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LOCALLY ANALYTIC REPRESENTATION THEORY OF p-ADIC REDUCTIVE GROUPS: A SUMMARY OF SOME RECENT DEVELOPMENTS

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The purpose of this short note is to summarize some recent progress in the theory of locally analytic representations of reductive groups over p-adic fields. This theory has begun to find applications to number theory, for example to the arithmetic theory of automorphic forms, as well as to the "p-adic Langlands programme" (see [3, 4, 5, 10, 11, 12]). I hope that this note can serve as an introduction to the theory for those interested in pursuing such applications.

The theory of locally analytic representations relies for its foundations on notions and techniques of functional analysis. We recall some of these notions in Section 1. In Section 2 we describe some important categories of locally analytic representations (originally introduced in [20], [23] and [8]). In Section 3, we discuss the construction of locally analytic representations by applying the functor "pass to locally analytic vectors" to certain continuous Banach space representations. In Section 4 we briefly describe the process of parabolic induction in the locally analytic situation, which allows one to pass from representations of a Levi subgroup of a reductive group to representations of the reductive group itself, and in Section 5 we describe the Jacquet module construction of [9], which provides functors mapping in the opposite direction. Parabolic induction and the Jacquet module functors are "almost" adjoint to one another. (See Theorem 5.19 for a precise statement.)

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1. Functional analysis

We begin by recalling some notions of non-archimedean functional analysis. A more detailed exposition of the basic concepts is available in [17], which provides an excellent introduction to the subject.

Let K be a complete discretely valued field of characteristic zero. A topological K-vector space V is said to be locally convex if its topology can be defined by a basis of neighbourhoods of the origin that are $\mathcal{O}_K$-submodules of V; or equivalently, by a collection of non-archimedean semi-norms. (We will often refer to V simply as a convex space, or a convex K-space if we which to emphasize the coefficient field K.) The space V is called complete if it is complete as a topological group under addition.

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If V is any locally convex K-space, then we may complete V to obtain a complete Hausdorff convex K-space $\hat{V}$, equipped with a continuous K-linear map $V \to \hat{V}$, which is universal for continuous K-linear maps from V to complete Hausdorff K-spaces. (See [17, Prop. 7.5] for a construction of $\hat{V}$. Note that in this reference $\hat{V}$ is referred to as the Hausdorff completion of V.)

If V is a convex K-space, then we let V' denote the space of K-valued continuous K-linear functionals on V, and let V'_b denote V' equipped with its strong topology (the "bounded-open" topology – see [17, Def., p. 58]; the subscript "b" stands for "bounded"). We refer to V'_b as the *strong dual* of V. There is a natural K-linear "double duality" map $V \to (V'_b)'$; we say that V is *reflexive* if this map induces a topological isomorphism $V \to (V'_b)'_b$.

If V and W are two convex K-spaces, then we always equip $V \otimes_K W$ with the projective tensor product topology. This topology is characterized by the requirement that the map $V \times W \to V \otimes_K W$ defined by $(v, w) \mapsto v \otimes w$ should be universal for continuous K-bilinear maps from $V \times W$ to convex K-spaces. (See [17, §17] for more details about the construction and properties of this topology.) We let $V \hat{\otimes}_K W$ denote the completion of $V \otimes_K W$.

A complete convex space V is called a Fréchet space if it is metrizable, or equivalently, if its topology can be defined by a countable set of seminorms. If the topology of the complete convex space V can be defined by a single norm, then we say that V is a Banach space. Note that we don't regard a Fréchet space or a Banach space as being equipped with any particular choice of metric, or norm.

If V and W are Banach spaces, then the space $\mathcal{L}(V, W)$ of continuous linear maps from V to W again becomes a Banach space, when equipped with its strong topology. (Concretely, if we fix norms defining the topologies of V and W respectively, then we may define a norm on $\mathcal{L}(V, W)$ as follows (we denote all norms by $|| \, ||$): for any $T \in \mathcal{L}(V, W)$, set $||T|| = \sup_{v \in V \text{ s.t. } ||v||=1} ||T(v)||$.) We say that an element $T \in \mathcal{L}(V, W)$ is compact if it may be written as a limit (with respect to the strong topology) of a sequence of maps with finite dimensional range (see [17, Rem. 18.10]).

If V is a Fréchet space, then completing V with respect to each of the members of an increasing sequence of semi-norms that define its topology, we obtain a projective sequence of Banach spaces $\{V_n\}_{n \geq 1}$, and an isomorphism of topological K-vector spaces

$$V \xrightarrow{\sim} \varprojlim_n V_n,$$

where $\{V_n\}_{n \geq 1}$ is a projective system of Banach spaces over K, and the right hand side is equipped with the projective limit topology. Conversely, any such projective limit is a Fréchet space over K.

Definition 1.1. A nuclear Fréchet space over K is a K-space which admits an isomorphism of topological K-vector spaces

$$V \xrightarrow{\sim} \varprojlim_n V_n,$$

where $\{V_n\}_{n \geq 1}$ is a projective system of Banach spaces over K with compact transition maps (and the right hand side is equipped with the projective limit topology).

In fact there is a more intrinsic definition of nuclearity for any convex space [17, Def., p. 120], which is equivalent to the above definition when applied to a Fréchet space (as follows from the discussion of [17, §16] together with [20, Thm. 1.3]).

Proposition 1.2. *Let V be a nuclear Fréchet space.*
(i) V is reflexive.
(ii) Any closed subspace or Hausdorff quotient of V is again a nuclear Fréchet space.

Proof. See [17, Prop. 19.4]. □

We now introduce another very important class of locally convex spaces.

Definition 1.3. We say that a convex K-space V is of compact type if there is an isomorphism of topological K-vector spaces

$$V \xrightarrow{\sim} \varinjlim_n V_n,$$

where $\{V_n\}_{n\geq 1}$ is an inductive system of Banach spaces over K with compact and injective transition maps (and the right hand side is equipped with the locally convex inductive limit topology).

Proposition 1.4. *Let V be a space of compact type.*
(i) V is complete and Hausdorff.
(ii) V is reflexive.
(iii) Any closed subspace or Hausdorff quotient of V is again of compact type.

Proof. See [20, Thm. 1.1, Prop. 1.2]. □

Proposition 1.5. *Passing to strong duals yields an anti-equivalence of categories between the category of spaces of compact type and the category of nuclear Fréchet spaces.*

Proof. This is [20, Thm. 1.3]. A proof can also be extracted from the discussion of [17, §16]. □

We now define an important class of topological algebras over K (originally introduced in [23]).

Definition 1.6. Let A be a topological K-algebra. We say that A is nuclear Fréchet-Stein algebra if we may find an isomorphism $A \xrightarrow{\sim} \varprojlim_n A_n$, where $\{A_n\}_{n\geq 1}$ is a sequence of Noetherian K-Banach algebras, for which the transition maps $A_{n+1} \to A_n$ are compact (as maps of K-Banach spaces) and flat (as maps of K-algebras), and such that each of the maps $A \to A_n$ has dense image (or equivalently, by [2, II §3.5 Thm. 1], such that each of the maps $A_{n+1} \to A_n$ has dense image).

If A is a nuclear Fréchet-Stein algebra over K, then A is certainly a nuclear Fréchet space. If A is a topological K-algebra, then any two representations of A as a projective limit as in Definition 1.6 are equivalent in an obvious sense. (See [8, Prop. 1.2.7].)

Example 1.7. Let us explain the motivating example of a nuclear Fréchet-Stein algebra. Suppose that $\mathbb{X}$ is a rigid analytic space over K that may be written as a union $\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n$, where $\{\mathbb{X}_n\}_{n\geq 1}$ is an increasing sequence of open affinoid subdomains of $\mathbb{X}$, for which the inclusions $\mathbb{X}_n \to \mathbb{X}_{n+1}$ are admissible and relatively compact (in the sense of [1, 9.6.2]), and such that for each n the restriction map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_{n+1}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$ has dense image. (Here $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$ denotes the Tate algebra of rigid analytic K-valued functions on $\mathbb{X}_n$.) We will say that such a rigid analytic space $\mathbb{X}$ is *strictly quasi-stein*. (If one omits the requirement that the inclusions be relatively compact, one obtains the notion of a quasi-stein rigid analytic space, as defined by Kiehl.) Since $\mathbb{X}_n \to \mathbb{X}_{n+1}$ is an admissible open immersion for each $n \geq 1$, the restriction map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_{n+1}, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$ is flat. The relative compactness assumption implies that it is furthermore compact, and hence that the space

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \xrightarrow{\sim} \varprojlim_n \mathcal{C}^{\mathrm{an}}(\mathbb{X}_n, K)$$

of rigid analytic functions on $\mathbb{X}$ is naturally a nuclear Fréchet-Stein algebra.

Definition 1.8. Let A be a nuclear Fréchet-Stein algebra over K, and write $A \xrightarrow{\sim} \varprojlim_n A_n$ as in Definition 1.6. We say that a Hausdorff topological A-module M is coadmissible if the following two conditions are satisfied:

(i) The tensor product $M_n := A_n \otimes_A M$ is a finitely generated A_n-Banach module, for each n. (We regard the tensor product $A_n \otimes_A M$ as being a quotient of $A_n \otimes_K M$, and endow it with the quotient topology induced by the projective tensor product topology on $A_n \otimes_K M$.)

(ii) The natural map $M \to \varprojlim_n M_n$ is an isomorphism of topological A-modules.

The preceding definition is a variation of [23, Def., p. 152], to which it is equivalent, as the results of [23, §3] show.

Theorem 1.9. *Let A be a nuclear Fréchet-Stein algebra over K.*

(i) Any coadmissible topological A-module is a nuclear Fréchet space.

(ii) Any A-linear map between coadmissible topological A-modules is automatically continuous, with closed image.

(iii) The category of coadmissible topological A-modules (with morphisms being A-linear maps, which by (ii) are automatically continuous) is closed under taking finite direct sums, passing to closed submodules, and passing to Hausdorff quotients.

Proof. This summarizes the results of [23, §3]. □

Remark 1.10. The category of all locally convex Hausdorff topological A-modules is an additive category that admits kernels, cokernels, images and coimages. More precisely, if $f : M \to N$ is a continuous A-linear morphism between such modules, then its categorical kernel is the usual kernel of f, its categorical image is the closure of its set-theoretic image (regarded as a submodule of N), its categorical coimage is its set-theoretical image (regarded as a quotient module of M), and its categorical cokernel is the quotient of N by its categorical image.

Part (ii) of Theorem 1.9 implies that if M and N in the preceding paragraph are coadmissible, then the image and coimage of f coincide. Part (iii) of the Theorem then implies that the kernel, cokernel, and image of f are again coadmissible. Thus the category of coadmissible topological A-modules is an abelian subcategory of the additive category of locally convex Hausdorff topological A-modules.

Remark 1.11. If B is a Noetherian K-Banach algebra (for example, one of the algebras A_n appearing in Definitions 1.6 and 1.8), then the results of [1, 3.7.3] show that the natural functor from the category of finitely generated B-Banach modules (with morphisms being continuous B-linear maps) to the abelian category of finitely generated B-modules, given by forgetting topologies, is an equivalence of categories. Theorem 1.9 is an analogue of this result for the nuclear Fréchet-Stein algebra A. It shows that forgetting topologies yields a fully faithful embedding of the category of coadmissible topological A-modules as an abelian subcategory of the abelian category of all A-modules. In light of this, one can suppress all mention of topologies in defining this category (as is done in the definitions of [23, p. 152]).

Definition 1.12. If A is a nuclear Fréchet-Stein algebra over K, we say that a topological A-module M is strongly coadmissible if it is a Hausdorff quotient of A^n, for some natural number n.

Since A is obviously a coadmissible module over itself, Theorem 1.9 implies that any strongly coadmissible topological A-module is a coadmissible topological A-module.

Example 1.13. Suppose that $\mathbb{X}$ is a strictly quasi-Stein rigid analytic space over K, as in Example 1.7. If $\mathcal{M}$ is any rigid analytic coherent sheaf on $\mathbb{X}$, then the space M of global sections of $\mathcal{M}$ is naturally a coadmissible $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$-module, and passing to global sections in fact yields an equivalence of categories between the category of coherent sheaves on $\mathbb{X}$ and the category of coadmissible $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$-modules. The $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$-module M of global sections of the coherent sheaf $\mathcal{M}$ is strongly coadmissible if and only if $\mathcal{M}$ is generated by a finite number of global sections.

Fix a complete subfield L of K. We close this section by recalling the definition of the space of locally analytic functions on a locally L-analytic manifold with values in a convex space. (More detailed discussions may be found in [14, §2.1.10], [20, p. 447], and [8, §2.1].)

Definition 1.14. If $\mathbb{X}$ is an affinoid rigid analytic space over L, and if W is a K-Banach space, then we write $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W) := \mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) \hat{\otimes}_K W$. (Here, as above, we let $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ denote the Tate algebra of K-valued rigid analytic functions on $\mathbb{X}$, equipped with its natural K-Banach algebra structure.)

If the set $X := \mathbb{X}(L)$ of L-valued points of $\mathbb{X}$ is Zariski dense in $\mathbb{X}$, then $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, W)$ may be identified with the space of W-valued functions on X that can be described by convergent power series with coefficients in W.

Now let X be a locally L-analytic manifold. A chart of X is a compact open subset X_0 of X together with a locally analytic isomorphism between X_0 and the set of L-valued points of a closed ball. We let $\mathbb{X}_0$ denote this ball (thought of as a rigid L-analytic space), so that $X_0 \xrightarrow{\sim} \mathbb{X}_0(L)$. By an analytic partition of X we mean a partition $\{X_i\}_{i \in I}$ of X into a disjoint union of charts X_i. We assume that X is paracompact; then any covering of X by charts may be refined to an analytic partition of X. (Here we are using a result of Schneider [18, Satz 8.6], which shows that any paracompact locally L-analytic manifold is in fact strictly paracompact, in the sense of the discussion of [20, p. 446].)

If V is a Hausdorff convex space, then we say that a function $f : X \to V$ is locally analytic if for each point $x \in X$, there is a chart X_0 containing x, a Banach space W equipped with a continuous K-linear map $\phi : W \to V$, and a rigid analytic function $f_0 \in \mathcal{C}^{\mathrm{an}}(\mathbb{X}_0, W)$ such that $f = \phi \circ f_0$. (Replacing W by its quotient by the kernel of ϕ, we see that it is no loss of generality to require that ϕ be injective.) We let $\mathcal{C}^{\mathrm{la}}(X, V)$ denote the K-vector space of locally analytic V-valued functions on X, and let $\mathcal{C}^{\mathrm{la}}_c(X, V)$ denote the subspace consisting of compactly supported locally analytic functions.

It follows from the definition that there are K-isomorphisms of vector spaces

$$\mathcal{C}^{\mathrm{la}}(X,V) \xrightarrow{\sim} \varinjlim_{\{X_i,W_i,\phi_i\}_{i\in I}} \prod_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, W_i)$$

and

$$\mathcal{C}_c^{\mathrm{la}}(X,V) \xrightarrow{\sim} \varinjlim_{\{X_i,W_i,\phi_i\}_{i\in I}} \bigoplus_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, W_i),$$

where in both cases the inductive limit is taken over the directed set of collections of triples $\{X_i, W_i, \phi_i\}_{i\in I}$, where $\{X_i\}_{i\in I}$ is an analytic partition of X, each W_i is a K-Banach space, and $\phi_i : W_i \to V$ is a continuous injection. We regard $\mathcal{C}^{\mathrm{la}}(X,V)$ and $\mathcal{C}_c^{\mathrm{la}}(X,V)$ as Hausdorff convex spaces by equipping them with the locally convex inductive limit topologies arising from the targets of these isomorphisms. Note that the inclusion $\mathcal{C}_c^{\mathrm{la}}(X,V) \to \mathcal{C}^{\mathrm{la}}(X,V)$ is continuous, but unless X is compact (in which case it is an equality) it is typically not a topological embedding.

Given any collection $\{X_i, W_i, \phi_i\}_{i\in I}$ as above, there is a natural map

$$\bigoplus_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, W_i) = \bigoplus_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \hat{\otimes}_K W_i$$
$$\xrightarrow{\oplus\, \mathrm{id}\, \hat{\otimes}\, \phi_i} \bigoplus_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \hat{\otimes}_K V \longrightarrow \Big(\bigoplus_{i\in I} \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K)\Big) \hat{\otimes}_K V.$$

(Note that if we were working with inductive, rather than projective, tensor product topologies, then the last map would be an isomorphism.) Passing to the inductive limit over all such collections yields a continuous map

$$\mathcal{C}_c^{\mathrm{la}}(X,V) \to \mathcal{C}_c^{\mathrm{la}}(X,K) \hat{\otimes}_K V. \tag{1.15}$$

Proposition 1.16. *If X is σ-compact (i.e. the union of a countable number of compact open subsets) and V is of compact type then the map (1.15) is a topological isomorphism and $\mathcal{C}_c^{\mathrm{la}}(X,V)$ is again of compact type.*

Proof. If X is compact (so that $\mathcal{C}_c^{\mathrm{la}}(X,V) = \mathcal{C}^{\mathrm{la}}(X,V)$) then this is [8, Prop. 2.1.28]. The proof in the general case is similar. □

2. Categories of locally analytic representations

Fix a finite extension L of $\mathbb{Q}_p$, for some prime p, as well a field K that extends L and is complete with respect to a discrete valuation extending that on L. Let G be a locally L-analytic group (an analytic group over L, in the sense of [25, p. LG 4.1]). The identity element of G then has a neighbourhood basis consisting of compact open subgroups of G [25, Cor. 2, p. LG 4.23].

If H is any compact open subgroup of G, then Proposition 1.16 shows that the space $\mathcal{C}^{\mathrm{la}}(H,K)$ of locally L-analytic K-valued functions on H is a compact type convex K-space, and hence its strong dual is a nuclear Fréchet space, which we will denote by $\mathcal{D}^{\mathrm{la}}(H,K)$. Any element $h \in H$ gives rise to a "Dirac delta function" supported at h, which is an element $\delta_h \in \mathcal{D}^{\mathrm{la}}(H,K)$. In this way we obtain an embedding $K[H] \to \mathcal{D}^{\mathrm{la}}(H,K)$ (where $K[H]$ denotes the group ring of H over K). The image of $K[H]$ is dense in $\mathcal{D}^{\mathrm{la}}(H,K)$, and the K-algebra structure on $K[H]$ extends (in a necessarily unique fashion) to a topological K-algebra structure on $\mathcal{D}^{\mathrm{la}}(H,K)$ [20, Prop. 2.3, Lem. 3.1].

Theorem 2.1. *The topological K-algebra $\mathcal{D}^{\mathrm{la}}(H,K)$ is a nuclear Fréchet-Stein algebra.*

Proof. This is the main result of [23]. A different proof is given in [8, §5.3]. □

We will now consider various convex K-spaces V equipped with actions of G by K-linear automorphisms. There are (at least) three kinds of continuity conditions on such an action that one can consider. Firstly, one may consider a situation in which G acts by continuous automorphisms of V. (Such an action is referred to as a *topological action* in [8]; note that this condition does not make any reference to the topology of G.) Secondly, one may consider the case when the action map $G \times V \to V$ is separately continuous. Thirdly, one may consider the case when the action map $G \times V \to V$ is continuous. If V is *barrelled* (see [17, Def., p. 39]; for example a Banach space, a Fréchet space, or a space of compact type) then any separately continuous action is automatically continuous, by the Banach-Steinhaus theorem.

Proposition 2.2. *If V is a compact type convex space, equipped with an action of G by continuous K-linear automorphisms, then the following are equivalent:*

(i) For some compact open subgroup H of G, the $K[H]$-module structure on V extends to a (necessarily unique) $\mathcal{D}^{\mathrm{la}}(H,K)$-module structure on V, for which the map $\mathcal{D}^{\mathrm{la}}(H,K) \times V \to V$ describing this module structure is separately continuous.

(i') For every compact open subgroup H of G, the $K[H]$-module structure on V extends to a (necessarily unique) $\mathcal{D}^{\mathrm{la}}(H,K)$-module structure on V, for which the map $\mathcal{D}^{\mathrm{la}}(H,K) \times V \to V$ describing this module structure is separately continuous.

(ii) For some compact open subgroup H of G, the $K[H]$-module structure on V'_b arising from the contragredient H-action on V'_b extends to a (necessarily unique) topological $\mathcal{D}^{\mathrm{la}}(H,K)$-module structure on V'_b.

(ii') For every compact open subgroup H of G, the $K[H]$-module structure on V'_b arising from the contragredient H-action on V'_b extends to a (necessarily unique) topological $\mathcal{D}^{\mathrm{la}}(H,K)$-module structure on V'_b.

(iii) There is a compact open subgroup H of G such that for any $v \in V$, the orbit map $o_v : H \to V$, defined via $h \mapsto hv$, lies in $\mathcal{C}^{\mathrm{la}}(H,V)$.

(iii') For any $v \in V$, the orbit map $o_v : G \to V$, defined via $g \mapsto gv$, lies in $\mathcal{C}^{\mathrm{la}}(G,V)$.

Proof. The uniqueness statement in each of the first four conditions is a consequence of the fact that $K[H]$ is dense in $\mathcal{D}^{\mathrm{la}}(H,K)$, for any compact locally analytic L-analytic group. The equivalence of (i), (ii) and (iii) follows from [20, Cor. 3.3] and the accompanying discussion at the top of p. 453 of this reference. The equivalence of (iii) and (iii') is straightforward. (See for example [8, Prop. 3.6.11].) Since (iii') is independent of H, we see that (i') and (ii') are each equivalent to the other four conditions. □

Definition 2.3. If V is a compact type convex space equipped with an action of G by continuous K-linear automorphisms, then we say that V is a locally analytic representation of G if the equivalent conditions of Proposition 2.2 hold.

We let $\mathrm{Rep}_{\mathrm{la.c}}(G)$ denote the category of compact type convex spaces equipped with a locally analytic representation of G (the morphisms being continuous G-equivariant K-linear maps).

Example 2.4. If G is compact, so that $\mathcal{C}^{\mathrm{la}}(G,K)$ is a compact type convex space (by Proposition 1.16), then the left regular action of G on $\mathcal{C}^{\mathrm{la}}(G,K)$ equips this space with a locally analytic G-representation. This is perhaps most easily seen by applying the criterion of Proposition 2.2 (ii). Indeed, the strong dual of $\mathcal{C}^{\mathrm{la}}(G,K)$ is equal to $\mathcal{D}^{\mathrm{la}}(G,K)$, and under the contragredient action to the left regular representation, an element $g \in G$ acts as left multiplication by δ_g on $\mathcal{D}^{\mathrm{la}}(G,K)$. Thus the required topological $\mathcal{D}^{\mathrm{la}}(G,K)$-module structure on the strong dual of $\mathcal{C}^{\mathrm{la}}(G,K)$ is obtained by regarding the topological algebra $\mathcal{D}^{\mathrm{la}}(G,K)$ as a left module over itself in the tautological manner.

Similarly, the right regular action of G on $\mathcal{C}^{\mathrm{la}}(G,K)$ makes $\mathcal{C}^{\mathrm{la}}(G,K)$ a locally analytic G-representation. (Indeed, the topological automorphism $f(g) \mapsto f(g^{-1})$ of $\mathcal{C}^{\mathrm{la}}(G,K)$ intertwines the left and right regular representations.)

Remark 2.5. If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(G)$, then since the orbit maps o_v lie in $\mathcal{C}^{\mathrm{la}}(G,V)$ for all $v \in V$ they are in particular continuous on G. Thus the G-action on V is separately continuous, and hence (as was remarked above) continuous, by the Banach-Steinhaus theorem. Furthermore, we may differentiate the G-action on V and so make G a module over the Lie algebra $\mathfrak{g}$ of G (or equivalently, over its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$). The action $\mathfrak{g} \times V \to V$ is again seen to be separately continuous (since the derivatives along the elements of $\mathfrak{g}$ of a function in $\mathcal{C}^{\mathrm{la}}(G,V)$ again lie in $\mathcal{C}^{\mathrm{la}}(G,V)$), and hence (applying the Banach-Steinhaus theorem once more) is continuous.

The $\mathrm{U}(\mathfrak{g})$-module structure on V admits an alternative description. Indeed, for any compact open subgroup H of G, there is a natural embedding $\mathrm{U}(\mathfrak{g}) \to \mathcal{D}^{\mathrm{la}}(H,K)$, given by mapping $X \in \mathrm{U}(\mathfrak{g})$ to the functional $f \mapsto (Xf)(e)$. (Here X acts on f as a differential operator,[1] and e denotes the identity of G.) Since V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(V)$, it is a $\mathcal{D}^{\mathrm{la}}(H,K)$-module (by part (i) of Theorem 2.1), and so in particular is a $\mathrm{U}(\mathfrak{g})$-module. This $\mathrm{U}(\mathfrak{g})$-module structure on V coincides with the one described in the preceding paragraph.

Now suppose that Z is a topologically finitely generated abelian locally L-analytic group. If E is any finite extension of L, then we may consider the set $\hat{Z}(E)$ of $E^{\times}$-valued locally L-analytic characters on Z.

Proposition 2.6. *There is a strictly quasi-stein rigid analytic space $\hat{Z}$ over L that represents the functor $E \mapsto \hat{Z}(E)$.*

Proof. This is [8, Prop. 6.4.5]. $\square$

Example 2.7. Suppose that $L = \mathbb{Q}_p$, and that Z is the group $\mathbb{Z}_p$. Then $\hat{Z}$ is isomorphic to the open unit disk centered at 1. (A character of $\hat{Z}$ may be identified with its value on the topological generator 1 of $\mathbb{Z}_p$.)

Example 2.8. Suppose that $L = \mathbb{Q}_p$, and that Z is the multiplicative group $\mathbb{Q}_p^{\times}$. There is an isomorphism

$$\mathbb{Q}_p^{\times} \xrightarrow{\sim} \mathbb{Z}_p^{\times} \times p^{\mathbb{Z}} \xrightarrow{\sim} \mu \times \Gamma \times p^{\mathbb{Z}},$$

[1] More precisely, the $\mathfrak{g}$ action on $\mathcal{C}^{\mathrm{la}}(H,K)$ that we have in mind is the one obtained via differentiating the right regular action of H on $\mathcal{C}^{\mathrm{la}}(H,K)$. (By applying Example 2.4 to H, we find that this H-action is locally analytic, and so may indeed be differentiated to yield a $\mathfrak{g}$-action.) It is given explicitly by the formula $(Xf)(h) = \frac{d}{dt}_{|t=0} f(h\exp(tX))$, for any $X \in \mathfrak{g}$.

where μ denotes the subgroup of roots of unity in $\mathbb{Q}_p^\times$, Γ denotes the subgroup of $\mathbb{Z}_p^\times$ consisting of elements congruent to 1 modulo p (respectively p^2 if $p = 2$), and $p^{\mathbb{Z}}$ denotes the cyclic group generated by $p \in \mathbb{Q}_p^\times$. The group Γ is isomorphic to $\mathbb{Z}_p$, and so there is an isomorphism

$$\hat{Z} \xrightarrow{\sim} \mathrm{Hom}(\mu, \mathbb{Q}_p^\times) \times \text{ open unit disk around 1 } \times \mathbb{G}_m.$$

Here $\mathrm{Hom}(\mu, \mathbb{Q}_p^\times)$ is the character group of the finite group μ, the open unit disk around 1 is the character group of Γ (see the preceding example), and $\mathbb{G}_m$ is the character group of $p^{\mathbb{Z}}$. (A character of the cyclic group $p^{\mathbb{Z}}$ may be identified with its value on p).

The discussion of Example 1.7 shows that the K-algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ of rigid analytic functions on $\hat{Z}$ is a nuclear Fréchet-Stein algebra. Evaluation of characters at elements of Z induces an embedding of K-algebras $K[Z] \to \mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, with dense image (by [8, Prop. 6.4.6] and [20, Lem. 3.1]), and we have the following analogue of Proposition 2.2.

Proposition 2.9. *If V is a compact type convex space, equipped with an action of Z by continuous K-linear automorphisms, then the following are equivalent:*

(i) The $K[Z]$-module structure on V extends to a (necessarily unique) $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-module structure on V, for which the map $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \times V \to V$ describing this module structure is separately continuous.

(ii) The $K[Z]$-module structure on V_b' arising from the contragredient Z-action on V_b' extends to a (necessarily unique) topological $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-module structure on V_b'.

Proof. This follows from [8, Prop. 6.4.7]. □

If the Z-action on V satisfies the equivalent conditions of the preceding proposition, then it is separately continuous (as follows from condition (i)), and so is in fact continuous.

If Z is a compact abelian locally L-analytic group (which is then necessarily topologically finitely generated [8, Prop. 6.4.1]), then we have the two nuclear Fréchet algebras $\mathcal{D}^{\mathrm{la}}(Z, K)$ and $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, each containing the group ring $K[Z]$ as a dense subalgebra.

Proposition 2.10. *If Z is a compact abelian locally L-analytic group, then there is an isomorphism of topological K-algebras $\mathcal{D}^{\mathrm{la}}(Z, K) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$, uniquely determined by the condition that it reduces to the identity on $K[Z]$ (regarded as a subalgebra of the source and target in the natural manner).*

Proof. This is [8, Prop. 6.4.6]. It is proved using the p-adic Fourier theory of [22]. □

We now wish to tie together the two strands of the preceding discussion. We begin with the following strengthening of Theorem 2.1.

Theorem 2.11. *If H is a compact locally L-analytic group and Z is a topological finitely generated abelian locally L-analytic group, then the completed tensor product $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$ (which by* [17, p. 107] *is a K-Fréchet algebra) is a nuclear Fréchet-Stein algebra.*

Proof. This follows from [8, Prop. 5.3.22], together with the remark following [8, Def. 5.3.21]. □

Suppose now that G is a locally L-analytic group, whose centre Z (an abelian locally L-analytic group) is topologically finitely generated.

Definition 2.12. We let $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{la.c}}(G)$ consisting of locally analytic representations V of G, the induced Z-action on which satisfies the equivalent conditions of Proposition 2.9.

It follows from Propositions 2.2 and 2.9 that if V is a compact type convex space equipped with an action of G by continuous K-linear automorphisms, then the following are equivalent:

(i) V is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$.

(ii) For some (equivalently, every) compact open subgroup H of G, the G-action on V induces a (uniquely determined) $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module structure on V for which the corresponding map

$$\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K) \times V \to V$$

is separately continuous.

(iii) For some (equivalently, every) compact open subgroup H of G, the contragredient G-action on V'_b induces a (uniquely determined) structure of topological $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module on V'_b.

We can now define some important subcategories of the category $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$.

Definition 2.13. Let V be an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$.

(i) We say that V is an essentially admissible locally analytic representation of G if V'_b is a coadmissible $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-module for some (equivalently, every) compact open subgroup H of G.

(ii) We say that V is an admissible locally analytic representation of G if V'_b is a coadmissible $\mathcal{D}^{\mathrm{la}}(H, K)$-module for some (equivalently, every) compact open subgroup H of G.

(iii) We say that V is a strongly admissible locally analytic representation of G if V'_b is a strongly coadmissible $\mathcal{D}^{\mathrm{la}}(H, K)$-module for some (equivalently, every) compact open subgroup H of G.

The equivalence of "some" and "every" in each of these definitions follows from the fact that if $H' \subset H$ is an inclusion of compact open subgroups of G then the algebra $\mathcal{D}^{\mathrm{la}}(H, K)$ is free of finite rank as a $\mathcal{D}^{\mathrm{la}}(H', K)$-module (since H' has finite index in H). Clearly, any strongly admissible locally analytic G-representation is admissible, and any admissible locally analytic G-representation is essentially admissible. The notion of strongly admissible (respectively admissible, respectively essentially admissible) locally analytic G-representation was first introduced in [20] (respectively [23], respectively [8]). (Let us remark that any object V of $\mathrm{Rep}_{\mathrm{la.c}}(G)$ for which V'_b satisfies condition (ii) of Definition 2.13 automatically lies in $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$, by [8, Prop. 6.4.10], and so the definitions of admissible and strongly admissible locally analytic representations of G given above do coincide with those of [23] and [20].)

We let $\mathrm{Rep}_{\mathrm{es}}(G)$ denote the full subcategory of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$ consisting of essentially admissible locally analytic representations, let $\mathrm{Rep}_{\mathrm{ad}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{es}}(G)$ consisting of admissible locally analytic representations, and let $\mathrm{Rep}_{\mathrm{sa}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{ad}}(G)$ consisting of strongly admissible locally analytic representations. These various categories lie in the following sequence of full embeddings:

$$\mathrm{Rep}_{\mathrm{sa}}(G) \subset \mathrm{Rep}_{\mathrm{ad}}(G) \subset \mathrm{Rep}_{\mathrm{es}}(G) \subset \mathrm{Rep}^{z}_{\mathrm{la.c}}(G) \subset \mathrm{Rep}_{\mathrm{la.c}}(G).$$

Both of the categories $\mathrm{Rep}_{\mathrm{la.c}}(G)$ and $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$ are closed under passing to countable direct sums (and more generally to Hausdorff countable locally convex inductive limits), closed subrepresentations, Hausdorff quotients, and completed tensor products [9, Lems. 3.1.2, 3.1.4].

Theorem 2.14. *Each of* $\mathrm{Rep}_{\mathrm{es}}(G)$ *and* $\mathrm{Rep}_{\mathrm{ad}}(G)$ *is an abelian category, closed under the passage to closed G-subrepresentations, and to Hausdorff quotient G-representations.*

Proof. This follows from Theorem 1.9. □

The subcategory $\mathrm{Rep}_{\mathrm{sa}}(G)$ of $\mathrm{Rep}_{\mathrm{ad}}(G)$ is closed under passing to finite direct sums and closed subrepresentations, but in general it is not closed under passing to Hausdorff quotients.

Remark 2.15. Let Z_0 denote the maximal compact subgroup of Z, and let H be a compact open subgroup of G. Replacing H by Z_0H if necessary, we may assume that H contains Z_0 (so that then $Z_0 = H \bigcap Z$). The K-algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}_0, K) \xrightarrow{\sim} \mathcal{D}^{\mathrm{la}}(Z_0, K)$ is a subalgebra of each of $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ and $\mathcal{D}^{\mathrm{la}}(H, K)$. If V is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$, then the two actions of $\mathcal{C}^{\mathrm{an}}(\hat{Z}_0, K)$ on each of V and V'_b (obtained by regarding it as a subalgebra of $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$ or $\mathcal{D}^{\mathrm{la}}(H, K)$ respectively) coincide (since both are obtained from the one action of Z_0 on V). Thus the $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-action on each of V and V'_b factors through the quotient algebra $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_{\mathcal{C}^{\mathrm{an}}(\hat{Z}_0, K)} \mathcal{D}^{\mathrm{la}}(H, K)$. We take particular note of two consequences of this remark.

Example 2.16. If Z is compact (and so equals Z_0), and if V lies in $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$, then the preceding remark shows that the $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-action on each of V and V'_b factors through $\mathcal{D}^{\mathrm{la}}(H, K)$. Thus any essentially admissible locally analytic G-representation is in fact admissible. Also, in this situation, the categories $\mathrm{Rep}_{\mathrm{la.c}}(G)$ and $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$ are equal. Thus if the centre Z of G is compact, it can be neglected entirely throughout the preceding discussion.

Example 2.17. If G is abelian, then $G = Z$. The preceding remark shows that if V lies in $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$, then the $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K) \hat{\otimes}_K \mathcal{D}^{\mathrm{la}}(H, K)$-action on each of V and V'_b factors through $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$. Example 1.13 then shows that passing to strong duals induces an antiequivalence of categories between the category $\mathrm{Rep}_{\mathrm{es}}(Z)$ and the category of coherent rigid analytic sheaves on $\hat{Z}$. Under this antiequivalence, the subcategory $\mathrm{Rep}_{\mathrm{ad}}(Z)$ of $\mathrm{Rep}_{\mathrm{es}}(Z)$ corresponds to the subcategory consisting of those coherent sheaves on $\hat{Z}$ whose pushforward to $\hat{Z}_0$ under the surjection $\hat{Z} \to \hat{Z}_0$ (induced by the inclusion $Z_0 \subset Z$) is again coherent. (The point is that on the level of global sections, this pushforward corresponds to regarding a $\mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$-module as a $\mathcal{C}^{\mathrm{an}}(\hat{Z}_0, K)$-module, via the embedding $\mathcal{C}^{\mathrm{an}}(\hat{Z}_0, K) \to \mathcal{C}^{\mathrm{an}}(\hat{Z}, K)$.)

Example 2.18. If G is compact, then $\mathcal{C}^{\mathrm{la}}(G, K)$ is an object of $\mathrm{Rep}_{\mathrm{sa}}(G)$, and furthermore, any object of $\mathrm{Rep}_{\mathrm{sa}}(G)$ is a closed subrepresentation of $\mathcal{C}^{\mathrm{la}}(G, K)^n$, for some $n \geq 0$. (This follows directly from Definitions 2.13 (iii) and 1.12, and the fact that passing to strong duals takes closed subrepresentations of $\mathcal{C}^{\mathrm{la}}(G, K)^n$ to Hausdorff quotient modules of $\mathcal{D}^{\mathrm{la}}(G, K)^n$.)

The following result connects the locally analytic representation theory discussed in this note with the more traditional theory of smooth representations of locally L-analytic groups.

Theorem 2.19. *If V is an admissible smooth representation of G on a K-vector space (in the usual sense), and if we equip V with its finest locally convex topology, then V becomes an element of $\mathrm{Rep}_{\mathrm{ad}}(G)$. Conversely, any object V of $\mathrm{Rep}_{\mathrm{ad}}(G)$ on which the G-action is smooth is an admissible smooth representation of G, equipped with its finest locally convex topology.*

Proof. See [8, Prop. 6.3.2] or [23, Thm. 6.5]. □

In the applications to the theory of automorphic forms, one typically assumes that G is the group of L-valued points of a connected reductive linear algebraic group $\mathbb{G}$ defined over L. (Any such group certainly has topologically finitely generated centre.) In this case, we can make the following definition.

Definition 2.20. If W is a finite dimensional algebraic representation of $\mathbb{G}$ defined over K, then we say that a representation of G on a K-vector space V is locally W-algebraic if, for each vector $v \in V$, there exists an open subgroup H of G, a natural number n, and an H-equivariant homomorphism $W^n \to V$ whose image contains the vector v.

When W is the trivial representation of V, we recover the notion of a smooth representation of G. The following result generalizes Theorem 2.19.

Theorem 2.21. *Suppose that $G = \mathbb{G}(L)$, for some connected reductive linear algebraic group over L. If V is an object of $\mathrm{Rep}_{\mathrm{ad}}(G)$ that is also locally W-algebraic, for some finite dimensional algebraic representation W of $\mathbb{G}$ over K, then V is isomorphic to a representation of the form $U \otimes_B W$, where B denotes the semi-simple K-algebra $\mathrm{End}_{\mathbb{G}}(W)$, and U is an admissible smooth representation of G defined over B, equipped with its finest locally convex topology. Conversely, any such tensor product is a locally W-algebraic representation in $\mathrm{Rep}_{\mathrm{ad}}(G)$.*

Proof. This is [8, Prop. 6.3.10]. □

Remark 2.22. Taking the tensor product of finite dimensional representations and smooth representations is something that is quite unthinkable in the classical theory of smooth representations of G (in which the field of coefficients typically is taken to be $\mathbb{C}$, or an ℓ-adic field, with $\ell \neq p$). In the arithmetic theory of automorphic forms, the role of smooth representations of p-adic reductive groups is to carry information about representations of the absolute Galois group of L on ℓ-adic vector spaces. (This is a very vague description of the local Langlands conjecture.) The consideration of locally algebraic representations of the type considered in Theorem 2.21 opens up the possibility of finding representations of p-adic reductive groups that can carry information about the representations of the absolute Galois group of L on p-adic vector spaces; in this optic, the role of the finite dimensional factor is to remember the "p-adic Hodge numbers" of such a representation. (See the introductory discussion of [3] for a lengthier account of this possibility.)

3. Locally analytic vectors in continuous admissible representations

Let L, K and G be as in the preceding section. In this section we discuss an important method for constructing strongly admissible locally analytic representations of G, which involves applying the functor "pass to locally analytic vectors" to certain Banach space representations of G. We will begin by defining that functor, but first we must recall the notion of an *analytic open subgroup* of G.

Suppose that H is a compact open subgroup of G that admits the structure of a "chart" of G; that is, a locally analytic isomorphism with the space of L-valued points of a closed ball. We let $\mathbb{H}$ denote the corresponding rigid analytic space (isomorphic to a closed ball) that has H as its space of L-valued points. If furthermore the group structure on H extends to a rigid analytic group structure on $\mathbb{H}$, then, suppressing the choice of chart structure on H, we will refer to H as an analytic open subgroup of G. Since G is locally L-analytic, it has a basis of neighbourhoods consisting of analytic open subgroups. (See the introduction of [8, §3.5] for a more detailed discussion of the notion of analytic open subgroup.)

Suppose now that U is a Banach space over K, equipped with a continuous G-action. If H is an analytic open subgroup of H, then we let $U_{\mathbb{H}-\mathrm{an}}$ denote the subspace of U consisting of vectors u for which the orbit map $o_u : H \to U$ defined by $o_u(h) = hu$ is (the restriction to H of) a rigid analytic U-valued function on $\mathbb{H}$. Via the association of o_u to a vector $u \in U_{\mathbb{H}-\mathrm{an}}$, we may regard $U_{\mathbb{H}-\mathrm{an}}$ as a subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, U)$, the Banach space of rigid analytic U-valued functions on $\mathbb{H}$.

Lemma 3.1. *For any analytic open subgroup H of G, the space $U_{\mathbb{H}-\mathrm{an}}$ is a closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, U)$.*

Proof. A rigid analytic function ϕ in $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, U)$ belongs to $U_{\mathbb{H}-\mathrm{an}}$ if and only if its restriction to H is in fact of the form o_u, for some $u \in U$ (which will then certainly lie in $U_{\mathbb{H}-\mathrm{an}}$). This is the case if and only if ϕ satisfies the equation $\phi(h) = h\phi(e)$ for all $h \in H$. (Here e denotes the identity element in H). These equations cut out a closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, U)$, as claimed. □

We will always regard $U_{\mathbb{H}-\mathrm{an}}$ as being endowed with the Banach space topology it inherits by being considered as a closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, U)$, as in the preceding lemma. The inclusion $U_{\mathbb{H}-\mathrm{an}} \to U$ is thus continuous, but typically is not a topological embedding.

Definition 3.2. We say that a vector u in U is locally analytic if the orbit map o_u lies in $\mathcal{C}^{\mathrm{la}}(G, U)$. (In fact, it suffices to require that o_u be locally analytic in a neighbourhood of the identity, since the G-action on U is by continuous automorphisms). We let U_{la} denote the subspace of U consisting of locally analytic vectors; the preceding parenthetical remark shows that $U_{\mathrm{la}} = \bigcup_H U_{\mathbb{H}-\mathrm{an}}$, where H runs over all analytic open subgroups of G. We topologize U_{la} by endowing it with the locally convex inductive limit topology arising from the isomorphism $U_{\mathrm{la}} \xrightarrow{\sim} \varinjlim_H U_{\mathbb{H}-\mathrm{an}}$ (the inductive limit being taken over the directed set of analytic open subgroups of G).

This definition exhibits $U_{\rm la}$ as the locally convex inductive limit of a sequence of Banach spaces (and thus $U_{\rm la}$ is a so-called LB-space). The inclusion $U_{\rm la} \to U$ is continuous, but typically is not a topological embedding.

The map $u \mapsto o_u$ defines a continuous injection

$$U_{\rm la} \to \mathcal{C}^{\rm la}(G, U). \tag{3.3}$$

Note that in [19] and [23], the topology on $U_{\rm la}$ is defined to be that induced by regarding it as a subspace of $\mathcal{C}^{\rm la}(G, U)$. In general, this is coarser than the inductive limit topology of Definition 3.2.

We next introduce some terminology related to lattices in convex spaces.

Definition 3.4. A separated, open lattice $\mathcal{L}$ in a convex K-space U is an open $\mathcal{O}_K$-submodule of U that is p-adically separated. We let $\mathcal{L}(U)$ denote the set of all separated open lattices in U.

Definition 3.5. If U is a convex space, then we say that two lattices $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}(U)$ are commensurable if $a\mathcal{L}_1 \subset \mathcal{L}_2 \subset a^{-1}\mathcal{L}_1$ for some $a \in K^\times$.

Clearly commensurability defines an equivalence relation on $\mathcal{L}(U)$.

Definition 3.6. If $\mathcal{L} \in \mathcal{L}(U)$ then we let $\{\mathcal{L}\}$ denote the commensurability class of $\mathcal{L}$ (i.e. the equivalence class of $\mathcal{L}$ under the relation of commensurability). We let $\overline{\mathcal{L}}(U)$ denote the set of commensurability classes of elements of $\mathcal{L}(U)$.

Example 3.7. If U is a Banach space over K, then $\mathcal{L}(U)$ is non-empty, and in fact the elements of $\mathcal{L}(U)$ form a neighbourhood basis of U. Furthermore, any two elements of $\mathcal{L}(U)$ are commensurable, and so $\overline{\mathcal{L}}(U)$ consists of a single element.

In general, if $\mathcal{L} \in \mathcal{L}(U)$, then $\mathcal{L}$ gives rise to a continuous norm $s_{\mathcal{L}}$ on U, its gauge, uniquely determined by the requirement that $\mathcal{L}$ is the unit ball of $s_{\mathcal{L}}$. We let $U_{\mathcal{L}}$ denote U equipped with the topology induced by $s_{\mathcal{L}}$, and let $\hat{U}_{\mathcal{L}}$ the Banach space obtained by completing $U_{\mathcal{L}}$ with respect to the norm $s_{\mathcal{L}}$. The identity map on the underlying vector space of U induces a continuous bijection $U \to U_{\mathcal{L}}$, and hence a continuous injection $U \to \hat{U}_{\mathcal{L}}$. Given a pair of elements $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}(U)$, the topologies on $U_{\mathcal{L}_1}$ and $U_{\mathcal{L}_2}$ coincide if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are commensurable.

Suppose now that U is equipped with a continuous G-action. There is then an induced action of G on $\mathcal{L}(U)$, defined by $(g, \mathcal{L}) \mapsto g\mathcal{L}$ for $g \in G$ and $\mathcal{L} \in \mathcal{L}(U)$. This action evidently respects the relation of commensurability, and so descends to an action on $\overline{\mathcal{L}}(U)$. We write $\mathcal{L}(U)^G$ (respectively $\overline{\mathcal{L}}(U)^G$) to denote the subset of $\mathcal{L}(U)$ (respectively of $\overline{\mathcal{L}}(U)$) consisting of elements that are fixed under the action of G. Passing to commensurability classes induces a map $\mathcal{L}(U)^G \to \overline{\mathcal{L}}(U)^G$.

Lemma 3.8. *If $\mathcal{L}$ is an element of $\mathcal{L}(U)$, then the G-action on U induces a continuous G-action on $U_{\mathcal{L}}$ (and hence on $\hat{U}_{\mathcal{L}}$) if and only if the commensurability class $\{\mathcal{L}\}$ is G-invariant.*

Proof. It is immediate from the definitions that G acts on $U_{\mathcal{L}}$ via continuous automorphisms if and only if $\{\mathcal{L}\}$ is G-invariant. Since the G-action on U is continuous by assumption, and since the natural bijection $U \to U_{\mathcal{L}}$ is continuous, the G-action on $U_{\mathcal{L}}$ automatically satisfies conditions (i) and (iii) of [8, Lem. 3.1.1]. It thus follows from that lemma that if G acts on $U_{\mathcal{L}}$ via continuous automorphisms, then the G-action on $U_{\mathcal{L}}$ is in fact continuous. □

Lemma 3.9. *Let H be an open subgroup of G.*

(i) If H is compact, then the map $\mathcal{L}(U)^H \to \overline{\mathcal{L}}(U)^H$ is surjective.

(ii) If $\mathcal{L} \in \mathcal{L}(U)$ is such that $\{\mathcal{L}\} \in \overline{\mathcal{L}}(U)^H$, then there is an open subgroup H' of H such that $\mathcal{L} \in \mathcal{L}(U)^{H'}$.

Proof. Suppose that $\mathcal{L} \in \overline{\mathcal{L}}(U)$ is H-invariant. The H-action on U then induces a continuous H-action on $U_{\mathcal{L}}$, by Lemma 3.8. Part (i) of the present lemma is now seen to follow from [8, Lem. 6.5.3], while part (ii) follows immediately from the fact that the H-action on $U_{\mathcal{L}}$ is continuous. □

In contrast to part (i) of the preceding lemma, if G is not compact then the map $\mathcal{L}(U)^G \to \overline{\mathcal{L}}(U)^G$ is typically not surjective. For example, if U is a Banach space, then $\overline{\mathcal{L}}(U)^G = \overline{\mathcal{L}}(U)$ (since the set on the right is a singleton). On the other hand, asking that $\mathcal{L}(U)^G$ be non-empty is a rather stringent condition.

Definition 3.10. A continuous representation of G on a Banach space is said to be unitary if $\mathcal{L}(U)^G \neq \emptyset$, that is, if U contains an open, separated lattice that is invariant under the entire group G (or equivalently, if its topology can be defined by a G-invariant norm).

Suppose now that $\mathcal{L} \in \mathcal{L}(U)^H$ for some open subgroup H of G. If π denotes a uniformizer of $\mathcal{O}_K$, then $\mathcal{L}/\pi\mathcal{L}$ is a vector space over the residue field $\mathcal{O}_K/\pi\mathcal{O}_K$, equipped with a smooth representation of H.

Definition 3.11. If U is a convex space, equipped with a continuous G-action of G, then we say that $\mathcal{L} \in \mathcal{L}(U)$ is admissible if it is H-invariant, for some compact open subgroup H of G, and if the resulting smooth H-representation on $\mathcal{L}/\pi\mathcal{L}$ is admissible.

Note that if $\mathcal{L} \in \mathcal{L}(U)$ is admissible, and if $H \subset G$ is a compact open subgroup that satisfies the conditions of the preceding definition with respect to $\mathcal{L}$, then any open subgroup $H' \subset H$ also satisfies these conditions.

Lemma 3.12. *If $\mathcal{L} \in \mathcal{L}(U)$ is admissible, then every lattice in $\{\mathcal{L}\}$ is admissible.*

Proof. Let H be a compact open subgroup of G that satisfies the conditions of Definition 3.11 with respect to $\mathcal{L}$. If $\mathcal{L}'$ is an element of $\{\mathcal{L}\}$, then by Lemma 3.9 (ii) (and replacing H by an open subgroup if necessary) we may assume that $\mathcal{L}'$ is again H-invariant. Since $\mathcal{L}'$ and $\mathcal{L}$ are commensurable, we may also assume (replacing $\mathcal{L}'$ by a scalar multiple if necessary) that $\pi^n\mathcal{L} \subset \mathcal{L}' \subset \mathcal{L}$ for some $n > 0$. Thus $\mathcal{L}'/\pi\mathcal{L}'$ is an H-invariant subquotient of $\mathcal{L}/\pi^{n+1}\mathcal{L}$. The latter H-representation is a successive extension of copies of $\mathcal{L}/\pi\mathcal{L}$, and so by assumption is an admissible smooth representation of H over $\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K$. Any subquotient of an admissible smooth H-representation over $\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K$ is again admissible. (This uses the fact that the category of such representations is anti-equivalent – via passing to $\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K$-duals – to the category of finitely generated modules over the completed group ring $(\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K)[[H]]$, together with a theorem of Lazard to the effect that this completed group ring is Noetherian [16, V.2.2.4].[2]) In particular we conclude that $\mathcal{L}'/\pi\mathcal{L}'$ is admissible. □

[2]Strictly speaking, this reference only applies to the case when $K = \mathbb{Q}_p$, so that $\mathcal{O}_K = \mathbb{Z}_p$. However, the result is easily extended to the case of general K; see for example the proof of [8, Thm. 6.2.8].

We say that a commensurability class $\{\mathcal{L}\} \in \overline{\mathcal{L}}(U)$ is admissible if one (or equivalently every, by Lemma 3.12) member of the class is admissible in the sense of Definition 3.11.

Proposition 3.13. *If U is an object of $\mathrm{Rep}_{\mathrm{es}}(G)$, then $\mathcal{L}(U)$ contains an admissible lattice if and only if U is strongly admissible. Furthermore, if U is strongly admissible, then for any compact open subgroup H of G, we may find an admissible H-invariant lattice in $\mathcal{L}(U)$.*

Proof. See [8, Prop. 6.5.9]. □

Definition 3.14. Let U be a Banach space over K, equipped with a continuous action of G. We say that U is an admissible continuous representation of G, or an admissible Banach space representation of G, if one (or equivalently every, by Lemma 3.12) lattice in $\mathcal{L}(U)$ is admissible, in the sense of the Definition 3.11.

Theorem 3.15. *The category of admissible continuous representations of G (with morphisms being continuous G-equivariant K-linear maps) is an abelian category, closed under passing to closed G-subrepresentations and Hausdorff quotient G-representations.*

Proof. This is the main result of [21]. (See [8, Cor. 6.2.16] for the case when K is not local.) The key point is that if H is any compact open subgroup of G, then the completed group ring $\mathcal{O}_K[[H]]$ is Noetherian [16, V.2.2.4].[3] □

We let $\mathrm{Rep}_{\mathrm{b.ad}}(G)$ denote the abelian category of admissible continuous representations of G. One important aspect of the preceding result is that maps in $\mathrm{Rep}_{\mathrm{b.ad}}(G)$ are necessarily strict, with closed image.

Example 3.16. If G is compact, then the space $\mathcal{C}(G, K)$ of continuous K-valued functions on G, made into a Banach space via the sup norm, and equipped with the left regular G-action, is an admissible continuous G-representation. Furthermore any object of $\mathrm{Rep}_{\mathrm{b.ad}}(G)$ is a closed subrepresentation of $\mathcal{C}(G, K)^n$ for some $n \geq 0$. (See [8, Prop.-Def. 6.2.3].)

If G is (the group of $\mathbb{Q}_p$-points of) a p-adic reductive group over $\mathbb{Q}_p$, then the admissible G-representations that are also unitary are perhaps the most important objects in the category $\mathrm{Rep}_{\mathrm{b.ad}}(G)$. In [3, §1.3], Breuil explains the role that he expects these representations to play in a hoped-for "p-adic local Langlands" correspondence, in the case of the group $\mathrm{GL}_2(\mathbb{Q}_p)$. For a discussion of how some of Breuil's ideas might generalize to the case of a general reductive group, see [24, §5].

The following result provides a basic technique for producing strongly admissible locally analytic representations of G.

Proposition 3.17. *If U is an object of $\mathrm{Rep}_{\mathrm{b.ad}}(G)$, then U_{la} is a strongly admissible locally analytic representation of G.*

Proof. This follows from the discussions of Examples 2.18 and 3.16, and the following two (easily verified) facts: (i) for any compact open subgroup H of G, there is a natural isomorphism $\mathcal{C}^{\mathrm{la}}(H, K) \xrightarrow{\sim} \mathcal{C}(H, K)_{\mathrm{la}}$ [8, Prop. 3.5.11]; (ii) if U and

[3] See the preceding note.

V are Banach spaces equipped with continuous G-representations, if $U \to V$ is a G-equivariant closed embedding, and if V_{la} is of compact type, then the diagram

$$\begin{array}{ccc} U_{\mathrm{la}} & \longrightarrow & V_{\mathrm{la}} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is Cartesian in the category of convex spaces; in particular, the map $U_{\mathrm{la}} \to V_{\mathrm{la}}$ is again a closed embedding [8, Prop. 3.5.10]. See [8, Prop. 6.2.4] for the details of the argument. □

A version of the preceding theorem, working with the topology obtained on U_{la} by regarding it as a closed subspace of $\mathcal{C}^{\mathrm{la}}(G, U)$, is given in [23, Thm. 7.1 (ii)]. We remark that if U is an object of $\mathrm{Rep}_{\mathrm{b.ad}}(G)$, then the map (3.3) is in fact a topological embedding (see [5, Rem. A.1.1]). Thus, for such U, the topology on U_{la} induced by regarding it as a subspace of $\mathcal{C}^{\mathrm{la}}(G, U)$ coincides with the inductive limit topology given by Definition 3.2.

Lemma 3.18. *If U is a convex space equipped with a continuous action of G, and if H is an open subgroup of G, then there exists a continuous H-equivariant injection $U \to W$ for some admissible continuous H-representation W if and only if $\overline{\mathcal{L}}(U)$ contains an H-invariant admissible commensurability class.*

Proof. Given such a map $U \to W$, the preimage of any lattice in W determines a commensurability class in $\overline{\mathcal{L}}(U)$ with the required properties. Conversely, given such a commensurability class $\{\mathcal{L}\}$, it follows from Lemma 3.8 that the H-action on U extends to a continuous H-action on $\hat{U}_{\mathcal{L}}$, and so we may take $W = \hat{U}_{\mathcal{L}}$. □

Definition 3.19. An object V of $\mathrm{Rep}_{\mathrm{ad}}(G)$ is called very strongly admissible if V admits a G-equivariant continuous K-linear injection into an object of $\mathrm{Rep}_{\mathrm{b.ad}}(G)$, or equivalently (by Lemma 3.18), if $\overline{\mathcal{L}}(V)$ contains a G-invariant admissible commensurability class.

We let $\mathrm{Rep}_{\mathrm{vsa}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{ad}}(G)$ consisting of very strongly admissible locally analytic G-representations. It is evidently closed under passing to subobjects and finite direct sums. Proposition 3.13 shows that it is a full subcategory of $\mathrm{Rep}_{\mathrm{sa}}(G)$.

It also follows from Proposition 3.13 that if G is compact, then every strongly admissible locally analytic G-representation is in fact very strongly admissible. The author knows no example of a strongly admissible, but not very strongly admissible, locally analytic G-representation (for any G).

The following theorem of Schneider and Teitelbaum is fundamental to the theory of admissible continuous representations.

Theorem 3.20. *If $L = \mathbb{Q}_p$ and if K is a finite extension of L then the map $U \mapsto U_{\mathrm{la}}$ yields an exact and faithful functor from the category $\mathrm{Rep}_{\mathrm{b.ad}}(G)$ to the category $\mathrm{Rep}_{\mathrm{vsa}}(G)$.*

Proof. See [23, Thm. 7.1]. (That the image of this functor lies in $\mathrm{Rep}_{\mathrm{vsa}}(G)$ follows from Proposition 3.17 and the definition of $\mathrm{Rep}_{\mathrm{vsa}}(G)$.) □

Given the exactness statement in the preceding result, the faithfulness statement is equivalent to the fact that U_{la} is dense as a subspace of U.

In the context of Theorem 3.20, the functor $U \mapsto U_{\mathrm{la}}$ is not full, in general, as we now explain. If U is an object of $\mathrm{Rep}_{\mathrm{b.ad}}(G)$, if $\mathcal{L}$ is an element of $\mathcal{L}(U)$, and if we write $\mathcal{L}_{\mathrm{la}} = \mathcal{L} \bigcap U_{\mathrm{la}}$, then $\{\mathcal{L}_{\mathrm{la}}\}$ is a G-invariant and admissible commensurability class in U_{la}, which is evidently well-defined independent of the choice of $\mathcal{L}$ (since all lattices in $\mathcal{L}(U)$ are commensurable).

Conversely, if V is an object of $\mathrm{Rep}_{\mathrm{vsa}}(G)$, equipped with a G-invariant and admissible commensurability class $\{\mathcal{M}\} \in \overline{\mathcal{L}}(V)$, then the completion $\hat{V}_{\mathcal{M}}$ is an object of $\mathrm{Rep}_{\mathrm{b.ad}}(G)$. In the case when $(V, \{\mathcal{M}\}) = (U_{\mathrm{la}}, \{\mathcal{L}_{\mathrm{la}}\})$ (in the notation of the previous paragraph), it follows from Theorem 3.20 (and the remark following that theorem) that $\hat{V}_{\mathcal{M}} \xrightarrow{\sim} U$.

Thus, if we let $\mathcal{C}$ denote the category whose objects consist of pairs $(V, \{\mathcal{M}\})$, where V is an object of $\mathrm{Rep}_{\mathrm{vsa}}(G)$ and $\{\mathcal{M}\} \in \overline{\mathcal{L}}(V)$ is a G-invariant admissible commensurability class (and whose morphisms are defined in the obvious way), then the preceding discussion shows that $U \mapsto (U_{\mathrm{la}}, \{\mathcal{L}_{\mathrm{la}}\})$ is a fully faithful functor $\mathrm{Rep}_{\mathrm{b.ad}}(G) \to \mathcal{C}$, to which the functor $(V, \{\mathcal{M}\}) \mapsto \hat{V}_{\mathcal{M}}$ is left adjoint, and left quasi-inverse. On the other hand, the obvious forgetful functor $\mathcal{C} \to \mathrm{Rep}_{\mathrm{vsa}}(G)$ (forget the commensurability class of lattices), while faithful, is not full. This amounts to the fact that a given very strongly admissible locally analytic representation of G can admit more than one G-invariant commensurability class of admissible lattices. Explicit examples are provided by the results of [3] (which show that the same irreducible admissible locally algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ can admit non-isomorphic admissible continuous completions, which are even unitary, in the sense of Definition 3.10).

4. Parabolic induction

This section provides a brief account of parabolic induction in the locally analytic context. We let L and K be as in the preceding sections, and we suppose that G is (the group of L-valued points of) a connected reductive linear algebraic group over L. We let P be a parabolic subgroup of G, and let M be the Levi quotient of P.

If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$ (regarded as a P-representation through the projection of P onto M), then we make the following definition:

$$\mathrm{Ind}_P^G V = \{f \in \mathcal{C}^{\mathrm{la}}(G, V) \mid f(pg) = pf(g) \text{ for all } p \in P, g \in G\},$$

equipped with its right regular G-action. (We topologize $\mathrm{Ind}_P^G V$ by regarding it as a closed subspace of $\mathcal{C}^{\mathrm{la}}(G, V)$.)

Proposition 4.1. *If V lies in $\mathrm{Rep}_{\mathrm{la.c}}(M)$ (respectively $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, $\mathrm{Rep}_{\mathrm{ad}}(M)$, $\mathrm{Rep}_{\mathrm{sa}}(M)$, $\mathrm{Rep}_{\mathrm{vsa}}(M)$), then $\mathrm{Ind}_P^G V$ lies in $\mathrm{Rep}_{\mathrm{la.c}}(G)$ (respectively $\mathrm{Rep}^z_{\mathrm{la.c}}(G)$, $\mathrm{Rep}_{\mathrm{ad}}(G)$, $\mathrm{Rep}_{\mathrm{sa}}(G)$, $\mathrm{Rep}_{\mathrm{vsa}}(G)$).*

Proof. Although the proof of each of these statements is straightforward, altogether they are a little lengthy, and we omit them. □

Locally analytic parabolic induction satisfies Frobenius reciprocity.

Proposition 4.2. *If U and V are objects of $\mathrm{Rep}_{\mathrm{la.c}}(G)$ and $\mathrm{Rep}_{\mathrm{la.c}}(M)$ respectively, then the P-equivariant map $\mathrm{Ind}_P^G V \to V$ induced by evaluation at the identity* *of G yields a natural isomorphism $\mathcal{L}_G(U, \mathrm{Ind}_P^G V) \xrightarrow{\sim} \mathcal{L}_P(U, V)$. (Here $\mathcal{L}_G(-,-)$ and $\mathcal{L}_P(-,-)$ denote respectively the space of continuous G-equivariant K-linear maps and the space of continuous P-equivariant K-linear maps between the indicated source and target.)*

Proof. This is a particular case of [14, Thm. 4.2.6], and also follows from [8, Prop. 5.1.1 (iii)]. □

Just as in other representation theoretic contexts, parabolic induction provides a way to obtain interesting new representations from old. The following result is due to H. Frommer [15]. (The case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ was first treated in [20].)

Theorem 4.3. *Suppose that $L = \mathbb{Q}_p$ and that G is split, and let G_0 be a hyperspecial maximal compact subgroup of G. If U is a finite dimensional irreducible object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$ for which $\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U'$ is irreducible as $\mathrm{U}(\mathfrak{g})$-module, then $\mathrm{Ind}_P^G U$ is topologically irreducible as a G_0-representation, and so in particular as a G-representation. (Here U' denotes the contragredient to U, and $\mathrm{U}(\mathfrak{p})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{p}$ of P.)*

One surprising aspect of this result is that it shows (in contrast to the cases of smooth representations of compact p-adic groups, and continuous representations of compact real Lie groups) that the compact group G_0 can admit topologically irreducible infinite dimensional locally analytic representations.

5. Jacquet modules

Let L, K and G be as in the previous section, let P be a parabolic subgroup of G, and choose an opposite parabolic $\overline{P}$ to P. The intersection $M := P \bigcap \overline{P}$ is then a Levi subgroup of each of P and $\overline{P}$. Let N denote the unipotent radical of P.

If U is an object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$, then let $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ denote the closed subspace of $\mathcal{C}_c^{\mathrm{la}}(N, U)$ consisting of compactly supported, locally constant (= smooth) U-valued functions on N. The projection map $G \to \overline{P}\backslash G$ restricts to an open immersion of locally analytic spaces $N \to \overline{P}\backslash G$, and this immersion allows us to identify $\mathcal{C}_c^{\mathrm{la}}(N, U)$ with the subspace of $\mathrm{Ind}_{\overline{P}}^G U$ consisting of functions whose support is contained in $\overline{P}N$. In this way $\mathcal{C}_c^{\mathrm{la}}(N, U)$ becomes a closed $(\mathrm{U}(\mathfrak{g}), P)$-submodule of $\mathrm{Ind}_{\overline{P}}^G U$, and $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ is identified with the closed P-submodule of $\mathcal{C}_c^{\mathrm{la}}(N, U)$ consisting of elements annihilated by $\mathfrak{n}$ (the Lie algebra of N).

Proposition 5.1. *If U is an object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$ then $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$.*

Proof. This follows from the identification of $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ with a closed P-invariant subspace of $\mathrm{Ind}_{\overline{P}}^G U$, which Proposition 4.1 shows to be an object of $\mathrm{Rep}_{\mathrm{la.c}}(G)$. □

The formation of $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ is clearly functorial in U, and so we obtain a functor $\mathcal{C}_c^{\mathrm{sm}}(N, -)$ from $\mathrm{Rep}_{\mathrm{la.c}}(M)$ to $\mathrm{Rep}_{\mathrm{la.c}}(P)$.

Proposition 5.2. *The restriction of $\mathcal{C}_c^{\mathrm{sm}}(N, -)$ to $\mathrm{Rep}_{\mathrm{la.c}}^z(M)$ (which is thus a functor from $\mathrm{Rep}_{\mathrm{la.c}}^z(M)$ to $\mathrm{Rep}_{\mathrm{la.c}}(P)$) admits a right adjoint.*

Proof. See [9, Thm. 3.5.6]. □

As usual, let δ denote the smooth character of M that describes how right multiplication by elements of M affects left-invariant Haar measure on P. Concretely, if $m \in M$, then $\delta(m)$ is equal to $[N_0 : mN_0m^{-1}]^{-1}$, for any compact open subgroup N_0 of N. If U is an object of $\mathrm{Rep}_{\mathrm{la.c}}^z(M)$, then let $U(\delta)$ denote the twist of U by δ.

Definition 5.3. We let J_P denote the functor from $\mathrm{Rep}_{\mathrm{la.c}}(P)$ to $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$ obtained by twisting by δ the right adjoint to the functor $\mathcal{C}^{\mathrm{sm}}_c(N,-)$. If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$, we refer to $J_P(V)$ as the Jacquet module of V.

Thus for any objects U of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$ and V of $\mathrm{Rep}_{\mathrm{la.c}}(P)$ there is a natural isomorphism

$$\mathcal{L}_P(\mathcal{C}^{\mathrm{sm}}_c(N,U),V) \xrightarrow{\sim} \mathcal{L}_M(U(\delta),J_P(V)). \tag{5.4}$$

Remark 5.5. If U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then the natural map $U(\delta) \to J_P(\mathcal{C}^{\mathrm{sm}}_c(N,U))$ in $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, corresponding via the adjointness isomorphism (5.4) to the identity automorphism of $\mathcal{C}^{\mathrm{sm}}_c(N,U)$, is an isomorphism [9, Lem. 3.5.2]. Thus the isomorphism (5.4) is induced by passing to Jacquet modules (i.e. applying the functor J_P).

Remark 5.6. Regarding a G-representation as a P-representation yields a forgetful functor from $\mathrm{Rep}_{\mathrm{la.c}}(G)$ to $\mathrm{Rep}_{\mathrm{la.c}}(P)$. Composing this functor with the functor J_P yields a functor from $\mathrm{Rep}_{\mathrm{la.c}}(G)$ to $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, which we again denote by J_P.

Theorem 5.7. *The functor J_P restricts to a functor* $\mathrm{Rep}_{\mathrm{es}}(G) \to \mathrm{Rep}_{\mathrm{es}}(M)$.

Proof. See [9, Thm. 0.5]. □

This theorem provides the primary motivation for introducing the notion of essentially admissible locally analytic representations. Indeed, even if V is an object of $\mathrm{Rep}_{\mathrm{ad}}(G)$, it need not be the case that $J_P(V)$ lies in $\mathrm{Rep}_{\mathrm{ad}}(M)$; however, see Corollary 5.24 below.

Example 5.8. If G is quasi-split (that is, has a Borel subgroup defined over L), and if we take P to be a Borel subgroup of G, then M is a torus, and so $\mathrm{Rep}_{\mathrm{es}}(M)$ is antiequivalent to the category of coherent sheaves on the rigid analytic space of characters $\hat{M}$. Thus if V is an object of $\mathrm{Rep}_{\mathrm{es}}(G)$, then we may regard $J_P(V)$ as giving rise to a coherent sheaf on $\hat{M}$. This fact underlies the approach followed in [10] to the construction of the eigencurve of [7], and of more general eigenvarieties.

Example 5.9. If V is an admissible smooth representation of G, then there is a natural isomorphism between $J_P(V)$ and V_N, the space of N-coinvariants of V [9, Prop. 4.3.4]. This space of coinvariants is what is traditionally referred to as the Jacquet module of V in the theory of smooth representations.

More generally, if $V = U \otimes_B W$ is an admissible locally W-algebraic representation of G, as in Theorem 2.21, then there is a natural isomorphism $J_P(V) \xrightarrow{\sim} U_N \otimes_B W^N$ (where W^N denotes the space of N-invariants in W) [9, Prop. 4.3.6]. Since U is an admissible smooth G-representation, the space U_N is an admissible smooth M-representation [6, Thm. 3.3.1]. Thus J_P takes admissible locally W-algebraic G-representations to admissible locally W^N-algebraic M-representations.

The remainder of this section is devoted to explaining the relation between the functor J_P on $\mathrm{Rep}_{\mathrm{la.c}}(G)$ and the process of locally analytic parabolic induction. We begin with the following remark.

Remark 5.10. If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(G)$, then the universal property of tensor products yields a natural isomorphism

$$\mathcal{L}_P(\mathcal{C}^{\mathrm{sm}}_c(N,U),V) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}^{\mathrm{sm}}_c(N,U),V). \tag{5.11}$$

Thus for such V, the adjointness isomorphism (5.4) induces an isomorphism

$$\mathcal{L}_{(\mathfrak{g},P)}(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N,U), V) \xrightarrow{\sim} \mathcal{L}_M(U(\delta), J_P(V).) \tag{5.12}$$

Definition 5.13. As above, we regard $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ as a closed subspace of $\mathrm{Ind}_P^G(U)$. We let $I_P^G(U)$ (respectively $I_{\mathfrak{p}}^{\mathfrak{g}}(U)$) denote the closed G-subrepresentation (respectively the $\mathrm{U}(\mathfrak{g})$-submodule) of $\mathrm{Ind}_P^G U$ that it generates.

Note that $I_{\mathfrak{p}}^{\mathfrak{g}}(U)$ admits the following alternative description: taking V to be $\mathrm{Ind}_P^G(U)$, the isomorphism (5.11), applied to the inclusion $\mathcal{C}_c^{\mathrm{sm}}(N,U) \subset \mathrm{Ind}_P^G U$, induces a $(\mathfrak{g},P)$-equivariant map

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N,U) \to \mathrm{Ind}_P^G(U),$$

whose image coincides with $I_{\mathfrak{p}}^{\mathfrak{g}}(U)$. In particular, there is a $(\mathfrak{g},P)$-equivariant surjection

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N,U) \to I_{\mathfrak{p}}^{\mathfrak{g}}(U). \tag{5.14}$$

Remark 5.15. The isomorphism of Remark 5.5 yields a closed embedding $U(\delta) \to J_P(I_P^G(U))$, and hence for each object V of $\mathrm{Rep}_{\mathrm{la.c}}(G)$, passage to Jacquet modules induces a morphism

$$\mathcal{L}_G(I_P^G(U), V) \to \mathcal{L}_M(U(\delta), J_P(V)), \tag{5.16}$$

which is injective, by the construction of $I_P^G(U)$. Restricting elements in the source of this map to $I_{\mathfrak{p}}^{\mathfrak{g}}(U)$ yields the left hand vertical arrow in the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}_G(I_P^G(U), V) & \xrightarrow{(5.16)} & \mathcal{L}_M(U(\delta), J_P(V)) \\ \downarrow & & \uparrow \sim \; (5.12) \\ \mathcal{L}_{(\mathfrak{g},P)}(I_{\mathfrak{p}}^{\mathfrak{g}}(U), V) & \longrightarrow & \mathcal{L}_{(\mathfrak{g},P)}(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N,U), V) \end{array}$$

whose bottom horizontal arrow is induced by composition with (5.14).

Definition 5.17. Let U and V be objects of $\mathrm{Rep}_{\mathrm{la.c}}^z(M)$ and $\mathrm{Rep}_{\mathrm{la.c}}(G)$ respectively, and suppose given an element $\psi \in \mathcal{L}_M(U(\delta), J_P(V))$, corresponding via the adjointness map (5.12) to an element $\phi \in \mathcal{L}_{(\mathfrak{g},P)}(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N,U), V)$. We say that ψ is balanced if ϕ factors through the surjection (5.14), and we let $\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$ denote the subspace of $\mathcal{L}_M(U(\delta), J_P(V))$ consisting of balanced maps. (Note that the property of a morphism being balanced depends not just on $J_P(V)$ as an M-representation, but on its particular realization as the Jacquet module of the G-representation V.)

Equivalently, $\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$ is the image of the injection

$$\mathcal{L}_{(\mathfrak{g},P)}(I_{\mathfrak{p}}^{\mathfrak{g}}(U), V) \to \mathcal{L}_M(U(\delta), J_P(V))$$

given by composing the right hand vertical arrow and bottom horizontal arrow in the commutative diagram of Remark 5.15. A consideration of this diagram thus shows that the image of (5.16) lies in $\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$.

Definition 5.18. Let U be an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$, and let $\mathcal{H}$ denote the space of linear M-equivariant endomorphisms of U. We say that U is allowable if for any pair of finite dimensional algebraic $\mathbb{M}$-representations W_1 and W_2, each element of $\mathcal{L}_{\mathcal{H}[M]}(U \otimes_K W_1, U \otimes_K W_2)$ is strict (i.e. has closed image). (Here each $U \otimes_K W_i$ is regarded as an $\mathcal{H}[M]$-module via the $\mathcal{H}$ action on the left hand factor along with the diagonal M-action.)

It is easily checked that if U is an object of $\mathrm{Rep}_{\mathrm{es}}(M)$ and W is a finite dimensional algebraic M-representation, then $M \otimes_K W$ is again an object of $\mathrm{Rep}_{\mathrm{es}}(M)$. Thus objects of $\mathrm{Rep}_{\mathrm{es}}(M)$ are allowable in the sense of Definition 5.18.

Theorem 5.19. *If U is an allowable object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$ (in the sense of Definition 5.18) and if V is an object of $\mathrm{Rep}_{\mathrm{vsa}}(G)$ (see Definition 3.19) then the morphism*

$$\mathcal{L}_G(I_{\overline{P}}^G(U), V) \to \mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$$

induced by (5.16) is an isomorphism.

The proof of Theorem 5.19 will appear in [13].

Remark 5.20. An equivalent phrasing of Theorem 5.19 is that (under the hypotheses of the theorem) the left hand vertical arrow in the commutative diagram of Remark 5.15 is an isomorphism.

Remark 5.21. If U and V are admissible smooth representations of M and G respectively, then $I_{\overline{P}}^G(U)$ coincides with the smooth parabolic induction of U, while any M-equivariant morphism $U(\delta) \to J_P(V)$ is balanced. The isomorphism of Theorem 5.19 in this case follows from Casselman's Duality Theorem [6, §4].

Example 5.22. We consider the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ in some detail. We take P (respectively $\overline{P}$) to be the Borel subgroup of upper triangular matrices (respectively lower triangular matrices) of G, so that M is the maximal torus consisting of diagonal matrices in G.

Let χ be a locally analytic K-valued character of $\mathbb{Q}_p^\times$, and let U denote the one dimensional representation of M over K on which M acts through the character $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \chi(a)$. Let $k \in K$ denote the derivative of the character χ.

Suppose first that k is a non-negative integer. Let W_k denote the irreducible representation $\mathrm{Sym}^k K^2$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over K, and let χ_k denote the highest weight of W_k with respect to P (so χ_k is the character $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto a^k$ of M). If $U(\chi_k^{-1})$ denotes the twist of U by the inverse of χ_k, then $U(\chi_k^{-1})$ is a smooth representation of M.

The G-representation $I_{\overline{P}}^G(U)$ is a proper subrepresentation of $\mathrm{Ind}_{\overline{P}}^G U$; it coincides with the subspace of functions that are locally polynomial of degree $\leq k$ when restricted to $N = \mathbb{Q}_p$ under the open immersion $N \to \overline{P} \backslash G = \mathbb{P}^1(\mathbb{Q})$, and may also be characterized more intrinsically as the subspace of locally algebraic vectors in $\mathrm{Ind}_{\overline{P}}^G U$. It decomposes as a tensor product in the following manner:

$$I_{\overline{P}}^G(U) \cong (\mathrm{Ind}_{\overline{P}}^G U(\chi_k^{-1}))_{\mathrm{sm}} \otimes_K W_k,$$

where the subscript "sm" indicates that we are forming the smooth parabolic induction of the smooth representation $U(\chi_k^{-1})$.

If V is any object of $\mathrm{Rep}_{\mathrm{vsa}}(G)$, then we let $V_{W_k-\mathrm{lalg}}$ denote the closed subspace of W_k-locally algebraic vectors in V. (See Proposition-Definition 4.2.2 and Proposition 4.2.10 of [8].) The closed embedding $V_{W_k-\mathrm{lalg}} \to V$ induces a corresponding morphism on Jacquet modules (which is again a closed embedding; see [9, Lem. 3.4.7 (iii)]), which in turn induces an injection $\mathcal{L}_M(U(\delta), J_P(V_{W_k-\mathrm{lalg}})) \to \mathcal{L}_M(U(\delta), J_P(V))$. It is not hard to check that $\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$ is precisely the image of this injection.

Now the space $V_{W_k-\mathrm{lalg}}$ admits a factorization $V_{W_k-\mathrm{lalg}} \cong X \otimes_K W_k$, where X is an admissible smooth locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation [8, Prop.4.2.4], and so by Example 5.9 there is an isomorphism $J_P(V_{W_k-\mathrm{lalg}}) \cong J_P(X)(\chi_k)$. Thus Theorem 5.19 reduces to the claim that the natural map

$$\mathcal{L}_G((\mathrm{Ind}_P^G U(\chi_k^{-1}))_{\mathrm{sm}} \otimes_K W_k, X \otimes_K W_k) \to \mathcal{L}_M(U(\delta), J_P(X)(\chi_k))$$

induced by passing to Jacquet modules is an isomorphism. This map sits in the commutative diagram

$$\begin{array}{ccc}
\mathcal{L}_G((\mathrm{Ind}_P^G U(\chi_k^{-1}))_{\mathrm{sm}} \otimes_K W_k, X \otimes_K W_k) & \longrightarrow & \mathcal{L}_M(U(\delta), J_P(X)(\chi_k)) \\
\downarrow \wr & & \downarrow \wr \\
\mathcal{L}_G((\mathrm{Ind}_P^G U(\chi_k^{-1}))_{\mathrm{sm}}, X) & \longrightarrow & \mathcal{L}_M(U(\chi_k^{-1})(\delta), J_P(X)),
\end{array}$$

where the bottom arrow is again induced by applying J_P. Thus we are reduced to considering the case of Theorem 5.19 when U and V are both smooth. As noted in the preceding remark, this case of Theorem 5.19 follows from Casselman's Duality Theorem.

If k is not a non-negative integer, on the other hand, then $I_P^G(U)$ coincides with $\mathrm{Ind}_P^G U$, and every element of $\mathcal{L}_M(U(\delta), J_P(V))$ is balanced. In this case the proof of Theorem 5.19 is given in [5, Prop. 2.1.4]. (More precisely, the cited result shows that the left hand vertical arrow of the commutative diagram of Remark 5.15 is an isomorphism.)

Corollary 5.23. *Suppose that G is quasi-split, and that P is a Borel subgroup of G. If V is an absolutely topologically irreducible*[4] *very strongly admissible locally analytic representation of G for which $J_P(V) \neq 0$, then V is a quotient of $I_P^G(\chi)$ for some locally L-analytic K-valued character χ of the maximal torus M of G.*

Proof. We sketch the proof; full details will appear in [13]. Since $J_P(V)$ is a non-zero object of $\mathrm{Rep}_{\mathrm{es}}(M)$, we may find a character $\psi \in \hat{M}(E)$ for some finite extension E of K for which the ψ-eigenspace of $J_P(V \otimes_K E)$ is non-zero. Taking U to be $\psi\delta^{-1}$ in Definition 5.17, we let W denote the image of the map $\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N, U) \to V \otimes_K E$ corresponding via (5.12) to the inclusion of $U(\delta)$ into $J_P(V \otimes_K E)$. If $d\psi$ denotes the derivative of ψ (regarded as a weight of the Lie algebra $\mathfrak{m}$ of M) then $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ is isomorphic to a direct sum of copies of $d\psi$ as a $\mathrm{U}(\mathfrak{p})$-module, and so W is a direct sum of copies of a quotient of the Verma module $\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} d\psi$.

Let $W[\mathfrak{n}]$ denote the set of elements of W killed by $\mathfrak{n}$; this space decomposes as a direct sum of weights of $\mathfrak{m}$. Furthermore, for every weight α of $\mathfrak{m}$ that appears,

[4]That is, $E \otimes_K V$ is topologically irreducible as a G-representation, for every finite extension E of K.

there is a corresponding character $\tilde{\psi}$ appearing in $J_P(V \otimes_K E)$ for which $d\tilde{\psi} = \alpha$. (Compare the proof of [9, Prop. 4.4.4].) The theory of Verma modules shows that we may find a weight α of $\mathfrak{m}$ appearing in $W[\mathfrak{n}]$ such that $\alpha - \beta$ does not appear in $W[\mathfrak{n}]$ for any element β in the positive cone of the root lattice of $\mathfrak{m}$. Let $\tilde{\psi}$ be a character of M appearing in $J_P(V \otimes_K E)$ for which $\alpha = d\tilde{\psi}$, and set $\tilde{U} = \tilde{\psi}\delta^{-1}$. Our choice of α ensures that the resulting inclusion $\tilde{U}(\delta) \to J_P(V \otimes_K E)$ is balanced, and so Theorem 5.19 yields a non-zero map $I_{\overline{P}}^G(\tilde{U}) \to V \otimes_K E$. Since $V \otimes_K E$ is irreducible by assumption, this map must be surjective. Since V is defined over K, a simple argument shows that $\tilde{\psi}\delta^{-1}$ must also be defined over K. □

Corollary 5.24. *Let G and P be as in Corollary 5.23. If V is an admissible locally analytic representation of G of finite length, whose composition factors are very strongly admissible, then $J_P(V)$ is a finite dimensional M-representation.*

Proof. The functor J_P is left exact (see [9, Thm. 4.2.32]), and so it suffices to prove the result for topologically irreducible objects of $\mathrm{Rep}_{\mathrm{vsa}}(G)$. One easily reduces to the case when V is furthermore an absolutely topologically irreducible object of $\mathrm{Rep}_{\mathrm{vsa}}(G)$. If $J_P(V)$ is non-zero then Corollary 5.23 yields a surjection $I_{\overline{P}}^G(\chi) \to V$ for some $\chi \in \hat{M}(K)$. Although J_P is not right exact in general, one can show that the induced map $J_P(I_{\overline{P}}^G(\chi)) \to J_P(V)$ is surjective. Thus it suffices to prove that the source of this map is finite dimensional. This is shown by a direct calculation. The details will appear in [13]. □

References

1. S. Bosch, U. Güntzer and R. Remmert, *Non-archimedean analysis*, Springer-Verlag, 1984.
2. N. Bourbaki, *General Topology. Chapters 1–4.*, Springer-Verlag, 1989.
3. C. Breuil, *Invariant $\mathcal{L}$ et série spéciale p-adique*, Ann. Scient. E.N.S. **37** (2004), 559–610.
4. C. Breuil, *Série spéciale p-adique et cohomologie étale complétée*, preprint (2003).
5. C. Breuil, M. Emerton, *Représentations p-adiques ordinaires de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et compatibilité local-global*, preprint (2005).
6. W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, unpublished notes distributed by P. Sally, draft May 7, 1993.
7. R. Coleman, B. Mazur, *The eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996) (A. J. Scholl and R. L. Taylor, eds.), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, 1998, pp. 1–113.
8. M. Emerton, *Locally analytic vectors in representations of locally p-adic analytic groups*, to appear in Memoirs of the AMS.
9. M. Emerton, *Jacquet modules for locally analytic representations of p-adic reductive groups I. Construction and first properties*, Ann. Scient. E.N.S. **39** (2006), 775–839.
10. M. Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. **164** (2006), 1–84.
11. M. Emerton, *p-adic L-functions and unitary completions of representations of p-adic reductive groups*, Duke Math. J. **130** (2005), 353–392.
12. M. Emerton, *A local-global compatibility conjecture in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbb{Q}}$, Pure and Appl. Math. Quarterly **2** (2006), no. 2 (Special Issue: In honor of John Coates, Part 2 of 2), 1–115.
13. M. Emerton, *Jacquet modules for locally analytic representations of p-adic reductive groups II. The relation to parabolic induction*, in preparation.
14. C. T. Féaux de Lacroix, *Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper*, Thesis, Köln 1997, Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, Heft 23 (1999), 1-111.
15. H. Frommer, *The locally analytic principal series of split reductive groups*, Preprintreihe des SFB 478 – Geometrische Strukturen in der Mathematik **265** (2003).
16. M. Lazard, *Groupes analytiques* $\mathfrak{p}$*-adiques*, Publ. Math. IHES **26** (1965).

17. P. Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Math., Springer-Verlag, 2002.
18. P. Schneider, *p-adische analysis*, Vorlesung in Münster, 2000; available electronically at `http://www.math.uni-muenster.de/math/u/schneider/publ/lectnotes`.
19. P. Schneider, J. Teitelbaum, *p-adic boundary values*, Astérisque **278** (2002), 51–123.
20. P. Schneider, J. Teitelbaum, *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Amer. Math. Soc. **15** (2001), 443–468.
21. P. Schneider, J. Teitelbaum, *Banach space representations and Iwasawa theory*, Israel J. Math. **127** (2002), 359–380.
22. P. Schneider, J. Teitelbaum, *p-adic Fourier theory*, Documenta Math. **6** (2001), 447–481.
23. P. Schneider, J. Teitelbaum, *Algebras of p-adic distributions and admissible representations*, Invent. Math **153** (2003), 145–196.
24. P. Schneider, J. Teitelbaum, *Banach-Hecke algebras and p-adic Galois representations*, Documenta Math. Extra Volume: John H. Coates' Sixtieth Birthday (2006), 631–684.
25. J.-P. Serre, *Lie algebras and Lie groups*, W. A. Benjamin, 1965.

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