus quotient of $\mathbb{G}'$. The natural map $\mathbb{S} \to \mathbb{S}'$ is an isogeny, and so if $S_\infty := \mathbb{S}(\mathbb{R})$ and $S'_\infty := \mathbb{S}'(\mathbb{R})$, then the induced map $S^\circ_\infty \to S'^\circ_\infty$ is an isomorphism. Let Z_∞ denote the centre of G_∞ (so Z_∞ contains S_∞, but is typically larger), and let Z^1_∞ denote the kernel of the natural map $Z_\infty \to S'_\infty$.

The spaces $Y(K_f)$ that we have defined are not the arithmetic quotients typically considered in the theory of automorphic forms; it is more usual to consider the quotients $Y^\circ(K_f) := Y(K_f)/S^\circ_\infty$, which have the advantage of being of finite volume.

(These are the arithmetic quotients considered in [18], for example.) The projection $Y(K_f) \to Y^\circ(K_f)$ makes the source an S_∞°-bundle over the target. Thus (since S_∞ is isomorphic to $\mathbb{R}^m$ for some m, and in particular is contractible) the local system $\mathcal{V}_W$ on $Y(K_f)$ attached to any algebraic $\mathbb{G}$-representation W on $Y(K_f)$ may be descended to $Y^\circ(K_f)$, and the cohomology and compactly supported cohomology spaces of $Y^\circ(K_f)$ with coefficients in $\mathcal{V}_W$ are isomorphic to the corresponding cohomology spaces of $Y(K_f)$ (up to a shift of degree, in the case of compactly supported cohomology).

In the theory of Shimura varieties it is customary to consider a different quotient again, namely $\tilde{Y}(K_f) := Y(K_f)/Z_\infty^\circ$. The projection map $Y^\circ(K_f) \to \tilde{Y}(K_f)$ makes the source a d-dimensional torus bundle over the target (in the topological sense: each fibre is a product of d circles), where d is the difference between the split rank of Z_∞ and the rank of S_∞ (i.e. the split rank of Z_∞^1). If W is an algebraic representation of $\mathbb{G}$, then the local system $\mathcal{V}_W$ on $Y(K_f)$ descends to $\tilde{Y}(K_f)$ if and only if Z_∞^1 acts trivially on W. (Note that if this condition does not hold, then the local system $\mathcal{V}_W$ has trivial cohomology on $Y(K_f)$; thus nothing is lost in any case by restricting attention to those W that satisfy this condition.)

For the remainder of this section, we assume that S_∞ is in fact the maximal split torus in Z_∞, so that $Y^\circ(K_f) = \tilde{Y}(K_f)$. This allows us to replace the quotients $Y(K_f)$ by the corresponding quotients $\tilde{Y}(K_f)$ in the results of Subsection 2.2. (Our assumption is a common one in the Shimura variety context, although it omits certain examples, such as Shimura curves and Hilbert modular varieties over totally real fields of degree greater than one. We adopt it more for simplicity than out of necessity, since we could get around it by working systematically with the quotients $\tilde{Y}(K_f)$ in Subsection 2.2, provided that we restricted our attention to representations W on which Z_∞^1 acts trivially.)

Suppose given a number field $F' \subset \mathbb{C}$, a projective system of schemes $\tilde{Y}(K_f)_{/F'}$ defined over F' indexed by the compact open subgroups K_f of $\mathbb{G}(\mathbb{A}_f)$, and an action of $\mathbb{G}(\mathbb{A}_f)$ defined on this projective system, such that when we take the $\mathbb{C}$-valued point of this projective system, the resulting projective system of spaces $\tilde{Y}(K_f)_{/F'}(\mathbb{C})$ with its $\mathbb{G}(\mathbb{A}_f)$-action is isomorphic to the projective system of arithmetic quotients $\tilde{Y}(K_f)$, with its $\mathbb{G}(\mathbb{A}_f)$-action. (More precisely, we should allow $\tilde{Y}(K_f)_{/F'}$ to be a stack, in the case when $\tilde{Y}(K_f)$ is an orbifold rather than a manifold. However, we will ignore this technicality, since we may restrict our attention to those K_f that are sufficiently small, in which case $\tilde{Y}(K_f)$ is a manifold.) If the group $\mathbb{G}'$ admits a Shimura datum with reflex field equal to F', then the theory of canonical models of Shimura varieties shows that such schemes do exist.

Now suppose that M is a finite G_0-module, for some compact open subgroup G_0 of G, and assume that the G_0-action on M is continuous, when M is endowed with its discrete topology. We may then follow the prescription of [24, §3] to define a compatible system of étale sheaves $\mathcal{V}_{M,\text{ét}}$ on the projective system of schemes $\tilde{Y}(K_f)_{/F'}$. More precisely, we can define the sheaf $\mathcal{V}_{M,\text{ét}}$ on $\tilde{Y}(K_f)_{/F'}$ provided that the projection of K_f onto the $\mathfrak{p}$th factor is contained in G_0. (This will certainly hold if K_f is small enough.) Assuming that K_f satisfies this hypothesis, we choose a compact open normal subgroup $K_f' \subset K_f$ such that the projection of K_f' onto the $\mathfrak{p}$th factor is contained in the pointwise stabilizer of M, and define $\mathcal{V}_{M,\text{ét}}$ to be the étale sheaf represented by the étale cover $M \times_{K_f} \tilde{Y}(K_f')_{/F'}$ of $\tilde{Y}(K_f)_{/F'}$. The resulting sheaf is, up to canonical isomorphism, independent of the choice of K_f'.

After passing to $\mathbb{C}$-valued points, the étale sheaf $\mathcal{V}_{M,\text{ét}}$ corresponds to the sheaf $\mathcal{V}_M$ of Definition 2.2.3.

The comparison isomorphism between étale and classical cohomology yields a canonical isomorphism

$$H^\bullet_{\text{ét}}(\mathbb{C} \times_{F'} \tilde{Y}(K_f)_{/F'}, \mathcal{V}_{M,\text{ét}}) \xrightarrow{\sim} H^\bullet(\tilde{Y}(K_f), \mathcal{V}_M).$$

If $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, then there is a natural action of $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$ on the left-hand side of this isomorphism, which may then be transported to the right-hand side.

If W is a representation of $\mathbb{G}$ defined over E, then the spaces $\hat{H}^\bullet(\mathcal{V}_W)$, $\tilde{H}^\bullet(\mathcal{V}_W)$, and $V_p H^\bullet(\mathcal{V}_W)$ constructed in Subsection 2.2 may all be defined as limits with respect to natural transition maps of various kinds of cohomology groups of the form $H^\bullet(\tilde{Y}(K_{\mathfrak{p}}K^{\mathfrak{p}}), \mathcal{V}_{W_0/p^s})$, for some compact open subgroup $K_{\mathfrak{p}}$ of G, some tame level $K^{\mathfrak{p}}$, and some $K_{\mathfrak{p}}$-invariant sublattice W_0 of W. Thus each of them is equipped (via the comparison isomorphism) with a continuous $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$-action, commuting with the $\mathbb{G}(\mathbb{A}_f)$-action.

By virtue of the natural sheaf theoretic nature of its construction, the short exact sequence of Theorem 2.2.16 (iv) is compatible with respect to these $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$-actions. Also, by the very definition of the étale sheaf $\mathcal{V}_{W_0/p^s}$, the isomorphism of Theorem 2.2.17 is compatible with respect to the comparison isomorphism, and hence with respect to the actions of $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$ on its source and target (the $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$-action on $\tilde{H}^\bullet \otimes_{F'} W$ being defined to be trivial on the second factor). Since passing to locally analytic vector is functorial, the Galois action on $\tilde{H}^\bullet$ induces a continuous $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$-action on $\tilde{H}^\bullet_{L-\text{la}}$, for each local field L intermediate between $F_{\mathfrak{p}}$ and $\mathbb{Q}_p$.

Proposition 2.4.1. *In the situation under consideration, the spectral sequence of Corollary 2.2.18 is* $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$*-equivariant.*

Proof. This spectral sequence is obtained by applying part (ii) of Theorem 2.1.5. Although the proof of that theorem uses simplicial methods, Proposition 2.1.11 gives an alternative construction of the same spectral sequence using just the Hochschild-Serre spectral sequence for cohomology with coefficients in the torsion local systems $\mathcal{V}_{W_0/p^s}$. One has such a Hochschild-Serre spectral sequence also for étale cohomology, which is compatible with the comparison maps between singular and étale cohomology with torsion coefficients. Thus the construction of Proposition 2.1.11 applies equally well in the étale cohomological context, and the present proposition follows. $\square$

Finally, suppose that $\mathbb{G}$ is quasi-split over $F_{\mathfrak{p}}$, so that we may form the eigenvariety $E(n, K^{\mathfrak{p}})$ for each cohomological degree n and each tame level $K^{\mathfrak{p}}$. Suppose furthermore that (2.3.6) is an isomorphism for some choice of n and $K^{\mathfrak{p}}$, and for all W. Theorem 0.7 yields a coherent sheaf $\mathcal{M}$ over $E(n, K^{\mathfrak{p}})$. Since this sheaf is constructed by applying the functor J_B to $\tilde{H}^n(K^{\mathfrak{p}})_{F_{\mathfrak{p}}-\text{la}}$, we see that it inherits a Galois action, and so forms a coherent sheaf of $\operatorname{Gal}(\overline{\mathbb{Q}}/F')$-modules on $E(n, K^{\mathfrak{p}})$ (and this action is compatible with the right $\mathcal{H}(K^{\mathfrak{p}})^{\text{ram}}$-module structure on $\mathcal{M}$).

3. $\mathfrak{p}$-adic automorphic representations

(3.1) In this subsection we put ourselves in the situation of Subsection 2.2, and we employ the notation introduced in that subsection. The given inclusion $F \subset \mathbb{C}$

determines a map

$$F_\infty := \mathbb{R} \otimes_{\mathbb{Q}} F \to \mathbb{C}, \tag{3.1.1}$$

and hence a map

$$G_\infty \to \mathbb{G}(\mathbb{C}). \tag{3.1.2}$$

We let $\mathfrak{z}(\mathfrak{g}_\infty)$ denote the centre of the universal enveloping algebra of the lie algebra $\mathfrak{g}_\infty$ of G_∞.

Definition 3.1.3. (i) Let χ be a complex-valued character of $\mathfrak{z}(\mathfrak{g}_\infty)$. We say that χ is allowable if there is an irreducible representation W of $\mathbb{G}$ over $\mathbb{C}$ such that the representation of G_∞ on W induced by the map (3.1.2) has infinitesimal character equal to χ. (Note that W is unique up to isomorphism if it exists, since irreducible representations of $\mathbb{G}_{/\mathbb{C}}$ are determined up to isomorphism by their infinitesimal characters.)

(ii) Let π_∞ be an irreducible admissible representation of the real reductive group G_∞. We say that π_∞ is allowable if the infinitesimal character of π_∞ is allowable in the sense of (i). If we wish to be precise, then we will say that π_∞ is W-allowable, where W is the representation of $\mathbb{G}_{\mathbb{C}}$ whose infinitesimal character (as a representation of G_∞) coincides with that of π_∞.

(iii) Let π be an automorphic representation of $\mathbb{G}(\mathbb{A})$, and write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is an admissible representation of G_∞, and π_f is a smooth representation of $\mathbb{G}(\mathbb{A}_f)$. We say that π is allowable if π_∞ is allowable, and if π_f can be defined over a finite extension of $\mathbb{Q}$.

For example, if $F = \mathbb{Q}$ then the map (3.1.1) above is simply the inclusion $\mathbb{R} \subset \mathbb{C}$. Thus in this case a character of $\mathfrak{z}(\mathfrak{g}_\infty)$ is allowable if and only if it is the infinitesimal character of an irreducible finite dimensional representation of $\mathbb{G}$ over $\mathbb{C}$.

For general F, let $\mathbb{G}'$ denote the restriction of scalars of $\mathbb{G}$ to $\mathbb{Q}$, so that $G_\infty = \mathbb{G}'(\mathbb{R})$. A character of $\mathfrak{z}(\mathfrak{g}_\infty)$ is allowable if it is the infinitesimal character of an irreducible algebraic representation of $\mathbb{G}'$ of a certain kind, namely one pulled back from a finite dimensional representation of $\mathbb{G}$ over $\mathbb{C}$ by the surjection $\mathbb{G}'_{/\mathbb{C}} \to \mathbb{G}_{/\mathbb{C}}$ induced by (3.1.1).

Note in particular that although we may regard an automorphic representation π of $\mathbb{G}(\mathbb{A}_F)$ equally well as an automorphic representation of $\mathbb{G}'(\mathbb{A}_{\mathbb{Q}})$ (where we have written $\mathbb{A}_F$ to denote the adèles of F and $\mathbb{A}_{\mathbb{Q}}$ to denote the adèles of $\mathbb{Q}$), the condition that π be allowable as a representation of $\mathbb{G}(\mathbb{A}_F)$ is not equivalent to the condition that it be allowable as a representation of $\mathbb{G}'(\mathbb{A}_{\mathbb{Q}})$. (The former implies the latter, but not conversely, in general.)[1]

Since the infinitesimal character of any irreducible algebraic representation of $\mathbb{G}'$ is regular, it follows that an allowable character of $\mathfrak{z}(\mathfrak{g}_\infty)$ is necessarily regular. It then follows from [10, Thm. 3.13] that for $\mathbb{G} = \mathrm{GL}_n$, the representation π_∞ being allowable implies that π_f is definable over a finite extension of $\mathbb{Q}$.

We recall the following simple lemma.

[1] For example, an automorphic representation of $\mathrm{GL}_{1/F}$ (that is, an idéle class character over F) is allowable if its restriction to the connected component of $F_\infty^\times$ (thought of as the component at infinity of $\mathbb{A}_F^\times$) is equal to the composite of the map $F_\infty^\times \to \mathbb{C}^\times$ induced by (3.1.1) and an algebraic character $z \to z^n$ of $\mathbb{C}^\times$. (Dirichlet's unit theorem shows that there are no such characters with $n \neq 0$ except in the case when $F = \mathbb{Q}$ or F is quadratic imaginary.) On the other hand, if $\mathbb{G}'$ denotes the restriction of scalars of $\mathrm{GL}_{1/F}$ from F to $\mathbb{Q}$, then any algebraic Hecke character gives an allowable automorphic representation of $\mathbb{G}'(\mathbb{A}_{\mathbb{Q}})$.

Lemma 3.1.4. *Let V be an irreducible admissible smooth representation of a p-adic reductive group H defined over an algebraically closed field Ω. If V_1 and V_2 are two smooth representations of H defined over a subfield E of Ω, equipped with isomorphisms $\Omega \otimes_E V_1 \xrightarrow{\sim} \Omega \otimes_E V_2 \xrightarrow{\sim} V$, then there is an isomorphism of $V_1 \xrightarrow{\sim} V_2$ of H-representations over E, uniquely determined up to multiplication by a non-zero element of E.*

Proof. Let U be a compact open subgroup of H such that $V^U \neq 0$. Note that V_1^U and V_2^U are then also non-zero; indeed $V^U \xrightarrow{\sim} \Omega \otimes_E V_1^U \xrightarrow{\sim} \Omega \otimes_E V_2^U$. Each of these spaces of invariants is a representation of the Hecke algebra $\mathcal{H}(H//U)$ defined over E of U double cosets in H. Furthermore, each of these spaces of invariants is finite dimensional over its field of definition (either Ω or E), since V is assumed admissible.

From [3, Cor. 3.9 (ii)] we see that we may (and do) choose U so that the functor $W \mapsto W^U$ induces an equivalence of categories between the category of smooth H-representations W (defined over either Ω or E) that are generated over H by W^U, and the category of $\mathcal{H}(H//U)$-modules (defined over either Ω or E). Our assumption may then be rephrased as saying that we have an Ω-valued point of the algebraic variety of $\mathcal{H}(H//U)$-equivariant isomorphisms between V_1^U and V_2^U. The Nullstellensatz assures us that we may then find a point of this variety defined over the algebraic closure of E, and thus we may assume that Ω is equal to this algebraic closure.

We now note that since V is irreducible, Schur's Lemma implies that $\mathrm{End}_H(V)$ is isomorphic to Ω. The lemma thus follows from Hilbert's Theorem 90. $\square$

Let π be an automorphic representation of $\mathbb{G}(\mathbb{A})$ that is allowable in the sense of Definition 3.1.3 (iii), and write $\pi_f = \pi_{\mathfrak{p}} \otimes \pi^{\mathfrak{p}}$, where $\pi_{\mathfrak{p}}$ is a smooth representation of G and $\pi^{\mathfrak{p}}$ is a smooth representation of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$. Let W be the finite dimensional irreducible representation of $\mathbb{G}$ over $\mathbb{C}$ for which π_∞ is W-allowable.

Via the isomorphism $\imath : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, we may regard $\pi_f = \pi_{\mathfrak{p}} \otimes \pi^{\mathfrak{p}}$ as an admissible representation of $\mathbb{G}(\mathbb{A}_f)$ on a $\overline{\mathbb{Q}}_p$-vector space, and we may also regard W as a $\overline{\mathbb{Q}}_p$-vector space equipped with a finite dimensional representation of $\mathbb{G}_{/\overline{\mathbb{Q}}_p}$, and so in particular with a representation of G.

We may descend π_f to a finite extension of $F_{\mathfrak{p}}$ (since we assumed that it may be descended even to a finite extension of $\mathbb{Q}$). Let E be a finite extension of $F_{\mathfrak{p}}$ to which π_f may be descended, and assume that E is chosen so large that $\mathbb{G}$ splits over E. This implies that W may also be descended to a representation of $\mathbb{G}$ defined over E.

Let π_f now denote a descent of π_f to E (uniquely determined up to isomorphism by Lemma 3.1.4) and W denote a descent of W to E (uniquely determined up to isomorphism by highest weight theory). Write $\tilde{\pi}_{\mathfrak{p}} := \pi_{\mathfrak{p}} \otimes_E W$ (equipped with the diagonal action of G), and (for notational consistency) set $\tilde{\pi}^{\mathfrak{p}} = \pi_f^{\mathfrak{p}}$.

We set $\Gamma = \mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$, so that $\mathbb{G}(\mathbb{A}_f) = G \times \Gamma$ is the product of a locally L-analytic group and a locally compact group satisfying the hypothesis of [13, §7.2]. When we apply the terminology and results of that reference to $\mathbb{G}(\mathbb{A}_f)$ we always regard it as being factored in this manner.

Definition 3.1.5. We write $\tilde{\pi} := \tilde{\pi}_{\mathfrak{p}} \otimes_E \tilde{\pi}^{\mathfrak{p}} = (\pi_{\mathfrak{p}} \otimes_E W) \otimes_E \pi_f^{\mathfrak{p}}$, regarded as a representation of $\mathbb{G}(\mathbb{A}_f) = G \times \Gamma$, and refer to $\tilde{\pi}$ as the classical $\mathfrak{p}$-adic automorphic representation of $\mathbb{G}(\mathbb{A}_f)$ over E attached to the allowable automorphic representation π.

If we equip $\tilde{\pi}$ with its finest convex topology then it becomes an admissible locally algebraic representation of the adèle group $\mathbb{G}(\mathbb{A}_f)$, in the sense of [13, Def. 7.2.15].

Let us mention an initial motivation for the introduction of classical $\mathfrak{p}$-adic automorphic representations. Common conditions that arise in the $\mathfrak{p}$-adic theory of automorphic representations, such as an automorphic representation π being ordinary or of non-critical slope, are not local at $\mathfrak{p}$; they depend on a comparison of invariants obtained from the local factors at the infinite places (typically, an infinity type) and from the local factor at $\mathfrak{p}$ (such as a Satake parameter). However, if we replace π by the associated classical $\mathfrak{p}$-adic automorphic representation $\tilde{\pi}$, then these properties often depend only on the local factor $\tilde{\pi}_{\mathfrak{p}}$ at $\mathfrak{p}$. (This is the case with the two examples mentioned.) Thus they can be analyzed by local $\mathfrak{p}$-adic representation theoretic techniques. It is this phenomenon that allows us to apply the local techniques of [13, 14] to the global problem of $\mathfrak{p}$-adic interpolation of systems of Hecke eigenvalues.

The basic example of allowable automorphic representations that we have in mind are those automorphic representations that contribute to cohomology, as we will now explain. As above, let $\mathbb{G}'$ denote the restriction of scalars of $\mathbb{G}$ from F to $\mathbb{Q}$. As in Subsection 2.4, let $\mathbb{S}$ denote the maximal split subtorus in the centre of $\mathbb{G}'$, and $\mathbb{S}'$ the the maximal split torus quotient of $\mathbb{G}'$; the natural map $\mathbb{S} \to \mathbb{S}'$ is an isogeny. Thus if $\mathfrak{s}$ denotes the Lie algebra of $\mathbb{S}$, and $\tilde{\mathfrak{g}}$ denotes the Lie algebra of the kernel of the the quotient map $\mathbb{G}' \to \mathbb{S}'$, then there is a natural isomorphism $\tilde{\mathfrak{g}} \bigoplus \mathfrak{s} \xrightarrow{\sim} \mathfrak{g}$. We write $\tilde{\mathfrak{g}}_\infty = \mathbb{R} \otimes_{\mathbb{Q}} \tilde{\mathfrak{g}}$. Also, as above, we fix a maximal compact subgroup K_∞ of G_∞, and write $\mathfrak{k}_\infty$ to denote the Lie algebra of K_∞. We let $\mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))$ denote the space of K_∞-finite automorphic forms on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})$. Let W be a finite dimensional representation of $\mathbb{G}$ (defined over $\mathbb{C}$), let χ denote the character through which $\mathfrak{z}(\mathfrak{g}_\infty)$ acts on $\check{W}$, let $I_\chi \subset \mathfrak{z}(\mathfrak{g}_\infty)$ denote the kernel of χ, and let ζ denote the character through which S_∞° (the group of connected components of $\mathbb{S}(\mathbb{R})$) acts on $\check{W}$ (so ζ is obtained by exponentiating the restriction of χ to $\mathfrak{s} \subset \mathfrak{z}(\mathfrak{g}_\infty)$). Let $\mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))[I_\chi^\infty]^{S_\infty^\circ=\zeta}$ denote the subspace of $\mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))$ consisting of vectors on which S_∞° acts via ζ, and which are annihilated by some power of I_χ. The main result of [18] shows that there is a natural isomorphism

$$H^n(\tilde{\mathfrak{g}}_\infty, \mathfrak{k}_\infty; W \otimes_{\mathbb{C}} \mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))[I_\chi^\infty]^{S_\infty^\circ=\zeta}) \xrightarrow{\sim} H^n(\mathcal{V}_W). \tag{3.1.6}$$

Suppose that π is an irreducible automorphic representation of $\mathbb{G}(\mathbb{A})$, written as a quotient $\pi = U/V$, where $V \subset U \subset \mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))[I_\chi^\infty]^{S_\infty^\circ=\zeta}$ are $\mathbb{G}(\mathbb{A})$-subrepresentations. (Note that since π is irreducible, $\mathfrak{z}(\mathfrak{g}_\infty)$ must then act on π via the character χ.) Suppose furthermore that the image of $H^n(\tilde{\mathfrak{g}}_\infty, \mathfrak{k}_\infty; W \otimes_{\mathbb{C}} U)$ in $H^n(\tilde{\mathfrak{g}}_\infty, \mathfrak{k}_\infty; W \otimes_{\mathbb{C}} \mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))[I_\chi^\infty]^{S_\infty^\circ=\zeta})$ is a proper subspace of the image of $H^n(\tilde{\mathfrak{g}}_\infty, \mathfrak{k}_\infty; W \otimes_{\mathbb{C}} V)$ in $H^n(\tilde{\mathfrak{g}}_\infty, \mathfrak{k}_\infty; W \otimes_{\mathbb{C}} \mathcal{A}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}))[I_\chi^\infty]^{S_\infty^\circ=\zeta})$. Then π_f (the finite component of π) appears as a subrepresentation of the quotient of the latter image by the former, and so in particular as an irreducible subquotient of $H^n(\mathcal{V}_W)$. We say that π *contributes to cohomology* . Conversely, the isomorphism (3.1.6) shows that any irreducible subquotient of $H^n(\mathcal{V}_W)$ arises as the finite component of such an automorphic representation π.

If an automorphic representation π contributes to the cohomology of $\mathcal{V}_W$, then π_∞ is $\check{W}$-allowable. Since π_f appears as a subquotient of $H^n(\mathcal{V}_W)$ (for an appropriate choice of W) it may be defined over a finite extension of $\mathbb{Q}$, and thus we see that π is in fact allowable. If we choose a finite extension E of $F_{\mathfrak{p}}$ such that both W and π_f can be defined over E, then we find that the corresponding classical $\mathfrak{p}$-adic automorphic representation $\tilde{\pi} := \pi_f \otimes_E \check{W}$ appears as an irreducible subquotient of the admissible locally algebraic $\mathbb{G}(\mathbb{A}_f)$-representation $H^n(\mathcal{V}_W) \otimes_E \check{W}$.

(3.2) As in Subsection 3.1, let $\mathbb{G}'$ denote the restriction of scalars of $\mathbb{G}$ from F to $\mathbb{Q}$. In this subsection we suppose that $\mathbb{G}'$ has the property that the maximal split torus in the centre of $\mathbb{G}'$ is a maximal split torus in $\mathbb{G}'$ over $\mathbb{R}$, so that we are in the situation of [19, Prop. 1.4]. We will construct a space of locally $F_{\mathfrak{p}}$-analytic $\mathfrak{p}$-adic automorphic forms attached to $\mathbb{G}$, in which the classical $\mathfrak{p}$-adic automorphic representations of $\mathbb{G}(\mathbb{A}_f)$ constructed in the preceding subsection appear as subrepresentations.

Our assumption implies that $Y^\circ(K_f) = \tilde{Y}(K_f)$ for any choice of level K_f. Also, there is an equality $K_\infty Z_\infty = G_\infty$ (as in Subsection 2.4, we let Z_∞ denote the centre of G_∞), so that $(K_\infty Z_\infty)^\circ = G_\infty^\circ$, and the quotient $G_\infty/(K_\infty Z_\infty)^\circ$ is equal to π_0, and hence finite. The quotients $\tilde{Y}(K_f)$ are thus also finite, and their only non-trivial cohomology is in degree zero. We will relate this cohomology to the notion of algebraic automorphic form described in [19].

We choose E so that $\mathbb{G}$ splits over E, and make the following variation on the definition of [19, p. 68].

Definition 3.2.1. If W is a finite dimensional representation of $\mathbb{G}$ defined over E then an algebraic automorphic form on $\mathbb{G}(\mathbb{A})$ with coefficients in W is a locally constant function $f : \mathbb{G}(\mathbb{A}) \to W$ such that $f(\gamma g) = \gamma f(g)$ for all $\gamma \in \mathbb{G}(F)$ and $g \in \mathbb{G}(\mathbb{A})$. We denote this space by $\mathcal{M}(W)$. It is endowed with a natural action of $\mathbb{G}(\mathbb{A})$, induced by the action by right translation of $\mathbb{G}(\mathbb{A})$ on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})$.

If $K^{\mathfrak{p}}$ is a tame level, then we let $\mathcal{M}(K^{\mathfrak{p}}, W)$ denote the $K^{\mathfrak{p}}$-fixed functions in $\mathcal{M}(W)$.

If W is the trivial representation of $\mathbb{G}$ on E then we omit reference to W, and write simply $\mathcal{M}$ and $\mathcal{M}(K^p)$ respectively.

Proposition 3.2.2. *There is a natural $\mathbb{G}(\mathbb{A})$-equivariant isomorphism $\mathcal{M}(W) \xrightarrow{\sim} H^0(\mathcal{V}_W)$. (Here we regard $H^0(\mathcal{V}_W)$ as a $\mathbb{G}(\mathbb{A})$-representation via the surjection $\mathbb{G}(\mathbb{A}) = G_\infty \times \mathbb{G}(\mathbb{A}_f) \to \pi_0 \times \mathbb{G}(\mathbb{A}_f)$.)*

Proof. Any $f \in \mathcal{M}(W)$ is constant on the right G_∞^0-cosets in $\mathbb{G}(\mathbb{A})$. Fix $f \in \mathcal{M}(W)$, and let K_f be a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$ such that f is constant on the right K_f-cosets in $\mathbb{G}(\mathbb{A})$. Then f gives rise to the section of $\mathbb{G}(\mathbb{A})/G_\infty^0 K_f \to W \times (\mathbb{G}(\mathbb{A})/G_\infty^0 K_f)$ defined by $g \mapsto (f(g), g)$. The assumption that f is an algebraic automorphic form implies that this section is $\mathbb{G}(F)$-equivariant, and thus gives an element of $H^0(Y(K_f), \mathcal{V}_W)$. Thus we obtain a $\mathbb{G}(\mathbb{A})$-equivariant map $\mathcal{M}(W) \to H^0(\mathcal{V}_W)$, which is immediately checked to be an isomorphism. $\square$

As a corollary we find that $\mathcal{M}(K^{\mathfrak{p}}, W)$ is isomorphic to $H^0(K^{\mathfrak{p}}, \mathcal{V}_W)$ for each choice of tame level $K^{\mathfrak{p}}$. Taking W to be the trivial representation E of $\mathbb{G}$, we see that $\mathcal{M}(K^{\mathfrak{p}})$ is isomorphic to $H^0(K^{\mathfrak{p}})$, and thus that $\tilde{H}^0(K^p)$ (being isomorphic to $\hat{H}^0(K^p)$ by Corollary 2.2.25) may be identified with the completion of the space

$\mathcal{M}(K^p)$ of algebraic automorphic forms on $\mathbb{G}(\mathbb{A})$ of tame level K^p with trivial coefficients, with respect to the sup norm.

Definition 3.2.3. For each choice of tame level $K^{\mathfrak{p}}$, we refer to $\tilde{H}^0(K^p)$ as the space of $\mathfrak{p}$-adic automorphic forms on $\mathbb{G}(\mathbb{A})$ of tame level $K^{\mathfrak{p}}$. It is an E-Banach space, equipped with an admissible continuous representation of $\pi_0 \times G$ by isometries. We refer to $\tilde{H}^0$ as the space of $\mathfrak{p}$-adic automorphic forms on $\mathbb{G}(\mathbb{A})$. It is equipped with an admissible continuous action of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$.

If L is a local field intermediate between $F_{\mathfrak{p}}$ and $\mathbb{Q}_p$, then we refer to the space $\tilde{H}^0{}_{L-\mathrm{la}}$ as the space of locally L-analytic p-adic automorphic forms on $\mathbb{G}(\mathbb{A}_f)$. It is a locally convex p-adic vector space of compact type, equipped with a strongly admissible locally L-analytic representation of $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$.

It seems worth remarking that $\tilde{H}^0$ (respectively $\tilde{H}^0{}_{L-\mathrm{la}}$) may be regarded as the space of continuous E-valued functions on $\mathbb{G}(\mathbb{A})$ that are locally constant on $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$-cosets (respectively, that are locally analytic on G-cosets and locally constant on $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$-cosets) and invariant under left translation by elements of $\mathbb{G}(F)$. This should serve to justify our use of the adjective automorphic.

Corollary 2.2.25 and Proposition 3.2.2 together show that for any finite dimensional representation W of $\mathbb{G}$ defined over E, there is a natural isomorphism $\mathcal{M}(W) \otimes_E \check{W} \xrightarrow{\sim} \tilde{H}^0_{W-\mathrm{lalg}}$. Consequently, unless W is trivial, we do not find automorphic representations appearing as irreducible closed subrepresentations of $\tilde{H}^0$; rather, as the following proposition shows, it is the classical $\mathfrak{p}$-adic automorphic representations that so appear.

Proposition 3.2.4. *The locally algebraic absolutely irreducible closed $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-subrepresentations of $\tilde{H}^0$ are (if we forget the π_0-action) precisely the classical $\mathfrak{p}$-adic automorphic representations attached to allowable automorphic representations of $\mathbb{G}(\mathbb{A})$ that are definable over E.*

Proof. Let us begin by remarking that π_0 is an abelian group [19, Prop. 2.4]. Thus the tensor product of a π_0-representation and a G_∞-representation is naturally a G_∞-representation, and also any absolutely irreducible $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation is absolutely irreducible as a $\mathbb{G}(\mathbb{A}_f)$-representation. Thus we may ignore the π_0-action, and we must show that the locally algebraic absolutely irreducible closed $\mathbb{G}(\mathbb{A}_f)$-subrepresentations of $\tilde{H}^0$ coincide with the classical $\mathfrak{p}$-adic automorphic representations attached to allowable automorphic representations of $\mathbb{G}(\mathbb{A})$ that are definable over E.

If W is an irreducible representation of $\mathbb{G}$ defined over E, then Corollary 2.2.25 yields a natural isomorphism $H^0(\mathcal{V}_W) \otimes_E \check{W} \xrightarrow{\sim} \tilde{H}^0_{\check{W}-\mathrm{lalg}}$. Thus the locally algebraic absolutely irreducible closed $\mathbb{G}(\mathbb{A}_f)$-subrepresentations of $\tilde{H}^0$ coincide with the absolutely irreducible $\mathbb{G}(\mathbb{A}_f)$-subrepresentations of $H^0(\mathcal{V}_W) \otimes_E \check{W}$, as W runs over all irreducible representations of $\mathbb{G}$ defined over E.

We use the isomorphism $\imath : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ to regard $\mathbb{C}$ as an extension of E, and so to form the base-changed representation $W_{/\mathbb{C}}$ of $\mathbb{G}_{/\mathbb{C}}$. Since G_∞ is compact modulo its centre, any automorphic representation π of $\mathbb{G}(\mathbb{A})$ whose infinity component π_∞ is $\check{W}_{/\mathbb{C}}$-allowable contributes to $H^0(\mathcal{V}_{W_{/\mathbb{C}}})$; furthermore, this cohomology space is semi-simple as a $\mathbb{G}(\mathbb{A}_f)$-representation. The discussion at the end of the preceding subsection therefore shows that the absolutely irreducible $\mathbb{G}(\mathbb{A}_f)$-subrepresentations of $H^0(\mathcal{V}_W)\otimes_E \check{W}$ are precisely the classical $\mathfrak{p}$-adic automorphic representations attached to the $\check{W}_{/\mathbb{C}}$-allowable automorphic representations of $\mathbb{G}(\mathbb{A})$ that are definable over E. This proves the proposition. $\square$

As we have already recalled, when $n = 0$, the map (2.3.7) is an isomorphism for any choice of W and any tame level $K^{\mathfrak{p}}$. Thus, if $\mathbb{G}$ is quasi-split over $F_{\mathfrak{p}}$, then we may form the eigenvariety $E(0, K^{\mathfrak{p}})$, for any choice of $K^{\mathfrak{p}}$, and Theorem 0.7 applies. Furthermore, it follows from [14, Thm. 4.2.36] that $E(0, K^{\mathfrak{p}})$ is equidimensional, of dimension equal to the dimension of a maximal torus of $\mathbb{G}_{/F_{\mathfrak{p}}}$.

(3.3) So as to give a completely concrete example, in this subsection we consider the case $\mathbb{G} = \mathbb{G}_m$ over $\mathbb{Q}$. In this context $F_{\mathfrak{p}} = \mathbb{Q}_p$ is the only possible choice for the local field L, and so we omit it from the notation.

Let $\hat{\mathbb{Z}}^p$ denote the prime-to-p profinite completion of $\mathbb{Z}$ (so that $\hat{\mathbb{Z}} \xrightarrow{\sim} \mathbb{Z}_p \times \hat{\mathbb{Z}}^p$). Choose a natural number N prime to p, and denote by $K(N)$ the kernel of the natural map $(\hat{\mathbb{Z}}^p)^\times \to (\mathbb{Z}/N)^\times$. Taking $K^p = K(N)$ in the constructions of Subsection 2.2, we find that $\tilde{H}^0(K(N))$ is an E-Banach algebra, canonically isomorphic to $\mathcal{C}(\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times, E)$. Thus if we pass to the limit over all such $K(N)$, we obtain an isomorphism

$$\tilde{H}^0 \xrightarrow{\sim} \mathcal{C}(\mathbb{Z}_p^\times, E) \otimes_E \mathcal{C}^{\mathrm{sm}}((\hat{\mathbb{Z}}^p)^\times, E).$$

The target is naturally isomorphic to the ring of E-valued functions on $\hat{\mathbb{Z}}$ that are continuous in the $\mathbb{Z}_p$-variable, and smooth in the $\hat{\mathbb{Z}}^p$-variable. Passing to locally analytic vectors yields an isomorphism

$$\tilde{H}^0_{\mathrm{la}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E) \otimes_E \mathcal{C}^{\mathrm{sm}}((\hat{\mathbb{Z}}^p)^\times, E).$$

Since $\mathbb{G}_m$ is a torus, it is certainly quasi-split, and is equal to its own Borel subgroup, and its own maximal torus. Thus we may form the eigenvariety $E(0, K(N))$ for any choice of tame level N, and Theorem 0.7 applies (since $n = 0$). In fact, we can described $E(0, K(N))$ explicitly: If we write $\mathcal{W} := (\mathbb{Z}_p^\times)\hat{}$ (the space of locally analytic characters of $\mathbb{Z}_p^\times$) and let $((\mathbb{Z}/N)^\times)\hat{}$ denote the space of characters of $(\mathbb{Z}/N)^\times$, then $E(0, K(N)) \xrightarrow{\sim} \mathcal{W} \times ((\mathbb{Z}/N)^\times)\hat{}$.

Since $\mathbb{T} = \mathbb{G} = \mathbb{G}_m$, we have $T = \mathbb{Q}_p^\times$, and the isomorphism $\mathbb{Q}_p^\times \xrightarrow{\sim} \mathbb{Z}_p^\times \times p^{\mathbb{Z}}$ induces an isomorphism $\hat{T} \xrightarrow{\sim} \mathcal{W} \times \mathbb{G}_m$. The finite map $E(0, K(N)) \to \hat{T}$ is the product of the identity map on $\mathcal{W}$ and the map $((\mathbb{Z}/N)^\times)\hat{} \to \mathbb{G}_m$ defined by taking a character of $(\mathbb{Z}/N)^\times$ to its value on p^{-1}.

4. The case of $\mathrm{GL}_{2/\mathbb{Q}}$

(4.1) As the title of this section indicates, we now take $F = \mathbb{Q}$ and $\mathbb{G} = \mathrm{GL}_2$. We fix a finite extension E of $\mathbb{Q}_p$ as our coefficient field. This is a case in which $Y^\circ(K_f) = \tilde{Y}(K_f)$ for any choice of level K_f, and so we will work with the quotients $\tilde{Y}(K_f)$ throughout this section. As we will briefly recall, these are the familiar modular curves.

Indeed $\mathbb{G}(F_\infty) = \mathrm{GL}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$, the group of $\mathbb{R}$-linear automorphisms of $\mathbb{C}$, the subgroup $Z_\infty K_\infty$ is equal to the group generated by $\mathbb{C}^\times$ $(= \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) \subset \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}))$ and complex conjugation (which we denote by c), and so $(Z_\infty K_\infty)^\circ$ is equal to $\mathbb{C}^\times$. The quotient $\mathrm{GL}_2(\mathbb{R})/(Z_\infty K_\infty)^\circ$ is isomorphic to the space of ordered pairs of $\mathbb{R}$-linearly independent complex numbers modulo scaling, which is in turn isomorphic to $\mathbb{C} \setminus \mathbb{R}$. If X denotes the upper half-plane in $\mathbb{C} \setminus \mathbb{R}$, and if K_f is a compact open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$, then strong approximation for SL_2, and the fact that $\mathbb{Z}$ is a PID, imply that $GL_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\hat{\mathbb{Z}})$, and hence that

$$\tilde{Y}(K_f) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z})\backslash(X \times \mathrm{GL}_2(\hat{\mathbb{Z}}))/K_f$$

(where of course $\mathrm{SL}_2(\mathbb{Z})$ acts on X by linear fractional transformations). (Compare the discussion of [12, §1.2].) The group π_0 has order two, and its non-trivial element acts on $\tilde{Y}(K_f)$ through the automorphism $\tau \mapsto -\tau^c$ of X. As is well known, the arithmetic quotients $\tilde{Y}(K_f)$ are affine algebraic curves over $\mathbb{C}$, the so-called modular curves.

Each finite dimensional representation W of GL_2 over E defines a corresponding family of local systems $\mathcal{V}_W$ on the curves $\tilde{Y}(K_f)$. The open Riemann surfaces $\tilde{Y}(K_f)$ have non-trivial cohomology in degrees 0 and 1, and we will consider the constructions of Subsection 2.2 in these cohomological degrees. Proposition 2.2.24, and our remark concerning the cohomological dimension of $\tilde{Y}(K_f)$, shows that V_pH^n vanishes for all n; the short exact sequence of Theorem 2.2.16 (iv) then implies that $\hat{H}^n \xrightarrow{\sim} \tilde{H}^n$ for every n. Thus we have two spaces to consider: namely, $\hat{H}^0$ and $\hat{H}^1$.

The compactly supported cohomology of $\tilde{Y}(K_f)$ is supported in degrees one and two. Proposition 2.2.26 and Corollary 2.2.27 show that $V_pH^n_c$ vanishes, and that the map $\hat{H}^n_c \to \tilde{H}^n_c$ is an isomorphism, for $n = 1, 2$. Thus we have the spaces $\hat{H}^1_c$ and $\hat{H}^2_c$ to consider.

As well as their cohomology and compactly supported cohomology, it will also be convenient to consider the parabolic cohomology of the modular curves.

Definition 4.1.1. For any finite dimensional representation W of GL_2 over E, we let $H^1_{\mathrm{par}}(\mathcal{V}_W)$ denote the image of the natural $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-equivariant map $H^1_c(\mathcal{V}_W) \to H^1(\mathcal{V}_W)$; it is a smooth $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-representation. If K^p is a choice of tame level, we let $H^1_{\mathrm{par}}(K_p, \mathcal{V}_W)$ and $H^1_{\mathrm{par}}(K_p, \mathcal{V}_W)$ denote the K^p-invariants of $H^1(\mathcal{V}_W)$. As usual, if W is the trivial representation, we will omit "$\mathcal{V}_W$" from the notation.

(4.2) If K_f is a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, then $\det(K_f)$ is a compact open subgroup of $\hat{\mathbb{Z}}^\times$, and it is well known (and follows directly from the adélic description) that the set of connected components of the modular curve $\tilde{Y}(K_f)$ is naturally isomorphic to $\hat{\mathbb{Z}}^\times/\det(K_f)$. Thus we see that $\hat{H}^0$ is isomorphic to the corresponding space for the group $\mathbb{G}_m$, considered in Subsection 3.3:

$$\hat{H}^0 \xrightarrow{\sim} \mathcal{C}(\mathbb{Z}_p^\times, E) \otimes_E \mathcal{C}^{\mathrm{sm}}((\hat{\mathbb{Z}}^p)^\times, E).$$

The group $\pi_0 \times \mathrm{GL}_2(\mathbb{A}_f)$ acts through the determinant map $\pi_0 \times \mathrm{GL}_2(\mathbb{A}_f) \xrightarrow{\det} \{\pm 1\} \times \mathbb{A}_f^\times$. (As an aside, let us remark that if W is irreducible of dimension greater than one, then $H^0(\mathcal{V}_W) = 0$, and hence its completion $\hat{H}^0(\mathcal{V}_W)$ also vanishes. On the other hand, $\tilde{H}^0(\mathcal{V}_W) = \tilde{H}^0 \otimes_E W \neq 0$. This illustrates the possibility remarked upon following the proof of Proposition 1.1.10.)

The space $\hat{H}^1$ is significantly more interesting then $\hat{H}^0$, since it contains information about p-power congruences of classical modular forms. Naturally, we cannot give such an explicit description of it. The remainder of this section is devoted to investigating some of the structure of this $\pi_0 \times \mathrm{GL}_2(\mathbb{A}_f)$-representation (and the various representations obtained from it by passing to locally analytic vectors, locally algebraic vectors, and so on). In particular, we will study the associated eigenvariety, and explain its relationship to the eigencurve of [11]. (See Proposition 4.4.15 below.)

We begin by considering the $\pi_0 \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant map

$$\hat{H}^0 \mathbin{\hat{\otimes}}_E \hat{H}^1 \to \hat{H}^1 \tag{4.2.1}$$

yielded by the cup-product and Proposition 2.2.21. This makes $\hat{H}^1$ a topological module over the topological E-algebra $\hat{H}^0$.

Let $\mathbb{T}$ denote the maximal torus of GL_2 consisting of diagonal matrices. As usual, we write $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $T = \mathbb{T}(\mathbb{Q}_p)$. Before going further in our analysis, it will help to introduce more notation. We write $T_0 = \{\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Z}_p^\times\}$ and $T_1 = \{\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \mid v \in \mathbb{Z}_p^\times\}$. We also write $\wp_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\wp_1 = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. If $\langle \wp_i \rangle$ denotes the infinite cyclic group generated by $\wp_i$ $(i = 1, 2)$, then there is a natural isomorphism

$$T_0 \times T_1 \times \langle \wp_0 \rangle \times \langle \wp_1 \rangle \xrightarrow{\sim} T. \tag{4.2.2}$$

The determinant map $\det : G \to \mathbb{Q}_p^\times$ induces an isomorphism $\det : T_0 \xrightarrow{\sim} \mathbb{Z}_p^\times$, and hence an embedding of rings

$$\mathcal{C}(T_0, E) \xrightarrow{\sim} \mathcal{C}(\mathbb{Z}_p^\times, E) \overset{\mathrm{id}\otimes 1}{\hookrightarrow} \mathcal{C}(\mathbb{Z}_p^\times, E) \otimes_E \mathcal{C}^{\mathrm{sm}}((\mathbb{Z}^p)^\times, E) \xrightarrow{\sim} \hat{H}^0. \tag{4.2.3}$$

Lemma 4.2.4. (i) *The image of the embedding (4.2.3) is a $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-invariant subalgebra of $\hat{H}^0$.*

(ii) *The $\pi_0 \times \mathbb{G}(\mathbb{A}_f)$-action on $\mathcal{C}(T_0, E)$ provided by (i), when restricted to $T_0 \subset \pi_0 \times \mathbb{G}(\mathbb{A}_f)$, agrees with the right regular action of T_0 on $\mathcal{C}(T_0, E)$.*

Proof. Both claims are straightforward. $\square$

Thus the map (4.2.1) restricts to a $\pi_0 \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant map

$$\mathcal{C}(T_0, E) \mathbin{\hat{\otimes}}_E \hat{H}^1 \to \hat{H}^1. \tag{4.2.5}$$

We let $(\hat{H}^1)^{T_0}$ denote the subspace of $\hat{H}^1$ consisting of T_0-fixed vectors. Clearly $(\hat{H}^1)^{T_0}$ is invariant under $\pi_0 \times T \times \mathrm{GL}_2(\mathbb{A}_f^p)$.

Proposition 4.2.6. *The map (4.2.5) restricts to a $\pi_0 \times T \times \mathrm{GL}_2(\mathbb{A}_f^p)$-equivariant topological isomorphism $\mathcal{C}(T_0, E) \mathbin{\hat{\otimes}}_E (\hat{H}^1)^{T_0} \xrightarrow{\sim} \hat{H}^1$.*

Proof. We have already observed that the map (4.2.5) has the stated equivariance properties, and thus so does its restriction

$$\mathcal{C}(T_0, E) \mathbin{\hat{\otimes}}_E (\hat{H}^1)^{T_0} \to \hat{H}^1. \tag{4.2.7}$$

It is a general fact that if G is a compact topological group, V is an E-Banach space equipped with a continuous G-action, and V is furthermore equipped with a G-equivariant $\mathcal{C}(G,E)$-Banach module structure (where G acts on $\mathcal{C}(G,E)$ through the right regular action), then the map $\mathcal{C}(G,E)\,\hat{\otimes}_E\,V^G \to V$ induced by the topological module structure on V is a topological isomorphism. Indeed, $\mathcal{C}(G,E)$ is naturally a Banach-Hopf-algebra, and our assumption on V implies that V is a Banach-Hopf-module over $\mathcal{C}(G,E)$. The main theorem on Hopf modules [34, Thm. 4.1.1, p. 84], adapted to the Banach-Hopf setting, then yields the required isomorphism.[2]

Applying this with $G = T_0$ and $V = (\hat{H}^1)^{K^p}$ for a compact open subgroup K^p of $\mathrm{GL}_2(\mathbb{A}_f^p)$ (taking into account Lemma 4.2.4 (ii)), and then passing to the inductive limit over all such K^p, we find that (4.2.7) is a topological isomorphism. □

The preceding proposition expresses the analogue, in our p-adically completed situation, of the fact that cohomology classes in one of the spaces $H^1(\mathcal{V}_W)$ may always be twisted by elements of H^0 so as to be invariant under T_0. Together with Proposition 4.2.8 below, it will be applied in Subsection 4.4 to obtain a corresponding decomposition of the eigenvariety for GL_2.

As observed in Proposition 2.2.23, the module structure (4.2.1) induces a module structure

$$\hat{H}^0_{\mathrm{la}}\,\hat{\otimes}_E\,\hat{H}^1_{\mathrm{la}} \to \hat{H}^1_{\mathrm{la}}.$$

(Since $\mathbb{Q}_p$ is the only field available with respect to which we may consider locally analytic vectors, here and below we will abbreviate "$\mathbb{Q}_p - \mathrm{la}$" by "la".) We may repeat the above discussion in the locally analytic context, and so obtain the following result.

Proposition 4.2.8. *There is a natural $\pi_0 \times T \times \mathrm{GL}_2(\mathbb{A}_f^p)$-equivariant topological isomorphism $\mathcal{C}^{\mathrm{la}}(T_0,E)\,\hat{\otimes}_E(\hat{H}^1_{\mathrm{la}})^{T_0} \xrightarrow{\sim} \hat{H}^1_{\mathrm{la}}$.*

Proof. This follows from the analogue for the compact type Hopf algebra $\mathcal{C}^{\mathrm{la}}(T_0,E)$ and its compact type topological Hopf module $\hat{H}^1_{\mathrm{la}}$ of the "main theorem on Hopf modules" recalled in the proof of Proposition 4.2.6. □

Analogous results to Propositions 4.2.6 and 4.2.8 hold with $\hat{H}^1$ replaced by $\hat{H}^1_c$.

(4.3) We turn to analyzing the spectral sequences of Corollary 2.2.18. If we fix a finite dimensional representation W of GL_2 over E, then we obtain the $\pi_0 \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant spectral sequence $E_2^{i,j} = \mathrm{Ext}^i_{\mathfrak{gl}_2}(\check{W}, \hat{H}^j_{\mathrm{la}}) \implies H^{i+j}(\mathcal{V}_W)$.

Let Z denote the centre of GL_2 (so that $Z \xrightarrow{\sim} \mathbb{G}_m$). Multiplication in G induces a homomorphism $Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \to G$, which is locally an isomorphism (that is, induces an isomorphism between sufficiently small neighbourhoods of the identity in its source and target).

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[2] When applying the main theorem on Hopf modules, note that the Banach comodule structure

$$c_G : V \to \mathcal{C}(G,E)\,\hat{\otimes}\,V \xrightarrow{\sim} \mathcal{C}(G,V)$$

(the isomorphism following by the example of [27, pp. 111-112]) is defined by $c_G(v)(g) = gv$ for all $g \in G$ and $v \in V$; that is, $c_G(v)$ is the orbit map of v. Thus

$$c_G(v) = 1\,\hat{\otimes}\,v = \text{ the constant function } v \text{ on } G$$

if and only $v \in V^G$.

Proposition 4.3.1. *If the centre Z of GL_2 acts on W via a character χ, then for any $i \geq 0$, there is a natural isomorphism of $\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$-representations*

$$\mathrm{Hom}_{\mathfrak{z}}(E(\chi^{-1}), \hat{H}^0_{\mathrm{la}}) \otimes_E H^i(\mathfrak{sl}_2, W) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}}).$$

(*Here $\mathfrak{z}$ denotes the Lie algebra of $Z(\mathbb{Q}_p)$ and $\mathrm{Hom}_{\mathfrak{z}}(E(\chi^{-1}), \hat{H}^0_{\mathrm{la}})$ denotes the space of $\mathfrak{z}$-equivariant maps from $E(\chi^{-1})$ (the one-dimensional representation on which $Z(\mathbb{Q}_p)$ acts by the character χ^{-1}) to $\hat{H}^0_{\mathrm{la}}$. It is a smooth $\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$-representation, while $H^i(\mathfrak{sl}_2, \check{W})$ is a smooth $\mathrm{SL}_2(\mathbb{Q}_p)$-representation.*)

Proof. The G-representation $\hat{H}^0$, when regarded as a representation of $\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$ via the natural map

$$\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p) \to \pi_0 \times \mathrm{GL}_2(\mathbb{A}_f),$$

may be regarded as the tensor product of itself, restricted to a representation of $\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$, and of the trivial representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. Similarly, the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\check{W}$, when regarded as a $Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$-representation, may be regarded as the tensor product of χ^{-1} and of itself, restricted to a representation of $\mathrm{SL}_2(\mathbb{Q}_p)$.

A version of the Künneth theorem (whose precise formulation and proof we leave to the reader) shows that the cup-product induces an isomorphism

$$\bigoplus_{a+b=i} \mathrm{Ext}^a_{\mathfrak{z}}(E(\chi^{-1}), \hat{H}^0_{\mathrm{la}}) \otimes \mathrm{Ext}^b_{\mathfrak{sl}_2}(\check{W}, E) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}}),$$

in which all the arrows are $\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$-equivariant. (We are using the fact that map $Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \to G$ is a local isomorphism, and so induces an isomorphism between the corresponding Lie algebras.)

Let H be a compact open subgroup of $Z(\mathbb{Q}_p)$, chosen so small that it maps isomorphically onto its image in $\mathbb{Z}_p^\times$ under the determinant. For any tame level K^p, the representation $\hat{H}^0(K^p)$ is isomorphic as an H-representation to a direct sum of copies of $\mathcal{C}(\mathbb{Z}_p^\times, E)$. By Proposition 1.1.12 (v) and Theorem 1.1.13, the spaces $\mathrm{Ext}^a_{\mathfrak{z}}(E(\chi^{-1}), \hat{H}^0(K^p))$ vanish if $a > 0$. Taking into account the isomorphism $\mathrm{Ext}^b_{\mathfrak{sl}_2}(\check{W}, E) \xrightarrow{\sim} H^b(\mathfrak{sl}_2, W)$, the lemma follows. $\square$

Corollary 4.3.2. *If W is an irreducible finite dimensional representation W of GL_2, then the space $\mathrm{Ext}^i_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}})$ vanishes except possibly when $i = 0$ and $i = 3$. Furthermore, $\mathrm{Hom}_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}})$ and $\mathrm{Ext}^3_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}})$ are isomorphic as $\pi_0 \times Z(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$-representations, and both vanish unless W is one-dimensional.*

Proof. Since $\mathfrak{sl}_2$ is a semi-simple Lie algebra, $H^1(\mathfrak{sl}_2, W)$ vanishes for every finite dimensional representation W of SL_2 over E. Since $\mathfrak{sl}_2$ is three-dimensional, Poincaré duality for Lie algebra cohomology implies that $H^2(\mathfrak{sl}_2, W)$ also vanishes, that $H^i(\mathfrak{sl}_2, W)$ vanishes if $i > 3$, and that $H^0(\mathfrak{sl}_2, W)$ and $H^3(\mathfrak{sl}_2, \check{W})$ are dual to one another.

If W is irreducible then $H^0(\mathfrak{sl}_2, W)$ is non-zero if and only if W is the trivial representation of SL_2. The corollary is now seen to follow from Proposition 4.3.1. $\square$

If we feed the information provided by Corollary 4.3.2 into the spectral sequence of Corollary 2.2.18, and also take into account the fact that $H^n(\mathcal{V}_W)$ vanishes for $n > 1$, we obtain the following natural isomorphisms:

$$H^0(\mathcal{V}_W) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}}), \tag{4.3.3}$$

$$H^1(\mathcal{V}_W) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{gl}_2}(\check{W}, \hat{H}^1_{\mathrm{la}}), \tag{4.3.4}$$

$$\mathrm{Ext}^1_{\mathfrak{gl}_2}(\check{W}, \hat{H}^1_{\mathrm{la}}) \xrightarrow{\sim} \mathrm{Ext}^3_{\mathfrak{gl}_2}(\check{W}, \hat{H}^0_{\mathrm{la}}). \tag{4.3.5}$$

We also see that $\mathrm{Ext}^i_{\mathfrak{gl}_2}(\check{W}, \hat{H}^j_{\mathrm{la}})$ vanishes for all values of i and j not appearing in the above list; that is, for $(i, j) \neq (0, 0), (0, 1), (1, 1), (3, 0)$. Furthermore, Corollary 4.3.2 shows that when (i, j) equals one of $(1, 1)$ or $(3, 0)$, we get a non-vanishing expression only when W is one-dimensional.

The isomorphism (4.3.4) shows that the hypotheses of Proposition 2.3.8 is satisfied for every representation W when $n = 1$. (The isomorphism (4.3.3) shows that it is also satisfied for $n = 0$. Of course, this is a special case of Corollary 2.2.25.)

In [21, §1, ex. 3], Ihara considers the projective limit of the $\mathbb{Z}_p$-homology of the connected components of the modular curves $\tilde{Y}(K_f)$, as K_f ranges over a sequence of groups of increasing p-power level and fixed prime-to-p level. This limit is equipped with a natural action of $\mathrm{SL}_2(\mathbb{Z}_p)$. This $\mathrm{SL}_2(\mathbb{Z}_p)$-module is closely related to the topological dual of the space $\hat{H}^1$, considered as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation. It may be that the methods of [21] could be of use in analyzing the structure of $\hat{H}^1$ as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation.

We now consider the case of compactly supported cohomology. The following result shows that this case is particularly simple.

Proposition 4.3.6. *The space $\hat{H}^2_c$ vanishes.*

Proof. Fix a tame level K^p. If K_p is a compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, then K_p contains an open subgroup K_p' such that the index $[K_p \bigcap \mathrm{SL}_2(\mathbb{Q}_p) : K_p' \bigcap \mathrm{SL}_2(\mathbb{Q}_p)]$ is divisible by p^r, for arbitrary $r > 0$. The map $\tilde{Y}(K_p'K^p) \to \tilde{Y}(K_pK^p)$ thus has degree divisible by p^r on each connected component of the source, and so the image of the pullback map $H^2_c(\tilde{Y}(K_pK^p), \mathcal{O}_E) \to H^2_c(\tilde{Y}(K_p'K^p), \mathcal{O}_E)$ lies in $p^r H^2_c(\tilde{Y}(K_p'K^p), \mathcal{O}_E)$. Since r was arbitrary, it follows that $\varinjlim_{K_p} H^2_c(\tilde{Y}(K_pK^p), \mathcal{O}_E)$ is p-divisible, and hence that its p-adic completion vanishes. This implies the proposition. □

From the spectral sequence of Corollary 2.2.18, we deduce the following isomorphisms:

$$H^1_c(\mathcal{V}_W) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{gl}_2}(\check{W}, \hat{H}^1_{c,\mathrm{la}}), \tag{4.3.7}$$

$$H^2_c(\mathcal{V}_W) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathfrak{gl}_2}(\check{W}, \hat{H}^1_{c,\mathrm{la}}), \tag{4.3.8}$$

and also that $\mathrm{Ext}^i_{\mathfrak{gl}_2}(\check{W}, \hat{H}^1_{c,\mathrm{la}})$ vanishes for $i > 1$. Since $H^2_c(\mathcal{V}_W)$ vanishes for irreducible W, except in the case when W is one-dimensional, the isomorphism (4.3.8) implies that when W is irreducible $\mathrm{Ext}^1_{\mathfrak{gl}_2}(\check{W}, \hat{H}^1_{c,\mathrm{la}})$ vanishes unless W is of dimension one.

We close this section with the following result, whose proof is similar to that of Proposition 4.3.6.

Proposition 4.3.9. *The natural map $\hat{H}^1_c \to \hat{H}^1$ is surjective.*

Proof. If K_f is a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, let $X(K_f)$ denote the completion of the curve $\tilde{Y}(K_f)$, and let $C(K_f)$ denote the complement of $\tilde{Y}(K_f)$ in $X(K_f)$ (the set of cusps). For any such K_f, the inclusion of $\tilde{Y}(K_f)$ in $X(K_f)$ induces a natural isomorphism $H^1(X(K_f), E) \xrightarrow{\sim} H^1_{\mathrm{par}}(K_f)$.

Fix a tame level K^p, and denote by $\hat{H}^1_{\mathrm{par}}(K^p)$ the completion of $H^1_{\mathrm{par}}(K_p)$ with respect to the gauge of its sublattice $\varinjlim_{K_p} H^1_{\mathrm{par}}(\tilde{Y}(K_pK^p), \mathcal{O}_E)$. The proposition will follow if we show that the map $\hat{H}^1_{\mathrm{par}}(K^p) \to \hat{H}^1(K^p)$ is an isomorphism. A consideration of the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\varinjlim_{K_p} H^1(X(K_pK^p), \mathcal{O}_E) \arrow[r] \arrow[d] & \varinjlim_{K_p} H^1(X(K_pK^p), E) \arrow[r, "\sim"] \arrow[d] & H^1_{\mathrm{par}}(K^p) \arrow[d] \\
\varinjlim_{K_p} H^1(\tilde{Y}(K_pK^p), \mathcal{O}_E) \arrow[r] \arrow[d] & \varinjlim_{K_p} H^1(\tilde{Y}(K_pK^p), E) \arrow[r, equal] \arrow[d] & H^1(K^p) \\
\varinjlim_{K_p} \mathrm{Div}^0(C(K_pK^p), \mathcal{O}_E) \arrow[r] \arrow[d] & \varinjlim_{K_p} \mathrm{Div}^0(C(K_pK^p), E) \arrow[d] & \\
0 & 0, &
\end{tikzcd}$$

in which $\mathrm{Div}^0(C(K_pK^p), R)$ denotes the R-module of divisors of degree zero supported on the set of cusps $C(K_f)$ with coefficients in R, and whose first two columns are exact, shows that it suffices to prove that the map $\varinjlim_{K_p} \mathrm{Div}^0(C(K_f), \mathcal{O}_E) \to \varinjlim_{K_p} \mathrm{Div}^0(C(K_f), E)$ is an isomorphism. This in turn follows from the fact that for any fixed compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Z}_p)$, we may find a normal open subgroup $K'_p \subset K_p$ such that the map $X(K'_pK^p) \to X(K_pK^p)$ is ramified over every cusp, with ramification degree divisible by any given power of p. □

(4.4) We fix a choice of tame level K^p of the form "$\Gamma_1(M)$", for some natural number M coprime to p; more precisely, we assume that

$$K^p := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \mid c \equiv 0 \pmod{M}, d \equiv 1 \pmod{M} \}.$$

We will write $E(1, M)$ rather than $E(1, K^p)$ to denote the degree one cohomological eigenvariety of tame level K^p, defined with respect to the Borel subgroup $\mathbb{B}$ of upper triangular matrices in GL_2. Similarly we will write $E(1, M)_{\mathrm{cl}}$ rather than $E(1, K^p)_{\mathrm{cl}}$ to denote the set of classical points in $E(1, M)$.

We also define a variant of $E(1,M)_{\mathrm{cl}}$, to be denoted $E(1,M)_{\mathrm{par,cl}}$, by replacing H^1 by H^1_{par} in the definition. We remark that $E(1,M)_{\mathrm{par,cl}}$ is obviously a subset of $E(1,M)_{\mathrm{cl}}$. We further define $E(1,M)_{\mathrm{cl},0}$ and $E(1,M)_{\mathrm{par,cl},0}$ to be the subsets of $E(1,M)_{\mathrm{cl}}$ and $E(1,M)_{\mathrm{par,cl}}$ respectively consisting of pairs $(\chi,\lambda)\in \left(\hat{T}\times \operatorname{Spec}\mathcal{H}(K^p)^{\mathrm{sph}}\right)(\overline{\mathbb{Q}}_p)$ such that χ is trivial when restricted to T_0. (Here and below we will use the notation introduced in Subsection 4.2.)

Let us give an explicit description of $\hat{T}\times\operatorname{Spec}\mathcal{H}(K^p)^{\mathrm{sph}}$. The isomorphism (4.2.2) yields an isomorphism $\hat{T}\xrightarrow{\sim}\mathcal{W}_0\times\mathcal{W}_1\times\mathbb{G}_{m,0}\times\mathbb{G}_{m,1}$, where $\mathbb{G}_{m,i}$ denotes $\widehat{\langle\wp_i\rangle}$, and $\mathcal{W}_i$ denotes $\hat{T}_i$ $(i=1,2)$. The isomorphism $\det: T_i\xrightarrow{\sim}\mathbb{Z}_p^\times$ induces an isomorphism $\mathcal{W}_i\xrightarrow{\sim}\mathcal{W}$ (where, as in Subsection 3.3, $\mathcal{W}$ denotes $\widehat{(\mathbb{Z}_p^\times)}$), while evaluation at $\wp_i$ yields an isomorphism $\mathbb{G}_{m,i}\xrightarrow{\sim}\mathbb{G}_m$. For each ℓ not dividing Mp, the local Hecke algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell)//\mathrm{GL}_2(\mathbb{Z}_\ell))$ is generated by the Hecke operators $T(\ell)$ and $T(\ell,\ell)^{\pm1}$, corresponding to the double cosets of $\begin{pmatrix}\ell&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}\ell^{\pm1}&0\\0&\ell^{\pm1}\end{pmatrix}$ respectively. Thus $\mathcal{H}(K^p)^{\mathrm{sph}}=E[\{T(\ell),T(\ell,\ell)^{\pm1}\}_{(\ell,Np)=1}]$.

We now describe the points of $E(1,M)_{\mathrm{par,cl},0}$ in classical terms.

Definition 4.4.1. By a p-stabilized newform of weight $k+2\geq 2$ we mean a cuspform f of weight $k+2$ satisfying one of the two following conditions:

(i) The modular form f is a normalized newform on $\Gamma_1(p^rC)$ for some natural numbers $r>0$ and C prime to p.

(ii) There is a normalized newform g on $\Gamma_1(C)$ for some natural number C prime to p, such that $f(\tau)=g(\tau)-\alpha_p g(p\tau)$ for one of the roots α_p of the pth Hecke polynomial of g.

We say that f has tame conductor C, and p-stabilized conductor Cp^r (in case (i)) or Cp (in case (ii)). Note that in either case, if Cp^r denotes the p-stabilized conductor of f, then f is a weight $k+2$ modular form on $\Gamma_1(Cp^r)$, and is an eigenform for the Atkin-Lehner U_p-operator. We say that f is of finite slope if its U_p-eigenvalue is non-zero.

For $k\geq 0$, let W_k denote the dual to the kth symmetric power of the standard two-dimensional representation of GL_2, and let ψ_k denote the highest weight of W_k with respect to $\mathbb{B}$. If (χ,λ) is a point of $E(1,K)_{\mathrm{par,cl},0}$, then we may factor χ as $\chi=\theta\psi$, where θ is a smooth $\overline{\mathbb{Q}}_p$-valued character of T, trivial on T_0, and $\psi=\psi_k$ for some $k\geq 0$. Write $\alpha=p\theta(\wp_0)$ and $\beta=\theta(\wp_1)$.

Proposition 4.4.2. *The set $E(1,M)_{\mathrm{par,cl},0}(\overline{\mathbb{Q}}_p)$ is in bijection with the set of p-stabilized newforms of finite slope whose tame conductor divides M. Let $(\chi,\lambda)\in E(1,M)_{\mathrm{par,cl},0}(\overline{\mathbb{Q}}_p)$, and let f be the corresponding p-stabilized newform. The asserted bijection is uniquely determined by the following properties:*

Let θ, ψ, k, α, and β be attached to χ as in the preceding discussion. Let ϵ denote the nebentypus of f, and factor $\epsilon=\epsilon_p\epsilon_M$ as the product of a character of p-power conductor and a character of conductor dividing M.

(*i*) *The weight of f is equal to $k+2$.*

(*ii*) *If we identify $\mathbb{Z}_p^\times$ and T_1 via $v\mapsto\begin{pmatrix}1&0\\0&v\end{pmatrix}$, then $\theta_{|T_1}$ and ϵ_p^{-1} coincide.*

(*iii*) *If ℓ is a prime not dividing Mp, then $\epsilon_M(\ell)=\lambda(T(\ell,\ell))\epsilon_p(\ell)^{-1}\ell^{-k}$, while $\epsilon_M(p)=\alpha\beta/p^{k+1}$.*

(*iv*) *The U_p-eigenvalue of f is equal to α.*

(*v*) *If ℓ is a prime not dividing Mp, then the T_ℓ-eigenvalue of f is equal to* $\lambda(T(\ell))$.

Proof. Write $N_0 := \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z}_p\}$, and write $P_0 := N_0 T_0$. A point $(\theta\psi_k, \lambda)$ (for some $k \geq 0$) appears in $E(1, M)_{\mathrm{par,cl},0}$ if and only if the $(T^+ = \theta, \mathcal{H}(K^p)^{\mathrm{sph}} = \lambda)$-eigenspace of $H^1_{\mathrm{par}}(K^p, \mathcal{V}_{\check{W}_k})^{P_0}$ is non-zero. (Here $T^+ := \{t \in T \mid tN_0t^{-1} \subset N_0\}$ acts via the Hecke operators $\pi_{N_0,t}$ as defined in [14, Def. 3.4.2]; also we have implicitly extended scalars from E to a subfield of $\overline{\mathbb{Q}}_p$ over which θ and λ are defined.) This follows from the relation between the Jacquet module $J_B(H^1_{\mathrm{par}}(K^p, \mathcal{V}_{\check{W}_k}))$ and parabolic induction, given by [14, Prop. 4.3.4]; we are also taking into account the fact that θ and ψ_k are trivial on T_0.

There is a natural isomorphism $H^1_{\mathrm{par}}(K^p, \mathcal{V}_{\check{W}_k})^{P_0} \xrightarrow{\sim} \varinjlim_r H^1_{\mathrm{par}}(Y_1(p^rM), \mathcal{V}_{\check{W}_k})$, where as usual we have written $Y_1(p^rM)$ to denote the modular curve corresponding to the congruence subgroup $\Gamma_1(p^rM)$. The proposition thus follows from the usual relation between the cohomology of modular curves and modular forms (Eichler-Shimura theory), together with an interpretation of the Hecke operators $\pi_{N_0,t}$ (for $t = \wp_0$, $t = \wp_0\wp_1$, or $t \in T_1$), as well as $T(\ell)$ and $T(\ell, \ell)$ (for ℓ a prime not dividing Mp), in classical terms.

More precisely, we may write $\varinjlim_r H^1_{\mathrm{par}}(Y_1(p^rM), \mathcal{V}_{\check{W}_k})$ as a direct sum of plus and minus eigenspaces under the action of π_0; after extending scalars to $\mathbb{C}$, the corresponding direct summands may be described by holomorphic and anti-holomorphic modular forms respectively. We will briefly sketch the holomorphic case; the anti-holomorphic case is treated in an identical fashion.

If K_f is a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, with the property that the determinant mapping maps K_f onto $\hat{\mathbb{Z}}^\times$, then the restriction of the projection

$$X \times \mathrm{GL}_2(\mathbb{A}_f) \to \mathrm{GL}_2(\mathbb{Q})^+\backslash(X \times \mathrm{GL}_2(\mathbb{A}_f))/K_f \xrightarrow{\sim} Y(K_f)$$

to $X \times 1$ (where 1 denotes the identity matrix in $\mathrm{GL}_2(\hat{\mathbb{Z}})$) remains surjective. (Here $\mathrm{GL}_2(\mathbb{Q})^+$ denotes the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ consisting of matrices with positive determinant.) If τ is an element of X, then we may form the element $(\tau, 1)$ of $\mathbb{C}^2$, and hence the element $(\tau, 1)^k$ of $\mathrm{Sym}^k(\mathbb{C}^2)$ (which equals $\check{W}_k$, after extending scalars from E to $\mathbb{C}$). A holomorphic class in $H^1_{\mathrm{par}}(Y(K_f), \mathcal{V}_{\check{W}_k})$ is represented by a holomorphic $\mathrm{Sym}^k(\mathbb{C}^2)$-valued one-form $\tau \mapsto f(\tau)d\tau(\tau, 1)^k$ on X, where $f(\tau)$ is a holomorphic function on X that vanishes at the cusps and satisfies the following invariance property: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+ \bigcap K_f$, then

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{k+2} f(\tau);$$

that is, the function $f(\tau)$ is a weight $k+2$ cuspform with respect to $\mathrm{GL}_2(\mathbb{Q})^+ \bigcap K_f$.

Thus the holomorphic part of $\varinjlim_r H^1_{\mathrm{par}}(Y_1(p^rM), \mathcal{V}_{\check{W}_k})$ may be identified with the inductive limit $\varinjlim_r S_{k+2}(\Gamma_1(p^rM))$, where $S_{k+2}(\Gamma)$ denotes the $\mathbb{C}$-vector space of cuspforms of weight $k+2$ on the congruence subgroup Γ. It remains to describe the action of $\pi_{N_0,t}$, $T(\ell)$, and $T(\ell, \ell)$ in classical terms.

Let K_f be as above. If $g \in \mathrm{GL}_2(\mathbb{A}_f)$, then multiplication on the right by g induces a homeomorphism $Y(gK_fg^{-1}) \xrightarrow{\sim} Y(K_f)$, and hence an isomorphism $H^1(Y(K_f), \mathcal{V}_{\check{W}_k}) \to H^1(Y(gK_fg^{-1}), \mathcal{V}_{\check{W}_k})$, that we denote by g^* (pullback via g). (This is just the action of g on $H^1(\mathcal{V}_{\check{W}_k})$, restricted to the subspace of K_f-invariant vectors.) We will now describe explicitly the action induced by g^* on the corresponding spaces of cuspforms. Although the necessary computations are straightforward and well-known, at the suggestion of the referee we will give the details, since they are crucial for comparing our representation theoretic approach to p-adic interpolation with the more classical approaches in the literature.

Let c_f be the cohomology class represented by a weight $k+2$ cuspform $f(\tau)$ on $\mathrm{GL}_2(\mathbb{Q})^+ \bigcap K_f$. We wish to compute g^*c_f. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$ is chosen so that $\gamma g \in K_f$, then $\gamma^{-1}K_f\gamma = gK_fg^{-1}$, giving $Y(\gamma^{-1}K_f\gamma) = Y(gK_fg^{-1})$, and the homeomorphism $Y(\gamma^{-1}K_f\gamma) = Y(gK_fg^{-1}) \xrightarrow{\sim} Y(K_f)$ induced by right multiplication by γ^{-1} coincides with the homeomorphism induced by right multiplication by g. Thus $g^*c_f = (\gamma^{-1})^*c_f$.

Let $\tau \in X$. The image of the $(\mathrm{GL}_2(\mathbb{Q})^+, gK_fg^{-1})$-double coset of $(\tau, 1)$ under right multiplication by γ^{-1} is equal to the $(\mathrm{GL}_2(\mathbb{Q})^+, K_f)$-double coset of (τ, γ^{-1}), which coincides with the $(\mathrm{GL}_2(\mathbb{Q})^+, K_f)$-double coset of $(\gamma\tau, 1)$. Similarly, the image of the $(\mathrm{GL}_2(\mathbb{Q})^+, gK_fg^{-1})$-double coset of $(\tau, \gamma, \check{w})$ (where $\check{w} \in \check{W}_k$; this is a typical element of $\mathcal{V}_{\check{W}_k}$ over $Y(gK_fg^{-1})$) is equal to the $(\mathrm{GL}_2(\mathbb{Q})^+, K_f)$-double coset of $(\gamma\tau, 1, \gamma\check{w})$ (which is an element of $\mathcal{V}_{\check{W}_k}$ over $Y(K_f)$). Thus, recalling that c_f is represented by the holomorphic one-form $f(\tau)d\tau(\tau, 1)^k$, and that $\gamma\tau = (a\tau + b)/(c\tau + d)$, we compute that $g^*c_f = (\gamma^{-1})^*c_f$ is represented by the holomorphic one-form $(ad - bc)(c\tau + d)^{-(k+2)}f(\dfrac{a\tau + b}{c\tau + d})d\tau(a\tau + b, c\tau + d)^k$, and hence (since $\gamma(\tau, 1)^k = (a\tau + b, c\tau + d)^k$) that $(ad - bc)(c\tau + d)^{-(k+2)}f(\dfrac{a\tau + b}{c\tau + d})$ is the cuspform on $\mathrm{GL}_2(\mathbb{Q})^+ \bigcap gK_fg^{-1}$ that corresponds to the cohomology class g^*c_f.

With this formula in hand we may now interpret our various representation theoretically defined Hecke operators in classical terms. Naturally we take K_f to be the open subgroup of $\mathrm{GL}_2(\hat{Z})$ corresponding to $\Gamma_1(p^rM)$-level structure for some $r \geq 1$; i.e. $K_f := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \mid c \equiv 0 \bmod p^rM, d \equiv 1 \bmod p^rM\}$.

We begin with the operator $\pi_{N_0,\wp_0}$, which maps a cohomology class c_f to the class $p^{-1}\sum_{i=0}^{p-1} g_i^*c_f$, where $g_i := \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p) \subset \mathrm{GL}_2(\mathbb{A}_f)$. (Here the inclusion denotes the embedding of $\mathrm{GL}_2(\mathbb{Q}_p)$ as the pth factor of $\mathrm{GL}_2(\mathbb{A}_f)$.) If we set $\gamma_i := \begin{pmatrix} 1/p & -i/p \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$, then $\gamma_i g_i \in K_f$. Since $U_pf := p^{-1}\sum_{i=0}^{p-1} f(\dfrac{\tau + i}{p})$, the above formula allows us to compute that $\pi_{N_0,\wp_0}c_f = p^{-1}c_{U_pf}$, as claimed (the factor p^{-1} arising from $\det \gamma_i = p^{-1}$).

The operator $\pi_{N_0,\wp_0\wp_1}$ acts via the element $g := \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$. Choose $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \pmod M$ and $\sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p^r}$. If we set $\gamma := \sigma \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$, then $\gamma g \in K_f$, and computing with the above formula shows that $\pi_{N_0,\wp_0\wp_1} c_f = p^k c_{\langle p\rangle_M f}$ (where $\langle p\rangle_M$ denotes the diamond operator corresponding to the residue class of p modulo M).

Similar computations show that if $t = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \in T_1$ then $\pi_{N_0,t} c_f = c_{\langle v^{-1}\rangle_p f}$ (where $\langle v^{-1}\rangle_p$ denotes the mod p-power diamond operator corresponding to $v^{-1} \in \mathbb{Z}_p^\times$), that $T(\ell)c_f = c_{T_\ell f}$, and that $T(\ell,\ell)c_f = \ell^k c_{\langle \ell\rangle_{Mp} f}$ (where $\langle \ell\rangle_{Mp}$ denotes the diamond operator corresponding to ℓ modulo M times a power of p). □

The following corollary shows that our use of the term "critical slope" agrees with the usual sense of this term, in the p-adic theory of modular forms.

Corollary 4.4.3. *Let f be a p-stabilized newform of weight $k+2$ and U_p-eigenvalue α attached to a point (χ,λ) of $E(1,M)_{\mathrm{par,cl},0}$. The character $\chi \in \hat{T}(\overline{\mathbb{Q}}_p)$ is of non-critical slope* (*in the sense of* [14, Def. 4.4.3]) *if and only if* $\mathrm{ord}_p(\alpha) < k+1$.

Proof. Write $\chi = \theta\psi_k$ as above, and let α_0 and α_1 denote a basis for the character lattice of $\mathbb{T}$, chosen so that α_0 denotes projection onto the upper left entry, and α_1 projection onto the lower right entry, of a matrix in $\mathbb{T}$. There is a unique positive root of $\mathbb{T}$, namely $\alpha_0 - \alpha_1$. The corresponding simple reflection s interchanges α_0 and α_1. If $h = \mathrm{ord}_p(\alpha)$, then $\mathrm{slope}\,\theta = (h-1)\alpha_0 + (k-h+1)\alpha_1$, while $\psi_k = -k\alpha_1$. (Here $\mathrm{slope}\,\theta$ is defined as in [14, Def. 1.4.2].) Finally, $\rho = (\alpha_0 - \alpha_1)/2$. Thus

$$\begin{aligned}\mathrm{slope}\,\theta + \rho + s(\psi_k + \rho) &\\ = (h - \tfrac{1}{2})\alpha_0 + (k - h + \tfrac{1}{2})\alpha_1 + (-k - \tfrac{1}{2})\alpha_0 + \tfrac{1}{2}\alpha_1 &\\ = (h-k-1)(\alpha_0 - \alpha_1),&\end{aligned}$$

and so χ is of non-critical slope if and only if $h < k+1$, as claimed. □

The points of $E(1,M)_{\mathrm{cl},0}$ that don't belong to $E(1,M)_{\mathrm{par},0}$ correspond to various classical Eisenstein series.

Recall from Definition 0.6 that $E(1,M)$ is defined to be the rigid analytic Zariski closure of $E(1,M)_{\mathrm{cl}}$ in $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$. Similarly, we define $E(1,M)_{\mathrm{par}}$ to be the rigid analytic Zariski closure of $E(1,M)_{\mathrm{par,cl}}$ in $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$, and define $E(1,M)_0$ and $E(1,M)_{\mathrm{par},0}$ to be the rigid analytic Zariski closures of $E(1,M)_{\mathrm{cl},0}$ and $E(1,M)_{\mathrm{par,cl},0}$ respectively in $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$.

The isomorphism (4.3.4) shows that Proposition 2.3.8 applies to our current situation. As in that subsection, let $\mathcal{E}$ denote the coherent sheaf on $\hat{T}$ that corresponds to the dual of $J_B(H^1(K^p)_{\mathrm{la}})$, and let $\mathcal{A}$ denote the commutative subring of $\mathrm{End}(\mathcal{E})$ generated by $\mathcal{H}(K^p)^{\mathrm{sph}}$. Proposition 2.3.8 shows that $\mathrm{Spec}\,\mathcal{A}$ (which we regard as a closed subspace of $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$) contains the eigenvariety $E(1,M)$. Similarly, let $\mathcal{E}_c$ be the coherent sheaf on $\hat{T}$ that corresponds to the dual of $J_B(H^1_c(K^p)_{\mathrm{la}})$, and let $\mathcal{A}_c$ denote the commutative subring of $\mathrm{End}(\mathcal{E})$ generated by $\mathcal{H}(K^p)^{\mathrm{sph}}$. The isomorphism (4.3.7), and an obvious analogue of Proposition 2.3.8 in the compact supports case, shows that $\mathrm{Spec}\,\mathcal{A}_c$ (regarded as a closed subspace of $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$) also contains the eigenvariety $E(1,M)$. (Here we are using the well-known fact that the collection of irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representations appearing in $H^1_c(\mathcal{V}_W)$ coincides with the collection of irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representations appearing in $H^1(\mathcal{V}_W)$, for any algebraic representation W of GL_2. Thus if we replace cohomology by compactly supported cohomology in the definition of $E(1,M)$, we obtain exactly the same subset of $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$.)

In the remainder of this subsection we indicate the relation between our constructions and the construction of [11] (and its generalization in [6]).

The inclusion of the trivial character into $\mathcal{W}_0$ induces an embedding

$$\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1} \to \mathcal{W}_0 \times \mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1} \xrightarrow{\sim} \hat{T}. \tag{4.4.4}$$

We let $\mathcal{E}_0$ denote the pullback of $\mathcal{E}$ to $\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$ via this embedding. (Thus $\mathcal{E}_0$ is a coherent sheaf on $\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$.) We let $\mathcal{A}_0$ denote the image of $\mathcal{A}$ in $\mathrm{End}(\mathcal{E}_0)$. Thus $\operatorname{Spec} \mathcal{A}_0$ is equal to the pullback of $\operatorname{Spec} \mathcal{A}$ along the embedding

$$\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1} \times \operatorname{Spec} \mathcal{H}(K^p)^{\mathrm{sph}} \to \hat{T} \times \operatorname{Spec} \mathcal{H}(K^p)^{\mathrm{sph}}$$

induced by (4.4.4).

We let $\mathcal{W}_\Delta$ denote the diagonal copy of $\mathcal{W}$ embedded in $\mathcal{W}_0 \times \mathcal{W}_1$. Since $\mathcal{W}$ is a rigid analytic group (the group structure given by multiplication of characters), there is a natural isomorphism

$$\mathcal{W}_\Delta \times \mathcal{W}_1 \xrightarrow{\sim} \mathcal{W}_0 \times \mathcal{W}_1, \tag{4.4.5}$$

given by $(w, w) \times w_1 \mapsto (w, ww_1)$.

Proposition 4.4.6. *There are natural isomorphisms*

$$\mathcal{E} \xrightarrow{\sim} \mathcal{O}_{\mathcal{W}_\Delta} \boxtimes \mathcal{E}_0 \tag{4.4.7}$$

and

$$\mathcal{A} \xrightarrow{\sim} \mathcal{O}_{\mathcal{W}_\Delta} \boxtimes \mathcal{A}_0. \tag{4.4.8}$$

(*Here the exterior products $\mathcal{O}_{\mathcal{W}_\Delta} \boxtimes \mathcal{E}_0$ and $\mathcal{A} \xrightarrow{\sim} \mathcal{O}_{\mathcal{W}_\Delta} \boxtimes \mathcal{A}_0$ are regarded at first as sheaves on $\mathcal{W}_\Delta \times \mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$, and are then transported to sheaves on $\mathcal{W}_0 \times \mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$ via the isomorphism (4.4.5).*)

Proof. Let $N_0 := \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z}_p\}$. Since N_0 acts trivially on $\hat{H}^0_{\mathrm{la}}$, we see that the isomorphism of Proposition 4.2.8 induces an isomorphism

$$\mathcal{C}^{\mathrm{la}}(T_0, E) \,\hat{\otimes}_E (\hat{H}^1(K^p)_{\mathrm{la}})^{T_0 N_0} \xrightarrow{\sim} (\hat{H}^1(K^p)_{\mathrm{la}})^{N_0}. \tag{4.4.9}$$

If we define T^+ as in the proof of Proposition 4.4.2, then the Hecke operators $\pi_{N_0,t}$ (as defined in [14, Def. 3.4.2]) determine an action of T^+ on each of $(\hat{H}^1(K^p)_{\mathrm{la}})^{T_0 N_0}$ and $(\hat{H}^1(K^p)_{\mathrm{la}})^{N_0}$. By Lemma 4.2.4 (i), the embedding (4.2.3) realises $\mathcal{C}^{\mathrm{la}}(T_0, E)$ as a T-invariant subspace of $\hat{H}^0(K^p)_{\mathrm{la}}$. The isomorphism (4.4.9) is T^+-equivariant, if we equip the source with the tensor product of the T^+-action on the first factor (obtained by restricting the T-action) and the Hecke T^+-action on the second factor, and if we equip the target with the Hecke T^+-action. By [14, Prop. 3.2.9] there is an isomorphism of locally analytic T-representations

$$\mathcal{C}^{\mathrm{la}}(T_0, E) \,\hat{\otimes}_E ((\hat{H}^1(K^p)_{\mathrm{la}})^{T_0 N_0})_{\mathrm{fs}} \xrightarrow{\sim} ((\hat{H}^1(K^p)_{\mathrm{la}})^{N_0})_{\mathrm{fs}} =: J_B(\hat{H}^1(K^p)_{\mathrm{la}}). \tag{4.4.10}$$

It follows from [14, Prop. 3.2.11] that there is a natural isomorphism

$$((\hat{H}^1(K^p)_{\mathrm{la}})^{T_0 N_0})_{\mathrm{fs}} \xrightarrow{\sim} J_B(\hat{H}^1(K^p)_{\mathrm{la}})^{T_0},$$

and hence we may rewrite (4.4.10) as an isomorphism

$$\mathcal{C}^{\mathrm{la}}(T_0, E) \hat{\otimes}_E J_B(\hat{H}^1(K^p)_{\mathrm{la}})^{T_0} \xrightarrow{\sim} J_B(\hat{H}^1(K^p)_{\mathrm{la}}).$$

Passing to duals, and reinterpreting this isomorphism sheaf theoretically, we obtain the isomorphism (4.4.7). The isomorphism (4.4.8) follows directly. □

The isomorphism (4.4.8) induces an isomorphism of rigid analytic spaces

$$\operatorname{Spec} \mathcal{A} \xrightarrow{\sim} \mathcal{W} \times \operatorname{Spec} \mathcal{A}_0. \tag{4.4.11}$$

This isomorphism induces in an analogous fashion isomorphisms

$$E(1, M) \xrightarrow{\sim} \mathcal{W} \times E(1, M)_0 \tag{4.4.12}$$

and

$$E(1, M)_{\mathrm{par}} \xrightarrow{\sim} \mathcal{W} \times E(1, M)_{\mathrm{par},0}. \tag{4.4.13}$$

Having eliminated one of the factors of $\mathcal{W}$ in the space $\hat{T}$ from consideration, let us explain how one can eliminate the corresponding factor of $\mathbb{G}_m$. The group $(\mathbb{Z}/M)^\times$ acts on $\mathcal{E}$ in the following way: an element $x \in (\mathbb{Z}/M)^\times$ acts via the coset $\begin{pmatrix} \tilde{x}^{-1} & 0 \\ 0 & \tilde{x}^{-1} \end{pmatrix} K^p \in \mathcal{H}(K^p)$, where $\tilde{x}$ is any lift of x to an element of $\hat{\mathbb{Z}}^p$. (This normalization of the action is chosen so that on classical points it corresponds to the action of the diamond operator $\langle x \rangle_M$, via the correspondence of Proposition 4.4.2.) Let us choose E so large that all the $\overline{\mathbb{Q}}_p^\times$-valued characters of $(\mathbb{Z}/M)^\times$ are defined over E. Then we find that $\mathcal{E} = \bigoplus_{\epsilon_M} \mathcal{E}^{\epsilon_M}$, where ϵ_M runs over the collection of such characters, and $(-)^{\epsilon_M}$ denotes the ϵ_M-eigenspace. Similarly, we have $\mathcal{E}_0 = \bigoplus_{\epsilon_M} \mathcal{E}_0^{\epsilon_M}$.

Fix a character ϵ_M of $(\mathbb{Z}/M)^\times$, and let S denote the support of $\mathcal{E}_0^{\epsilon_M}$, in the "scheme theoretic" sense, so that S is the closed rigid analytic subvariety of $\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$, cut out by the annihilator ideal sheaf of $\mathcal{E}_0^{\epsilon_M}$. Then the restriction of $\mathcal{E}_0^{\epsilon_M}$ to S is a coherent rigid analytic sheaf whose pushforward to $\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$ is naturally isomorphic to $\mathcal{E}_0^{\epsilon_M}$.

Proposition 4.4.14. (*i*) *The closed subvariety S of $\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1}$ maps isomorphically onto its "scheme theoretic" image in $\mathcal{W}_1 \times \mathbb{G}_{m,1}$.*

(*ii*) *The projection map $S \to \mathcal{W}_1$ has discrete fibres; in particular, S is at most one-dimensional.*

Proof. The global sections of $\mathcal{E}_0^{\epsilon_M}$ are equal to $(J_B(\hat{H}^1(K^p)_{\mathrm{la}})')^{T_0, \epsilon_M}$. To prove part (i), it suffices to show that $\mathcal{C}^{\mathrm{an}}(\mathcal{W}_1 \times \mathbb{G}_{m,0} \times \mathbb{G}_{m,1})$ and $\mathcal{C}^{\mathrm{an}}(\mathcal{W}_1 \times \mathbb{G}_{m,1})$ have the same image in the ring of endomorphisms of this space.

Let us consider how the element $\wp_0 \in T$ acts on $(J_B(\hat{H}^1(K^p)_{\mathrm{la}})')^{T_0, \epsilon_M}$. If we take into account that $Z(\mathbb{Q})_{>0}$ (by which we mean the group of scalar matrices with positive rational number entries, regarded as a subgroup of $Z(\mathbb{A}_f)$) acts trivially on this space, we find that $\wp_0$ acts through the scalar $\epsilon_M(p)$ times the action of $\wp_1^{-1}$. (Here we write $\epsilon_M(x)$ to denote ϵ_M evaluated on the scalar matrix $\begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}$.) Thus we see that indeed $\mathcal{C}^{\mathrm{an}}(\mathbb{G}_{m,0} \times \mathbb{G}_{m,1} \times \mathcal{W}_0)$ and $\mathcal{C}^{\mathrm{an}}(\mathbb{G}_{m,0} \times \mathcal{W}_0)$ have the same image in the ring of endomorphisms of $(J_B(\hat{H}^1(K^p)_{\mathrm{la}})')^{T_0 K^p, \epsilon_M}$, proving (i).

The claim of (ii) follows from Proposition 2.3.3 (ii), which shows that the support of $\mathcal{E}$ maps onto $\mathcal{W}_0 \times \mathcal{W}_1$ with discrete fibres. Together with Proposition 4.4.6, this implies that the support of $\mathcal{E}_0$, and so also the support of $\mathcal{E}_0^{\epsilon_M}$, maps onto $\mathcal{W}_1$ with discrete fibres. $\square$

The preceding proposition shows that $\mathcal{E}_0^{\epsilon_M}$ may be regarded as a coherent sheaf on $\mathcal{G}_{m,1} \times \mathcal{W}_1$, whose support maps to $\mathcal{W}_1$ with discrete fibres.

Let $\mathcal{A}_0^{\epsilon_M}$ denote the image of $\mathcal{A}$ in $\mathrm{End}(\mathcal{E}_0^{\epsilon_M})$. Then $\mathrm{Spec}\,\mathcal{A}_0^{\epsilon_M}$ is a closed subspace of $\mathrm{Spec}\,\mathcal{A}_0$, and the Proposition 4.4.14 allows us to regard $\mathrm{Spec}\,\mathcal{A}_0^{\epsilon_M}$ as a closed subspace of $\mathcal{W}_1 \times \mathbb{G}_{m,1} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$. In fact if ℓ is a prime not dividing Mp, then $T(\ell,\ell)$ acts on $\mathrm{Spec}\,E_0^{\epsilon_M}$ via $\epsilon_M(\ell)$ times the action of the element ℓ^{-1}, regarded as a section of $\mathcal{O}_{\mathcal{W}_1}$. Thus in fact $\mathrm{Spec}\,\mathcal{A}_0^{\epsilon_M}$ embeds as a closed subspace of the product

$$\mathcal{W}_1 \times \mathbb{G}_{m,1} \times \prod_{(\ell,Mp)=1} \mathrm{Spec}\,E[T(\ell)].$$

If we let $E(1,M)_0^{\epsilon_M}$ denote the the intersection of $E(1,M)_0$ and $\mathrm{Spec}\,\mathcal{A}^{\epsilon_M}$, then $E(1,M)_0 = \coprod_{\epsilon_M} E(1,M)_0^{\epsilon_M}$, and each $E(1,M)^{\epsilon_M}$ embeds as a closed subspace of

$$\mathcal{W}_1 \times \mathbb{G}_{m,1} \times \prod_{(\ell,Mp)=1} \mathrm{Spec}\,E[T(\ell)].$$

In fact the union of these embeddings is again an embedding of $E(1,M)_0$ into $\mathcal{W}_1 \times \mathbb{G}_{m,1} \times \prod_{(\ell,Mp)=1} \mathrm{Spec}\,E[T(\ell)]$. This follows from the fact that the eigenvalues of the Hecke operators $T(\ell)$ parameterize the traces of Frobenius of a family of Galois pseudorepresentations over $E(1,M)_0$, from whose determinant the tame nebentypus ϵ_M may be recovered.

In short, a point of $E(1,M)_0$ is determined by a "weight" (i.e. an element of $\mathcal{W}_1$), a U_p-eigenvalue (i.e. an element of $\mathbb{G}_{m,1}$ – more precisely, if this element is denoted a, then the corresponding U_p-eigenvalue equals $p\epsilon_M(p)/a$, where ϵ_M denotes the tame nebentypus, as one sees from the statement of Proposition 4.4.2), and a collection of T_ℓ-eigenvalues, one for each prime ℓ not dividing Mp.

It follows from its construction, and from Proposition 4.4.13 (ii), that $E(1,M)_0$ is a reduced rigid analytic space whose connected components are either zero-dimensional or equidimensional of dimension one. Since $E(1,M)_0$ is equal to the Zariski closure of $E(1,M)_{\mathrm{cl},0}$, we see that any zero-dimensional component of $E(1,M)_0$ must consist of classical points. Since "slope of the U_p-eigenvalue" is a locally constant function on $E(1,M)_0$, we see that any point of $E(1,M)_{\mathrm{cl}}$ that is not isolated in $E(1,M)_0$ can be written as the limit of a sequence of points of classical weight and of non-critical slope, which are hence themselves classical. Thus we see that each one-dimensional irreducible component of $E(1,M)_0$ contains a Zariski dense set of classical points of non-critical slope.

As we will now explain, one can in fact show that $E(1,M)_0$ is isomorphic to the reduced eigencurve of [11] (or more precisely, to its tame level M generalization considered in [6]). This will show in particular that every component of $E(1,M)_0$ has dimension one (i.e. that $E(1,M)_0$ contains no isolated points).

In Part II of [6] Buzzard applies his "eigenvariety machine" to spaces of overconvergent modular forms of tame level M (for any $M \geq 1$ prime to p), equipped with the action of the compact operator U_p and the Hecke algebra $\mathbf{T}$ of prime-to-p Hecke operators, to construct a rigid analytic curve that he calls "D", which in the case when $M = 1$ coincides with the reduced eigencurve constructed in [11].

In general, the curve that Buzzard constructs will not be reduced (because he works with the full Hecke algebra $\mathbf{T}$, and because of the presence of oldforms). However, one may form the analogue of Buzzard's construction after replacing $\mathbf{T}$ by its subalgebra $\mathbf{T}'$ of prime-to-Np Hecke operators, to obtain a curve that we will denote by $D(M)$. Since we omit the T_ℓ for ℓ dividing M, the curve $D(M)$ *will be* reduced. (Compare Proposition 3.9 and the proof of Proposition 4.8 of [9].)

The curve $D(M)$ parameterizes prime-to-M systems of eigenvalues arising from finite slope overconvergent p-adic Hecke eigenforms of tame level M and arbitrary weight. Thus a point of $D(M)$ is determined by giving a point of $\mathcal{W}$ (the weight), a point of $\mathbb{G}_m$ (the non-zero U_p-eigenvalue), and for each prime ℓ not dividing Mp a point of $\mathbb{A}^1$ (the ℓth Hecke eigenvalue). Thus we obtain an immersion of $D(M)$ into $\mathcal{W} \times \mathbb{G}_m \times \prod_{(\ell,Mp)} \mathbb{A}^1$. We noted above that $E(1,M)_0$ also embeds as a closed subspace of this product.

Proposition 4.4.15. *The embedding of $D(M)$ in $\mathcal{W} \times \mathbb{G}_m \times \prod_{(\ell,Mp)=1} \mathbb{A}^1$ discussed above induces an isomorphism of $D(M)$ with $E(1,M)_0$.*

Proof. Proposition 4.4.2 (and its extension to the non-cuspidal case, the details of which we leave to the reader) shows that $D(M)$ contains all the points of $E(1,M)_{\mathrm{cl}}$, and these points are Zariski dense in $D(M)$ by an evident generalization of [11, Thm. F]. Thus the proposition will follows if we show that $D(M)$ embeds as a Zariski closed subvariety of $\mathcal{W} \times \mathbb{G}_m \times \prod_{(\ell,Mp)=1} \mathbb{A}^1$. This follows from the fact that (by its very construction) the projection $D(M) \to \mathcal{W} \times \mathbb{G}_m$ is a finite morphism. $\square$

Corollary 4.4.16. *The rigid analytic space $E(1,M)_0$ is equidimensional of dimension one.*

Proof. This follows from the corresponding statement for $D(M)$ (which is a particular case of [6, Lem. 5.8]). $\square$

Since the isomorphisms (4.4.12) and (4.4.13) show that the varieties $E(1,M)$ and $E(1,M)_{\mathrm{par}}$ are obtained from $E(1,M)_0$ and $E(1,M)_{\mathrm{par},0}$ by allowing all wild twists, we deduce from this corollary that $E(1,M)$ and $E(1,M)_{\mathrm{par}}$ are equidimensional of dimension two.

We let $\mathcal{M}$ denote the localization of $\mathcal{E}$ over $\operatorname{Spec} \mathcal{A}$. It is a coherent sheaf on $\mathcal{E}$. It will be useful to describe explicitly its fibre over a point of $E(1,M)_{\mathrm{par,cl},0}$ of non-critical slope and tame conductor M.

Definition 4.4.17. If f is a p-stabilized newform of weight $k+2$ and p-stabilized conductor Mp^r, defined over a finite extension E' of E, then let H_f denote the subspace of $E' \otimes_E H^1(Y_1(Mp^r), \mathcal{V}_{\check{W}_k})$ on which the classical Hecke and diamond operators of level Mp^r act through the corresponding eigenvalues of f.

Let us remark that if in the preceding definition we were to replace H^1 by either H^1_c or H^1_{par} then we could analogously define spaces $H_{c,f}$ and $H_{\mathrm{par},f}$. However, since f is a cuspform, the natural maps

$$H_{c,f} \to H_{\mathrm{par},f} \to H_f$$

are both isomorphisms. Thus we write H_f to denote any of these canonically isomorphic spaces.

Proposition 4.4.18. *Suppose that $x = (\chi, \lambda) \in E(1, M)_{\mathrm{par,cl},0}(\overline{\mathbb{Q}}_p)$ corresponds (via Proposition 4.4.2) to a p-stabilized newform f of weight $k+2$, of tame conductor M, and of U_p-eigenvalue α. If χ is of non-critical slope (equivalently, by Corollary 4.4.3, if $\mathrm{ord}_p(\alpha) < k+1$), then the fibre $\mathcal{M}_x$ of $\mathcal{M}$ over x is canonically dual to the space $H_f \otimes_E (W_k)^N$. (Here N denotes the unipotent radical of B, so that $(W_k)^N$ is the (one-dimensional) highest weight space of W_k with respect to N.) It is two-dimensional, and the $\pm$-eigenspaces under the action of π_0 are each one-dimensional.*

Proof. Extending scalars from E to a finite extension over which x is defined, we may assume that x is defined over E. As in the statement and proof of Proposition 4.4.2, write $\chi = \theta\psi_k$ with θ smooth. The fibre $\mathcal{M}_{c,x}$ is by Proposition 2.3.3 (iii) dual to the $(T = \chi, \mathcal{H}(K^p) = \lambda)$-eigenspace of $J_B(\hat{H}^1(K^p)_{\mathrm{la}})$. This eigenspace is identified via the canonical lifting of [14, (3.4.8)] with the subspace of $\hat{H}^1(K^p)^{P_0}$ on which $\mathcal{H}(K^p)^{\mathrm{sph}}$ acts through λ, and the Hecke operators $\pi_{N_0,t}$ act through $\chi(t)$, for $t \in T^+$. (Here we are using the notation introduced in the proof of Proposition 4.4.2.) Since χ is of non-critical slope, this eigenspace consists of locally W_k-algebraic vectors [14, Thm. 4.4.5], and so (by the isomorphism (4.3.4)) is isomorphic to $H^1(K^p)^{P_0, T^+=\theta, \mathcal{H}(K^p)=\lambda} \otimes_E (W_k)^N$. By assumption (χ, λ) lies in $E(1, M)_{\mathrm{par,cl},0}$, and so this space is equal to $H^1_{\mathrm{par}}(K^p)^{P_0, T^+=\theta, \mathcal{H}(K^p)=\lambda} \otimes_E (W_k)^N$. If $\alpha = p\theta(\wp_0)$, then α is equal to the U_p-eigenvalue of f, and the discussion in the proof of Proposition 4.4.2 (together with strong multiplicity one for newforms) identifies $H^1_{\mathrm{par}}(K^p)^{P_0, T^+=\theta, \mathcal{H}(K^p)=\lambda}$ with the space H_f. Eichler-Shimura theory shows that this space is two-dimensional, and that it splits as the direct sum of $\pm$-eigenspaces under the action of π_0, each one-dimensional. □

If we take into account the fact that the modular curves $\tilde{Y}(K_f)$ admit canonical models defined over $\mathbb{Q}$, then the discussion of Subsection 2.4 shows that that $\mathcal{M}$ is equipped with a natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, commuting with its $\mathcal{H}_{\mathrm{fs}}(K^p)$-module structure.

Proposition 4.4.19. *If x is a point of $E(1, M)_{\mathrm{par,cl},0}$ of non-critical slope, corresponding (via Proposition 4.4.2) to a p-stabilized newform f of weight $k+2$ and tame conductor M, then the fibre $\mathcal{M}_x$ is "the" two-dimensional Galois representation attached to f. More precisely, this action is unramified at primes not dividing Mp, and if ℓ is such a prime, then the characteristic polynomial of Frob_ℓ (the arithmetic Frobenius at ℓ) is equal to $X^2 - a_\ell X + \epsilon(\ell)\ell^{k+1}$, where a_ℓ denotes the eigenvalue of T_ℓ on f, and ϵ denotes the nebentypus character of f.*

Proof. This follows from Proposition 4.4.18 and the known properties of the Galois representations on the spaces $H^1_{\mathrm{par}}(K^p, \mathcal{V}_{\check{W}_k})$, together with the fact that the isomorphism of Proposition 4.4.18 is Galois equivariant (as follows from Proposition 2.4.1). □

Proposition 4.4.20. *The coherent sheaf $\mathcal{M}$ is locally free of rank two in the neighbourhood of each point of non-critical slope and tame conductor M in $E(1,M)_{\mathrm{par,cl}}$.*

Proof. The isomorphism (4.4.7) gives rise to a corresponding isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{O}_{\mathcal{W}} \boxtimes \mathcal{M}_{|\operatorname{Spec}\mathcal{A}_0}$, compatible with the isomorphisms (4.4.11) and (4.4.12). Thus it suffices to check the statement of the corollary with $E(1,M)$ replaced by $E(1,M)_0$.

Proposition 4.4.18 shows that fibre of $\mathcal{M}$ over each classical point of non-critical slope and tame conductor M in $E(1,M)_{\mathrm{par},0}$ is two-dimensional. This already implies the statement of the corollary in the neighbourhood of any such point that is isolated in $E(1,M)_0$ (although Corollary 4.4.16 shows that there are no such points). Thus it suffices to check the statement of the corollary at classical points of non-critical slope that lie in one-dimensional components of $E(1,M)_0$. As was noted above, the neighbourhood of any such point x contains an infinite number of other such points. As x corresponds to a cuspidal newform of tame conductor M, a consideration of associated Galois representations shows that it cannot be written as the p-adic limit of Eisenstein series, or of newforms of tame conductor less than M. Thus each neighbourhood of x contains infinitely many points of $E(1,M)_{\mathrm{par,cl},0}$ of non-critical slope and tame conductor M. Since $E(1,M)_0$ is reduced, and since $\mathcal{M}$ has fibre rank two at all such points, we conclude that indeed $\mathcal{M}$ is locally free of rank two in the neighbourhood of x. □

(**4.5**) We close Section 4 by explaining how the results of the forthcoming paper [17] relating Jacquet functors and parabolic induction (which are summarized in [16, §5]) may be applied to construct a two-variable p-adic L-function over the eigencurves $E(1,M)_{\mathrm{par},0}$ considered in Subsection 4.4.

One-variable p-adic L-functions, following [26]. In [26] the authors define p-adic L-functions attached to p-stabilized newforms of non-critical slope. We will briefly recall their construction, using the notation and point of view adopted in this paper (which differs slightly from that of [26]).

If $r \in \mathbb{P}^1(\mathbb{Q})$, let $\{r,\infty\}$ denote an arc in the upper half-plane X of $\mathbb{C}\setminus\mathbb{R}$ joining r to ∞. (If $r=\infty$ then we understand this arc to be trivial.) Let K_f be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, and fix $k \geq 0$. Given any element of $H^1_c(\tilde{Y}(K_f), \mathcal{V}_{\check{W}_k})$, we may pull this element back to X and then integrate it along $\{r,\infty\}$ (for any $r \in \mathbb{P}^1(\mathbb{Q})$) and so obtain an element of $\check{W}_k$. This may in turn be paired against any element of W_k so as to obtain a scalar; thus each arc $\{r,\infty\}$ induces an element in the dual space to $H^1_c(\tilde{Y}(K_f), \mathcal{V}_{\check{W}_k}) \otimes_E W_k$, which we again denote by $\{r,\infty\}$.

Let f be a p-stabilized newform f of weight $k+2$, of p-stabilized conductor Mp^r, of U_p-eigenvalue α, and of non-critical slope (i.e. for which $\mathrm{ord}_p(\alpha) < k+1$). The discussion of the preceding paragraph allows us in particular to regard $\{r,\infty\}$ (for $r \in \mathbb{P}^1(\mathbb{Q})$) as an element in the dual space to $H_f \otimes_E W_k$. Equivalently, we obtain a map $\phi : W_k \times \mathbb{P}^1(\mathbb{Q}) \to \check{H}_f$ that is linear in the first variable.

Recall that $\check{W}_k$ is the kth symmetric power of the standard representation of GL_2. Let $(0,1)$ be the indicated element of E^2, and write $\check{w}_0 := (0,1)^k \in \mathrm{Sym}^k E^2 = \check{W}_k$. (If $\overline{\mathbb{B}}$ denotes the Borel subgroup of GL_2 consisting of lower triangular matrices, then $\check{w}_0$ is a highest weight vector of $\check{W}_k$ with respect to $\overline{\mathbb{B}}$.) The map $\mathbb{Q}_p \times W_k \to E$ defined by $(z,w) \mapsto \langle \check{w}_0, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} w\rangle$ identifies W_k with the space of polynomial functions on $\mathbb{Q}_p$ of degree $\leq k$ with coefficients in E. Having made this identification, the map ϕ of the preceding paragraph is seen to be a variant of the map $\phi(f,-,-)$ defined in [26, §1]. More precisely, the latter map is obtained by composing the map ϕ we have constructed with the map $\check{H}_f \to \mathbb{C}$ obtained by pairing elements of $\check{H}_f$ against the element of H_f defined by the holomorphic de Rham cohomology class associated to f as in the proof of Proposition 4.4.2 (multiplied by $2\pi i$), and replacing the polynomial $P(z)$ by $P(-z)$.

We may now use the recipe of [26] to define the p-adic L-function attached to x. This p-adic L-function is defined by a locally analytic distribution on $\mathbb{Z}_p^\times$ that we will denote by μ_f. We first define the values of μ_f on locally polynomial functions of degrees $\leq k$. Let $P(z)$ be a polynomial of degree $\leq k$, let $a \in \mathbb{Z}_p^\times$, and let p^m be some positive power of p. We define

$$\int_{a+p^m\mathbb{Z}_p} P(z)d\mu_f(z) := \alpha^{-m}\phi(P(p^m z + a), \frac{a}{p^m}) \in \check{H}_f. \tag{4.5.1}$$

(Here $P(p^m z + a)$ is the indicated polynomial, regarded as an element of W_k via the above identification.) The integral of any locally polynomial function of degrees $\leq k$ over $\mathbb{Z}_p^\times$ with respect to μ_f can obviously be written as a linear combination of such integrals, and so by linearity μ_f is defined on the space of all such functions. Since f is of non-critical slope, the Theorem of Vishik and Amice-Vélu [26, Thm., p. 13] shows that μ_f extends to a locally analytic $\check{H}_f$-valued distribution on $\mathbb{Z}_p^\times$. The p-adic L-function attached to f is the $\check{H}_f$-valued rigid analytic L-function on weight space $\mathcal{W}$ defined via $L(f,\chi) := \int_{\mathbb{Z}_p^\times} \chi d\mu_f$, for $\chi \in \mathcal{W}(\overline{\mathbb{Q}}_p)$. (The p-adic Fourier theory of Amice [1] shows that this formula does indeed yield a rigid analytic function on $\mathcal{W}$.)

Two-variable p-adic L-functions. We now explain our construction of two-variable p-adic L-functions. We will use an obvious variation of the constructions of Subsection 4.4, in which we work with compactly supported cohomology rather than cohomology. We let $\mathcal{M}_c$ denote the sheaf on $\operatorname{Spec} \mathcal{A}_c$ obtained by localizing $\mathcal{E}_c$. The natural map $\hat{H}^1_c(K^p)_{\mathrm{la}} \to \hat{H}^1(K^p)_{\mathrm{la}}$ induces a corresponding map of Jacquet modules, and hence a map $\mathcal{E} \to \mathcal{E}_c$ of coherent sheaves on $\hat{T}$. This in turn induces a map

$$\mathcal{M}_{|E(1,M)_{\mathrm{par},0}} \to \mathcal{M}_{c|E(1,M)_{\mathrm{par},0}}. \tag{4.5.2}$$

Lemma 4.5.3. *The map (4.5.2) is an isomorphism in the neighbourhood of each point of tame conductor M and non-critical slope.*

Proof. Proposition 4.4.20, and its obvious analogue in the case of compactly supported cohomology, show that the source and target of (4.5.2) are each locally free of rank two in the neighbourhood of points of tame conductor M and non-critical slope in $E(1,M)_{\mathrm{par},0}$. Furthermore, this map induces an isomorphism on fibres at each such point (as follows from Proposition 4.4.18, its analogue in the compactly supported case, and the observations preceding its statement). The lemma follows. □

Definition 4.5.4. Let E' be a finite extension of E. We say that a point $x = (\chi,\lambda) \in E(1,M)_{\mathrm{par},0}(E')$ is bad if the character χ is locally algebraic, of the form $\chi = \theta\psi_k$ for some smooth character θ and some $k \geq 0$, and if the point $(r^{-k-1}\chi,\lambda)$ also lies in $\operatorname{Spec} \mathcal{A}_c$; here r denotes the positive simple root of GL_2 with respect to $\mathbb{B}$.

Proposition 4.5.5. *The set of bad points (defined over some finite extension of E) is a discrete subset of $E(1,M)_{\mathrm{par},0}$. It does not contain any point of $E(1,M)_{\mathrm{par,cl},0}$ that is of non-critical slope.*

Proof. If $x \in E(1,M)_{\mathrm{par},0}$ is a classical point of non-critical slope, then an argument like that in the proof of [14, Thm. 4.4.5] shows that x does not satisfy Definition 4.5.4. This completes the proof of the second claim of the proposition. Since "slope" is a locally constant function on $E(1,M)_{\mathrm{par},0}$, we see that any point of $E(1,M)_{\mathrm{par},0}$ that does satisfy Definition 4.5.4 has a neighbourhood all of whose points of classical weight, other than x itself, are in fact of non-critical slope. This proves the first claim of the proposition. $\square$

Corollary 4.4.16 shows that $E(1,M)_{\mathrm{par},0}$ is a union of irreducible components that are each of dimension one. Since the fibre of $E(1,M)_{\mathrm{par}}$ over any point of $\mathcal{W}$ is discrete, it follows that each component of $E(1,M)_{\mathrm{par},0}$ is flat over $\mathcal{W}$.

Definition 4.5.6. We let $E(1,M)^*_{\mathrm{par},0}$ denote the complement of the set of bad points in $E(1,M)_{\mathrm{par},0}$, and let $\mathcal{M}_c^*$ denote the quotient of the restriction of $\mathcal{M}_c$ to $E(1,M)^*_{\mathrm{par},0}$ by its maximal $\mathcal{O}_{\mathcal{W}}$-torsion submodule.

Let $x \in E(1,M)_{\mathrm{par,cl},0}$ be a point of non-critical slope, corresponding via Proposition 4.4.2 to a p-stabilized newform f of tame conductor M. Proposition 4.4.20 and Lemma 4.5.3 imply that $\mathcal{M}_c$ is locally free of rank two in a neighbourhood of x, and hence (by the flatness result discussed prior to Definition 4.5.6) that $\mathcal{M}_c$ is $\mathcal{O}_{\mathcal{W}}$-torsion free in the neighbourhood of x. Thus $\mathcal{M}_c$ and $\mathcal{M}_c^*$ coincide in a neighbourhood of x. Proposition 4.4.18 (and Lemma 4.5.3) then imply that the fibre $\mathcal{M}^*_{c,x}$ of $\mathcal{M}_c^*$ over x is canonically dual to $H_f \otimes_E (W_k)^N$. (As in the statement of Proposition 4.4.18, we let $(W_k)^N$ denote the highest weight space of W_k with respect to $\mathbb{B}$.) There is a unique basis element of $(W_k)^N$ that satisfies $\langle \check{w}_0, w\rangle = 1$ (where as above $\check{w}_0$ denotes the vector $(0,1)^k \in \mathrm{Sym}^k E^2 = \check{W}_k$). This basis element determines an isomorphism $(W_k)^N \xrightarrow{\sim} E$, and hence an isomorphism $\mathcal{M}^*_{c,x} \xrightarrow{\sim} \check{H}_f$.

Our goal in the remainder of this subsection is to prove the following theorem.

Theorem 4.5.7. *There is a rigid analytic section of the coherent sheaf $\mathcal{O}_{\mathcal{W}} \boxtimes \mathcal{M}_c^*$ on $\mathcal{W} \times E(1,M)^*_{\mathrm{par},0}$, which we denote by L, with the property that at each point $x \in E(1,M)_{\mathrm{par,cl},0}$ of non-critical slope, corresponding via Proposition 4.4.2 to a p-stabilized newform f of tame conductor M, the fibre L_x (which is a rigid analytic section of $\mathcal{O}_W \otimes \mathcal{M}^*_{c,x} \xrightarrow{\sim} \mathcal{O}_W \otimes \check{H}_f$, the isomorphism being provided by the preceding discussion) is equal to the p-adic L-function L_f constructed above.*

Proof. We let U denote the compact type space that is topologically dual to the Fréchet space of global sections of $\mathcal{M}_c^*$. The natural locally closed immersion $E(1,M)^*_{\mathrm{par},0} \to \hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^p)^{\mathrm{sph}}$ allows us to regard the space of global sections of $\mathcal{M}_c^*$ as a topological module over $\mathcal{C}^{\mathrm{an}}(\hat{T},E) := \Gamma(\hat{T},\mathcal{O}_{\hat{T}})$, with a commuting action of $\mathcal{H}(K^p)^{\mathrm{sph}}$. Thus U is a locally analytic T-representation, equipped with a commuting $\mathcal{H}(K^p)^{\mathrm{sph}}$-action. The locally analytic T-representation U is allowable in the sense of [16, Def. 5.18], since it is dual to the space of global sections of a coherent sheaf on a quasi-Stein rigid analytic variety.

We may restrict global sections of $\mathcal{M}_c$ to $E(1,M)^*_{\mathrm{par},0}$ and so obtain global sections of $\mathcal{M}_c^*$. The dual of this restriction map is a continuous $T \times \mathcal{H}(K^p)^{\mathrm{sph}}$-

equivariant map of compact type spaces

$$U \to J_B(\hat{H}^1_c(K^p)_{\mathrm{la}}). \tag{4.5.8}$$

Lemma 4.5.12 below shows that (4.5.8) is balanced, in the sense of [16, Def. 5.17].

As usual, let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and let $\overline{B} := \overline{\mathbb{B}}(\mathbb{Q}_p)$ denote the Borel subgroup of lower triangular matrices. Let $I^G_{\overline{B}}(U(\delta^{-1}))$ be the closed subspace of the locally analytic induction $\mathrm{Ind}^G_{\overline{B}} U(\delta^{-1})$ defined in [16, Def. 5.13]. Lemma 4.5.12 below shows that in fact $I^G_{\overline{B}}(U(\delta^{-1})) = \mathrm{Ind}^G_{\overline{B}} U(\delta^{-1})$. Thus [16, Thm. 5.19] shows that the map (4.5.8) induces a continuous $G \times \mathcal{H}(K^p)^{\mathrm{sph}}$-equivariant map

$$\mathrm{Ind}^G_{\overline{B}} U(\delta^{-1}) \to \hat{H}^1_c(K^p)_{\mathrm{la}}. \tag{4.5.9}$$

As above, let N denote the unipotent radical of B (the Borel of upper triangular matrices). It is naturally isomorphic to $\mathbb{Q}_p$ (via $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$), and embeds as an open subset of $G/\overline{B}$. (See the discussion at the beginning of [16, §5].) In particular, we obtain an embedding $\mathbb{Z}_p^\times \to G/\overline{B}$, and a corresponding closed embedding of compact type spaces

$$\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E) \,\hat{\otimes}_E\, U \to \mathrm{Ind}^G_{\overline{B}} U(\delta^{-1}). \tag{4.5.10}$$

(The source is a closed subspace of $\mathcal{C}^{\mathrm{la}}_c(N, U(\delta))$, which in turn is a closed subspace of the target.) Recall from [1] that the dual to $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E)$ is isomorphic to the space $\mathcal{C}^{\mathrm{an}}(\mathcal{W}, E) := \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})$. Thus composing (4.5.10) with (4.5.9), and then passing to topological duals, we obtain a $\mathcal{H}(K^p)^{\mathrm{sph}}$-equivariant morphism

$$(\hat{H}^1_c(K^p)_{\mathrm{la}})' \to \Gamma(\,\mathcal{W} \times E(1, M)^*_{\mathrm{par},0}\,,\, \mathcal{O}_W \boxtimes \mathcal{M}^*_c\,). \tag{4.5.11}$$

(Here the source is the topological dual to the space $\hat{H}^1_c(K^p)_{\mathrm{la}}$, and the right hand side denotes the Fréchet space of global sections of the coherent sheaf $\mathcal{O}_{\mathcal{W}} \boxtimes \mathcal{M}^*_c$ on $\mathcal{W} \times E(1, M)^*_{\mathrm{par},0}$.)

One way to construct elements in the source of (4.5.11) is via the arcs $\{r, \infty\}$ for $r \in \mathbb{P}^1(\mathbb{Q})$. Indeed, integrating along such an arc gives rise to a homomorphism $H^1_c(\tilde{Y}(K_f), A) \to A$ for any level K_f and any ring A, compatibly with change of level and change of ring. Consequently $\{r, \infty\}$ induces an element of the topological dual to $\hat{H}^1_c(K^p)$, and so also (simply by composing with the continuous injection $\hat{H}^1_c(K^p)_{\mathrm{la}} \to \hat{H}^1_c(K^p)$) an element of $(\hat{H}^1_c(K^p)_{\mathrm{la}})'$. Applying (4.5.11) to the element in its source corresponding to $\{0, \infty\}$, we obtain a global section of $\mathcal{O}_{\mathcal{W}} \times \mathcal{M}^*$ over $\mathcal{W} \times E(1, M)^*_{\mathrm{par},0}$, which we denote by L.

If $x \in E(1, M)^*_{\mathrm{par},0}(\overline{\mathbb{Q}}_p)$, then let L_x denote the restriction of L to the closed subspace $\mathcal{W} \times x$ of $\mathcal{W} \times E(1, M)_{\mathrm{par},0}$; it is a section of $\mathcal{O}_{\mathcal{W}} \times \mathcal{M}^*_{c,x}$. Let μ_x denote the $\check{H}_f$-valued locally analytic distribution on $\mathbb{Z}_p^\times$ that corresponds via the p-adic Fourier theory of [1] to the rigid analytic function L_x on $\mathcal{W}$.

If x is a classical point of non-critical slope, corresponding to a p-stabilized newform f of tame conductor M, then by its construction the distribution μ_x coincides with the distribution μ_g appearing in the statement of Proposition 4.9 of [15] (taking the p-stabilized newform g of that proposition to be f). That proposition shows that L_x coincides with the p-adic L-function of f, and so completes the proof of the present theorem. $\square$

Lemma 4.5.12. *(i) The map (4.5.8) is balanced.*

(ii) The closed embedding $I_{\overline{B}}^{G}(U(\delta^{-1})) \to \mathrm{Ind}_{\overline{B}}^{G} U(\delta^{-1})$ is an isomorphism.

Proof. By [14, Prop. 3.5.6], the map (4.5.8) induces a B-equivariant map

$$\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, U(\delta^{-1})) \to \hat{H}_c^1(K^p)_{\mathrm{la}},$$

and thus a $(\mathfrak{gl}_2, B)$-equivariant map

$$U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, U(\delta^{-1})) \to \hat{H}_c^1(K^p)_{\mathrm{la}}. \tag{4.5.13}$$

(As in the proof of Theorem 4.5.7, we are identifying $\mathbb{Q}_p$ with the unipotent radical N of B.)

Let $\mathcal{C}_c^{\mathrm{lp}}(\mathbb{Q}_p, U(\delta^{-1}))$ denote the space of compactly supported locally polynomial $U(\delta^{-1})$-valued functions on $\mathbb{Q}_p$. The inclusion $\mathbb{Q}_p \xrightarrow{\sim} N \to G/\overline{B}$ allows us to regard $\mathcal{C}_c^{\mathrm{lp}}(\mathbb{Q}_p, U(\delta^{-1}))$ as a $(\mathfrak{gl}_2, B)$-invariant subspace of $\mathrm{Ind}_{\overline{B}}^{G} U(\delta^{-1})$. The inclusion of $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, U(\delta^{-1}))$ in $\mathcal{C}_c^{\mathrm{lp}}(\mathbb{Q}_p, U(\delta^{-1}))$ thus induces a $(\mathfrak{gl}_2, B)$-equivariant map

$$U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, U(\delta^{-1})) \to \mathcal{C}_c^{\mathrm{lp}}(\mathbb{Q}_p, U(\delta^{-1})). \tag{4.5.14}$$

There are isomorphisms

$$\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, U(\delta^{-1})) \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, E) \otimes U(\delta^{-1})$$

and

$$\mathcal{C}_c^{\mathrm{lp}}(\mathbb{Q}_p, U(\delta^{-1})) \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, E) \otimes U(\delta^{-1})[z]$$

(where $U(\delta^{-1})[z]$ denotes the space of polynomials in z with coefficients in $U(\delta^{-1})$), and (4.5.14) may be factored as the tensor product of the identity automorphism of $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, E)$ and a map

$$U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{b})} U \to U[z] \tag{4.5.15}$$

(we drop the twists by δ from our notation, since we will consider only Lie algebra actions from now on, and δ is smooth), defined by the following explicit formula: Let n_- and h denote the elements $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\mathfrak{gl}_2$. Then $U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{b})} U = \bigoplus_i (n_-)^i U$, and (4.5.15) is defined by the formula

$$(n_-)^i u \mapsto ((h - i + 1) \ldots (h-1) h u)\, z^i. \tag{4.5.16}$$

Referring to the definitions of [16, §5], we find that part (i) of the lemma is equivalent to showing that the kernel of (4.5.13) contains the kernel of (4.5.14). Consider the map

$$U \to (\hat{H}_c^1(K^p)_{\mathrm{la}})^{N_0} \tag{4.5.17}$$

obtained as the composition of (4.5.8) with the canonical lift $J_B(\hat{H}_c^1(K^p)_{\mathrm{la}}) \to \hat{H}_c^1(K^p)_{\mathrm{la}}$. From the explicit formula (4.5.16), we see that part (i) holds provided that (4.5.17) takes any element u of U on which h acts via an integer $k \geq 0$ to a locally W_k-algebraic element of $\hat{H}^1_c(K^p)_{\mathrm{la}}$. Note that since the T_0-action on U is fixed to be trivial, the $\mathfrak{t}$-action on any such point u under (4.5.17) is completely determined. The fibre of $\operatorname{Spec}\mathcal{A}_c$ over a given character of $\check{\mathfrak{t}}$ is discrete, and so we may assume (after extending scalars if necessary) that such a point u is a generalized eigenvector for the action of T, with respect to some character $\chi \in \hat{T}(E)$, as well as a generalized eigenvector for the action of $\mathcal{H}(K^p)^{\mathrm{sph}}$, with respect to some character $\lambda \in (\operatorname{Spec}\mathcal{H}(K^p)^{\mathrm{sph}})(E)$. If v denotes the image of u under (4.5.17), then v is a generalized χ-eigenvector for the action of the Hecke operators $\pi_{N_0,t}$ (for $t \in T^+$), as well as a generalized λ-eigenvector for the action of $\mathcal{H}(K^p)^{\mathrm{sph}}$. A minor variant of [14, Prop. 4.4.4] shows that $(n_-)^{k+1}v$ lies in the generalized $(r^{-k-1}\chi, \lambda)$-eigenspace for the action of the operators $\pi_{N_0,t}$ $(t \in T^+)$ and the Hecke algebra $\mathcal{H}(K^p)^{\mathrm{sph}}$. From the construction of U, we see that the point (χ, λ) must not be a bad point of $\operatorname{Spec}\mathcal{A}_c$, and thus that $(r^{-k-1}\chi, \lambda)$ must not lie on $\operatorname{Spec}\mathcal{A}_c$. Consequently the generalized $(r^{-k-1}\chi, \lambda)$-eigenspace of $(\hat{H}^1_c(K^p)_{\mathrm{la}})^{N_0}$ vanishes, and so $(n_-)^{k+1}$ annihilates v. Hence v is W_k-locally algebraic, as we wanted to show.

Again referring to the definitions of [16, §5], we see that part (ii) of the lemma will follow if we show that (4.5.15) (and hence (4.5.14)) is surjective. Since U is dual to the space of global sections of a flat $\mathcal{O}_{\mathcal{W}}$-sheaf, it is divisible as a module under the action of the polynomial ring $E[h]$. Formula (4.5.16) thus implies that (4.5.15) is indeed surjective, as required. $\square$

The global section L of $\mathcal{O}_{\mathcal{W}} \boxtimes \mathcal{M}_c^*$ may be regarded as a two-variable p-adic L-function: one variable is provided by the factor $\mathcal{O}_{\mathcal{W}}$ (with an appropriate normalization this becomes the classical "s"-variable), the other by the punctured eigencurve $E(1, M)^*_{\mathrm{par},0}$.

If we decompose the coherent sheaf $\mathcal{M}_c^*$ into its $\pm$ eigenspaces under the action of π_0, then the section L decomposes into the sum of two sections $L^{\pm}$, whose fibres at classical points of non-critical slope correspond to the usual $+$ and $-$ p-adic L-functions (which vanish when evaluated on odd and even characters respectively).

Let us remark that Stevens [33] has previously constructed two-variable p-adic L-functions along the eigencurve, via a method quite different to ours. He uses his theory of rigid analytic modular symbols to construct a coherent sheaf (which we will denote by $\mathcal{S}$) along the eigencurve $E(1, M)_{\mathrm{par},0}$, whose sections are certain modular symbols with values in the space of locally analytic distributions on $\mathbb{Z}_p^{\times}$. The fibre of $\mathcal{S}$ at a classical point of non-critical slope maps isomorphically, via a specialization map, to the dual of the fibre of $\mathcal{M}_c$ at such a point. If $\mathcal{U}$ is an open subset of $E(1, M)_{\mathrm{par},0}$ for which $\mathcal{S}_{|\mathcal{U}}$ is free of rank two, then by choosing a basis of each of the $\pm$-parts of $\mathcal{S}_{|\mathcal{U}}$, evaluating these modular symbols on the arc $\{0, \infty\}$, and passing via the p-adic Fourier theory of [1] from distributions on $\mathbb{Z}_p^{\times}$ to rigid analytic functions on $\mathcal{W}$, Stevens obtains p-adic L-functions $L^{\pm}$ as rigid analytic functions on $\mathcal{W} \times \mathcal{U}$.

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