# VARIATION OF IWASAWA INVARIANTS IN HIDA FAMILIES

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## 1. Introduction

Let $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ be an absolutely irreducible modular Galois representation over a finite field $k$ of characteristic $p$. Assume further that $\bar{\rho}$ is $p$-ordinary and $p$-distinguished in the sense that the restriction of $\bar{\rho}$ to a decomposition group at $p$ is reducible and non-scalar. The *Hida family* $\mathcal{H}(\bar{\rho})$ of $\bar{\rho}$ is the set of all $p$-ordinary $p$-stabilized newforms $f$ with mod $p$ Galois representation isomorphic to $\bar{\rho}$. (If $\bar{\rho}$ is unramified at $p$, then one must also fix an unramified line in $\bar{\rho}$ and require that the ordinary line of $f$ reduces to this fixed line.) These newforms are a dense set of points in a certain $p$-adic analytic space of overconvergent eigenforms, consisting of an intersecting system of branches (i.e. irreducible components) $\mathbf{T}(\mathfrak{a})$ indexed by the minimal primes $\mathfrak{a}$ of a certain Hecke algebra.

To each modular form $f \in \mathcal{H}(\bar{\rho})$ one may associate the *Iwasawa invariants* $\mu^{\mathrm{an}}(f)$, $\lambda^{\mathrm{an}}(f)$, $\mu^{\mathrm{alg}}(f)$ and $\lambda^{\mathrm{alg}}(f)$. The analytic (resp. algebraic) $\lambda$-invariants are the number of zeroes of the $p$-adic $L$-function (resp. of the characteristic power series of the dual of the Selmer group) of $f$, while the $\mu$-invariants are the exponents of the powers of $p$ dividing the same objects. In this paper we prove the following results on the behavior of these Iwasawa invariants as $f$ varies over $\mathcal{H}(\bar{\rho})$.

**Theorem 1.** *Fix* $* \in \{\mathrm{alg}, \mathrm{an}\}$. *If* $\mu^*(f_0) = 0$ *for some* $f_0 \in \mathcal{H}(\bar{\rho})$, *then* $\mu^*(f) = 0$ *for all* $f \in \mathcal{H}(\bar{\rho})$. *(We then write simply* $\mu^*(\bar{\rho}) = 0$.*)*

It is conjectured by Greenberg that if a $p$-ordinary modular form $f$ of weight two has a residually irreducible Galois representation, then $\mu^{\mathrm{an}}(f) = \mu^{\mathrm{alg}}(f) = 0$; Theorem 1 thus shows that this conjecture is equivalent to the corresponding conjecture for modular forms of arbitrary weight.

**Theorem 2.** *Fix* $* \in \{\mathrm{alg}, \mathrm{an}\}$ *and assume that* $\mu^*(\bar{\rho}) = 0$. *Let* $f_1, f_2 \in \mathcal{H}(\bar{\rho})$ *lie on the branches* $\mathbf{T}(\mathfrak{a}_1), \mathbf{T}(\mathfrak{a}_2)$ *respectively. Then*

$$\lambda^*(f_1) - \lambda^*(f_2) = \sum_{\ell \mid N_1 N_2} e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1); \tag{1.1}$$

*here the sum is over all primes dividing the tame level of* $f_1$ *or* $f_2$ *and* $e_\ell(\mathfrak{a}_j)$ *is a certain explicit non-negative invariant of the branch* $\mathbf{T}(\mathfrak{a}_j)$ *and the prime* $\ell$.

The first (resp. second) theorem corresponds to Theorems 3.7.5 and 4.3.3 (resp. Theorems 3.7.7 and 4.3.4) with trivial twist by the mod $p$ cyclotomic character.

Note that the right-hand side of (1.1) is identical both algebraically and analytically. In particular, using work of Kato on the main conjecture of Iwasawa theory for modular forms, we obtain the following result.

**Corollary 1.** *Assume that* $\mu^{\mathrm{alg}}(\bar{\rho}) = \mu^{\mathrm{an}}(\bar{\rho}) = 0$. *If the main conjecture holds for some* $f_0 \in \mathcal{H}(\bar{\rho})$, *then it holds for all* $f \in \mathcal{H}(\bar{\rho})$.

The corollary in particular reduces the main conjecture to the case of modular forms of weight two, together with the conjecture on the vanishing of the $\mu$-invariant. We also obtain the following result on the variation of $\lambda$-invariants in a Hida family.

**Corollary 2.** *Fix* $* \in \{\mathrm{alg}, \mathrm{an}\}$ *and assume that* $\mu^*(\bar{\rho}) = 0$.

(1) $\lambda^*(\cdot)$ *is constant on branches of* $\mathcal{H}(\bar{\rho})$.
(2) $\lambda^*(\cdot)$ *is minimized on the branches of* $\mathcal{H}(\bar{\rho})$ *of minimal tame level.*

Although we give an extensive summary of our methods in the following sections, we give a brief indication now of our approach. On the analytic side, our results are obtained by specialization of a two-variable $p$-adic $L$-function. Our construction is inspired by those of [22, 28], although we are able to obtain somewhat better control over the canonical periods. This control of canonical periods in families and over varying tame level (see Section 3 especially Theorem 3.6.2) is the key technical input needed for our Iwasawa theoretic applications. On the algebraic side we use a theory of residual Selmer groups to relate the Iwasawa invariants of the modular forms in $\mathcal{H}(\bar{\rho})$.

This work was motivated by the paper [15] of Greenberg and Vatsal where Theorems 1 and 2 were obtained for $p$-ordinary modular forms of weight two. Our results here help to illuminate the results of [15]; indeed, if two congruent elliptic curves have different $\lambda$-invariants, it follows from these results that they must lie on two branches of the associated Hida family with different ramification behavior. One may thus think of the change of $\lambda$-invariants in terms of "jumps" as one moves from one branch to another at crossing points (which necessarily occur at non-classical[1] eigenforms).

## 1.1. **Iwasawa theory.**

The papers [12] of Greenberg and [27] of Mazur introduce the following point of view on Iwasawa theory: If $X$ is a $p$-adic analytic family of $p$-adic representations of $G_{\mathbf{Q}}$ interpolating a collection of motivic representations, then there should exist an analytic function (or perhaps a section of an invertible sheaf) $L$, which we call a $p$-adic $L$-function, defined on $X$ and a coherent sheaf $H$ on $X$, such that the zero locus of $L$ coincides with the codimension one part of the support of $H$ (thought of as a Cartier divisor on $X$). The function $L$ should $p$-adically interpolate the (suitably modified) values at $s = 1$ of the classical $L$-functions attached to the Galois representations at motivic points of $X$, while at such a motivic point $x \in X$, the fiber $H_x$ should coincide with an appropriately defined Selmer group of the motive attached to $x$.

The $L$-function considered above is what is usually known as the *analytic* $p$-adic $L$-function in classical Iwasawa theory; the equation of the codimension one part of the support of $H$ is usually known as the *algebraic* $p$-adic $L$-function. The equality of the zero locus of $L$ with the divisorial part of the support of $H$ is thus a statement of the *main conjecture of Iwasawa theory* for the family of Galois representations $X$.

In this paper, we restrict to the case of two dimensional nearly ordinary modular representations, in which case the space $X$ can be described via Hida theory. Consider a nearly ordinary residual representation; such a representation may be written in the form $\bar{\rho} \otimes \omega^i : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$, where $\bar{\rho}$ is as above, $\omega$ denotes the

[1] To avoid circumlocutions, we adopt the convention that weight one is a non-classical weight.

mod $p$ cyclotomic character and $0 \leq i \leq p-1$. As $\bar{\rho}$ is ordinary, we may (and do) fix a *$p$-stabilization* of $\bar{\rho} \otimes \omega^i$: a choice of one dimensional $G_p$-invariant quotient of $\bar{\rho} \otimes \omega^i$.

Consider the universal deformation space $Y_\Sigma^{\mathrm{n.ord}}$ parameterizing all nearly ordinary deformations of the $p$-stabilized representation $\bar{\rho} \otimes \omega^i$ which are unramified at primes in $\Sigma \cup \{p\}$ for some fixed finite set of primes $\Sigma$. Any such deformation is of the form $\rho \otimes \omega^i \otimes \chi$, where $\rho$ is an ordinary deformation of $\rho$ and $\chi$ is a character of $\Gamma := 1 + p\mathbf{Z}_p$. (We regard $\chi$ as a Galois character by composing it with the projection onto $\Gamma$ of the $p$-adic cyclotomic character.) If we let $Y_\Sigma^{\mathrm{ord}}$ denote the universal deformation space parameterizing all ordinary deformations of the $p$-stabilized representation $\bar{\rho}$ which are unramified at primes in $\Sigma \cup \{p\}$, then $Y_\Sigma^{\mathrm{n.ord}}$ is equal to the product (as formal schemes) of $Y_\Sigma^{\mathrm{ord}}$ and $\mathrm{Spf}\,(\mathbf{Z}_p[[\Gamma]])$. By results of Wiles and Taylor-Wiles [37, 34], as strengthened by Diamond [6], the deformation spaces $Y_\Sigma^{\mathrm{n.ord}}$ can be identified with various local pieces of the universal ordinary Hecke algebra constructed by Hida (at least under mild hypotheses on $\bar{\rho}$).

If we allow $\Sigma$ to vary, the spaces $Y_\Sigma^{\mathrm{n.ord}}$ form a formal Ind-scheme over $W(k)$ parameterizing nearly ordinary deformations of $\bar{\rho}$ of arbitrary conductor. Let $Y^{\mathrm{n.ord}}$ be an irreducible component of this formal Ind-scheme; we may write $Y^{\mathrm{n.ord}}$ as the product of $Y^{\mathrm{ord}}$ (an irreducible component of the the formal Ind-scheme parameterizing ordinary deformations of $\bar{\rho}$) and $\mathrm{Spf}\,(\mathbf{Z}_p[[\Gamma]])$. The motivic points are dense on $Y^{\mathrm{n.ord}}$ and all have the same tame conductor, which we will denote by $N$. If $\mathbf{T}_N^{\mathrm{new}}$ denotes the new-quotient of the Hida Hecke algebra of level $N$, then $Y^{\mathrm{ord}}$ corresponds to a minimal prime $\mathfrak{a}$ of $\mathbf{T}_N^{\mathrm{new}}$; we set $X^{\mathrm{ord}} = \mathrm{Spf}\,(\mathbf{T}_N^{\mathrm{new}}/\mathfrak{a})$ and $X^{\mathrm{n.ord}} = \mathrm{Spf}\,(\mathbf{T}_N^{\mathrm{new}}/\mathfrak{a}\,[[\Gamma]])$. (We point out that $X^{\mathrm{ord}}$ is nearly the same as $Y^{\mathrm{ord}}$ – and hence $X^{\mathrm{n.ord}}$ is nearly the same as $Y^{\mathrm{n.ord}}$. Indeed, $X^{\mathrm{ord}}$ is a finite cover of $Y^{\mathrm{ord}}$ and maps isomorphically to it after inverting $p$.)

The generalities of Iwasawa theory discussed above apply to this space $X^{\mathrm{n.ord}}$. In Section 3 we give a construction of the analytic $p$-adic $L$-function on $X^{\mathrm{n.ord}}$; as is usual, we regard it as a "function of two-variables" – one variable moving along $X^{\mathrm{ord}}$ and the other along $\mathrm{Spf}\,(\mathbf{Z}_p[[\Gamma]])$. An important feature of the situation we consider is that the nearly ordinary deformation space contains many different irreducible components. The main goal of this paper is to describe how to pass Iwasawa theoretic information (such as knowledge of the main conjecture) between components.

In our discussion of Selmer groups we do not go so far as to construct a coherent sheaf of Selmer groups over the space $X^{\mathrm{n.ord}}$. For this, the reader should refer to recent and forthcoming work of Ochiai [30] in which a "two-variable main conjecture" is formulated for each fixed tame level. Moreover, Ochiai shows that this two-variable main conjecture is equivalent to the one-variable main conjecture for each classical modular form in the family. In particular, he deduces our Corollary 1 for modular forms of a fixed tame level. His method has the advantage that he does not need any assumption on the $\mu$-invariant, but has the disadvantage that it only applies to forms on a fixed branch of the Hida family.

1.2. **Two-variable $p$-adic $L$-functions.** To establish the analytic parts of Theorems 1 and 2, we make use of a two-variable $p$-adic $L$-function (playing the role of $L$ above). As already remarked upon in 1.1, one variable is a parameter of wild character space (i.e. the standard cyclotomic variable) and the second variable runs through the Hida family (which is a finite cover of weight space) of some fixed residual representation $\bar{\rho}$. Of the many constructions of two-variable $p$-adic $L$-functions (see [14], [33], [31]), our construction follows most closely that of Mazur [28] and Kitigawa [22], with two main differences.

The first difference is that by using the fact that the Hecke algebras are known to be Gorenstein (as in [37]) we may construct these $L$-function with fewer assumptions on $\bar{\rho}$. The second difference is that we do not limit ourselves to working solely on a part of the Hida family parameterizing newforms of some fixed tame level.

For each branch $Y$ of the Hida family we construct a two-variable $L$-function along $Y$, which at each classical point specializes to the $p$-adic $L$-function of the corresponding newform computed with respect to its canonical period. (Actually, the two-variable $L$-function is defined not on $Y$, but on the partial normalization $X$ of $Y$ discussed above in 1.1.) However, for any finite set of primes $\Sigma$ not containing $p$, we also construct a two-variable $L$-function along $Y_\Sigma$ (the Hida family of $\bar{\rho}$ with ramification only at the primes of $\Sigma$). If we specialize this two-variable $p$-adic $L$-function at a classical point, we obtain a non-primitive $p$-adic $L$-function attached to the corresponding newform (i.e. the usual $p$-adic $L$-function stripped of its Euler factors at primes of $\Sigma$). In fact, we prove that this non-primitivity occurs in families: the $p$-adic $L$-function on the Hida family $Y_\Sigma$, restricted to some branch $Y$, is equal to the $p$-adic $L$-function of that branch stripped of its two variable Euler factors at the primes of $\Sigma$.

With this construction in hand, the analytic part of Theorem 1 follows immediately. Indeed, the vanishing of the analytic $\mu$-invariant for any particular form $f$ in the Hida family is equivalent to the vanishing of the $\mu$-invariant of the two-variable $p$-adic $L$-function along the branch $Y$ containing $f$, which in turn is equivalent to the vanishing of the $\mu$-invariant of the non-primitive two-variable $L$-function on $Y_\Sigma$ for any sufficiently large choice of $\Sigma$ (i.e. such that $Y_\Sigma$ contains $Y$). This last condition is independent of $f$ and so is equivalent to the vanishing of the $\mu$-invariant for every modular form in the family.

The analytic part of Theorem 2 also follows from this construction. The analytic $\lambda$-invariant of a modular form $f$ is equal to the $\lambda$-invariant of the two-variable $L$-function attached to the branch containing $f$. To compare the $\lambda$-invariants of two branches, we choose $\Sigma$ large enough so that $Y_\Sigma$ contains both of these branches. We then relate each $\lambda$-invariant to the $\lambda$-invariant of the two-variable $p$-adic $L$-function attached to $Y_\Sigma$. As explained above, the difference of the two $\lambda$-invariants is then realized in terms of the $\lambda$-invariants of certain Euler factors. The quantity $e_\ell(\mathfrak{a}_i)$ appearing in the formula of Theorem 2 is precisely the $\lambda$-invariant of the Euler factor at $\ell$ along the branch corresponding to $\mathfrak{a}_i$.

1.3. **Residual Selmer groups.** Let $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ be as above. For any $f \in \mathcal{H}(\bar{\rho})$, say with Fourier coefficients in the finite extension $K$ of $\mathbf{Q}_p$, Greenberg has defined the Selmer group $\mathrm{Sel}(\mathbf{Q}_\infty, \rho_f)$ as the subgroup of $H^1(\mathbf{Q}_\infty, A_f)$ cut out by local conditions, all of which are as strong as possible except for the condition at $p$; here $\mathbf{Q}_\infty$ is the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$ and $A_f$ is $(K/\mathcal{O}_K)^2$ with Galois action via $\rho_f$. If one defines $\mathrm{Sel}(\mathbf{Q}_\infty, \bar{\rho})$ in the analogous way, one is confronted with the fundamental problem that the $\pi$-torsion on $\mathrm{Sel}(\mathbf{Q}_\infty, \rho_f)$ may be larger than $\mathrm{Sel}(\mathbf{Q}_\infty, \bar{\rho})$; this is, of course, precisely the reason why congruent modular forms need not have isomorphic Selmer groups.

In [15] this issue is overcome by introducing non-primitive Selmer groups of $\bar{\rho}$; essentially, if $f_1, f_2 \in \mathcal{H}(\bar{\rho})$ have tame levels $N_1$ and $N_2$ respectively, then the $\pi$-torsion on $\mathrm{Sel}(\mathbf{Q}_\infty, \rho_{f_1})$ and $\mathrm{Sel}(\mathbf{Q}_\infty, \rho_{f_2})$ can both be compared to the non-primitive Selmer group of $\bar{\rho}$ obtained by ignoring the local conditions at primes dividing $N_1 N_2$.

Although this approach can also be made to work in the higher weight case, we proceed somewhat differently. Following Mazur [26], we allow non-strict, but not necessarily vacuous, local conditions $\mathcal{S}$ on the cohomology of $\bar{\rho}$, resulting in a family of residual Selmer groups $\mathrm{Sel}_{\mathcal{S}}(\mathbf{Q}_\infty, \bar{\rho})$. We show that for any $f \in \mathcal{H}(\bar{\rho})$ there is a local condition $\mathcal{S}(f)$ such that

$$\mathrm{Sel}_{\mathcal{S}(f)}(\mathbf{Q}_\infty, \bar{\rho}) \cong \mathrm{Sel}(\mathbf{Q}_\infty, \rho_f)[\pi].$$

It follows that the Iwasawa invariants of $f$ can be recovered from the residual Selmer group $\mathrm{Sel}_{\mathcal{S}(f)}(\mathbf{Q}_\infty, \bar{\rho})$.

In fact, if the tame level of $f$ coincides with the conductor of $\bar{\rho}$, then we show that $\mathrm{Sel}_{\mathcal{S}(f)}(\mathbf{Q}_\infty, \bar{\rho})$ agrees with the naive residual Selmer group $\mathrm{Sel}(\mathbf{Q}_\infty, \bar{\rho})$. Since such $f$ always exist by level lowering, we are then able to use duality results to show that the difference

$$\dim_k \mathrm{Sel}_{\mathcal{S}(f)}(\mathbf{Q}_\infty, \bar{\rho}) - \dim_k \mathrm{Sel}(\mathbf{Q}_\infty, \bar{\rho})$$

is precisely given by the dimensions of the local conditions $\mathcal{S}(f)$. The main algebraic results follow from this.

We note that, unlike on the analytic side, Hida theory plays little overt role in these algebraic computations. In particular, in [36], these methods have been applied more generally for algebraic groups other than $\mathrm{GL}_2$. The key inputs are automorphic descriptions of the Galois representation at ramified primes, level lowering results and the fact that the Selmer groups of interest are $\Lambda$-cotorsion; the remainder of the argument is essentially formal.

1.4. ***L*-functions modulo *p*.** In the paper [25], Mazur raises the question of whether one can define a mod $p$ $L$-function attached to the residual representation $\bar{\rho}$ and a choice of tame conductor $N$ (divisible by the tame conductor of $\bar{\rho}$). The construction discussed in 1.2 gives a positive answer to this question, with the caveat that the appropriate extra data is not simply a choice of tame conductor $N$, but rather the more precise data of a component $Y$ of the universal deformation space of $\bar{\rho}$. We may then specialize the $p$-adic $L$-function at the closed point of the partial normalization $X$ of $Y$ on which it is defined, so as to obtain a mod $p$ $L$-function attached to $\bar{\rho}$ and $Y$.

We also show that the dimension of $\mathrm{Sel}_{\mathcal{S}(f)}(\mathbf{Q}_\infty, \bar{\rho})$ over $k$ depends only on the component $Y$ of the Hida family that contains $f$; we thus simply write $\lambda^{\mathrm{alg}}(Y)$ to denote this dimension. Assuming that both the analytic and algebraic $\mu$-invariant of $\bar{\rho}$ vanish, Theorem 2 then shows that the main conjecture for any member of $\mathcal{H}(\bar{\rho})$ is equivalent to the following mod $p$ main conjecture (for one, or equivalently every, choice of $Y$):

**Mod *p* Main Conjecture.** *Let $Y$ be a branch of $\mathcal{H}(\bar{\rho})$. Then the mod $p$ $L$-function $L_p(Y)$ of $Y$ is non-zero, $\lambda^{\mathrm{alg}}(Y)$ is finite and*

$$\lambda\big(L_p(Y)\big) = \lambda^{\mathrm{alg}}(Y).$$

1.5. **Overview of the paper.** In the following section, we recall the Hida theory used in this paper. In particular, we discuss the Hida family attached to a residual representation $\bar{\rho}$, its decomposition into irreducible components and the various Galois representations attached to this family with an emphasis on the integral behavior.

In the third section, we construct two-variable $p$-adic $L$-functions on the Hida family of $\bar{\rho}$ and on each of its irreducible components. By relating these two constructions to each other and to the construction of classical $p$-adic $L$-functions, we obtain proofs of the analytic parts of Theorems 1 and 2.

In the fourth section, we prove the algebraic parts of Theorems 1 and 2 via the theory of residual Selmer groups and the use of level lowering to reduce to the minimal case.

In the final section we give applications to the main conjecture. We discuss some explicit examples to illustrate the general theory, including the congruence modulo 11 between the elliptic curve $X_0(11)$ and the modular form $\Delta$.

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**Notation.** We fix an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$ and write $G_{\mathbf{Q}}$ to denote the Galois group of $\bar{\mathbf{Q}}$ over $\mathbf{Q}$. If $\Sigma$ is a finite set of places of $\mathbf{Q}$, we write $\mathbf{Q}_\Sigma$ for the maximal extension of $\mathbf{Q}$ in $\bar{\mathbf{Q}}$ unramified away from $\Sigma$. We fix an odd prime $p$ and let $\mathbf{Q}_\infty$ denote the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$ in $\bar{\mathbf{Q}}$. For each prime $\ell$ of $\mathbf{Q}$ (resp. place $v$ of $\mathbf{Q}_\infty$) fix a decomposition group $G_\ell \hookrightarrow G_{\mathbf{Q}}$ (resp. $G_v \hookrightarrow G_{\mathbf{Q}_\infty}$) with inertia group $I_\ell$ (resp. $I_v$) such that $G_v \subseteq G_\ell$ whenever $v$ divides $\ell$; note that $I_\ell = I_v$ for a place $v$ dividing a prime $\ell \neq p$. Let $v_p$ denote the unique place of $\mathbf{Q}_\infty$ above $p$.

We write $\varepsilon : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ for the $p$-adic cyclotomic character. We let $\Gamma$ denote the group of 1-units in $\mathbf{Z}_p^\times$; the cyclotomic character induces a canonical isomorphism $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\cong} \Gamma$. If $\mathcal{O}$ is a $\mathbf{Z}_p$-algebra, we let $\Lambda_{\mathcal{O}}$ denote the completed group ring $\mathcal{O}[[\Gamma]]$; we simply write $\Lambda$ for $\Lambda_{\mathbf{Z}_p}$. We denote the natural map $\mathbf{Z}_p^\times \to \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ by $\gamma \mapsto \langle\gamma\rangle_p$; this restricts to the natural map $\Gamma \to \Lambda^\times$.

Recall that $\Lambda$ is a complete regular local ring of dimension two, non-canonically isomorphic to the power series ring $\mathbf{Z}_p[[T]]$. (Such an isomorphism is obtained by choosing a topological generator $\gamma$ of $\Gamma$ and mapping $\langle\gamma\rangle_p$ to $T+1$.) We let $\mathcal{L}$ denote the field of fractions of $\Lambda$.

We let $\Delta$ denote the group of cyclotomic units of $\mathbf{Z}_p$; there is then an isomorphism $\Gamma \times \Delta \cong \mathbf{Z}_p^\times$. We let $\omega$ denote the inclusion of $\Delta$ in $\mathbf{Z}_p^\times$, so that

$$\mathrm{Hom}(\Delta, \mathbf{Z}_p^\times) = \{\omega^i \,|\, 0 \leq i \leq p-2\}.$$

If $\mathbf{Z}_{p,(i)}$ denotes $\mathbf{Z}_p$ regarded as a $\Delta$-module via the character $\omega^i$, then there is a natural isomorphism of $\mathbf{Z}_p[\Delta]$-algebras $\mathbf{Z}_p[\Delta] \cong \prod_{i=0}^{p-2} \mathbf{Z}_{p,(i)}$. Combining this with the natural isomorphism $\mathbf{Z}_p[[\mathbf{Z}_p^\times]] \cong \mathbf{Z}_p[\Delta] \otimes_{\mathbf{Z}_p} \Lambda$, we obtain an isomorphism of $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$-algebras

$$\mathbf{Z}_p[[\mathbf{Z}_p^\times]] \cong \prod_{i=0}^{p-2} \Lambda_{(i)},$$

where $\Lambda_{(i)}$ denotes a copy of $\Lambda$, enhanced to a $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$-algebra by having $\Delta$ act through $\omega^i$.

If $N$ is a natural number prime to $p$, then we will write $\mathbf{Z}_{p,N}^\times := (\mathbf{Z}/N)^\times \times \mathbf{Z}_p^\times$. We will have occasion to consider the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^\times]]$. In this situation, we will extend the diamond bracket notation introduced above and write $x \mapsto \langle x \rangle_{Np}$ to denote the natural map $\mathbf{Z}_{p,N}^\times \to \mathbf{Z}_p[[\mathbf{Z}_{p,N}^\times]]^\times$.

## 2. Hida theory

2.1. **The universal ordinary Hecke algebra.** Fix a positive integer $N$ relatively prime to $p$. We begin by briefly recalling the basic properties of $\mathbf{T}_N$, the universal ordinary Hecke algebra of tame level $N$ (denoted by $h^0(N, \mathbf{Z}_p)$ in [17]) acting on ordinary $p$-adic modular forms of tame level $N$, arbitrary $p$-power level and arbitrary weight. We first fix some notation.

For each integer $k \geq 2$, let $S_k(Np^\infty, R)$ denote the union over all $r \geq 0$ of the spaces of weight $k$ cusp forms on $\Gamma_1(Np^r)$ whose Fourier coefficients lie in a fixed $\mathbf{Z}_p$-algebra $R$. We make $S_k(Np^\infty, R)$ into a $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$-module by letting $\mathbf{Z}_p^\times$ act via the product of the nebentypus action and the character $\gamma \mapsto \gamma^k$. Let $S_k(Np^\infty, R)^{\mathrm{ord}}$ be the subspace of these cusps forms on which $U_p$ acts invertibly. If $\kappa$ is a homomorphisms from $\Gamma = 1 + p\mathbf{Z}_p$ to $R^\times$, we let $S_k(Np^\infty, R)^{\mathrm{ord}}[\kappa]$ denote the $R$-submodule of $S_k(Np^\infty, R)^{\mathrm{ord}}$ on which $\Gamma$ acts by the character $\kappa$.

If $\wp$ is a height one prime of $\Lambda$, then we set $\mathcal{O}(\wp) := \Lambda/\wp$. Let $\kappa_\wp : \Gamma \to \mathcal{O}(\wp)^\times$ be the character induced by the embedding $\Gamma \to \Lambda^\times$. We say that $\wp$ is *classical of weight* $k \geq 2$, if it is of residue characteristic zero and if there is a finite index subgroup $\Gamma'$ of $\Gamma$ such that $\kappa_\wp$, when restricted to $\Gamma'$, coincides with the character $\gamma \mapsto \gamma^k \in \mathbf{Z}_p^\times \subset \mathcal{O}(\wp)^\times$.

More generally, if $\wp$ is a height one prime ideal of a finite $\Lambda$-algebra $\mathbf{T}$, then we again write $\mathcal{O}(\wp) := \mathbf{T}/\wp$. We say $\wp$ is *classical of weight* $k$, if $\wp' := \wp \bigcap \Lambda$ is classical of weight $k$ and write $\kappa_\wp := \kappa_{\wp'} : \Gamma \to \mathcal{O}(\wp')^\times \subset \mathcal{O}(\wp)^\times$.

The algebra $\mathbf{T}_N$ is defined to be commutative algebra of endomorphisms of $S_k(Np^\infty, R)^{\mathrm{ord}}$ generated over $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ by the Hecke operators $T_\ell$ (resp. $U_\ell$) for primes $\ell \nmid Np$ (resp. primes $\ell \mid Np$), together with the diamond operators $\langle a \rangle$ for $a \in (\mathbf{Z}/N)^\times$. By definition $\mathbf{T}_N$ is a $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$-algebra. The map $(\mathbf{Z}/N)^\times \to \mathbf{T}_N^\times$ defined via $a \mapsto \langle a \rangle$ allows us to extend this to a structure of $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^\times]]$-algebra on $\mathbf{T}_N$.

**Theorem 2.1.1.**

(1) *The algebra $\mathbf{T}_N$ is free of finite rank over $\Lambda = \mathbf{Z}_p[[\Gamma]]$.*
(2) *If $\wp$ is a classical height one prime in $\Lambda$ of weight $k$, then $\mathbf{T}_N/\wp\mathbf{T}_N$ is canonically identified with the the quotient of the full Hecke algebra that acts faithfully on $S_k(Np^\infty, \mathcal{O}(\wp))^{\mathrm{ord}}[\kappa_\wp]$.*

*Proof.* See [18, Thm. 3.1, Cor. 3.2] and [17, Thm. 1.1, Thm. 1.2]. □

**Remark 2.1.2.** It follows from part (2) of the above theorem, together with the usual duality between spaces of modular forms and Hecke algebras, that classical height one primes in $\mathbf{T}_N$ are in one-to-one correspondence with Galois conjugacy classes of $p$-ordinary normalized Hecke eigenforms of tame level $N$ and weight $k \geq 2$ (defined over an algebraic closure of $\bar{\mathbf{Q}}_p$). For such an eigenform $f$, we denote the corresponding prime ideal by $\wp_f$ and for a classical height one prime ideal $\wp$ in $\mathbf{T}_N$, we denote the corresponding eigenform by $f_\wp$.

The new quotient $\mathbf{T}_N^{\mathrm{new}}$ of $\mathbf{T}_N$ is defined to be the quotient of $\mathbf{T}_N$ that acts faithfully on the space of newforms. The following theorem of Hida summarizes its basic properties.

**Theorem 2.1.3.**

1. *$\mathbf{T}_N^{\mathrm{new}}$ is a finite and reduced torsion free $\Lambda$-algebra.*
2. *The classical height one primes of $\mathbf{T}_N^{\mathrm{new}}$ correspond (under pull-back) to those classical height one primes $\wp$ of $\mathbf{T}_N$ for which the corresponding normalized eigenform $f_\wp$ is of tame conductor $N$.*
3. *If $\wp$ is a classical height one prime ideal of $\Lambda$, then $\mathbf{T}_N^{\mathrm{new}} \otimes_\Lambda \Lambda_\wp$ is a finite étale extension of the discrete valuation ring $\Lambda_\wp$.*

*Proof.* Parts (1) and (2) follow from the results of [18]. See, in particular, Corollaries 3.3 and 3.7. Part (3) is a rewording of [17, Cor. 1.4]. □

2.2. **Galois representations.** In this section we recall the basic facts regarding Galois representations attached to Hida families. As above, we fix a tame level $N$ prime to $p$. Recall that $\mathcal{L}$ denotes the field of fractions of $\Lambda$.

**Theorem 2.2.1.** *There is a continuous Galois representation*

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}_N^{\mathrm{new}} \otimes_\Lambda \mathcal{L}),$$

*characterized by the following properties:*

1. *$\rho$ is unramified away from $Np$.*
2. *If $\ell$ is a prime not dividing $Np$ then $\rho(\mathrm{Frob}_\ell)$ has characteristic polynomial equal to $X^2 - T_\ell X + \langle \ell \rangle_{Np} \ell^{-1} \in \mathbf{T}_N^{\mathrm{new}}[X]$. (Recall that $\langle \ell \rangle_{Np}$ denotes the unit in $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^\times]]$ corresponding to the element $\ell \in \mathbf{Z}_{p,N}^\times$.)*

*The representation $\rho$ satisfies the following additional properties:*

3. *$\rho$ is absolutely irreducible.*
4. *The determinant of $\rho$ is equal to the following character:*

$$G_{\mathbf{Q}} \to \hat{\mathbf{Z}}^\times \to \mathbf{Z}_{p,N}^\times \xrightarrow{x \mapsto \langle x \rangle_{Np} x_p^{-1}} \mathbf{Z}_p[[\mathbf{Z}_{p,N}^\times]]^\times \subset (\mathbf{T}_N^{\mathrm{new}})^\times.$$

   *(Here the first arrow is the full cyclotomic character, and $x_p$ denotes the image of $x$ under the projection $\mathbf{Z}_{p,N}^\times \to \mathbf{Z}_p^\times$.)*
5. *The space of $I_p$-coinvariants of $\rho$ is free of rank one over $\mathbf{T}_N^{\mathrm{new}} \otimes_\Lambda \mathcal{L}$ and $\mathrm{Frob}_p$ acts on this space through the eigenvalue $U_p \in \mathbf{T}_N^{\mathrm{new}}$.*

*Proof.* See [17, Thm. 2.1] and [14, Thm. 2.6]. □

**Remark 2.2.2.** The representation $\rho$ encodes all of the representations attached to the classical modular forms occurring in the Hida family determined by $\mathbf{T}_N^{\mathrm{new}}$. Indeed, if $\hat{\mathbf{T}}_N^{\mathrm{new}}$ denotes the normalization of $\mathbf{T}_N^{\mathrm{new}}$, then we may descend $\rho$ to a two dimensional representation defined over $\hat{\mathbf{T}}_N^{\mathrm{new}}$ (see [18, Theorem 2.1]). If $\wp$ is a classical height one prime ideal in $\mathbf{T}_N^{\mathrm{new}}$, then we have an isomorphism $(\mathbf{T}_N^{\mathrm{new}})_\wp \cong (\hat{\mathbf{T}}_N^{\mathrm{new}})_\wp$, by part (3) of Theorem 2.1.3, and consequently we may descend $\rho$ to a two dimensional representation over $(\mathbf{T}_N^{\mathrm{new}})_\wp$. Reducing this representation modulo the maximal ideal of this local ring, we obtain a representation $\bar{\rho}_\wp : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}(\wp)[1/p])$. Part (2) of Theorem 2.2.1 shows that this is the usual absolutely irreducible two dimensional Galois representation attached to the newform $f_\wp$.

From the Galois representation $\rho$, we may also construct various related Galois representations corresponding to the minimal and maximal primes of $\mathbf{T}_N^{\text{new}}$.

We consider first the case of minimal primes. Note that $\mathbf{T}_N^{\text{new}} \otimes_\Lambda \mathcal{L} \cong \prod_{\mathfrak{a}} (\mathbf{T}_N^{\text{new}})_{\mathfrak{a}}$ where the product is taken over all minimal primes of $\mathbf{T}_N^{\text{new}}$. For $\mathfrak{a}$ a fixed minimal prime ideal of $\mathbf{T}_N^{\text{new}}$, let $\rho_{\mathfrak{a}}$ denote the representation

$$\rho_{\mathfrak{a}} : G_{\mathbf{Q}} \to \mathrm{GL}_2((\mathbf{T}_N^{\text{new}})_{\mathfrak{a}})$$

obtained by composing $\rho$ with the projection $\mathbf{T}_N^{\text{new}} \otimes_\Lambda \mathcal{L} \to (\mathbf{T}_N^{\text{new}})_{\mathfrak{a}}$.

Since $\rho_{\mathfrak{a}}$ takes values in a field, we may define its tame conductor by the usual formula. That is, if $\rho_{\mathfrak{a}}$ acts on the two dimensional $(\mathbf{T}_N^{\text{new}})_{\mathfrak{a}}$-vector space $\mathcal{V}$, then the exponent of a prime $\ell \neq p$ in the tame conductor is equal to

$$\dim \mathcal{V} - \dim \mathcal{V}_{I_\ell} + \int_0^\infty \left(\dim \mathcal{V} - \dim \mathcal{V}_{I_\ell^u}\right) du,$$

where $\{I_\ell^u\}_{u \geq 0}$ denotes the usual filtration of $I_\ell$ by higher ramification groups, indexed by the upper numbering. (Note that this formula is often expressed in terms of invariants under the ramification groups, rather than coinvariants. One obtains the same value with either formulation, since $\dim \mathcal{V}_{I_\ell^u} = \dim \mathcal{V}^{I_\ell^u}$ for any value of $u$.)

**Proposition 2.2.3.** *If $\mathfrak{a}$ is any minimal prime of $\mathbf{T}_N^{\text{new}}$, then the tame conductor of $\rho_{\mathfrak{a}}$ is equal to $N$.*

*Proof.* Write $\hat{\mathbf{T}}$ to denote the normalization of $\mathbf{T}_N^{\text{new}}/\mathfrak{a}$; this is the component of the normalization of $\mathbf{T}_N^{\text{new}}$ cut out by the minimal prime $\mathfrak{a}$. Since it is Cohen-Macaulay (being normal and of dimension two) and finite over $\Lambda$, it is in fact finite flat over $\Lambda$ by [24, Thm. 23.1].

Let $\mathcal{V}$ denote a two dimensional vector space over the fraction field of $\mathbf{T}_N^{\text{new}}/\mathfrak{a}$ on which $\rho_{\mathfrak{a}}$ acts and let $M$ be a free rank two $\hat{\mathbf{T}}$-lattice in $\mathcal{V}$ invariant under $G_{\mathbf{Q}}$. If $\wp$ is a classical height one prime of $\mathbf{T}_N^{\text{new}}$ containing $\mathfrak{a}$, then as was observed above, the injection

$$(\mathbf{T}_N^{\text{new}}/\mathfrak{a})_\wp \to \hat{\mathbf{T}}_\wp$$

is an isomorphism. Moreover, $G_{\mathbf{Q}}$ acts on the $\mathcal{O}(\wp)[1/p]$-vector space $V(\wp) := (M/\wp M)[1/p]$ via the usual Galois representations $\bar{\rho}_\wp$ attached to newform $f_\wp$ of tame conductor $N$.

The ratio of the tame conductor of $\rho_{\mathfrak{a}}$ and the tame conductor of $\bar{\rho}_\wp$ (that is, $N$) is equal to

$$\prod_{\ell \neq p} \ell^{\dim V(\wp)_{I_\ell} - \dim \mathcal{V}_{I_\ell}}; \tag{2.1}$$

furthermore, each of the exponents appearing in this expression is non-negative (i.e. $\dim V(\wp)_{I_\ell}$ bounds $\dim \mathcal{V}_{I_\ell}$ from above). (See [23, §1]). Thus to prove the proposition, we must show that $\dim V(\wp)_{I_\ell} = \dim \mathcal{V}_{I_\ell}$ for each prime $\ell \neq p$.

If $\dim V(\wp)_{I_\ell} = 2$, then $\ell$ does not divide $N$. In this case, $\rho_{\mathfrak{a}}$ is unramified at $\ell$, and so $\dim \mathcal{V}_{I_\ell} = 2$ as well. Conversely, if $\dim \mathcal{V}_{I_\ell} = 2$, then the same is true of $\dim V(\wp)_{I_\ell}$ (since the latter dimension bounds the former dimension from above). If $\dim V(\wp)_{I_\ell} = 0$, then also $\dim \mathcal{V}_{I_\ell} = 0$ (again, since the latter dimension is bounded above by the former). Thus it remains to show that if $\dim V(\wp)_{I_\ell} > 0$, then $\dim \mathcal{V}_{I_\ell} > 0$. This we now do.

Note that the formula (2.1) (and the fact that the tame conductor of $V(\wp)$ is equal to $N$, and thus is independent of the particular choice of $\wp$ containing $\mathfrak{a}$) shows that $\dim V(\wp)_{I_\ell}$ is independent of the choice of the classical height one prime $\wp$ containing $\mathfrak{a}$. Thus we may assume that $\dim V(\wp)_{I_\ell}$ is positive for every classical height one prime $\wp$ containing $\mathfrak{a}$. There is a natural map $I_\ell \to \mathbf{Z}_p(1)$, given by projection onto the $p$-Sylow subgroup of the tame quotient of $I_\ell$. Let $J_\ell$ denote the kernel of this projection. Since $J_\ell$ has order prime to $p$, it acts on $M$ through a finite quotient of order prime to $p$ and so certainly $\dim \mathcal{V}_{J_\ell} = \dim V(\wp)_{J_\ell}$ for any classical height one prime $\wp$. Let $\sigma$ denote a topological generator of $\mathbf{Z}_p(1)$ and consider the matrix $\rho_{\mathfrak{a}}(\sigma) - I$ acting on $\mathcal{V}_{J_\ell}$. (Here $I$ denotes the identity matrix.) By assumption, the determinant of this matrix (which lies in $\hat{\mathbf{T}}$) lies in $\wp\hat{\mathbf{T}}$ for every classical height one prime $\wp$ of $\hat{\mathbf{T}}$. Thus it lies in $\wp'\hat{\mathbf{T}}$ for every classical height one prime $\wp'$ of $\Lambda$. (Recall that these primes are unramified in $\hat{\mathbf{T}}$, by part (3) of Theorem 2.1.3.) Since $\hat{\mathbf{T}}$ is finite flat over $\Lambda$, we see that this determinant vanishes and thus $\rho_{\mathfrak{a}}(\sigma)$ admits a non-zero coinvariant quotient of $\mathcal{V}_{J_\ell}$. This proves that $\dim \mathcal{V}_{I_\ell} > 0$ as required. □

As a byproduct of the proof of the preceding proposition, we also obtain the following useful result.

**Proposition 2.2.4.** *Let $\wp$ be a classical height one prime ideal in $\mathbf{T}_N^{\mathrm{new}}$ and let $L$ denote a choice of two dimensional $(\mathbf{T}_N^{\mathrm{new}})_\wp$-lattice on which $\rho$ acts. (In particular, $L/\wp L$ is a two dimensional $\mathcal{O}(\wp)[1/p]$-vector space on which $\bar{\rho}_\wp$ acts.) If $\ell$ is any prime distinct from $p$, then $L_{I_\ell}$ is a free $(\mathbf{T}_N^{\mathrm{new}})_\wp$-module and there are natural isomorphisms $L_{I_\ell} \otimes_\Lambda \mathcal{L} \to (L \otimes_\Lambda \mathcal{L})_{I_\ell}$ and $L_{I_\ell}/\wp L_{I_\ell} \to (L/\wp L)_{I_\ell}$.*

*Proof.* We let $\mathfrak{a}$ denote the minimal ideal contained in $\wp$ (unique since $(\mathbf{T}_N^{\mathrm{new}})_\wp$ is a discrete valuation ring) and employ the notation introduced in the proof of the preceding proposition. The lattice $L$ is a free rank two $(\mathbf{T}_N^{\mathrm{new}})_\wp$-module, which we may regard as being embedded in a $G_{\mathbf{Q}}$-equivariant fashion in $\mathcal{V}$.

For general reasons, the composite surjection $L \to L/\wp L \to (L/\wp L)_{I_\ell}$ induces an isomorphism $L_{I_\ell}/\wp L_{I_\ell} \cong (L/\wp L)_{I_\ell}$. Thus a minimal set of generators for $L_{I_\ell}$ as a $(\mathbf{T}_N^{\mathrm{new}})_\wp$-module contains at most $\dim(L/\wp L)_{I_\ell}$ elements. On the other hand the surjection $\mathcal{V} \to \mathcal{V}_{I_\ell}$ induces a surjection $L_{I_\ell} \otimes_\Lambda \mathcal{L} \to \mathcal{V}_{I_\ell}$ and hence $L_{I_\ell} \otimes_\Lambda \mathcal{L}$ is of dimension at least $\dim \mathcal{V}_{I_\ell}$. The proof of Proposition 2.2.3 shows that $\dim(L/\wp L)_{I_\ell} = \dim \mathcal{V}_{I_\ell}$ and hence (taking into account the fact that $(\mathbf{T}_N^{\mathrm{new}})_\wp$ is a discrete valuation ring) we may conclude that $L_{I_\ell}$ is free over $(\mathbf{T}_N^{\mathrm{new}})_\wp$ of rank equal to $\dim \mathcal{V}_{I_\ell}$. Thus the surjection $L_{I_\ell} \otimes_\Lambda \mathcal{L} \to \mathcal{V}_{I_\ell}$ is an isomorphism as claimed. □

We now turn to the Galois representations attached to maximal ideals. If $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_N^{\mathrm{new}}$, then the localization $(\mathbf{T}_N^{\mathrm{new}})_\mathfrak{m}$ is a direct factor of $\mathbf{T}_N^{\mathrm{new}}$ and so $(\mathbf{T}_N^{\mathrm{new}})_\mathfrak{m} \otimes_\Lambda \mathcal{L}$ is a direct factor of $\mathbf{T}_N^{\mathrm{new}} \otimes_\Lambda \mathcal{L}$. We let $\rho_\mathfrak{m}$ denote the representation

$$\rho_\mathfrak{m} : G_{\mathbf{Q}} \to \mathrm{GL}_2((\mathbf{T}_N^{\mathrm{new}})_\mathfrak{m} \otimes_\Lambda \mathcal{L})$$

obtained by composing $\rho$ with the projection $\mathbf{T}_N^{\mathrm{new}} \otimes_\Lambda \mathcal{L} \to (\mathbf{T}_N^{\mathrm{new}})_\mathfrak{m} \otimes_\Lambda \mathcal{L}$.

The next result describes the residual representation $\bar{\rho}_\mathfrak{m}$ attached to a maximal ideal of $\mathbf{T}_N^{\mathrm{new}}$.

**Theorem 2.2.5.** *If $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_N^{\mathrm{new}}$, then attached to $\mathfrak{m}$ is a semi-simple representation $\bar{\rho}_\mathfrak{m} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}_N^{\mathrm{new}}/\mathfrak{m})$, uniquely determined by the properties:*

(1) $\bar{\rho}_{\mathfrak{m}}$ *is unramified away from* $Np$.
(2) *If* $\ell$ *is a prime not dividing* $Np$ *then* $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\ell)$ *has characteristic polynomial equal to* $X^2 - T_\ell X + \langle \ell \rangle_{Np} \ell^{-1} \bmod \mathfrak{m} \in (\mathbf{T}_N^{\mathrm{new}}/\mathfrak{m})[X]$.

*Furthermore,* $\bar{\rho}_{\mathfrak{m}}$ *satisfies the following condition:*

(3) *The restriction of* $\bar{\rho}_{\mathfrak{m}}$ *to* $G_p$ *has the following shape (with respect to a suitable choice of basis):*
$$\begin{pmatrix} \chi & * \\ 0 & \psi \end{pmatrix},$$
*where* $\chi$ *and* $\psi$ *are* $(\mathbf{T}_N^{\mathrm{new}}/\mathfrak{m})^\times$*-valued characters of* $G_p$ *such that* $\psi$ *is unramified and* $\psi(\mathrm{Frob}_p) = U_p \bmod \mathfrak{m}$.

*Proof.* The representation $\bar{\rho}_{\mathfrak{m}}$ is constructed in the usual way, by choosing an integral model for $\rho$ over $\hat{\mathbf{T}}_N^{\mathrm{new}}$, reducing this model modulo a maximal ideal $\hat{\mathfrak{m}}$ lifting $\mathfrak{m}$, semi-simplifying and then descending (if necessary) from $\hat{\mathbf{T}}_N^{\mathrm{new}}/\hat{\mathfrak{m}}$ to $\mathbf{T}_N^{\mathrm{new}}/\mathfrak{m}$. The stated properties follow from the corresponding properties of $\rho$. □

**Proposition 2.2.6.** *If* $\mathfrak{m}$ *is a maximal ideal of* $\mathbf{T}_N^{\mathrm{new}}$ *for which the associated residual representation* $\bar{\rho}_{\mathfrak{m}}$ *is irreducible, then* $\rho_{\mathfrak{m}}$ *admits a model over* $(\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{m}}$ *(which we denote by the same symbol)*
$$\rho_{\mathfrak{m}} : G_{\mathbf{Q}} \to \mathrm{GL}_2((\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{m}}),$$
*unique up to isomorphism.*

*Proof.* This follows from the irreducibility of $\bar{\rho}_{\mathfrak{m}}$ and the fact that the traces of $\rho_{\mathfrak{m}}$ lie in $(\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{m}}$. (See [2].) □

**Definition 2.2.7.** If $k$ is a field of characteristic $p$ and $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ is a continuous representation for which $\bar{\rho}_{|G_p}$ is reducible, then we say that $\bar{\rho}$ is $p$-distinguished if the semisimplification of $\bar{\rho}_{|G_p}$ is non-scalar (i.e. if the two characters appearing as Jordan-Hölder factors of the reducible representation $\bar{\rho}_{|G_p}$ are distinct).

Let $\langle \varepsilon \rangle_p : G_{\mathbf{Q}} \to \mathbf{Z}_p[[\mathbf{Z}_p^\times]]^\times$ denote the character obtained by composing the $p$-adic cyclotomic character $\varepsilon : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ with the map $\langle\,\rangle_p : \mathbf{Z}_p^\times \to \mathbf{Z}_p[[\mathbf{Z}_p^\times]]^\times$.

**Proposition 2.2.8.** *Let* $\mathfrak{m}$ *be a maximal ideal of* $\mathbf{T}_N^{\mathrm{new}}$ *for which the associated residual representation* $\bar{\rho}_{\mathfrak{m}}$ *is irreducible and let* $L$ *denote a choice of a free rank two* $(\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{m}}$*-module on which the representation* $\rho_{\mathfrak{m}}$ *acts. (Such an* $L$ *exists by Proposition 2.2.6.) Let* $L_0$ *denote the maximal submodule of* $L$ *on which* $I_p$ *acts through the character* $\langle \varepsilon \rangle_p \varepsilon^{-1}$. *If* $\bar{\rho}_{\mathfrak{m}}$ *is p-distinguished, then each of* $L_0$ *and* $L/L_0$ *is free of rank one over* $(\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{m}}$ *and the* $G_p$*-action on* $L/L_0$ *is unramified.*

*Proof.* Theorem 2.2.1 shows that $L_0 \otimes_\Lambda \mathcal{L}$ is a one-dimensional direct summand of $L \otimes_\Lambda \mathcal{L}$ and that $G_p$ acts on this space and on $(L \otimes_\Lambda \mathcal{L})/(L_0 \otimes_\Lambda \mathcal{L})$ through a character. Thus $G_p$ acts on each of $L_0$ and $L/L_0$ through a character. By assumption, $L/\mathfrak{m}L$ is a $p$-distinguished representation of $G_p$ and so $(L/L_0)/\mathfrak{m}(L/L_0)$ must be one-dimensional over $\mathbf{T}_N/\mathfrak{m}$. The proposition now follows by Nakayama's Lemma. □

Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_N^{\mathrm{new}}$ satisfying the hypothesis of the preceding proposition. As in the statement of that proposition, let $L$ denote a choice of free rank two $(\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{m}}$-module on which the representation $\rho_{\mathfrak{m}}$ acts and let $L_0$ be the maximal submodule of $L$ on which $I_p$ acts through the character $\langle \varepsilon \rangle_p \varepsilon^{-1}$. The group $G_{\mathbf{Q}}$ then acts on the quotient $L/\mathfrak{m}L$ via the residual representation $\bar{\rho}_{\mathfrak{m}}$.

Since the space $L/L_0$ is a free rank one quotient of $L$ on which the $G_p$-action is unramified (by Proposition 2.2.8), we see that $(L/L_0)/\mathfrak{m}(L/L_0)$ is a one dimensional unramified quotient of $L/\mathfrak{m}L$.

**Definition 2.2.9.** If $k$ is a finite field of characteristic $p$ and $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ is a continuous Galois representation acting on a two dimensional $k$-vector space $V$, then a *$p$-stabilization* of $\bar{\rho}$ is a choice of a one dimensional quotient of $V$ on which the $G_p$-action is unramified.

**Definition 2.2.10.** The discussion preceding Definition 2.2.9 shows that if $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_N^{\mathrm{new}}$ for which $\bar{\rho}_{\mathfrak{m}}$ is irreducible and $p$-distinguished, then the quotient $(L/L_0)/\mathfrak{m}(L/L_0)$ (in the notation of that discussion) forms a $p$-stabilization of $\bar{\rho}_{\mathfrak{m}}$. We refer to this as the *canonical $p$-stabilization* of $\bar{\rho}_{\mathfrak{m}}$ attached to the maximal ideal $\mathfrak{m}$.

2.3. **The reduced Hida algebra.**

**Definition 2.3.1.** For any level $N$, we let $\mathbf{T}'_N$ denote the $\Lambda$-subalgebra of $\mathbf{T}_N$ generated by the Hecke operators $T_\ell$ for $\ell$ prime to $Np$, together with the operator $U_p$.

Since $\mathbf{T}'_N$ is a subalgebra of the finite flat $\Lambda$-algebra $\mathbf{T}_N$, it is certainly finite and torsion free over $\Lambda$. It turns out that $\mathbf{T}'_N$ is also reduced. (Being reduced is a standard property of Hecke algebras in which we omit the operators indexed by the primes dividing the level.) In fact, we can be somewhat more precise.

If $M$ is a divisor of $N$, then restricting the action of the prime-to-$N$ Hecke operators to $p$-ordinary forms of level dividing $M$ yields a surjective map of $\Lambda$-algebras $\mathbf{T}'_N \to \mathbf{T}'_M$. Composing this with the composite $\mathbf{T}'_M \subset \mathbf{T}_M \to \mathbf{T}_M^{\mathrm{new}}$ yields a map

$$\mathbf{T}'_N \to \mathbf{T}_M^{\mathrm{new}}. \tag{2.2}$$

Taking the product of these maps over all divisors $M$ of $N$, we obtain a map

$$\mathbf{T}'_N \to \prod_{M|N} \mathbf{T}_M^{\mathrm{new}}. \tag{2.3}$$

**Proposition 2.3.2.**

(1) *The map (2.3) is injective and induces an isomorphism after localizing at any classical height one prime ideal of $\Lambda$ (and hence also after tensoring over $\Lambda$ with its fraction field $\mathcal{L}$).*

(2) *If $\wp$ is a classical height one prime ideal of $\Lambda$, then $\mathbf{T}'_N \otimes_\Lambda \Lambda_\wp$ is a finite étale extension of the discrete valuation ring $\Lambda_\wp$.*

*Proof.* Part (1) follows from the theory of newforms for ordinary families developed in [18]. Part (2) then follows from part (1) together with part (3) of Theorem 2.1.3. □

Taking the product of the Galois representations

$$G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}_M^{\mathrm{new}} \otimes_\Lambda \mathcal{L})$$

given by Theorem 2.2.1, as $M$ ranges over all divisors of $N$, and taking into account part (1) of the preceding proposition, we obtain a Galois representation

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}'_N \otimes_\Lambda \mathcal{L})$$

satisfying the analogue of Theorem 2.2.1.

Just as in the preceding section, we may reduce $\rho$ modulo a height one prime ideal $\wp$ or a maximal ideal $\mathfrak{m}$ of $\mathbf{T}'_N$. We denote the corresponding residual representations by $\bar{\rho}_\wp$ and $\bar{\rho}_\mathfrak{m}$ respectively. Similarly, we may localize $\mathbf{T}'_N$ at any maximal ideal $\mathfrak{m}$ and obtain a corresponding representation

$$\rho_\mathfrak{m} : G_\mathbf{Q} \to \mathrm{GL}_2((\mathbf{T}'_N)_\mathfrak{m} \otimes_\Lambda \mathcal{L}).$$

If $\bar{\rho}_\mathfrak{m}$ is irreducible, then the analogue of Proposition 2.2.6 holds (by the same appeal to the results of [2]) and so we obtain a uniquely determined representation

$$\rho_\mathfrak{m} : G_\mathbf{Q} \to \mathrm{GL}_2((\mathbf{T}'_N)_\mathfrak{m}).$$

2.4. **The reduced Hida algebras attached to $\bar{\rho}$.** If $k$ is a finite field of characteristic $p$ and $\bar{\rho} : G_\mathbf{Q} \to \mathrm{GL}_2(k)$ is a continuous two dimensional Galois representation defined over $k$, then we say that $\bar{\rho}$ is $p$-ordinary if $\bar{\rho}$ restricted to $G_p$ has an unramified quotient of dimension one over $k$. Clearly $\bar{\rho}$ admits a $p$-stabilization, in the sense of Definition 2.2.9, if and only if $\bar{\rho}$ is $p$-ordinary. If $\bar{\rho}$ is furthermore $p$-distinguished (in the sense of Definition 2.2.7), then $\bar{\rho}$ admits at most two choices of $p$-stabilization (and does admit two such choices precisely when the determinant of $\bar{\rho}$ is unramified at $p$).

Let us now fix such a representation $\bar{\rho} : G_\mathbf{Q} \to \mathrm{GL}_2(k)$ and let $\overline{V}$ be a two dimensional $k$-vector space on which $\bar{\rho}$ acts. We assume that $\bar{\rho}$ is irreducible, odd, $p$-ordinary and $p$-distinguished and we fix a choice of $p$-stabilization of $\bar{\rho}$. We assume that $k$ is equal to the field generated by the traces of $\bar{\rho}$. (If it were not, we could replace $k$ by this field of traces and descend $\bar{\rho}$ to the smaller field – see [29, Section 6].) Finally, we suppose that $\bar{\rho}$ is modular (i.e. that it arises as the residual representation attached to a modular form of some weight and level defined over $\bar{\mathbf{Q}}_p$). Our goal in this section is to define, for $\Sigma$ a finite set of primes, the reduced ordinary Hecke algebra $\mathbf{T}_\Sigma(\bar{\rho})$ attached to a modular $p$-ordinary residual representation and to describe its basic properties.

We let $N(\bar{\rho})$ denote the tame conductor of $\bar{\rho}$. If $\ell \neq p$ is prime, then define

$$m_\ell = \dim_k \overline{V}_{I_\ell}$$

and for any finite set of primes $\Sigma$ that does not contain $p$, write

$$N(\Sigma) = N(\bar{\rho}) \prod_{\ell \in \Sigma} \ell^{m_\ell}.$$

**Theorem 2.4.1.** *There is a unique maximal ideal $\mathfrak{m}$ of $\mathbf{T}'_{N(\Sigma)}$ such that $\bar{\rho}_\mathfrak{m}$, with its canonical $p$-stabilization, is isomorphic to $\bar{\rho}$, with its given $p$-stabilization.*

*Proof.* The uniqueness is clear. The existence follows from the assumption that $\bar{\rho}$ is modular and the results of [5]. □

**Proposition 2.4.2.** *If $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{T}'_{N(\Sigma)}$ of the preceding theorem, then there is a unique maximal ideal $\mathfrak{n}$ of $\mathbf{T}_{N(\Sigma)}$ satisfying the following conditions:*

1. *$\mathfrak{n}$ lifts $\mathfrak{m}$.*
2. *$T_\ell \in \mathfrak{n}$ for each $\ell \in \Sigma$.*
3. *The natural map of localizations $(\mathbf{T}'_{N(\Sigma)})_\mathfrak{m} \to (\mathbf{T}_{N(\Sigma)})_\mathfrak{n}$ is an isomorphism of $\Lambda$-algebras.*

*In particular,* $(\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}}$ *is a finite flat* $\Lambda$*-algebra. Also, the image of* $T_\ell$ *in the localization* $(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ *vanishes for each* $\ell \in \Sigma$*.*

*Proof.* This is a variant of [37, Prop. 2.1.5] and is proved in an analogous manner. The second to last claim follows from the rest of the proposition together with part (1) of Theorem 2.1.1. ☐

**Definition 2.4.3.** We let $\mathbf{T}_\Sigma(\bar\rho)$ (or simply $\mathbf{T}_\Sigma$, if $\bar\rho$ is understood) denote the localization of $\mathbf{T}'_{N(\Sigma)}$ at the maximal ideal whose existence is guaranteed by Theorem 2.4.1. We let $\rho_\Sigma : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}_\Sigma)$ denote the Galois representation attached to this local factor of $\mathbf{T}'_{N(\Sigma)}$ as discussed in Section 2.3. Recall that $\rho_\Sigma$ is characterized by the following property: If $\ell$ is a prime not dividing $N(\Sigma)p$, then $\rho_\Sigma(\mathrm{Frob}_\ell)$ has trace equal to $T_\ell \in \mathbf{T}_\Sigma$.

Taking into account Proposition 2.4.2, we see that $\mathbf{T}_\Sigma$ is a reduced and finite flat $\Lambda$-algebra. Note that if $\Sigma \subset \Sigma'$, then $N(\Sigma) \mid N(\Sigma')$ and the natural surjection $\mathbf{T}'_{N(\Sigma')} \to \mathbf{T}'_{N(\Sigma)}$ induces a surjection $\mathbf{T}_{\Sigma'} \to \mathbf{T}_\Sigma$. The Galois representations $\rho_{\Sigma'}$ and $\rho_\Sigma$ are evidently compatible with this surjection.

We refer to $\operatorname{Spec} \mathbf{T}_\Sigma$ as the universal $p$-ordinary family of newforms, or sometimes simply "the Hida family" attached to $\bar\rho$ and our chosen $p$-stabilization that is minimally ramified outside $\Sigma$. Localizing $\rho_\Sigma$ over $\operatorname{Spec} \mathbf{T}_\Sigma$, we obtain a two dimensional vector bundle on which $G_{\mathbf{Q}}$ acts, which we refer to as the universal family of Galois representations over $\operatorname{Spec} \mathbf{T}_\Sigma$. If $\Sigma \subset \Sigma'$, then the surjection $\mathbf{T}_\Sigma \to \mathbf{T}_{\Sigma'}$ induces a closed embedding $\operatorname{Spec} \mathbf{T}_\Sigma \to \operatorname{Spec} \mathbf{T}_{\Sigma'}$ and the universal family of Galois representations on the target pulls back to the universal family of Galois representations on the source. If we consider all $\Sigma$ simultaneously, then we obtain an Ind-scheme

$$\text{“}\varinjlim_{\Sigma}\text{” } \operatorname{Spec} \mathbf{T}_\Sigma$$

and a family of two dimensional Galois representations lying over it. We refer to this Ind-scheme as the universal $p$-ordinary family of newforms, or simply "the Hida family" attached to $\bar\rho$ and its chosen $p$-stabilization.

The Hida family corresponding to $\Sigma = \emptyset$ will play a special role; we refer to it as the minimal Hida family attached to $\bar\rho$ and our chosen $p$-stabilization.

When $\bar\rho$ is irreducible after restriction to the quadratic field of discriminant $(-1)^{(p-1)/2}p$, the results of Wiles and Taylor–Wiles [37, 34], as strengthened by Diamond [6], in fact allow us to identify the rings $\mathbf{T}_\Sigma$, and their accompanying Galois representations $\rho_\Sigma$, with certain universal deformation rings, and their accompanying universal Galois representations, attached to the residual representation $\bar\rho$. Specifically, let $R_\Sigma$ denote the universal deformation ring parameterizing lifts of $\bar\rho$ which are $p$-ordinary and whose tame conductor coincides with that of $\bar\rho$ at primes not in $\Sigma \cup \{p\}$. The representation $\rho_\Sigma$ induces a map $R_\Sigma \to \mathbf{T}_\Sigma$, which by [6] is an isomorphism after tensoring with the quotient of $\Lambda$ by any classical height one prime; it follows that in fact $R_\Sigma \to \mathbf{T}_\Sigma$ is an isomorphism, as claimed.

2.5. **Branches.** We now prove some results concerning the irreducible components of the Hida family attached to $\bar\rho$.

**Definition 2.5.1.** If $\mathfrak{a}$ is a minimal prime ideal in $\mathbf{T}_\Sigma$ for any $\Sigma$ as above, then we write $\mathbf{T}(\mathfrak{a}) := \mathbf{T}_\Sigma/\mathfrak{a}$. Note that $\mathbf{T}(\mathfrak{a})$ is a local domain, finite over $\Lambda$. We write $\mathcal{K}(\mathfrak{a})$ to denote the fraction field of $\mathbf{T}(\mathfrak{a})$. (Thus there is an isomorphism $\mathcal{K}(\mathfrak{a}) \cong \mathbf{T}(\mathfrak{a}) \otimes_\Lambda \mathcal{L}$.) We let $\rho(\mathfrak{a})$ denote the Galois representation

$$\rho(\mathfrak{a}) : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}(\mathfrak{a}))$$

induced by $\rho_\Sigma$.

**Proposition 2.5.2.** *If $\mathfrak{a}$ is a minimal prime of $\mathbf{T}_\Sigma$ for any $\Sigma$ as above, then there is a unique divisor $N(\mathfrak{a})$ of $N(\Sigma)$ and a unique minimal prime $\mathfrak{a}' \subseteq \mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}$ sitting over $\mathfrak{a}$ such that*

$$\begin{tikzcd}
\mathbf{T}_\Sigma \arrow[r] \arrow[d] & \mathbf{T}'_{N(\Sigma)} \arrow[r] & \prod_{M \mid N(\Sigma)} \mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})} \arrow[d] \\
\mathbf{T}_\Sigma/\mathfrak{a} \arrow[r, "="] & \mathbf{T}(\mathfrak{a}) \arrow[r] & \mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}'
\end{tikzcd}$$

*commutes.*

*Proof.* Since $\mathbf{T}_\Sigma$ is finite over $\Lambda$, the minimal primes of $\mathbf{T}_\Sigma$ are in bijection with the local components of $\mathbf{T}_\Sigma \otimes_\Lambda \mathcal{L}$. Since $\mathbf{T}_\Sigma$ is a local factor of $\mathbf{T}'_{N(\Sigma)}$, these local components are included in the local components of $\mathbf{T}'_{N(\Sigma)} \otimes_\Lambda \mathcal{L}$. By part (1) of Proposition 2.3.2, we have that

$$\mathbf{T}'_{N(\Sigma)} \otimes_\Lambda \mathcal{L} \cong \prod_{M \mid N(\Sigma)} \mathbf{T}^{\mathrm{new}}_M \otimes_\Lambda \mathcal{L}.$$

The local components of $\prod_{M \mid N(\Sigma)} \mathbf{T}^{\mathrm{new}}_M \otimes_\Lambda \mathcal{L}$ are in one-to-one correspondence with its minimal primes. Thus our given minimal prime $\mathfrak{a}$ gives rise to a minimal prime of this ring. However, any such minimal prime corresponds to a minimal prime $\mathfrak{a}'$ in $\mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}$ for some $N(\mathfrak{a}) \mid N(\Sigma)$, which establishes the proposition. ☐

**Definition 2.5.3.** For a minimal prime ideal $\mathfrak{a}$ of $\mathbf{T}_\Sigma$, we refer to $N(\mathfrak{a})$ (of the preceding proposition) as the tame conductor attached to $\mathfrak{a}$ (or to the irreducible component of $\operatorname{Spec} \mathbf{T}_\Sigma$ corresponding to $\mathfrak{a}$). Moreover, set $\mathbf{T}(\mathfrak{a})^\circ := \mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}'$.

**Remark 2.5.4.** Proposition 2.5.2 gives rise to an embedding of local domains $\mathbf{T}(\mathfrak{a}) \to \mathbf{T}(\mathfrak{a})^\circ$. (Note that $\mathbf{T}(\mathfrak{a})^\circ$ is local since it is a complete finite $\Lambda$-algebra and hence a product of local rings. Being a domain, it must be local.)

We next observe that at the generic point of the $\mathfrak{a}$-component of $\operatorname{Spec} \mathbf{T}_\Sigma$, the universal Galois representation has conductor equal to the conductor of $\mathfrak{a}$.

**Corollary 2.5.5.** *The tame conductor of the Galois representation $\rho(\mathfrak{a})$ equals $N(\mathfrak{a})$. (Here we define the tame conductor by regarding $\rho(\mathfrak{a})$ as a Galois representation defined over the field of fractions $\mathcal{K}(\mathfrak{a})$ of $\mathbf{T}(\mathfrak{a})$.)*

*Proof.* This follows from Propositions 2.2.3 and 2.5.2. ☐

The next result deals with the classical height one primes in $\mathbf{T}_\Sigma$.

**Proposition 2.5.6.** *Let $\wp$ be any classical height one prime ideal of $\mathbf{T}_\Sigma$.*

(1) *The ring $\mathbf{T}_\Sigma$ is étale over $\Lambda$ (and so regular) in a neighborhood of $\wp$; consequently, $\wp$ contains a unique minimal prime $\mathfrak{a}$ of $\mathbf{T}_\Sigma$ and the natural map of localizations $(\mathbf{T}_\Sigma)_\wp \to \mathbf{T}(\mathfrak{a})_\wp$ is an isomorphism.*

(2) *Thinking of $\wp$ as a height one prime of $\mathbf{T}(\mathfrak{a})$, the map $\mathbf{T}(\mathfrak{a}) \to \mathbf{T}(\mathfrak{a})^\circ$ is an isomorphism in a neighborhood of $\wp$. Consequently, there is a unique height one prime $\wp'$ of $\mathbf{T}(\mathfrak{a})^\circ$ that pulls back to $\wp$ under this map and the map of localizations $\mathbf{T}(\mathfrak{a})_\wp \to \mathbf{T}(\mathfrak{a})^\circ_{\wp'}$ is an isomorphism.*

*Proof.* Both claims follow directly from Proposition 2.3.2. □

In the situation of the preceding proposition, we write $\mathcal{O}(\wp)^\circ := \mathcal{O}(\wp')$; this is a finite extension of $\mathcal{O}(\wp)$. Recall that we have defined the classical newform $f_{\wp'}$ attached to the height one prime ideal $\wp'$ of $\mathbf{T}(\mathfrak{a})^\circ$ (thought of as a classical height one prime of $\mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}$). We write $f_\wp := f_{\wp'}$; part (2) of Theorem 2.1.3 implies that $f_\wp$ lies in $S_k(N(\mathfrak{a})p^\infty, \mathcal{O}(\wp)^\circ)^{\mathrm{ord}}_{\mathrm{new}}[\kappa_\wp]$.

2.6. **Hecke eigenvalues.** In this section, we will give two lemmas that describe Hecke eigenvalues at primes dividing the level in terms of the corresponding Galois representation. The first lemma treats the case of a single modular form and the second discusses the case of a Hida family.

Let $f$ denote a classical newform over $\bar{\mathbf{Q}}_p$ of some weight and of tame level $N$. Let $\rho_f$ denote the $p$-adic Galois representation attached to $f$, let $V_f$ denote its underlying space and let $a_\ell(f)$ denote its Hecke eigenvalues. If $\ell$ does not divide $N$, then $\rho_f$ is unramified at $\ell$ and $a_\ell(f)$ is equal to the trace of $\rho_f(\mathrm{Frob}_\ell)$. The following lemma describes $a_\ell(f)$ for $\ell$ dividing $N$ in terms of $\rho_f$.

**Lemma 2.6.1.** *If $\ell$ is a prime dividing $N$, then the following are equivalent:*
(1) *$a_\ell(f)$ is a non-unit.*
(2) *$a_\ell(f) = 0$.*
(3) *$V_{I_\ell} = 0$.*

*If these equivalent conditions do not hold, then we have that $V_{I_\ell}$ is one dimensional and $a_\ell(f)$ is equal to the eigenvalue of $\mathrm{Frob}_\ell$ acting on this line.*

*Proof.* This is standard. □

Fix a tame level $N$ and a maximal ideal $\mathfrak{n}$ in $\mathbf{T}^{\mathrm{new}}_N$. Let $\mathcal{V}$ denote the free rank two $\mathbf{T}^{\mathrm{new}}_N \otimes_\Lambda \mathcal{L}$-module on which $\rho_{\mathfrak{n}}$ acts. The following lemma describes $T_\ell$ for $\ell$ dividing $N$ in terms of $\rho_{\mathfrak{n}}$.

**Lemma 2.6.2.** *If $\ell$ is a prime dividing $N$, then the following are equivalent:*
(1) *$T_\ell$ lies in the maximal ideal $\mathfrak{n}$.*
(2) *$T_\ell \equiv 0 \bmod \wp$ for some classical height one prime ideal $\wp$ of $(\mathbf{T}^{\mathrm{new}}_N)_{\mathfrak{n}}$.*
(3) *$T_\ell \equiv 0 \bmod \wp$ for every classical height one prime ideal $\wp$ of $(\mathbf{T}^{\mathrm{new}}_N)_{\mathfrak{n}}$.*
(4) *$T_\ell = 0$ in $(\mathbf{T}^{\mathrm{new}}_N)_{\mathfrak{n}}$.*
(5) *$\mathcal{V}_{I_\ell} = 0$.*

*If these equivalent conditions do not hold, then we have that $\mathcal{V}_{I_\ell}$ is free of rank one over $(\mathbf{T}^{\mathrm{new}}_N)_{\mathfrak{n}} \otimes_\Lambda \mathcal{L}$ and $T_\ell$ is equal to the eigenvalue of $\mathrm{Frob}_\ell$ acting on this free rank one module.*

*Proof.* It is clear that (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). Also, Condition (1) implies that $a_\ell(f_\wp)$ is contained in the maximal ideal of $\mathcal{O}(f_\wp) = \mathcal{O}(\wp)$ for each classical height one prime $\wp$ of $(\mathbf{T}^{\mathrm{new}}_N)_{\mathfrak{n}}$. By Lemma 2.6.1, $a_\ell(f_\wp) = 0$ for each such prime and thus (1) implies (3). Proposition 2.2.4, together with Lemma 2.6.1, shows that (3) and (5) are equivalent. Lastly, Lemma 2.6.3 (below), together with Theorem 2.1.3, establishes that (3) implies (4).

Suppose now that these equivalent conditions do not hold. Then, Proposition 2.2.4 and Lemma 2.6.1 show that the space $\mathcal{V}_{I_\ell}$ is free of rank one over $(\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{n}} \otimes_\Lambda \mathcal{L}$ as claimed. Now let $A_\ell \in (\mathbf{T}_N^{\mathrm{new}})_{\mathfrak{n}} \otimes_\Lambda \mathcal{L}$ denote the eigenvalue of $\mathrm{Frob}_\ell$ acting on this module. By Proposition 2.2.4, $A_\ell \in (\mathbf{T}_N^{\mathrm{new}})_\wp$ for each classical height one prime $\wp$ of $\mathbf{T}_N^{\mathrm{new}}$. By Lemma 2.6.1, we see that $T_\ell - A_\ell$ lies in the maximal ideal of this ring for each such prime $\wp$. Then, by Lemma 2.6.3 (below) and Theorem 2.1.3, $T_\ell = A_\ell$. □

**Lemma 2.6.3.** *Let $\mathbf{T}$ be a reduced finite free $\Lambda$-algebra such that $\mathbf{T} \otimes_\Lambda \Lambda_\wp$ is an étale extension of $\Lambda_\wp$ for every classical height one prime $\wp$ of $\Lambda$. If $x \in \mathbf{T} \otimes \mathcal{L}$ lies in the maximal ideal of $\mathbf{T}_{\wp'}$ for every classical height one prime $\wp'$ of $\mathbf{T}$, then $x = 0$.*

*Proof.* Let $\wp$ be a classical height one prime of $\Lambda$. Since $\mathbf{T} \otimes_\Lambda \Lambda_\wp$ is étale over $\Lambda$, we see that $\wp\mathbf{T} \otimes_\Lambda \Lambda_\wp = \bigcap_{\wp'} \wp'\mathbf{T}_{\wp'}$, where the intersection runs over all height one primes $\wp'$ of $\mathbf{T}$ lying over $\wp$. Our assumption on $x$ thus implies that $x \in \wp\mathbf{T} \otimes_\Lambda \Lambda_\wp$ for all classical height one primes $\wp$ of $\Lambda$. As $\mathbf{T}$ is free over $\Lambda$, the lemma follows once we note that $\bigcap_\wp \wp\Lambda_\wp = 0$ (the intersection taking place in $\mathcal{L}$). Indeed, the numerator of any element in this intersection lies in an infinite number of height one primes of $\Lambda$ and so vanishes. □

2.7. **$\Lambda$-adic modular forms and Euler factors.** For each minimal prime $\mathfrak{a}$ of $\mathbf{T}_\Sigma$, we can define a formal $q$-expansion along the partial normalization $\operatorname{Spec} \mathbf{T}(\mathfrak{a})^\circ$ of the component $\operatorname{Spec} \mathbf{T}(\mathfrak{a})$ of $\operatorname{Spec} \mathbf{T}_\Sigma$ that interpolates the $q$-expansions of the newforms $f_\wp$ obtained from the classical primes $\wp$ in $\operatorname{Spec} \mathbf{T}(\mathfrak{a})$. Namely, if we write $T(\mathfrak{a})^\circ = \mathbf{T}_{N(\mathfrak{a})}^{\mathrm{new}}/\mathfrak{a}'$, as in Definition 2.5.3, then we define $f(\mathfrak{a}, q) \in \mathbf{T}(\mathfrak{a})^\circ[[q]]$ via

$$f(\mathfrak{a}, q) = \sum_{n \geq 1} (T_n \bmod \mathfrak{a}') q^n.$$

(Here we have written $T_n$ rather than $U_p^r T_{n'}$, when $n$ is of the form $n = n'p^r$ with $(n', p) = 1$, for the sake of uniformity of notation.)

An alternative way of describing this formal $q$-expansion along $\operatorname{Spec} \mathbf{T}(\mathfrak{a})^\circ$ is to describe the corresponding Euler factors.

**Definition 2.7.1.** Let $\mathfrak{a}$ be a minimal prime of $\mathbf{T}_\Sigma$. As above, write $\mathbf{T}(\mathfrak{a})^\circ \cong \mathbf{T}_{N(\mathfrak{a})}^{\mathrm{new}}/\mathfrak{a}'$, For each prime $\ell \neq p$, define the reciprocal Euler factor $E_\ell(\mathfrak{a}, X) \in \mathbf{T}(\mathfrak{a})^\circ[X]$ via the usual formula:

$$E_\ell(\mathfrak{a}, X) := \begin{cases} 1 - (T_\ell \bmod \mathfrak{a}')X + \langle \ell \rangle_{N(\mathfrak{a})p} \ell^{-1} X^2 & \text{if } \ell \text{ is prime to } N(\mathfrak{a}) \\ 1 - (T_\ell \bmod \mathfrak{a}')X & \text{otherwise.} \end{cases}$$

For the sake of completeness, define $E_p(\mathfrak{a}, X) = (1 - (U_p \bmod \mathfrak{a}')X)$.

In terms of these reciprocal Euler factors, the formal $q$-expansion $f(\mathfrak{a}, q)$ is characterized by the fact that its "formal Mellin transform" is equal to the formal Dirichlet series

$$\prod_\ell E_\ell(\mathfrak{a}, \ell^{-s})^{-1}.$$

We will now give a Galois theoretic description of these reciprocal Euler factors.

**Proposition 2.7.2.** *Let $\mathfrak{a}$ be a minimal prime of $\mathbf{T}_\Sigma$ and let $\mathcal{V}$ denote a two dimensional $\mathcal{K}(\mathfrak{a})$-vector space on which the Galois representation $\rho(\mathfrak{a})$ acts. If $\ell \neq p$ is prime, then the Euler factor $E_\ell(\mathfrak{a}, X) \in \mathcal{K}(\mathfrak{a})[X]$ is equal to the determinant* $\det(\mathrm{Id} - \mathrm{Frob}_\ell X \mid \mathcal{V}_{I_\ell})$.

*Proof.* This follows from Lemma 2.6.2. □

Since $\mathbf{T}(\mathfrak{a})^\circ \subset \mathbf{T}(\mathfrak{a})[1/p]$, we may in particular regard the reciprocal Euler factors $E_\ell(\mathfrak{a}, X)$ and the formal $q$-expansion $f(\mathfrak{a}, q)$ as varying over $\operatorname{Spec}\mathbf{T}(\mathfrak{a})[1/p]$. Let us close this section by signaling a phenomena which will be fundamental to all that follows: the reciprocal Euler factors $E_\ell(\mathfrak{a}, X)$ (or equivalently the formal $q$-expansions $f(\mathfrak{a}, q)$) do not extend in a well-defined fashion over $\operatorname{Spec}\mathbf{T}_\Sigma[1/p]$. In general, if $\wp$ is a (necessarily non-classical) height one prime lying in the intersection of two different components of $\operatorname{Spec}\mathbf{T}_\Sigma[1/p]$, say $\operatorname{Spec}\mathbf{T}(\mathfrak{a}_1)[1/p]$ and $\operatorname{Spec}\mathbf{T}(\mathfrak{a}_2)[1/p]$, then $E_\ell(\mathfrak{a}_1, X)$ and $E_\ell(\mathfrak{a}_2, X)$ (and hence $f(\mathfrak{a}_1, q)$ and $f(\mathfrak{a}_2, q)$) may have different specializations in $\mathcal{O}(\wp)[1/p][X]$ (resp. $\mathcal{O}(\wp)[1/p][[q]]$) for certain values of $\ell$ dividing $N(\mathfrak{a}_1)$ or $N(\mathfrak{a}_2)$.

## 3. Two-variable $p$-adic $L$-functions

3.1. **Canonical periods.** For any level $M$, we let $X_1(M)$ denote the closed modular curve of level $\Gamma_1(M)$ and let $C_1(M)$ denote its set of cusps. Let $L_k(R)$ denote the set of polynomials of degree less than or equal to $k-2$ with coefficients in $R$ and let $\tilde{L}_k(R)$ denote the associated local system on $X_1(M)$.

For $N$ prime to $p$, let $\mathbf{T}^*_{N,r,k}$ (resp. $\mathbf{T}_{N,r,k}$) denote the Hecke algebra corresponding to $p$-ordinary modular forms (resp. cusp forms) on $\Gamma_1(Np^r)$ of weight $k$. To ease notation, we write

$$H_1(Np^r; \tilde{L}_k(\mathbf{Z}_p)) := H_1(X_1(Np^r); \tilde{L}_k(\mathbf{Z}_p))$$

and

$$H_1(Np^r, \{\text{cusps}\}; \tilde{L}_k(\mathbf{Z}_p)) := H_1(X_1(Np^r); C_1(Np^r); \tilde{L}_k(\mathbf{Z}_p)).$$

We then have a short exact sequence of $\mathbf{T}^*_{N,r,k}$-modules:

$$0 \to H_1(Np^r; \tilde{L}_k(\mathbf{Z}_p)) \to H_1(Np^r, \{\text{cusps}\}; \tilde{L}_k(\mathbf{Z}_p)) \to \tilde{H}_0(C_1(Np^r); \tilde{L}_k(\mathbf{Z}_p)) \to 0.$$

(This is a part of the relative homology sequence of the pair $(X_1(Np^r), C_1(Np^r))$; the tilde over the $H_0$ denotes reduced homology.)

If we localize the above sequence at a maximal ideal $\mathfrak{m}$ in $\mathbf{T}^*_{N,r,k}$ corresponding to an irreducible residual $G_{\mathbf{Q}}$-representation, then the localization of $\tilde{H}_0(C_1(Np^r); \mathbf{Z}_p)$ will vanish and the two $H_1$ terms will become isomorphic. Passing to ordinary parts (an exact functor) yields an isomorphism

$$H_1(Np^r; \tilde{L}_k(\mathbf{Z}_p))^{\text{ord}}_{\mathfrak{m}} \cong H_1(Np^r, \{\text{cusps}\}; \tilde{L}_k(\mathbf{Z}_p))^{\text{ord}}_{\mathfrak{m}} \tag{3.1}$$

of $(\mathbf{T}_{N,r,k})_\mathfrak{m} \cong (\mathbf{T}^*_{N,r,k})_\mathfrak{m}$-modules. We write $(\mathfrak{M}_{N,r,k})_\mathfrak{m}$ to denote these isomorphic modules.

There is an action of complex conjugation on $X_1(Np^r)$ which induces an action on these homology group. We will use a superscript of $\pm$ to indicate the $\pm$-eigenspaces for this action; since $p$ is odd, we have $(\mathfrak{M}_{N,r,k})_\mathfrak{m} = (\mathfrak{M}_{N,r,k})^+_\mathfrak{m} \oplus (\mathfrak{M}_{N,r,k})^-_\mathfrak{m}$.

The following proposition is fundamental to the construction of the canonical period.

**Proposition 3.1.1.** *Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}_{N,r,k}$ whose residual representation is irreducible and $p$-distinguished. Then $(\mathfrak{M}_{N,r,k})^{\pm}_{\mathfrak{m}}$ is a free $(\mathbf{T}_{N,r,k})_{\mathfrak{m}}$-module of rank 1. In particular, there exists an isomorphism of $(\mathbf{T}_{N,r,k})_{\mathfrak{m}}$-modules*

$$(\mathfrak{M}_{N,r,k})^{\pm}_{\mathfrak{m}} \overset{\alpha_{N,r,k}}{\cong} (\mathbf{T}_{N,r,k})_{\mathfrak{m}} \tag{3.2}$$

*(although $\alpha_{N,r,k}$ is only well-defined up to a $p$-adic unit).*

*Proof.* We consider first the case $k = 2$ and $r = 1$. Set $\mathfrak{M} := \mathfrak{M}_{N,1,2}$ and $\mathbf{T} := \mathbf{T}_{N,1,2}$. Let $J_1(Np)$ denote the Jacobian of $X_1(Np)$. By [37, Thm. 2.1], the ordinary $\mathfrak{m}$-torsion $J_1(Np)(\bar{\mathbf{Q}})[\mathfrak{m}]^{\mathrm{ord}}$ is free of rank 2 over $\mathbf{T}/\mathfrak{m}$. Since

$$J_1(Np)(\bar{\mathbf{Q}})[\mathfrak{m}] \cong H^1(Np; \mathbf{F}_p)[\mathfrak{m}],$$

by dualizing, we have that $(\mathfrak{M}_N)_{\mathfrak{m}} \otimes \mathbf{T}/\mathfrak{m}$ is free of rank 2 over $\mathbf{T}/\mathfrak{m}$. As $(\mathfrak{M}_N)_{\mathfrak{m}} \otimes \mathbf{Q}_p$ is free of rank 2 over $\mathbf{T}_{\mathfrak{m}} \otimes \mathbf{Q}_p$, it follows that $(\mathfrak{M}_N)_{\mathfrak{m}}$ is free of rank 2 over $\mathbf{T}_{\mathfrak{m}}$. Passing to $\pm$-subspaces establishes the proposition for $k = 2$.

The case $k > 2$ or $r > 1$ can be deduced from the case $k = 2$ and $r = 1$ by Hida theory. We will give the details in Section 3.3 immediately after Proposition 3.3.1. □

Fix a $p$-ordinary and $p$-distinguished eigenform $f$ defined over $\bar{\mathbf{Q}}_p$ of weight $k$ on $\Gamma_1(Np^r)$ whose corresponding maximal ideal in $\mathbf{T}_{N,r,k}$ is $\mathfrak{m}$. By the duality between spaces of cusp forms and Hecke algebras, the eigenform $f$ gives rise to a morphism of $\mathbf{Z}_p$-algebras

$$(\mathbf{T}_{N,r,k})_{\mathfrak{m}} \to \mathcal{O}(f); \tag{3.3}$$

here $\mathcal{O}(f)$ is the finite extension of $\mathbf{Z}_p$ in $\bar{\mathbf{Q}}_p$ generated by the Fourier coefficients of $f$. Composing (3.3) with the fixed map $\alpha_{N,r,k}$ of (3.2) then yields an $\mathcal{O}(f)$-valued functional

$$H_1(Np^r; \tilde{L}_k(\mathcal{O}(f)))^{\pm}_{\mathfrak{m}} \xrightarrow{\delta_f^{\pm}} \mathcal{O}(f).$$

On the other hand, integration yields a map

$$H_1(Np^r; \tilde{L}_k(\mathbf{C}))^{\pm} \xrightarrow{\omega_f^{\pm}} \mathbf{C}.$$

If we fix an isomorphism $\imath : \bar{\mathbf{Q}}_p \cong \mathbf{C}$, then this integration map factors through $H_1(Np^r; \tilde{L}_k(\mathbf{C}))^{\pm}_{\mathfrak{m}'}$, where $\mathfrak{m}'$ is the maximal ideal in $\mathbf{T}_{N,r,k} \otimes_{\mathbf{Z}_p,\imath} \mathbf{C}$ obtained as the kernel of the $\mathbf{C}$-algebra map

$$\mathbf{T}_{N,r,k} \otimes_{\mathbf{Z}_p,\imath} \mathbf{C} \to \mathbf{C}$$

induced by composing (3.3) with $\imath$. By the multiplicity one theorem, the functionals $\delta_f^{\pm}$ (extended to $\mathbf{C}$) and $\omega_f^{\pm}$ differ by a non-zero constant and we have

$$\omega_f^{\pm} = \Omega_f^{\pm} \cdot \delta_f^{\pm}$$

with $\Omega_f^{\pm} \in \mathbf{C}^{\times}$. The constant $\Omega_f^{\pm}$ is the canonical period of $f$. It is well-defined up to a $p$-adic unit. (The ambiguity arises from the choice of the map $\alpha_{N,r,k}$ in (3.2)).

**Remark 3.1.2.** The constructions given here are dual (but equivalent) to the ones given by Vatsal in [35], since here we are working with homology while Vatsal used cohomology.

3.2. **One variable $p$-adic $L$-functions.** The advantage of describing $(\mathfrak{M}_{N,r,k})_{\mathfrak{m}}$ as a relative homology group is that the corresponding relative homology classes admit a description via modular symbols. More precisely, for fixed $r$, there is a map

$$\mathbf{P}^1(\mathbf{Q}) \to H_1(Np^r, \{\text{cusps}\}; \tilde{L}_k(\mathbf{Z}_p))$$

defined by sending the element $a \in \mathbf{P}^1(\mathbf{Q})$ to the homology class corresponding to the image of the path $[\infty, a]$ on the modular curve $X_1(Np)$. By projecting to ordinary parts and localizing at $\mathfrak{m}$, we obtain a map

$$\mathbf{P}^1(\mathbf{Q}) \to (\mathfrak{M}_{N,r,k})_{\mathfrak{m}},$$

which we denote by $a \mapsto \{\infty, a\}$. This map allows us to define an $(\mathfrak{M}_{N,r,k})_{\mathfrak{m}}$-valued measure on $\mathbf{Z}_p^\times$ in the usual way.

**Definition 3.2.1.** For any open subset $a + p^r\mathbf{Z}_p$ of $\mathbf{Z}_p^\times$, we define

$$\mu_k(a + p^r\mathbf{Z}_p) = U_p^{-r}\{\infty, a/p^r\} \in (\mathfrak{M}_{N,r,k})_{\mathfrak{m}}.$$

**Proposition 3.2.2.** *The function $\mu_k$ is a measure (i.e. it is additive).*

*Proof.* This is standard. □

For $f$ an eigenform of weight $k$ on $\Gamma_1(Np^r)$, let $\wp_f$ denote the corresponding height one prime ideal of $\mathbf{T}_{N,r,k}$ and let $\mathfrak{m}$ be the unique maximal ideal containing $\wp_f$. By decomposing $\mu_k$ into its $\pm$-parts and applying the map $\alpha_{N,r,k}$ of (3.2), we may view $\mu_k^\pm$ as taking values in $\mathbf{T}_{N,r,k}$. Thus $\mu_k^\pm \bmod \wp_f$ is $\mathcal{O}(f)$-valued and moreover, it is precisely the $p$-adic $L$-function attached to $f$ computed with respect to the canonical period $\Omega_f^\pm$. That is, $\mu_k^\pm \bmod \wp_f$ interpolates the special values $L(f, \chi, 1)/\Omega_f^\pm$ where $\chi$ is a finite order character of $\mathbf{Z}_p^\times$ such that $\chi(-1) = \pm 1$.

3.3. **Canonical periods in families.** In [35], it is proven that if $f$ and $g$ are $p$-ordinary eigenforms of weight $k$ whose residual representations are isomorphic, then their respective $p$-adic $L$-functions are congruent. More precisely, it is proven that one can simultaneously choose canonical periods $\Omega_f$ and $\Omega_g$ so that the $p$-adic $L$-functions computed with respect to these periods are congruent.

We will now describe how one can simultaneously choose canonical periods for all modular forms in a Hida family so that the associated single variable $p$-adic $L$-functions vary $p$-adic analytically in $k$. These constructions are originally due to Mazur [28] and Kitigawa [22]. Using results of Wiles in [37], we have extended these constructions to a wider class of modular forms.

Let $\mathbf{T}_N$ be the universal ordinary Hecke algebra defined in Section 2 and let $\mathbf{T}_N^*$ be the corresponding Hecke algebra defined with respect to all ordinary modular forms rather than just cusp forms. Let $\mathfrak{m} \subseteq \mathbf{T}_N^*$ be a maximal ideal whose residual representation is irreducible.

For each $r \geq 0$, we have an isomorphism

$$H_1(Np^r; \mathbf{Z}_p)_{\mathfrak{m}}^{\text{ord}} \cong H_1(Np^r, \{\text{cusps}\}; \mathbf{Z}_p)_{\mathfrak{m}}^{\text{ord}}$$

and passing to the projective limit in $r$, we obtain a corresponding isomorphism of $(\mathbf{T}_N)_{\mathfrak{m}} \cong (\mathbf{T}_N)^*_{\mathfrak{m}}$-modules

$$\varprojlim_r H_1(Np^r; \mathbf{Z}_p)_{\mathfrak{m}}^{\text{ord}} \cong \varprojlim_r H_1(Np^r, \{\text{cusps}\}; \mathbf{Z}_p)_{\mathfrak{m}}^{\text{ord}}.$$

We denote these isomorphic modules by $(\mathfrak{M}_N)_\mathfrak{m}$. As before, there is an action of complex conjugation that yields a decomposition $(\mathfrak{M}_N)_\mathfrak{m} = (\mathfrak{M}_N)^+_\mathfrak{m} \oplus (\mathfrak{M}_N)^-_\mathfrak{m}$.

The following proposition is the Hida-theoretic analogue of Proposition 3.1.1.

**Proposition 3.3.1.** *If $\bar{\rho}_\mathfrak{m}$ is irreducible and $p$-distinguished, then $(\mathfrak{M}_N)^\pm_\mathfrak{m}$ is a free $(\mathbf{T}_N)_\mathfrak{m}$-module of rank 1.*

*Proof.* Let $\wp_2$ denote the product of all classical height one primes $\wp$ of weight 2 in $\mathbf{T}_N$ such that $\kappa_\wp$ is trivial. Then, by [17, Thm. 3.1],

$$(\mathfrak{M}_N)_\mathfrak{m} \otimes \mathbf{T}_N/\wp_2 \cong (\mathfrak{M}_{N,1,2})_{\mathfrak{m}_2} \tag{3.4}$$

where $\mathfrak{m}_2$ is the maximal ideal in $\mathbf{T}_{N,1,2}$ induced by $\mathfrak{m}$. By Proposition 3.1.1 (applied in the case $k=2$ and $r=1$), we know that $(\mathfrak{M}_{N,1,2})_{\mathfrak{m}_2}$ is a free $\mathbf{T}_{N,1,2} \cong \mathbf{T}_N/\wp_2$-module of rank 2. Since $(\mathfrak{M}_N)_\mathfrak{m} \otimes \mathbf{Q}_p$ is free of rank 2 over $(\mathbf{T}_N)_\mathfrak{m} \otimes \mathbf{Q}_p$, it follows that $(\mathfrak{M}_N)_\mathfrak{m}$ is free of rank 2 over $(\mathbf{T}_N)_\mathfrak{m}$. Passing to $\pm$-subspaces then establishes the proposition. □

Using Proposition 3.3.1, we now complete the proof of Proposition 3.1.1 for arbitrary $k$ and $r$. (Note that the proof of Proposition 3.3.1 only relied upon Proposition 3.1.1 in the already established case of $k=2$ and $r=1$ and so these arguments are not circular.)

*Proof of Proposition 3.1.1.* Let $\wp_{N,r,k}$ denote the product of all classical height one primes $\wp$ of weight $k$ in $\mathbf{T}_N$ such that $\kappa_\wp$ restricted to $1+p^r\mathbf{Z}_p$ is trivial. Then, by [19, Theorem 1.9],

$$(\mathfrak{M}_N)_\mathfrak{m} \otimes \mathbf{T}_N/\wp_{N,r,k} \cong (\mathfrak{M}_{N,r,k})_{\mathfrak{m}_{N,r,k}} \tag{3.5}$$

where $\mathfrak{m}_{N,r,k}$ is the maximal ideal in $\mathbf{T}_{N,r,k}$ induced by $\mathfrak{m}$. By Proposition 3.3.1, $(\mathfrak{M}_N)_\mathfrak{m}$ is free of rank two over $\mathbf{T}_N$. Thus $(\mathfrak{M}_{N,r,k})_{\mathfrak{m}_{N,r,k}}$ is free of rank two over $\mathbf{T}_{N,r,k} \cong \mathbf{T}_N/\wp_{N,r,k}$. Passing to $\pm$-subspace then completes the proof. □

We now assume that the hypotheses of Proposition 3.3.1 are satisfied. Thus we may choose an isomorphism

$$(\mathbf{T}_N)_\mathfrak{m} \overset{\alpha_N}{\cong} (\mathfrak{M}_N)^\pm_\mathfrak{m}. \tag{3.6}$$

Using the notation of the above proof, if we consider $\alpha_N$ modulo $\wp_{N,r,k}$, we get an isomorphism

$$(\mathbf{T}_{N,r,k})_{\mathfrak{m}_{N,r,k}} \overset{\alpha_{N,r,k}}{\cong} (\mathfrak{M}^\pm_{N,r,k})_{\mathfrak{m}_{N,r,k}}.$$

For a given modular form $f$ of level $Np^r$ and of weight $k$ in the Hida family corresponding to $(\mathbf{T}_N)_\mathfrak{m}$, we then define its canonical period to be the one computed with respect to $\alpha_{N,r,k}$.

Note that the usual ambiguity of the canonical period coming from the choice of isomorphism $\alpha_{N,r,k}$ is controlled across the entire Hida family by the single choice of $\alpha_N$ identifying $(\mathfrak{M}_N)^\pm_\mathfrak{m}$ with $(\mathbf{T}_N)_\mathfrak{m}$. In the following section, we show that under these choices of canonical periods, the $p$-adic $L$-functions enjoy good congruence properties; that is, they interpolate into a two-variable $p$-adic $L$-function.

3.4. **Two variable $p$-adic $L$-functions.** Recall, that for each $r \geq 1$, we have a map

$$\mathbf{P}^1(\mathbf{Q}) \to H_1(Np^r, \{\text{cusps}\}; \mathbf{Z}_p).$$

This map is compatible with varying $r$ and so projecting to ordinary parts, localizing at $\mathfrak{m}$ and passing to the limit, we obtain a map

$$\mathbf{P}^1(\mathbf{Q}) \to (\mathfrak{M}_N)_{\mathfrak{m}},$$

which we again denote by $a \mapsto \{\infty, a\}$.

**Definition 3.4.1.** For any open subset $a + p^r\mathbf{Z}_p$ of $\mathbf{Z}_p^\times$, we define

$$\mu(a + p^r\mathbf{Z}_p) = U_p^{-r}\{\infty, a/p^r\} \in (\mathfrak{M}_N)_{\mathfrak{m}}.$$

**Proposition 3.4.2.** *The function $\mu$ is a measure (i.e. it is additive).*

*Proof.* This is standard. □

Recall that the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ may naturally be regarded as the space of $\mathbf{Z}_p$-valued measures on $\mathbf{Z}_p^\times$. We may thus regard $\mu$ as defining an element

$$L(\mathfrak{m}, N) \in (\mathfrak{M}_N)_{\mathfrak{m}} \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\mathbf{Z}_p^\times]].$$

(Here the tensor product is completed with respect to the usual (i.e. profinite topology) on $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ and the $\mathfrak{m}$-adic topology on $(\mathfrak{M}_N)_{\mathfrak{m}}$.) We may decompose $L(\mathfrak{m}, N)$ under the action of complex conjugation to get a pair of elements

$$L^\pm(\mathfrak{m}, N) \in (\mathfrak{M}_N)_{\mathfrak{m}}^\pm \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\mathbf{Z}_p^\times]].$$

We now assume that the hypotheses of Proposition 3.3.1 are satisfied. Thus we may identify $(\mathbf{T}_N)_{\mathfrak{m}}$ with $(\mathfrak{M}_N)_{\mathfrak{m}}^\pm$ and so regard

$$L^\pm(\mathfrak{m}, N) \in (\mathbf{T}_N)_{\mathfrak{m}} \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\mathbf{Z}_p^\times]] \cong (\mathbf{T}_N)_{\mathfrak{m}}[[\mathbf{Z}_p^\times]].$$

We will refer to $L^\pm(\mathfrak{m}, N)$ as the *two-variable $p$-adic $L$-function attached to* $(\mathbf{T}_N)_{\mathfrak{m}}$. Note that it is well-defined up to a unit in $(\mathbf{T}_N)_{\mathfrak{m}}$.

If we fix a tame character $\omega^i$ for some $0 \leq i \leq p-2$, then we may project $L^\pm(\mathfrak{m}, N)$ onto the "$\omega^i$-part" of $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ and obtain

$$L(\mathfrak{m}, N, \omega^i) \in (\mathbf{T}_N)_{\mathfrak{m}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}.$$

(Here we choose the sign $\pm$ to equal $(-1)^i$, since it is well-known, and easily checked, that otherwise the projection is trivial.)

For any height one prime $\wp$ in $(\mathbf{T}_N)_{\mathfrak{m}}$, we may reduce $L^\pm(\mathfrak{m}, N, \omega^i)$ modulo $\wp$ to obtain an element

$$L^\pm(\mathfrak{m}, N, \omega^i)(\wp) \in ((\mathbf{T}_N)_{\mathfrak{m}}/\wp) \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)} \cong \mathcal{O}(\wp) \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)},$$

which is well-defined up to a unit in $\mathcal{O}(\wp)$. Also, we may reduce modulo $\mathfrak{m}$ to obtain an element

$$\bar{L}^\pm(\mathfrak{m}, N, \omega^i) \in ((\mathbf{T}_N)_{\mathfrak{m}}/\mathfrak{m}) \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)} \cong k \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)},$$

which is well-defined up to an element of $k^\times$.

The following proposition describes the manner in which the two-variable $p$-adic $L$-function $L^\pm(\mathfrak{m}, N, \omega^i)$ interpolates the one-variable $p$-adic $L$-functions of the modular forms in the Hida family corresponding to $(\mathbf{T}_N)_{\mathfrak{m}}$.

**Proposition 3.4.3.** *If $\wp$ is a classical height one prime ideal in $(\mathbf{T}_N)_{\mathfrak{m}}$, then*

$$L^{\pm}(\mathfrak{m}, N, \omega^i)(\wp) \in \mathcal{O}(\wp)\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{(i)}$$

*is the usual analytic $p$-adic $L$-function attached to the corresponding normalized eigenform*

$$f_{\wp} \in S_k(Np^{\infty}, \mathcal{O}(\wp))^{\mathrm{ord}}[\kappa_{\wp}]$$

*(computed with respect to the canonical period described in Section 3.3).*

*Proof.* This comparison follows immediately from our given constructions of one and two-variable $p$-adic $L$-functions together with the isomorphism in (3.5). □

**Remark 3.4.4.** Note that the normalized eigenform $f_{\wp}$ appearing in the statement of Proposition 3.4.3 need not be a newform and thus $L^{\pm}(\mathfrak{m}, N, \omega^i)(\wp)$ may not be a "primitive" $p$-adic $L$-function. This issue will be explored in the following section.

**Remark 3.4.5.** The following remark answers a question asked by the referee; it will not be used elsewhere in the paper.

Note that the formula of Definition 3.4.1 works equally well to define an $(\mathfrak{M}_N)_{\mathfrak{m}}$-valued measure on $\mathbf{Z}_p$. Integrating the constant function 1 on $\mathbf{Z}_p$ with respect to this measure yields an element of $(\mathfrak{M}_N)_{\mathfrak{m}}$; explicitly, this is just the element $\{\infty, 1\} \in (\mathfrak{M}_N)_{\mathfrak{m}}$. This element in fact lies in $(\mathfrak{M}_N)^+_{\mathfrak{m}}$, and thus (via our chosen identification of $(\mathfrak{M}_N)^+_{\mathfrak{m}}$ with $(\mathbf{T}_N)_{\mathfrak{m}}$) gives rise to an element of $(\mathbf{T}_N)_{\mathfrak{m}}$, which we denote by $L^*(\mathfrak{m}, N, 1)$. Any easy computation (using the fact that $\mathbf{Z}_p = \mathbf{Z}_p^{\times} \bigcup p\mathbf{Z}_p$) shows that $(1 - U_p^{-1})L^*(\mathfrak{m}, N, 1) = L^+(\mathfrak{m}, N, 1)$. (Note that the '1' in the third argument stands for the trivial character $\omega^0$.) Thus $L^*(\mathfrak{m}, N, 1)$ coincides with the "improved $p$-adic $L$-function" of [14], computed with respect to a canonical period.

3.5. **Two variable $p$-adic $L$-functions on branches of the Hida family.** Fix an irreducible modular Galois representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ that is both $p$-ordinary and $p$-distinguished. We choose also a $p$-stabilization on $\bar{\rho}$ (which we will suppress in our discussion and notation). Furthermore, fix a finite set of primes $\Sigma$ and consider the Hecke algebra $\mathbf{T}_{\Sigma}$ associated to $\bar{\rho}$ as in Section 2.4.

Proposition 2.4.2 yields an isomorphism $\mathbf{T}_{\Sigma} \cong (\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ for a certain maximal ideal $\mathfrak{n}$ of $\mathbf{T}_{N(\Sigma)}$. The construction of the preceding section thus defines a two-variable $p$-adic $L$-function

$$L(\mathfrak{n}, N(\Sigma), \omega^i) \in (\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{(i)}.$$

The isomorphism $\mathbf{T}_{\Sigma} \cong (\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ then yields elements

$$\begin{aligned} L_{\Sigma}(\bar{\rho}, \omega^i) &\in \mathbf{T}_{\Sigma}\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{(i)}, \\ L_{\Sigma}(\bar{\rho}, \omega^i)(\wp) &\in \mathcal{O}(\wp) \otimes_{\mathbf{Z}_p} \Lambda_{(i)}, \\ \overline{L}_{\Sigma}(\bar{\rho}, \omega^i) &\in k\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{(i)}, \end{aligned}$$

as defined in the previous section, which are well-defined up to units in $\mathbf{T}_{\Sigma}$, $\mathcal{O}(\wp)$ and $k$ respectively.

We now note that there are two different ways that one can naturally attach a modular form to a classical height one prime ideal $\wp$ of $\mathbf{T}_{\Sigma}$. First, via the isomorphism of Proposition 2.4.2, this prime corresponds to a classical height one prime ideal of $\mathbf{T}_{N(\Sigma)}$ which we again call $\wp$. By part (2) of Theorem 2.1.1, this prime gives rise to a normalized eigenform $f_{\wp}$ of tame level $N(\Sigma)$. On the other hand, by Proposition 2.5.6, $\wp$ also corresponds to a classical height one prime of $\mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}$

which we denote by $\wp^{\mathrm{new}}$. (Here $\mathfrak{a}$ is the unique minimal prime of $\mathbf{T}_\Sigma$ contained in $\wp$.) By part (2) of Theorem 2.1.3, this prime gives rise to a normalized newform $f_\wp^{\mathrm{new}} := f_{\wp^{\mathrm{new}}}$ of tame level $N(\mathfrak{a})$.

In general, $f_\wp$ and $f_\wp^{\mathrm{new}}$ are not equal. Indeed, they are eigenforms for different Hecke algebras: $f_\wp^{\mathrm{new}}$ is an eigenform for $\mathbf{T}_{N(\mathfrak{a})}$ while $f_\wp$ is an eigenform for $\mathbf{T}_{N(\Sigma)}$. Proposition 2.4.2 makes it clear how they are related: $f_\wp$ is the normalized oldform obtained from $f_\wp^{\mathrm{new}}$ by "removing the Euler factor" at each of the primes $\ell \in \Sigma$.

Note that Proposition 3.4.3 shows that $L_\Sigma(\bar{\rho}, \omega^i)$ interpolates the $p$-adic $L$-functions attached to the modular forms $f_\wp$ for $\wp$ a classical height one prime of $\mathbf{T}_\Sigma$. In particular, these are "non-primitive" $p$-adic $L$-functions.

We now seek to construct a two variable $p$-adic $L$-function for each minimal prime $\mathfrak{a}$ of $\mathbf{T}_\Sigma$, which interpolates the $p$-adic $L$-functions of the modular forms $f_\wp^{\mathrm{new}}$ as $\wp^{\mathrm{new}}$ runs through the classical height one primes of $\mathbf{T}(\mathfrak{a})$. For this, we recall the natural isomorphism $\mathbf{T}(\mathfrak{a})^\circ \cong \mathbf{T}_{N(\mathfrak{a})}^{\mathrm{new}}/\mathfrak{a}'$ of Definition 2.5.3, which gives rise to the composite surjection

$$\mathbf{T}_{N(\mathfrak{a})} \to \mathbf{T}_{N(\mathfrak{a})}^{\mathrm{new}} \to \mathbf{T}_{N(\mathfrak{a})}^{\mathrm{new}}/\mathfrak{a}' \cong \mathbf{T}(\mathfrak{a})^\circ. \tag{3.7}$$

If we let $\mathfrak{m}$ denote the maximal ideal of $\mathbf{T}_{N(\mathfrak{a})}$ obtained as the preimage of the maximal ideal of $\mathbf{T}(\mathfrak{a})^\circ$ under (3.7), then the construction of the preceding section yields an $L$-function $L(\mathfrak{m}, N(\mathfrak{a}), \omega^i) \in \mathbf{T}_{N(\mathfrak{a})} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}$. The surjection (3.7) induces a corresponding surjection

$$\mathbf{T}_{N(\mathfrak{a})} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)} \to \mathbf{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}.$$

We then denote the image of $L(\mathfrak{m}, N(\mathfrak{a}), \omega^i)$ under this surjection by

$$L(\bar{\rho}, \mathfrak{a}, \omega^i) \in \mathbf{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}$$

and we refer to $L(\bar{\rho}, \mathfrak{a}, \omega^i)$ as the *two variable $p$-adic $L$-function attached to* $\mathbf{T}(\mathfrak{a})$.

The following proposition describes the interpolation property of $L(\bar{\rho}, \mathfrak{a}, \omega^i)$.

**Proposition 3.5.1.** *If $\wp^{\mathrm{new}}$ is a classical height one prime ideal in $\mathbf{T}(\mathfrak{a})$,*

$$L(\bar{\rho}, \mathfrak{a}, \omega^i)(\wp^{\mathrm{new}}) \in \mathcal{O}(\wp^{\mathrm{new}}) \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}$$

*equals the $p$-adic $L$-function of $f_\wp^{\mathrm{new}} := f_{\wp^{\mathrm{new}}}$ computed with respect to a canonical period.*

*Proof.* This proposition follows immediately from Proposition 3.4.3. □

3.6. **Comparisons.** We now have a two-variable $p$-adic $L$-function $L_\Sigma(\bar{\rho}, \omega^i)$ defined over the entire Hida algebra $\mathbf{T}_\Sigma$ and, for each minimal prime $\mathfrak{a}$ of $\mathbf{T}_\Sigma$, we have a two-variable $p$-adic $L$-function $L(\bar{\rho}, \mathfrak{a}, \omega^i)$ defined on the branch $\mathbf{T}(\mathfrak{a})$. It is natural then to compare $L(\bar{\rho}, \mathfrak{a}, \omega^i)$ with $L_\Sigma(\bar{\rho}, \omega^i)$ modulo $\mathfrak{a}$.

Each of these $p$-adic $L$-functions lies in $\mathbf{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}$. The interpolation property satisfied by classical $p$-adic $L$-functions shows that for any height one prime $\wp$ of $\mathbf{T}_\Sigma$ containing $\mathfrak{a}$, the $L$-functions $L_p(f_\wp^{\mathrm{new}})$ and $L_p(f_\wp)$ agree up to multiplication by a certain product of Euler factors (reflecting the Euler factors removed from $f_\wp^{\mathrm{new}}$ to obtain $f_\wp$) and an element of $\mathcal{O}(\wp)[1/p]^\times$ (reflecting the ratio of the canonical period of $f_\wp^{\mathrm{new}}$ and the canonical period of $f_\wp$).

We will show that in fact the ratio of these canonical periods lies in $(\mathcal{O}(\wp)^\circ)^\times$. Hence, $L_p(f_\wp^{\mathrm{new}})$ and $L_p(f_\wp)$ agree up to multiplication by a product of Euler factors and the inevitable ambiguity of a unit of $\mathcal{O}(\wp)^\circ$. Furthermore, we will show that this occurs uniformly in Hida families; in other words, that $L_\Sigma(\bar{\rho}, \omega^i) \bmod \mathfrak{a}$ and $L(\bar{\rho}, \mathfrak{a}, \omega^i)$ agree up to multiplication by a product of Euler factors and a unit in $\mathbf{T}(\mathfrak{a})^\circ$.

We begin by defining the relevant Euler factors. Recall the Euler factors $E_\ell(\mathfrak{a}, X)$ in $\mathbf{T}(\mathfrak{a})^\circ[X]$ defined in Section 2.7. Since $\ell$ is prime to $p$, we may regard $\ell$ as an element of $\mathbf{Z}_p^\times$ and so obtain a corresponding unit element $\langle \ell \rangle_p \in \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$. Thus

$$E_\ell(\mathfrak{a}, \langle \ell \rangle_p^{-1}) \in \mathbf{T}(\mathfrak{a})^\circ[[\mathbf{Z}_p^\times]] \cong \mathbf{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\mathbf{Z}_p^\times]].$$

For $0 \le i \le p-2$, we write $\langle \ell \rangle_{p,i}$ to denote the projection of $\langle \ell \rangle_p \in \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ under the surjection $\mathbf{Z}_p[[\mathbf{Z}_p^\times]] \to \Lambda_{(i)}$. We write $E_\ell(\mathfrak{a}, \langle \ell \rangle_{p,i}^{-1})$ to denote the corresponding element of $\mathbf{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}$.

**Definition 3.6.1.** If $\mathfrak{a}$ is a minimal prime of $\mathbf{T}_\Sigma$, then we write

$$E_\Sigma(\mathfrak{a}) := \prod_{\ell \in \Sigma} E_\ell(\mathfrak{a}, \langle \ell \rangle_p^{-1}) \in T(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$$

and

$$E_\Sigma(\mathfrak{a}, \omega^i) := \prod_{\ell \in \Sigma} E_\ell(\mathfrak{a}, \langle \ell \rangle_{p,i}^{-1}) \in T(\mathfrak{a})^\circ \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{(i)}.$$

Since $L_\Sigma(\bar{\rho}, \omega^i)$ and $L(\bar{\rho}, \mathfrak{a}, \omega^i)$ are constructed using modular symbols of levels $N(\Sigma)$ and $N(\mathfrak{a})$ respectively, in order to compare them, we will need to be able to compare the corresponding Hecke algebras. Recall that by construction $\mathbf{T}_\Sigma = (\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'}$ for a certain maximal ideal $\mathfrak{m}'$ of $\mathbf{T}'_{N(\Sigma)}$ and that $\mathfrak{n}$ is a certain maximal ideal of $\mathbf{T}_{N(\Sigma)}$ lying over $\mathfrak{m}'$ with the property that the map $(\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'} \to (\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ is an isomorphism. We let $\mathfrak{m}$ denote the preimage in $\mathbf{T}_{N(\mathfrak{a})}$, under the surjection (3.7), of the maximal ideal of $\mathbf{T}(\mathfrak{a})^\circ$. Altogether, we have the following diagram of maps between the various Hecke algebras:

$$\begin{tikzcd} (\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'} \arrow[r, "\cong"] \arrow[d] & (\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}} \\ (\mathbf{T}_{N(\mathfrak{a})})_{\mathfrak{m}} \arrow[r] & \mathbf{T}(\mathfrak{a})^\circ. \end{tikzcd} \tag{3.8}$$

By inverting the upper horizontal isomorphism, we obtain a map $(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}} \to \mathbf{T}(\mathfrak{a})^\circ$. This is a homomorphism of $\mathbf{T}'_{N(\Sigma)}$-algebras when we equip the source and target with the $\mathbf{T}'_{N(\Sigma)}$-algebra structure provided by the diagram (3.8).

We write $(\mathfrak{M}_{N(\Sigma)})_{\mathfrak{n}}$ and $(\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}}$ to denote the modules of $p$-adic modular symbols constructed in Section 3.3 for the indicated choice of tame level and localized at the indicated maximal ideal. The following result provides the key to comparing the $L$-functions $L_\Sigma(\bar{\rho}, \omega^i) \bmod \mathfrak{a}$ and $L(\bar{\rho}, \mathfrak{a}, \omega^i)$. We postpone its proof to the Section 3.8.

**Theorem 3.6.2.** *There is an isomorphism of* $\mathbf{T}(\mathfrak{a})^\circ$*-modules*

$$\mathbf{T}(\mathfrak{a})^\circ \otimes_{(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}} (\mathfrak{M}_{N(\Sigma)})_{\mathfrak{n}} \cong \mathbf{T}(\mathfrak{a})^\circ \otimes_{(\mathbf{T}_{N(\mathfrak{a})})_{\mathfrak{m}}} (\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}},$$

*compatible with the action of complex conjugation.*

*Moreover, the induced map on the corresponding spaces of measures taking values in these modules sends the measure* $L(\mathfrak{n}, N(\Sigma))$*, which takes values in the source of* *the isomorphism, to the measure $L(\mathfrak{m}, N(\mathfrak{a}))E_\Sigma(\mathfrak{a})$, which takes values in the target of the isomorphism.*

**Corollary 3.6.3.** *There is a unit $u \in \mathbf{T}(\mathfrak{a})^\circ$ such that*

$$L_\Sigma(\bar\rho, \omega^i) \equiv u \cdot L(\bar\rho, \mathfrak{a}, \omega^i) E_\Sigma(\mathfrak{a}, \omega^i) \pmod{\mathfrak{a}}.$$

*Proof.* This follows immediately from the preceding theorem. □

3.7. **Iwasawa invariants.** If we choose an isomorphism

$$\Lambda \cong \mathbf{Z}_p[[T]], \tag{3.9}$$

and hence an isomorphism $(\mathbf{T}_N)_\mathfrak{m} \hat\otimes_{\mathbf{Z}_p} \Lambda_{(i)} \cong (\mathbf{T}_N)_\mathfrak{m}[[T]]$, we may then regard $L(\mathfrak{m}, N, \omega^i)$ as an element of $\mathbf{T}_\mathfrak{m}[[T]]$. We are, of course, interested in the Iwasawa invariants of such power series. We begin this section by reviewing the definitions and basic properties of such invariants.

**Definition 3.7.1.** If $R$ is a ring and $f(T) \in R[[T]]$ is a one-variable power series with coefficients in $R$, then we define the *content* of $f(T)$ to be the ideal $I(f(T)) \subset R$ generated by the coefficients of $f(T)$.

The proof of the next lemma is straightforward.

**Lemma 3.7.2.** *Let $f(T) \in R[[T]]$.*

(1) *If $\alpha$ is any automorphism of $R[[T]]$, then $I(f(T)) = I(\alpha(f(T)))$.*
(2) *If $u(T) \in R[[T]]^\times$, then $I(u(T)f(T)) = I(f(T))$.*
(3) *If $f(T) = g(T)h(T)$ in $R[[T]]$, then if any two of these three power series have unit content, so does the third.*
(4) *If $\phi : R \to S$ is a morphism of rings then $I(\phi(f(T))) = \phi(I(f(T)))$.*

In particular, regarding $L(\mathfrak{m}, N, \omega^i)$ as a power series, we may consider its content $I(L(\mathfrak{m}, N, \omega^i))$, an ideal of $(\mathbf{T}_N)_\mathfrak{m}$. Parts (1) and (2) of the preceding lemma show that this ideal is in fact independent of the choice of the isomorphisms (3.6) and (3.9).

For each height one prime $\wp$, we may likewise form the corresponding ideal $I(L(\mathfrak{m}, N, \omega^i)(\wp))$ in $\mathcal{O}(\wp)$. By part (4) of Lemma 3.7.2, this ideal equals the image of $I(L(\mathfrak{m}, N, \omega^i))$ under the surjection $(\mathbf{T}_N)_\mathfrak{m} \to (\mathbf{T}_N)_\mathfrak{m}/\wp \cong \mathcal{O}(\wp)$. We also have that $I(\bar L(\mathfrak{m}, N, \omega^i))$ is an ideal of $k$ and is thus either zero or all of $k$. Again, we have that this ideal equals the image of $I(L(\mathfrak{m}, N, \omega^i))$ under the surjection $(\mathbf{T}_N)_\mathfrak{m} \to (\mathbf{T}_N)_\mathfrak{m}/\mathfrak{m} \cong k$.

The next result is an immediate consequence of the fact that $(\mathbf{T}_N)_\mathfrak{m}$ is a local ring.

**Proposition 3.7.3.** *Fix $i$ such that $0 \le i \le p-2$. The following are equivalent:*

*(1)* $I(L(\mathfrak{m}, N, \omega^i)) = (\mathbf{T}_N)_\mathfrak{m}$.
*(2)* $I(L(\mathfrak{m}, N, \omega^i)(\wp)) = \mathcal{O}(\wp)$ *for some height one prime $\wp$ of* $(\mathbf{T}_N)_\mathfrak{m}$.
*(3)* $I(L(\mathfrak{m}, N, \omega^i)(\wp)) = \mathcal{O}(\wp)$ *for every height one prime $\wp$ of* $(\mathbf{T}_N)_\mathfrak{m}$.
*(4)* $I(\bar L(\mathfrak{m}, N, \omega^i))$ *is non-zero (and hence equal to $k$).*

We remark that if $\wp$ is a classical height one prime ideal in $(\mathbf{T}_N)_\mathfrak{m}$, then the length of the quotient $\mathcal{O}(\wp)/I(L(\mathfrak{m}, N, \omega^i)(\wp))$ is related to the usual $\mu$-invariant of the $p$-adic $L$-function $L(\mathfrak{m}, N, \omega^i)(\wp) \in \mathcal{O}(\wp)[[T]]$. In particular, this $\mu$-invariant vanishes if and only if $I(L(\mathfrak{m}, N, \omega^i)(\wp))$ is the unit ideal of $\mathcal{O}(\wp)$.

Analogously, we have well-defined ideals

$$I(L_\Sigma(\bar\rho,\omega^i)), I(L_\Sigma(\bar\rho,\omega^i)(\wp)) \text{ and } I(\overline{L}_\Sigma(\bar\rho,\omega^i))$$

of $\mathbf{T}_\Sigma$, $\mathcal{O}(\wp)$ and $k$ respectively. The above result applies equally well to these ideals since they are constructed out of $I(L(\mathfrak{m},N,\omega^i))$ for some choice of $\mathfrak{m}$ and $N$.

We now compute the content of the reciprocal Euler factor $E_\Sigma(\mathfrak{a},\omega^i)$ defined in Section 3.6.

**Lemma 3.7.4.** *The element $E_\Sigma(\mathfrak{a},\omega^i)$ of $T(\mathfrak{a})^\circ\hat\otimes_{\mathbf{Z}_p}\Lambda_{(i)}$ has unit content.*

*Proof.* Part (3) of Lemma 3.7.2 shows that it suffices to prove that each of the reciprocal Euler factors $E_\ell(\mathfrak{a},\langle\ell\rangle_{p,i}^{-1})$ has unit content. If $\gamma$ is a topological generator of $\Gamma$, then we may write $\ell^{-1}\omega(\ell)=\gamma^{up^n}$ for some $u\in\mathbf{Z}_p^\times$ and some integer $n\geq 0$. If we choose our isomorphism (3.9) so that $\langle\gamma\rangle_p\mapsto 1+T$, then we see that

$$E_\ell(\mathfrak{a},\langle\ell\rangle_{p,i}^{-1}) = \begin{cases} (1-(T_\ell \bmod \mathfrak{a}')\omega^{-i}(\ell)(1+T)^{up^n}+\langle\ell\rangle_{N(\mathfrak{a})p}\omega^{-2i}(\ell)\ell^{-1}(1+T)^{2up^n}) \\ \quad \text{if } \ell \text{ is prime to } N(\mathfrak{a}) \\ (1-(T_\ell \bmod \mathfrak{a}')\omega^{-i}(\ell)(1+T)^{up^n}) \text{ otherwise.} \end{cases}$$

In the second case either the constant term is a unit (if $T_\ell$ is not a unit) or else the coefficient of $T^{p^n}$ is a unit (if $T_\ell$ is a unit). In the first case, we may compute the content after making the substitution $(1+T)^u\mapsto 1+T$, in which case one sees immediately that the coefficient of $T^{2p^n}$ is a unit. $\square$

The following theorem establishes that if the $\mu$-invariant vanishes for one form in a Hida family, then it vanishes for every form in that family.

**Theorem 3.7.5.** *The following are equivalent:*

1. $\mu(L_p(f,\omega^i))=0$ *for some newform $f$ in the Hida family of $\bar\rho$.*
2. $\mu(L_p(f,\omega^i))=0$ *for every newform $f$ in the Hida family of $\bar\rho$.*
3. $L(\bar\rho,\mathfrak{a},\omega^i)$ *has unit content for some irreducible component $\mathbf{T}(\mathfrak{a})$ of the Hida family of $\bar\rho$.*
4. $L(\bar\rho,\mathfrak{a},\omega^i)$ *has unit content for every irreducible component $\mathbf{T}(\mathfrak{a})$ of the Hida family of $\bar\rho$.*

*Proof.* Let $\mathbf{T}(\mathfrak{a})$ be an irreducible component of the Hida family of $\bar\rho$ and let $\wp$ be a classical height one prime ideal of $\mathbf{T}(\mathfrak{a})$. Since $L(\bar\rho,\mathfrak{a},\omega^i)$ modulo $\wp$ equals $L_p(f_\wp,\omega^i)$ and $\mathbf{T}(\mathfrak{a})^\circ$ is a local ring, we deduce from Proposition 3.7.3 that the following are equivalent: $L(\bar\rho,\mathfrak{a},\omega^i)$ has unit content; $L_p(f_\wp,\omega^i)$ has unit content (i.e. trivial $\mu$-invariant) for one classical height one prime $\wp$ of $\mathbf{T}(\mathfrak{a})$; $L_p(f_\wp,\omega^i)$ has unit content for every classical height one prime $\wp$ of $\mathbf{T}(\mathfrak{a})$.

To complete the proof of the proposition, it suffices to show that if $\mathbf{T}(\mathfrak{a}_1)$ and $\mathbf{T}(\mathfrak{a}_2)$ are two irreducible components of the Hida family of $\bar\rho$, then $L(\bar\rho,\mathfrak{a}_1,\omega^i)$ has unit content if and only if the same holds true for $L(\bar\rho,\mathfrak{a}_2,\omega^i)$. We may choose $\Sigma$ large enough so that each of $\mathbf{T}(\mathfrak{a}_1)$ and $\mathbf{T}(\mathfrak{a}_2)$ is an irreducible component of $\mathbf{T}_\Sigma$. Corollary 3.6.3, together with Lemmas 3.7.2 (3) and 3.7.4, then shows that each of $L(\bar\rho,\mathfrak{a}_1,\omega^i)$ and $L(\bar\rho,\mathfrak{a}_2,\omega^i)$ have unit content if and only if the same is true of $L_\Sigma(\bar\rho,\omega^i)$. $\square$

If the equivalent conditions of the preceding theorem hold, then we write

$$\mu^{\mathrm{an}}(\bar{\rho}, \omega^i) = 0.$$

(As usual, our notation suppresses the choice of $p$-stabilization of $\bar{\rho}$.) In the case when the $\mu$-invariant vanishes we can further study the $\lambda$-invariant of these power series. (Note that for a local ring which is not a discrete valuation ring one can only define the $\lambda$-invariant for power series of unit content.)

**Definition 3.7.6.** If $A$ is a local ring and $f(T) \in A[[T]]$ is a power series having unit content, then we define the $\lambda$-invariant $\lambda(f(T))$ to be the smallest degree in which $f(T)$ has a unit coefficient.

We remark that if $\phi : A \to B$ is a local morphism of complete local rings and if $f(T)$ is an element of $A[[T]]$ having unit content, then $\lambda(f(T)) = \lambda(\phi(f(T)))$.

If $\mathbf{T}(\mathfrak{a})$ is an irreducible component of the Hida family of $\bar{\rho}$ with $\mu^{\mathrm{an}}(\bar{\rho}, \omega^i) = 0$, then Theorem 3.7.5 shows that $L(\bar{\rho}, \mathfrak{a}, \omega^i)$ has unit content. We can therefore define the analytic $\lambda$-invariant of a branch by

$$\lambda^{\mathrm{an}}(\bar{\rho}, \mathfrak{a}, \omega^i) = \lambda(L(\bar{\rho}, \mathfrak{a}, \omega^i)).$$

Our main results on analytic $\lambda$-invariants in Hida families are as follows.

**Theorem 3.7.7.** *Assume that* $\mu^{\mathrm{an}}(\bar{\rho}, \omega^i) = 0$.

(1) *For any given irreducible component* $\mathbf{T}(\mathfrak{a})$ *of the Hida family of* $\bar{\rho}$*, the* $\lambda$*-invariant of* $L_p(f_\wp, \omega^i)$ *takes on the constant value of* $\lambda^{\mathrm{an}}(\mathfrak{a}, \omega^i)$ *as* $\wp$ *varies over all classical height one primes of* $\mathbf{T}(\mathfrak{a})$*.*

(2) *For any two irreducible components* $\mathbf{T}(\mathfrak{a}_1)$*,* $\mathbf{T}(\mathfrak{a}_2)$ *of the Hida family of* $\bar{\rho}$*, we have that*

$$\lambda^{\mathrm{an}}(\mathfrak{a}_1, \omega^i) - \lambda^{\mathrm{an}}(\mathfrak{a}_2, \omega^i) = \sum_{\ell \neq p} (e_\ell(\mathfrak{a}_2, \omega^i) - e_\ell(\mathfrak{a}_1, \omega^i))$$

*where* $e_\ell(\mathfrak{a}, \omega^i) = \lambda(E_\ell(\mathfrak{a}, \langle \ell \rangle_{p,i}^{-1}))$*.*

*Proof.* The first part follows from Proposition 3.5.1 and Definition 3.7.6. For the second part, choose $\Sigma$ large enough so that both $\mathbf{T}(\mathfrak{a}_1)$ and $\mathbf{T}(\mathfrak{a}_2)$ are irreducible components of $\mathbf{T}_\Sigma$. Then, by Corollary 3.6.3, we have that

$$\lambda(L_\Sigma(\bar{\rho}, \omega^i)) = \lambda^{\mathrm{an}}(\mathfrak{a}_j, \omega_i) + \sum_{\ell \in \Sigma} e_\ell(\mathfrak{a}_j, \omega^i)$$

for $j = 1, 2$. Our formula then follows since $e_\ell(\mathfrak{a}_1, \omega^i) = e_\ell(\mathfrak{a}_2, \omega^i)$ for $\ell \notin \Sigma$. $\square$

3.8. **Proof of Theorem 3.6.2.** In this section, we present the proof of Theorem 3.6.2. We will utilize the notation introduced prior to the statement of the theorem and begin the proof by introducing some additional notation.

If $M \geq 1$ is an integer, $d$ is a divisor of $M$ and $d'$ is a divisor of $d$, then we let $B_{d,d'} : X_1(M) \to X_1(M/d)$ denote the map induced by the map $\tau \mapsto d'\tau$ on the upper half-plane. This map induces a corresponding map

$$(B_{d,d'})_* : H_1(M, \{\mathrm{cusps}\}; \mathbf{Z}_p) \to H_1(M/d, \{\mathrm{cusps}\}; \mathbf{Z}_p)$$

on relative homology groups.

If $\ell$ is any prime distinct from $p$, then we let $e_\ell$ denote the largest power of $\ell$ dividing $N(\Sigma)/N(\mathfrak{a})$. We have the inequality $0 \le e_\ell \le 2$. Also, $e_\ell = 0$ unless $\ell \in \Sigma$, while $e_\ell = 2$ only if $\ell \in \Sigma$ and $\ell$ is prime to $N(\mathfrak{a})$. For each such prime $\ell$, we write

$$\epsilon(\ell) := \begin{cases} 1 & \text{if } e_\ell = 0 \\ ((B_{\ell,1})_* - \ell^{-1}T_\ell\,(B_{\ell,\ell})_*) & \text{if } e_\ell = 1 \\ ((B_{\ell^2,1})_* - \ell^{-1}T_\ell\,(B_{\ell^2,\ell})_* + \ell^{-3}\langle \ell\rangle_{N(\mathfrak{a})p}(B_{\ell^2,\ell^2})_*) & \text{if } e_\ell = 2. \end{cases}$$

Now write $\Sigma = \{\ell_1, \ldots, \ell_n\}$ and for any $r \ge 1$ define

$$\epsilon_r : H_1(N(\Sigma)p^r, \{\text{cusps}\}; \mathbf{Z}_p)^{\text{ord}} \to H_1(N(\mathfrak{a})p^r, \{\text{cusps}\}; \mathbf{Z}_p)^{\text{ord}}$$

by $\epsilon_r = \epsilon(\ell_n) \circ \cdots \circ \epsilon(\ell_1)$.

Let us explain this formula. For any $i = 1, \ldots, n$, write $N_i = N(\Sigma)/\ell_1^{e_{\ell_1}} \cdots \ell_i^{e_{\ell_i}}$. In the formula for $\epsilon_r$, the map $\epsilon(\ell_i)$ is taken to be the map

$$H_1(N_{i-1}p^r, \{\text{cusps}\}; \mathbf{Z}_p)^{\text{ord}} \to H_1(N_i p^r, \{\text{cusps}\}; \mathbf{Z}_p)^{\text{ord}}$$

given by the stated formula for $\epsilon(\ell_i)$. (The symbol $T_{\ell_i}$ in the formula for $\epsilon(\ell_i)$ is understood to stand for the corresponding Hecke operator acting in level $N_i p^r$.) It is easily verified that the map $\epsilon_r$ is in fact independent of the choice of ordering of the elements of $\Sigma$.

For any tame level $M$ we let $(\mathbf{T}^*_M)'$ denote the $\Lambda$-subalgebra of the ordinary Hecke algebra $\mathbf{T}^*_M$ generated by the Hecke operators prime to $M$. If we regard the source and target of $\epsilon_r$ as $(\mathbf{T}^*_{N(\Sigma)})'$-modules via the inclusion $(\mathbf{T}^*_{N(\Sigma)})' \subset \mathbf{T}^*_{N(\Sigma)}$ and the natural map $(\mathbf{T}^*_{N(\Sigma)})' \to (\mathbf{T}^*_{N(\mathfrak{a})})' \subset \mathbf{T}^*_{N(\mathfrak{a})}$, then $\epsilon_r$ is immediately seen to be $(\mathbf{T}^*_{N(\Sigma)})'$-linear.

As $r$ varies, the sources and targets of the maps $\epsilon_r$ each form a projective system and the maps $\epsilon_r$ are evidently compatible with the projection maps on source and target. Thus, passing to the limit in $r$, we obtain a $(\mathbf{T}^*_{N(\Sigma)})'$-linear map

$$\epsilon_\infty : \varprojlim_r H_1(N(\Sigma)p^r, \{\text{cusps}\}; \mathbf{Z}_p)^{\text{ord}} \to \varprojlim_r H_1(N(\mathfrak{a})p^r, \{\text{cusps}\}; \mathbf{Z}_p)^{\text{ord}}.$$

We denote the source and target of this map by $\mathfrak{M}_{N(\Sigma)}$ and $\mathfrak{M}_{N(\mathfrak{a})}$ respectively.

We may regard each of the maximal ideals $\mathfrak{m}'$, $\mathfrak{n}$ and $\mathfrak{m}$ equally well as maximal ideals of $(\mathbf{T}^*_{N(\Sigma)})'$, $\mathbf{T}^*_{N(\Sigma)}$ and $\mathbf{T}^*_{N(\mathfrak{a})}$. If we localize $\epsilon_\infty$ with respect to $\mathfrak{m}'$, we obtain a map

$$(\mathfrak{M}_{N(\Sigma)})_{\mathfrak{m}'} \to (\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}'}. \tag{3.10}$$

Now $\mathfrak{n}$ and $\mathfrak{m}$ each pull back to $\mathfrak{m}'$ under the natural maps $\mathbf{T}'_{N(\Sigma)} \to \mathbf{T}_{N(\Sigma)}$ and $\mathbf{T}'_{N(\Sigma)} \to \mathbf{T}_{N(\mathfrak{a})}$ and so the localizations $(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ and $(\mathbf{T}_{N(\mathfrak{a})})_{\mathfrak{m}}$ are local factors of the complete semi-local rings $(\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'} \otimes_{\mathbf{T}'_{N(\Sigma)}} \mathbf{T}_{N(\Sigma)}$ and $(\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'} \otimes_{\mathbf{T}'_{N(\Sigma)}} \mathbf{T}_{N(\mathfrak{a})}$. Thus the localizations $(\mathfrak{M}_{N(\Sigma)})_{\mathfrak{n}}$ and $(\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}}$ appear naturally as direct factors of $(\mathfrak{M}_{N(\Sigma)})_{\mathfrak{m}'}$ and $(\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}'}$ respectively, and so the map (3.10) induces a map

$$(\mathfrak{M}_{N(\Sigma)})_{\mathfrak{n}} \to (\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}}.$$

Tensoring the source of this map with $\mathbf{T}(\mathfrak{a})^\circ$ over $(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ and the target with $\mathbf{T}(\mathfrak{a})^\circ$ over $(\mathbf{T}_{N(\mathfrak{a})})_{\mathfrak{m}}$, we obtain a $\mathbf{T}(\mathfrak{a})^\circ$-linear map

$$\mathbf{T}(\mathfrak{a})^\circ \otimes_{(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}} (\mathfrak{M}_{N(\Sigma)})_{\mathfrak{n}} \to \mathbf{T}(\mathfrak{a})^\circ \otimes_{(\mathbf{T}_{N(\mathfrak{a})})_{\mathfrak{m}}} (\mathfrak{M}_{N(\mathfrak{a})})_{\mathfrak{m}}. \tag{3.11}$$

We claim that this map satisfies the requirements of Theorem 3.6.2.

We first observe that this map satisfies the claimed property with regard to the $L$-functions. This follows from an explicit calculation of the effect of the maps $\epsilon_\ell$ on modular symbols. In fact, one easily shows that for any character $\chi$ of conductor $p^n$ and $\alpha = \sum_{a\in(\mathbf{Z}/p^n)^\times} \chi(a)\,\{a/p^n,\infty\}$, we have that

$$\epsilon_r(\alpha) = \prod_{i \text{ s.t. } e_{\ell_i}=1} (1-\chi^{-1}(\ell)\ell^{-1}T_\ell) \prod_{i \text{ s.t. } e_{\ell_i}=2} (1-\chi^{-1}(\ell)\ell^{-1}T_\ell + \chi^{-2}(\ell)\ell^{-3}\langle\ell\rangle_{N(\mathfrak{a})p})\cdot\alpha.$$

(In the left hand side of this equation, the modular symbols $\{a/p^n,\infty\}$ are regarded as lying in $H_1(N(\Sigma)p^r,\{\text{cusps}\};\mathbf{Z}_p)$. On the right hand side, they are regarded as lying in $H_1(N(\mathfrak{a})p^r,\{\text{cusps}\};\mathbf{Z}_p)$.) Passing to the limit in $r$, and taking into account the fact that $\chi$ is an arbitrary Dirichlet character of $p$-power conductor, we conclude the the isomorphism (3.11) has the required effect on $L$-functions.

We now turn to showing that (3.11) is an isomorphism. Note that Proposition 3.3.1 shows that both source and target are free of rank two over $\mathbf{T}(\mathfrak{a})^\circ$. Thus to see that this map is an isomorphism, it suffices to check that it induces a surjection after being reduced modulo the maximal ideal of $\mathbf{T}(\mathfrak{a})^\circ$. To do this, we first mod out by a classical prime of weight two and then by the full maximal ideal.

Let $\wp$ denote the classical height one prime in $\Lambda$ of weight two for which $\kappa_\wp$ is trivial. If we tensor each side of (3.11) by $\Lambda/\wp$ over $\Lambda$, we obtain the map

$$\mathbf{T}(\mathfrak{a})^\circ \otimes_{(\mathbf{T}_{N(\Sigma)})_\mathfrak{n}} H_1(N(\Sigma)p;\mathbf{Z}_p)^{\text{ord}}_\mathfrak{n} \to \mathbf{T}(\mathfrak{a})^\circ \otimes_{(\mathbf{T}_{N(\mathfrak{a})})_\mathfrak{m}} H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\text{ord}}_\mathfrak{m} \tag{3.12}$$

induced by localizing the source and target of $\epsilon_1$ at $\mathfrak{n}$ and $\mathfrak{m}$ respectively and then extending scalars to $\mathbf{T}(\mathfrak{a})^\circ$. (Here, as in Proposition 3.3.1, we are using [17, Thm. 3.1].)

Thus the reduction modulo the maximal ideal of $\mathbf{T}(\mathfrak{a})^\circ$ of (3.11) coincides with the map

$$\begin{aligned}\left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H_1(N(\Sigma)p;\mathbf{Z}_p)^{\text{ord}}/\mathfrak{n}H_1(N(\Sigma)p;\mathbf{Z}_p)^{\text{ord}}\right)& \\ \to H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\text{ord}}/\mathfrak{m}H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\text{ord}}&\end{aligned} \tag{3.13}$$

of $\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}$-vector spaces induced by (3.12).

Rather than showing directly that (3.13) is surjective, we will show that the corresponding dual map

$$H^1(N(\mathfrak{a})p;\mathbf{F}_p)^{\text{ord}}[\mathfrak{m}] \to \left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H^1(N(\Sigma)p;\mathbf{F}_p)^{\text{ord}}[\mathfrak{n}]\right) \tag{3.14}$$

is injective. (In writing the dual of (3.13) in this form, we have implicitly fixed an isomorphism of one dimensional $\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}$-vector spaces between $\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}$ and its $\mathbf{T}_{N(\Sigma)}/\mathfrak{m}'$-linear dual. We will suppress this choice of isomorphism here and below.) This map may be written as a composite

$$\begin{aligned} H^1(N(\mathfrak{a})p;\mathbf{F}_p)^{\text{ord}}[\mathfrak{m}] &\to \left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H^1(N(\mathfrak{a})p;\mathbf{F}_p)^{\text{ord}}[\mathfrak{m}']\right) \\ &\to \left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H^1(N(\Sigma)p;\mathbf{F}_p)^{\text{ord}}[\mathfrak{m}']\right) \\ &\to \left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H^1(N(\Sigma)p;\mathbf{F}_p)^{\text{ord}}[\mathfrak{n}]\right). \end{aligned} \tag{3.15}$$

To explain this, we first recall that since the localization $(\mathbf{T}_{N(\Sigma)})_\mathfrak{n}$ is a local factor of the tensor product $(\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'} \otimes_{\mathbf{T}'_{N(\Sigma)}} \mathbf{T}_{N(\Sigma)}$ for which the natural map $(\mathbf{T}'_{N(\Sigma)})_{\mathfrak{m}'} \to$

$(\mathbf{T}_{N(\Sigma)})_{\mathfrak{n}}$ is an isomorphism, the residue field $\mathbf{T}_{N(\Sigma)}/\mathfrak{n}$ is a local factor of the Artin local ring $\mathbf{T}_{N(\Sigma)}/\mathfrak{m}'\mathbf{T}_{N(\Sigma)}$; write

$$\mathbf{T}_{N(\Sigma)}/\mathfrak{m}'\mathbf{T}_{N(\Sigma)} \cong \mathbf{T}_{N(\Sigma)}/\mathfrak{n} \times A,$$

where $A$ denotes the product of the remaining local factors. This decomposition induces a corresponding decomposition

$$B \cong B[\mathfrak{n}] \times A \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{m}'\mathbf{T}_{N(\Sigma)}} B \tag{3.16}$$

for any $\mathbf{T}_{N(\Sigma)}/\mathfrak{m}'\mathbf{T}_{N(\Sigma)}$-module $B$. The third arrow of (3.15) is precisely projection onto the the first of the two factors in (3.16) with

$$B = \left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H^1(N(\Sigma)p;\mathbf{F}_p)^{\mathrm{ord}}[\mathfrak{m}']\right).$$

The second arrow is induced by dualizing the reduction modulo $\mathfrak{m}'$ of the map (3.10) and the first arrow is induced from the obvious inclusion, or if the reader prefers, is obtained by dualizing the surjection

$$\begin{aligned}\left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} &\left(H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\mathrm{ord}}/\mathfrak{m}'H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\mathrm{ord}}\right)\\ &\to H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\mathrm{ord}}/\mathfrak{m}H_1(N(\mathfrak{a})p;\mathbf{Z}_p)^{\mathrm{ord}}.\end{aligned}$$

**Lemma 3.8.1.** *The map (3.14) is injective if and only if the composite of the first two arrows of (3.15) is injective.*

*Proof.* The only if statement is clear. In order to prove the other statement, we first observe that the second arrow of (3.15) is given by the cohomological version of the map $\epsilon_1$. More precisely, if for each $\ell \in \Sigma$ we define

$$\epsilon_\ell^* := \begin{cases} 1 & \text{if } e_\ell = 0 \\ B_{\ell,1}^* - B_{\ell,\ell}^*\,\ell^{-1}T_\ell & \text{if } e_\ell = 1 \\ B_{\ell^2,1}^* - B_{\ell^2,\ell}^*\,\ell^{-1}T_\ell + B_{\ell^2,\ell^2}^*\,\ell^{-3}\langle\ell\rangle_{N(\mathfrak{a})p} & \text{if } e_\ell = 2 \end{cases}$$

(where $B_{d,d'}^*$ denotes the map on cohomology induced by the degeneracy map $B_{d,d'}$) and define

$$\epsilon^* = \epsilon_{\ell_1}^* \circ \cdots \circ \epsilon_{\ell_n}^* : H^1(N(\mathfrak{a})p;\mathbf{F}_p)^{\mathrm{ord}} \to H^1(N(\Sigma)p;\mathbf{F}_p)^{\mathrm{ord}},$$

then the second arrow of (3.15) is obtained from the map $\epsilon^*$ by passing to the kernel of $\mathfrak{m}'$ in the source and target and then extending scalars from $\mathbf{T}_{N(\Sigma)}/\mathfrak{n}$ to $\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}$.

One then checks that any element in the image of $\epsilon^*$ is annihilated by the Hecke operators $T_\ell$, for those $\ell \in \Sigma$. On the other hand, Proposition 2.4.2 shows that the maximal ideal $\mathfrak{n}$ is uniquely characterized by the property of containing these operators. Thus the image of the second arrow of (3.15) lies in the local factor

$$\left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H_1(N(\Sigma)p;\mathbf{F}_p)^{\mathrm{ord}}[\mathfrak{n}]\right)$$

of

$$\left(\mathbf{T}_{N(\mathfrak{a})}/\mathfrak{m}\right) \otimes_{\mathbf{T}_{N(\Sigma)}/\mathfrak{n}} \left(H_1(N(\Sigma)p;\mathbf{F}_p)^{\mathrm{ord}}[\mathfrak{m}']\right).$$

This proves the lemma. □

By the preceding lemma, we are reduced to proving that the composite of the first two arrows of (3.15) is injective. It will be notationally easier to deal with each of the maps $\epsilon_{\ell_i}$ separately and so we put ourselves in the following more general situation. We consider a natural number $M$ prime to $p$ and a prime $\ell$ distinct from

$p$. We define $n_\ell = 1$ or $2$ according to whether or not $\ell$ divides $M$ and write $N := \ell^{n_\ell} M$. We let $\mathfrak{m}$ denote a maximal ideal of $\mathbf{T}_M$ for which $\bar{\rho}_{\mathfrak{m}}$ is irreducible and $p$-distinguished and let $\mathfrak{m}'$ denote the pull back of $\mathfrak{m}$ under the natural map $\mathbf{T}'_N \to \mathbf{T}_M$. The map $\epsilon_\ell^*$ defined in the proof of Lemma 3.8.1 induces a $\mathbf{T}_M/\mathfrak{m}$-linear map

$$H^1(Mp; \mathbf{F}_p)^{\mathrm{ord}}[\mathfrak{m}] \to (\mathbf{T}_M/\mathfrak{m}) \otimes_{\mathbf{T}'_N/\mathfrak{m}'} \left(H^1(Np; \mathbf{F}_p)^{\mathrm{ord}}[\mathfrak{m}']\right). \tag{3.17}$$

**Lemma 3.8.2.** *The map (3.17) is injective.*

The proof of Lemma 3.8.1 shows that elements in the image of (3.17) are annihilated by $T_\ell$ and hence are in fact annihilated by a maximal ideal of the full Hecke algebra $\mathbf{T}_N$. Thus we may apply Lemma 3.8.2 inductively to establish the injectivity of the composite of the first two arrows of (3.15) and thereby complete the proof of the theorem.

*Proof of Lemma 3.8.2.* It will be convenient to bring the residue field $\mathbf{T}_M/\mathfrak{m}$ inside the coefficients of cohomology. To do this, we choose a finite field $k$ containing $\mathbf{T}_M/\mathfrak{m}$ and tensor the source and target of (3.17) with $k$ over $\mathbf{F}_p$ to obtain a map of $k \otimes_{\mathbf{F}_p} \mathbf{T}_M/\mathfrak{m}$-modules. The chosen inclusion of $\mathbf{T}_M/\mathfrak{m}$ in $k$ determines a projection

$$k \otimes_{\mathbf{F}_p} \mathbf{T}_M/\mathfrak{m} \to k, \tag{3.18}$$

which realizes $k$ as a local factor of $k \otimes_{\mathbf{F}_p} \mathbf{T}_M/\mathfrak{m}$. Projecting onto this local factor, we recover our original map (3.17), but rewritten as

$$H^1(Mp; k)^{\mathrm{ord}}[\mathfrak{m}_k] \to H^1(Np; k)^{\mathrm{ord}}[\mathfrak{m}'_k]. \tag{3.19}$$

Here we regard the source as a $W(k) \otimes_{\mathbf{Z}_p} \mathbf{T}_M$-module and the target as a $W(k) \otimes_{\mathbf{Z}_p} \mathbf{T}'_N$-module. Also, we have written $\mathfrak{m}_k$ to denote the the maximal ideal of $W(k) \otimes_{\mathbf{Z}_p} \mathbf{T}_M$ that is the kernel of the composite

$$W(k) \otimes_{\mathbf{Z}_p} \mathbf{T}_M \to k \otimes_{\mathbf{F}_p} \mathbf{T}_M/\mathfrak{m} \to k$$

(where the second arrow is given by the projection (3.18)) and $\mathfrak{m}'_k$ to denote the maximal ideal of $W(k) \otimes_{\mathbf{Z}_p} \mathbf{T}'_N$ that is the kernel of the composite

$$W(k) \otimes_{\mathbf{Z}_p} \mathbf{T}'_N \to k \otimes_{\mathbf{F}_p} \mathbf{T}'_N/\mathfrak{m}' \to k \otimes_{\mathbf{F}_p} \mathbf{T}_M/\mathfrak{m} \to k$$

(where the second arrow is obtained by tensoring the injection $\mathbf{T}'_N/\mathfrak{m}' \to \mathbf{T}_M/\mathfrak{m}$ by $k$ over $\mathbf{F}_p$ and the third arrow is given by the projection (3.18)). We must show that the map (3.19) is injective.

Our argument will rely on the results of [37, §2.2] which extend a well-known result of Ihara. Recall that the standard practice for analyzing the map (3.19) is to factor the map $\epsilon_\ell^*$ and so to write (3.19) as a composite

$$H^1(Mp; k)^{\mathrm{ord}}[\mathfrak{m}_k] \xrightarrow{\alpha_\ell} (H^1(Mp; k)^{\mathrm{ord}}[\mathfrak{m}_k])^{n_\ell+1} \xrightarrow{\beta_\ell} H^1(Np; k)^{\mathrm{ord}}[\mathfrak{m}'_k],$$

where

$$\alpha_\ell = \begin{cases} (1, -\ell^{-1} T_\ell) & \text{if } n_\ell = 1 \\ (1, -\ell^{-1} T_\ell, \ell^{-3} \langle \ell \rangle_{Mp}) & \text{if } n_\ell = 2 \end{cases}$$

and

$$\beta_\ell = \begin{cases} B^*_{\ell,1}\pi_1 + B^*_{\ell,\ell}\pi_2 & \text{if } n_\ell = 1 \\ B^*_{\ell^2,1}\pi_1 + B^*_{\ell^2,\ell}\pi_2 + B^*_{\ell^2,\ell^2}\pi_3 & \text{if } n_\ell = 2 \end{cases}$$

with $\pi_i$ denoting projection onto the $i$th factor of the product

$$(H^1(Np;k)^{\mathrm{ord}}[\mathfrak{m}_k])^{n_\ell+1}.$$

The map $\alpha_\ell$ is manifestly injective and, in the case $n_\ell = 2$, Wiles has shown that the map $\beta_\ell$ is also injective. (See in particular the discussion at the top of [37, p. 497] or the discussion of Wiles' results provided by [7, §§4.3] and [3, §§4.4, 4.5].) Thus when $n_\ell = 2$ the lemma is proved.

However, if $n_\ell = 1$, then the situation is more complicated. Lemma 2.5 of [37] provides an exact sequence

$$H^1((M/\ell)p;k)^{\mathrm{ord}}_{\mathfrak{m}'_k} \xrightarrow{\gamma_\ell} (H^1(Mp;k)^{\mathrm{ord}}_{\mathfrak{m}'_k})^2 \xrightarrow{\beta_\ell} H^1(Np;k)^{\mathrm{ord}}_{\mathfrak{m}'_k},$$

where $\gamma_\ell = (B^*_{\ell,\ell}, -B^*_{\ell,1})$. (Here the subscript $\mathfrak{m}'_k$ denotes that we have localized at this maximal ideal.)

Let $x \in H^1(Mp;k)[\mathfrak{m}_k]$ be an element in the kernel of (3.19). Then (3.8) shows that we may find $y \in H^1((M/\ell)p;k)$ such that $(x, -T_\ell x) = (B^*_{\ell,\ell}y, -B^*_{\ell,1}y)$. In particular, $B^*_{\ell,\ell}y = x$ is annihilated by $\mathfrak{m}_k$ and so is an eigenvector for the full Hecke algebra $\mathbf{T}_M$. The following result shows that $B^*_{\ell,\ell}y = 0$ and hence that $x = 0$. This completes the proof of the lemma. $\square$

**Lemma 3.8.3.** *Let $D$ be a natural number prime to $p$, let $\ell$ be a prime distinct from $p$ and consider the map $B^*_{\ell,\ell} : H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}} \to H^1(D\ell p;\bar{\mathbf{F}}_p)^{\mathrm{ord}}$. If $y$ is a class in the domain with the property that $B^*_{\ell,\ell}y$ is an eigenvector for the full Hecke ring $\mathbf{T}_{D\ell}$, corresponding to a maximal ideal for which the attached Galois representation into $\mathrm{GL}_2(\bar{\mathbf{F}}_p)$ is irreducible and $p$-distinguished, then $B^*_{\ell,\ell}y = 0$.*

*Proof.* We will prove the lemma by comparing the map $B^*_{\ell,\ell}$ on cohomology classes with the corresponding map on modular forms $\bmod p$. For such modular forms, the analogue of the lemma follows immediately from a consideration of $q$-expansions. For the comparison with modular forms mod $p$, we follow the discussion in the proof of [37, Thm. 2.1].

Let us make some initial reductions in the situation of the lemma. We let $\mathfrak{m}$ denote the maximal ideal in $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_{D\ell}$ describing the action of the Hecke operators on the eigenvector $B^*_{\ell,\ell}y$. Let $\mathfrak{m}'$ denote the intersection of $\mathfrak{m}$ with the subring $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}'_{D\ell}$ of $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_{D\ell}$. We may regard each of $H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}$ and $H^1(D\ell p;\bar{\mathbf{F}}_p)^{\mathrm{ord}}$ as $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}'_{D\ell}$-modules (the former via the natural map $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}'_{D\ell} \to \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}'_D$) and the map $B^*_{\ell,\ell}$ is $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}'_{D\ell}$-linear. If $y_{\mathfrak{m}'}$ denotes the projection of $y$ onto the localization $H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{m}'}$ of $H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}$ at $\mathfrak{m}'$, then $B^*_{\ell,\ell}y_{\mathfrak{m}'} = B^*_{\ell,\ell}y$ (since $B^*_{\ell,\ell}y$ is assumed to be annihilated by $\mathfrak{m}$ and so in particular by $\mathfrak{m}'$). Also, there is a natural isomorphism

$$H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{m}'} \cong \prod_{\mathfrak{n} \supset \mathfrak{m}'(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_D)} H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{n}},$$

where $\mathfrak{n}$ ranges over all maximal ideals of $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_D$ containing $\mathfrak{m}'(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_D)$. Thus to show that $B^*_{\ell,\ell}y = 0$, it suffices to show that $B^*_{\ell,\ell}y_{\mathfrak{n}} = 0$ for each such maximal ideal $\mathfrak{n}$, where $y_{\mathfrak{n}}$ denotes the projection of $y$ onto $H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{n}}$. Thus for the duration of the proof we assume that $y \in H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{n}}$, for some maximal ideal $\mathfrak{n}$ of $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_D$ lying over $\mathfrak{m}'$.

As in the proof of [37, Thm. 2.1], we consider two cases: that in which the mod $p$ diamond operators have non-trivial image in $(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_{D\ell})/\mathfrak{m}$ and that in which they have trivial image.

In order to treat the first case, we begin by noting that the commutative diagram

$$\begin{tikzcd} H^1(Dp;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{n}} \arrow[r, "B^*_{\ell,\ell}"] \arrow[d, "\cong"] & H^1(D\ell p;\bar{\mathbf{F}}_p)^{\mathrm{ord}}_{\mathfrak{m}} \arrow[d, "\cong"] \\ (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})_{\mathfrak{n}} \arrow[r, "B^*_{\ell,\ell}"] & (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})_{\mathfrak{m}}, \end{tikzcd} \tag{3.20}$$

in which the lower horizontal arrow is induced by the map of Jacobians arising by Picard functoriality applied to the degeneracy map $B_{\ell,\ell} : X_1(D\ell p) \to X_1(Dp)$, allows us to replace curves with coefficients in $\bar{\mathbf{F}}_p$ with a consideration of $p$-torsion in the corresponding Jacobians (tensored by $\bar{\mathbf{F}}_p$ over $\mathbf{F}_p$). (The superscript ord on $J_1(Dp)[p]$ and $J_1(D\ell p)[p\ell]$ denotes the localization of these $p$-torsion modules at the $p$-ordinary part of the Hecke algebra, or equivalently, the image of the $p$-torsion modules under Hida's idempotent $e^{\mathrm{ord}}$.)

Let $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_p)$ denote the residual Galois representation attached to each of $\mathfrak{m}$ and $\mathfrak{n}$. From [37, Thm. 2.1] and its proof we conclude that there are isomorphisms of Galois modules

$$(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_D)_{\mathfrak{n}}/p(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_D)_{\mathfrak{n}} \otimes_{\bar{\mathbf{F}}_p} \bar{\rho} \cong (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})_{\mathfrak{n}}$$

and

$$(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_{D\ell})_{\mathfrak{m}}/p(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} \mathbf{T}_{D\ell})_{\mathfrak{m}} \otimes_{\bar{\mathbf{F}}_p} \bar{\rho} \cong (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})_{\mathfrak{m}}.$$

Also, passing to $p$-torsion in the short exact sequence [37, (2.2)] for each of the maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ yields short exact sequences

$$0 \to (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})^0_{\mathfrak{n}} \to (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})_{\mathfrak{n}} \to (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})^E_{\mathfrak{n}} \to 0$$

and

$$0 \to (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})^0_{\mathfrak{m}} \to (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})_{\mathfrak{m}} \to (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})^E_{\mathfrak{m}} \to 0.$$

The lower horizontal arrow $B^*_{\ell,\ell}$ of (3.20) is $G_{\mathbf{Q}}$-equivariant and also induces a morphism between these short-exact sequences.

If $B^*_{\ell,\ell}y$ is non-zero, then, since $\bar{\rho}$ is an irreducible $G_{\mathbf{Q}}$-module, we may find $\sigma \in G_{\mathbf{Q}}$ such that $\sigma(B^*_{\ell,\ell}y)$ has non-zero image in $(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})^E_{\mathfrak{m}}$. Replacing $y$ by the image of $\sigma(y)$ in $(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})^E_{\mathfrak{n}}$, we are reduced to proving the following claim.

**Claim:** If $y \in (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})^E_{\mathfrak{n}}$ is such that $B^*_{\ell,\ell}y \in (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})^E_{\mathfrak{n}}$ is annihilated by $\mathfrak{m}$, then $B^*_{\ell,\ell}y$ vanishes.

Passing to the special fiber of the Néron model of each of $J_1(Dp)$ and $J_1(D\ell p)$ over $\mathbf{Z}_p[\zeta_p]$ (where $\zeta_p$ denotes a primitive $p$th root of unity), the discussion in the proof of [37, Thm. 2.1] yields a commutative diagram in which the horizontal arrows are isomorphisms:

$$\begin{tikzcd}
(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(Dp)[p]^{\mathrm{ord}})^E_{\mathfrak{n}} \arrow[r] \arrow[d, "B^*_{\ell,\ell}"] & \big(H^0(\Sigma_1(D)^{\mu}, \Omega^1) \oplus H^0(\Sigma_1(D)^{\text{ét}}, \Omega^1)\big)^{\mathrm{ord}}_{\mathfrak{n}} \arrow[d, "B^*_{\ell,\ell}"] \\
(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell p)[p]^{\mathrm{ord}})^E_{\mathfrak{m}} \arrow[r] & \big(H^0(\Sigma_1(D\ell)^{\mu}, \Omega^1) \oplus H^0(\Sigma_1(D\ell)^{\text{ét}}, \Omega^1)\big)^{\mathrm{ord}}_{\mathfrak{m}} .
\end{tikzcd}$$

(For any $M$ prime to $p$, we denote by $\Sigma_1(M)^{\mu}$ and $\Sigma_1(M)^{\text{ét}}$ the base-change from $\mathbf{F}_p$ to $\bar{\mathbf{F}}_p$ of the two components of the special fiber of the canonical model of $X_1(Mp)$ over $\mathbf{Z}_p[\zeta_p]$.) Thus to prove the claim, it suffices to show that if $z$ is a non-zero element of $\big(H^0(\Sigma_1(D)^{\mu}, \Omega^1) \oplus H^0(\Sigma_1(D)^{\text{ét}}, \Omega^1)\big)^{\mathrm{ord}}_{\mathfrak{n}}$, then $B^*_{\ell,\ell}$ cannot lie in the $\mathfrak{m}$-eigenspace of $\big(H^0(\Sigma_1(D\ell)^{\mu}, \Omega^1) \oplus H^0(\Sigma_1(D\ell)^{\text{ét}}, \Omega^1)\big)^{\mathrm{ord}}_{\mathfrak{m}}$. This follows from the $q$-expansion principal; more precisely, the $q$-expansion of $B^*_{\ell,\ell}z$ at the cusp $\infty$ involves only powers of $q^{\ell}$ and so cannot be an eigenform for the full Hecke algebra $T_{D\ell}$.

The second case (when the mod $p$ diamond operators are trivial modulo $\mathfrak{m}$) is treated similarly, using the corresponding results from the proof of [37, Thm. 2.1]. $\square$

## 4. Algebraic Iwasawa invariants

4.1. **Selmer groups of modular forms.** Let $f = \sum a_n q^n$ be a $p$-ordinary and $p$-stabilized newform of weight $k \geq 2$, tame level $N$ and character $\chi$. Let $K$ denote the finite extension of $\mathbf{Q}_p$ generated by the Fourier coefficients of $f$ and let $\mathcal{O}$ denote the ring of integers of $K$; we write $k$ for the residue field and fix also a uniformizer $\pi$ of $\mathcal{O}$. Let

$$\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(K)$$

be the corresponding Galois representation, characterized by the fact that the characteristic polynomial under $\rho_f$ of an arithmetic Frobenius at a prime $\ell \nmid Np$ is

$$X^2 - a_{\ell} X + \chi(\ell)\ell^{k-1}.$$

By [14, Thm. 2.6] the restriction of $\rho_f$ to $G_p$ is of the form

$$\rho|_{G_p} \cong \begin{pmatrix} \varepsilon^{k-1}\chi\varphi^{-1} & * \\ 0 & \varphi \end{pmatrix} \tag{4.1}$$

with $\varphi : G_p \to \mathcal{O}^{\times}$ the unramified character sending an arithmetic Frobenius to $a_p$.

We assume that the semisimple residual representation

$$\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$$

is absolutely irreducible. It follows that, up to conjugation by $\mathrm{GL}_2(\mathcal{O})$, there is a unique integral model

$$\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$$

of $\rho_f$, which we now fix. For $0 \leq i \leq p-2$, let $A_{f,i}$ denote a cofree $\mathcal{O}$-module of corank 2 with $G_{\mathbf{Q}}$-action via $\rho_f \otimes \omega^i$. We obtain from (4.1) and [16, Prop. 12.1] an $\mathcal{O}[G_p]$-equivariant exact sequence

$$0 \to (K/\mathcal{O})(\varepsilon^{k-1}\chi\omega^i\varphi^{-1}) \to A_{f,i} \to (K/\mathcal{O})(\omega^i\varphi) \to 0. \tag{4.2}$$

We write $A'_{f,i}$ (resp. $A''_{f,i}$) for the submodule (resp. quotient module) of $A_{f,i}$ in the above sequence.

For a place $v$ of $\mathbf{Q}_\infty$ define

$$H^1_s(\mathbf{Q}_{\infty,v}, A_{f,i}) = \begin{cases} H^1(\mathbf{Q}_{\infty,v}, A_{f,i}) & v \neq v_p; \\ \mathrm{im}\big(H^1(\mathbf{Q}_{\infty,v_p}, A_{f,i}) \to H^1(I_{v_p}, A''_{f,i})\big) & v = v_p. \end{cases}$$

Following [10], the *Selmer group* of $A_{f,i}$ is defined by

$$\begin{aligned} \mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i}) &= \ker\big(H^1(\mathbf{Q}_\infty, A_{f,i}) \to \prod_v H^1_s(\mathbf{Q}_{\infty,v}, A_{f,i})\big) \\ &= \ker\big(H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, A_{f,i}) \to \prod_{v\in\Sigma} H^1_s(\mathbf{Q}_{\infty,v}, A_{f,i})\big) \end{aligned}$$

for any finite set of places $\Sigma$ containing $v_p$, all archimedean places and all places dividing $N$. We regard $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ as a $\Lambda_{\mathcal{O}}$-module via the natural action of $\Gamma$.

Recall that $\mu^{\mathrm{alg}}(f,\omega^i)$ (resp. $\lambda^{\mathrm{alg}}(f,\omega^i)$) is defined to be the largest power of $\pi$ dividing (resp. the number of zeroes of) the characteristic power series of the $\Lambda_{\mathcal{O}}$-dual of $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ (assuming that this Selmer group is $\Lambda_{\mathcal{O}}$-cotorsion).

**Theorem 4.1.1.** *Let $f$ be a $p$-ordinary and $p$-stabilized newform with $\bar\rho_f$ absolutely irreducible. Then $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ is co-finitely generated, $\Lambda_{\mathcal{O}}$-cotorsion and has no proper $\Lambda_{\mathcal{O}}$-submodules of finite index. Furthermore, $\mu^{\mathrm{alg}}(f,\omega^i)$ vanishes if and only if $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi]$ is finite. If this is the case, then $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ is $\mathcal{O}$-divisible and*

$$\lambda^{\mathrm{alg}}(f,\omega^i) = \dim_k \mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi].$$

*Proof.* It is shown in [10, Prop. 6] that Selmer groups are always co-finitely generated. The fact that they are also $\Lambda_{\mathcal{O}}$-cotorsion for modular forms is proven in [20]. The equivalence of the vanishing of $\mu^{\mathrm{alg}}(f,\omega^i)$ and the finiteness of $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi]$ is now an immediate consequence of the structure theory of $\Lambda_{\mathcal{O}}$-modules.

The proof of [13, Prop. 4.14] easily adapts to show that $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ has no proper $\Lambda_{\mathcal{O}}$-submodules of finite index. (As $A_{f,i}$ need not be self-dual, this also requires the fact that $\mathrm{Sel}(\mathbf{Q}_\infty, A^*_{f,i})$ is $\Lambda_{\mathcal{O}}$-cotorsion, which follows since $A^*_{f,i}$ is also modular.) When $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi]$ is finite, the maximal $\mathcal{O}$-divisible $\Lambda_{\mathcal{O}}$-submodule of $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ has finite index, so that it must coincide with $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$; that is, $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ is divisible. It now follows again from the structure theory of $\Lambda_{\mathcal{O}}$-modules that as $\mathcal{O}$-modules

$$\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i}) \cong (K/\mathcal{O})^{\lambda^{\mathrm{alg}}(f,\omega^i)},$$

which proves the last statement. □

We close this section with a useful result on the local invariants $H^0(G_v, A_{f,i})$ for places $v$ dividing primes $\ell \neq p$. Let $\mathrm{cond}_\ell(\bar\rho_f)$ denote the exponent of the highest power of $\ell$ that divides the conductor of $\bar\rho_f$.

**Lemma 4.1.2.** *Let $v$ be a place of $\mathbf{Q}_\infty$ dividing a prime $\ell \neq p$. If $\mathrm{cond}_\ell(\bar\rho_f) = \mathrm{ord}_\ell(N)$, then $H^0(G_v, A_{f,i})$ is $\mathcal{O}$-divisible for all $i$.*

*Proof.* By the invariance of the Swan conductor under reduction (see [23, §1]), we have

$$\mathrm{cond}_\ell(\rho_f) - \mathrm{cond}_\ell(\bar\rho_f) = \dim_k A_f[\pi]^{I_\ell} - \dim_K V_f^{I_\ell}$$

where $V_f$ is a two dimensional $K$-vector space with $G_{\mathbf{Q}}$-action via $\rho_f$. Since $\mathrm{cond}_\ell(\rho_f)$ equals $\mathrm{ord}_\ell(N)$ by [1], we see that the hypothesis of the lemma is equivalent to the equality

$$\dim_k A_f[\pi]^{I_\ell} = \dim_K V_f^{I_\ell}.$$

It follows easily from this that $A_f^{I_\ell} = A_f^{I_v}$ is $\mathcal{O}$-divisible. As the $G_v/I_v$-invariants of an $\mathcal{O}$-divisible $G_v/I_v$-module are again $\mathcal{O}$-divisible, we conclude that $H^0(G_v, A_f)$ is divisible, as claimed. Since $\omega$ is unramified at $\ell$, the above argument works for $A_{f,i}$ as well. □

4.2. **Residual Selmer groups.** Let $\bar{A}$ denote a two-dimensional $k$-vector space equipped with a continuous $k$-linear action of $G_{\mathbf{Q}}$ and a $k[G_p]$-equivariant exact sequence

$$0 \to \bar{A}' \to \bar{A} \to \bar{A}'' \to 0 \tag{4.3}$$

with $\bar{A}'$ and $\bar{A}''$ one-dimensional. We make the following assumptions on this data.

(1) $\bar{A}$ is absolutely irreducible as a $k[G_{\mathbf{Q}}]$-module;
(2) The $k$-vector space $\bar{A}''$ is unramified as a $G_p$-module;
(3) $\bar{A}$ is *p-distinguished* in the sense that the representation of $G_p$ on $\bar{A}$ is non-scalar;
(4) $\bar{A}$ is *modular*: there exists a totally ramified extension $K'$ of the field of fractions of the Witt vectors of $k$, a $p$-stabilized newform $f \in \mathcal{O}'[[q]]$ of weight $k \geq 2$ and a $k[G_{\mathbf{Q}}]$-isomorphism $\bar{A} \cong A_f[\pi']$ identifying $\bar{A}'$ with $A'_f[\pi']$ in the notation of the previous section; here $\pi'$ is a uniformizer of the ring of integers $\mathcal{O}'$ of $K'$.

We will study Selmer groups of $\bar{A}$ and its cyclotomic twists $\bar{A} \otimes \omega^i$ with respect to various local conditions.

**Definition 4.2.1.** A *finite/singular structure* $\mathcal{S}$ on $\bar{A}\otimes\omega^i$ is a choice of $k$-subspaces

$$H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i) \hookrightarrow H^1(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i) \tag{4.4}$$

for each place $v$ of $\mathbf{Q}_\infty$, subject to the restrictions:

(1) $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i) = 0$ for almost all $v$;
(2) $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,v_p}, \bar{A} \otimes \omega^i) = \ker\big(H^1(\mathbf{Q}_{\infty,v_p}, \bar{A} \otimes \omega^i) \to H^1(I_{v_p}, \bar{A}'' \otimes \omega^i)\big)$.

(Note that we are not allowing any variation in the choice of condition at $v_p$). We define $H^1_{s,\mathcal{S}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i)$ as the cokernel of (4.4). The *$\mathcal{S}$-Selmer group* of $\bar{A} \otimes \omega^i$ is

$$\begin{aligned}\mathrm{Sel}_{\mathcal{S}}(\mathbf{Q}_\infty, \bar{A} \otimes \omega^i) &= \ker\Big(H^1(\mathbf{Q}_\infty, \bar{A} \otimes \omega^i) \to \prod_v H^1_{s,\mathcal{S}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i)\Big) \\ &= \ker\Big(H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, \bar{A} \otimes \omega^i) \to \prod_{v\in\Sigma} H^1_{s,\mathcal{S}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i)\Big)\end{aligned}$$

for any finite set of places $\Sigma$ containing $p$, all archimedean places, all places at which $\bar{A}$ is ramified and all places for which $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i)$ does not vanish.

We will be especially interested in two kinds of finite/singular structures. First, the *minimal structure* $\mathcal{S}_{\min}$ on $\bar{A} \otimes \omega^i$ is given by

$$H^1_{f,\mathcal{S}_{\min}}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i) = 0$$

for $v \neq v_p$.

Next let $f$ be a newform as in (4) above. We define the *induced structure* $\mathcal{S}(f,i)$ on $\bar{A}\otimes\omega^i$ by setting

$$H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)=\ker\bigl(H^1(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)\to H^1_s(\mathbf{Q}_{\infty,v},A_{f,i})\bigr)$$

for all places $v$ of $\mathbf{Q}_\infty$. Note that by the definition of $H^1_s(\mathbf{Q}_{\infty,v_p},A_{f,i})$ we have that $H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v_p},\bar{A}\otimes\omega^i)$ is the kernel of the composite map

$$H^1(\mathbf{Q}_{\infty,v_p},\bar{A}\otimes\omega^i)\to H^1(I_{v_p},\bar{A}''\otimes\omega^i)\to H^1(I_{v_p},A''_{f,i});$$

in fact, since $H^0(I_{v_p},A''_{f,i})$ is either $A''_f$ (for $i=0$) or else zero (for $i\neq 0$), the latter map is injective, so that

$$H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v_p},\bar{A}\otimes\omega^i)=\ker\bigl(H^1(\mathbf{Q}_{\infty,v_p},\bar{A}\otimes\omega^i)\to H^1(I_{v_p},\bar{A}''\otimes\omega^i)\bigr)\tag{4.5}$$

as required.

**Lemma 4.2.2.** *For $v\nmid p$ we have*

$$H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)=\operatorname{im}\bigl(A_{f,i}^{G_v}/\pi\hookrightarrow H^1(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)\bigr)$$

*In particular, $H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)=0$ if $\bar{A}\otimes\omega^i$ is unramified at $v\neq v_p$ so that $\mathcal{S}(f,i)$ is a finite/singular structure. The natural map*

$$\mathrm{Sel}_{\mathcal{S}(f,i)}(\mathbf{Q}_\infty,\bar{A}\otimes\omega^i)\to\mathrm{Sel}(\mathbf{Q}_\infty,A_{f,i})[\pi]$$

*is an isomorphism.*

*Proof.* The lemma follows from various exact sequences in cohomology coming from the exact sequence

$$0\to\bar{A}\otimes\omega^i\to A_{f,i}\xrightarrow{\pi}A_{f,i}\to 0$$

together with Lemma 4.1.2. We leave the details to the reader. □

**Proposition 4.2.3.** *Let $\bar{A}$ be as above. Then there exists a newform $f$ as in (4) above such that the finite/singular structures $\mathcal{S}_{\min}$ and $\mathcal{S}(A_{f,i})$ coincide under the isomorphism $\bar{A}\otimes\omega^i\cong A_{f,i}[\pi']$ for any $i$.*

*Proof.* Let $N$ denote the tame conductor of the Galois representation $\bar{A}$. As $f$ is $p$-distinguished, by [5, Thm. 6.4] there exists a finite totally ramified extension $K'$ of $K$ and a (not necessarily unique) $p$-stabilized newform $f\in K'[[q]]$ of tame level $N$ and weight 2 satisfying the condition of (4). (The final condition in (4) is in fact already automatic from the more standard level lowering result [5, Thm. 1.1] unless $\bar{A}$ is an unramified $G_p$-module, in which case the Selmer case of [5, Thm. 6.4] ensures the existence of such an $f$.)

Since the tame conductor of $f$ equals the conductor of $\bar{A}$, by Lemma 4.1.2, $A_{f,i}^{G_v}$ is divisible for any place $v\neq v_p$. It then follows from Lemma 4.2.2 that the structure $\mathcal{S}(A_{f,i})$ on $A_{f,i}[\pi']\cong\bar{A}$ agrees with $\mathcal{S}_{\min}$, as desired. □

The next proposition, which follows from a result of Greenberg, is crucial to our method.

**Proposition 4.2.4.** *For $f$ be as above there is an exact sequence*

$$0\to\mathrm{Sel}_{\mathcal{S}(f,i)}(\mathbf{Q}_\infty,\bar{A}\otimes\omega^i)\to H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty,\bar{A}\otimes\omega^i)\to\prod_{v\in\Sigma}H^1_{s,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)\to 0$$

*for any finite set of places $\Sigma$ containing $v_p$, all archimedean places and all places dividing the tame level of $f$.*

*Proof.* Since $A_{f,i}$ is odd and $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ is $\Lambda_{\mathcal{O}}$-cotorsion, by [15, Prop. 2.1] there is an exact sequence

$$0 \to \mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i}) \to H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, A_{f,i}) \to \prod_{v\in\Sigma} H^1_s(\mathbf{Q}_{\infty,v}, A_{f,i}) \to 0$$

with $\Sigma$ as above. As $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ is $\mathcal{O}$-divisible, the $\pi$-torsion of this sequence is an exact sequence

$$0 \to \mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi] \to H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, A_{f,i})[\pi] \to \prod_{v\in\Sigma} H^1_s(\mathbf{Q}_{\infty,v}, A_{f,i})[\pi] \to 0.$$

By Lemma 4.2.2 and the definition of the induced structure, we see that this sequence identifies with that of the proposition. (Note that (4.5) shows that the local conditions agree at the place $v_p$.) □

Our main algebraic theorems are consequences of the next result.

**Corollary 4.2.5.** *Let $\bar{A}$ be as above and let $f$ be a $p$-stabilized newform of tame level $N_f$ such that $A_f[\pi] \cong \bar{A}$ as in (4). Then the sequence*

$$0 \to \mathrm{Sel}_{\mathcal{S}_{\min}}(\mathbf{Q}_\infty, \bar{A}\otimes\omega^i) \to \mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi] \to \prod_{v\mid N_f} A_{f,i}^{G_v}/\pi \to 0 \tag{4.6}$$

*is exact for any $i$.*

*Proof.* The exactness of (4.6) follows from the definitions and Lemma 4.2.2 except for the surjectivity of the last map. For this, by Proposition 4.2.3 there exists a totally ramified extension $K'/K$ and a $p$-stabilized newform $f_0$ over $K'$ such that $(\bar{A}\otimes\omega^i, \mathcal{S}_{\min}) \cong (A_{f_0,i}[\pi'], \mathcal{S}(f_0,i))$. The surjectivity is then a formal consequence of the exact sequence of Proposition 4.2.4 applied to both $A_{f_0,i}$ and $A_{f,i}$ with $\Sigma = \{v \mid N_f p\}$. □

4.3. **Algebraic Iwasawa invariants.** Let $\bar{A}$ be as in the previous section. We write $\bar\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ for the corresponding Galois representation. We now use Corollary 4.2.5 to study the relations between Selmer groups of newforms in the Hida family of $\bar\rho$. We first consider the behavior of the corresponding finite/singular structures on $\bar{A}$.

**Lemma 4.3.1.** *Let $f$ be a newform on the branch $\mathbf{T}(\mathfrak{a})$ of the Hida family of $\bar\rho$. Then for any place $v$, the dimension of $H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v}, \bar{A}\otimes\omega^i)$ depends only on the branch $\mathbf{T}(\mathfrak{a})$ and $i$.*

**Remark 4.3.2.** It was pointed out to the authors by Ralph Greenberg that although the dimension of the local condition is constant along branches of Hida families, it is possible (when the branch is special at $v$) that the actual subspace $H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v}, \bar{A}\otimes\omega^i)$ varies as $f$ varies along the branch $\mathbf{T}(\mathfrak{a})$. It would be interesting to determine if such an example actually exists.

*Proof.* Fix a place $v \neq v_p$ dividing the rational prime $\ell$. Consider first the unramified $G_\ell$-representation $(V_f)_{I_\ell}$. By the proof of Proposition 2.2.3 the dimension of this space is constant on the branch $\mathbf{T}(\mathfrak{a})$. It follows that, as a $G_\ell$-representation, the residual representation of $(V_f)_{I_\ell}$ is independent of $f$ on $\mathbf{T}(\mathfrak{a})$.

Consider now the exact sequence

$$0 \to A_{f,i}^{G_v}/\pi \to H^1(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i) \to H^1(\mathbf{Q}_{\infty,v}, A_{f,i})[\pi] \to 0. \tag{4.7}$$

By [15, Prop. 2.4], $H^1(\mathbf{Q}_{\infty,v}, A_{f,i})[\pi]$ has $k$-dimension equal to the multiplicity of $\omega^{1-i}$ in the residual representation of the unramified $G_\ell$-representation $(V_f)_{I_\ell}$. By the above discussion this is independent of $f$ on the branch $\mathbf{T}(\mathfrak{a})$, as claimed. □

For a branch $\mathbf{T}(\mathfrak{a})$ of the Hida family of $\bar{\rho}$ and a place $v \neq v_p$, we may now define

$$\delta_v(\mathfrak{a}, \omega^i) := \dim_k H^1_{f,\mathcal{S}(f,i)}(\mathbf{Q}_{\infty,v}, \bar{A} \otimes \omega^i) = \dim_k A_{f,i}^{G_v}/\pi$$

for any form $f$ lying on the branch $\mathbf{T}(\mathfrak{a})$.

We say that

$$\mu^{\mathrm{alg}}(\bar{\rho}, \omega^i) = 0$$

if $\mathrm{Sel}_{\mathcal{S}_{\min}}(\mathbf{Q}_\infty, \bar{A} \otimes \omega^i)$ is finite dimensional over $k$; if this is the case we define

$$\lambda^{\mathrm{alg}}(\bar{\rho}, \omega^i) = \dim_k \mathrm{Sel}_{\mathcal{S}_{\min}}(\mathbf{Q}_\infty, \bar{A} \otimes \omega^i).$$

**Theorem 4.3.3.** *Let $\bar{\rho}$ be as above. Let $f$ be any newform in the Hida family of $\bar{\rho}$. Then $\mu^{\mathrm{alg}}(\bar{\rho}, \omega^i) = 0$ if and only if $\mu^{\mathrm{alg}}(f, \omega^i) = 0$.*

*Proof.* By Theorem 4.1.1, $\mu^{\mathrm{alg}}(f, \omega^i)$ vanishes if and only if $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})[\pi]$ is finite dimensional. By Corollary 4.2.5 this is equivalent to the finite dimensionality of $\mathrm{Sel}_{\mathcal{S}_{\min}}(\mathbf{Q}_\infty, \bar{A} \otimes \omega^i)$, as claimed. □

**Theorem 4.3.4.** *Let $\bar{\rho}$ be as above. Assume that $\mu^{\mathrm{alg}}(\bar{\rho}, \omega^i) = 0$.*

(1) *Let $f$ be a newform of level $N_f$ lying on the branch $\mathbf{T}(\mathfrak{a})$ of the Hida family of $\bar{\rho}$. Then*

$$\lambda^{\mathrm{alg}}(f, \omega^i) = \lambda^{\mathrm{alg}}(\bar{\rho}, \omega^i) + \sum_{v \mid N_f} \delta_v(\mathfrak{a}, \omega^i).$$

*In particular, $\lambda^{\mathrm{alg}}(f, \omega^i)$ depends only on the branch $\mathbf{T}(\mathfrak{a})$ of $f$; we write $\lambda^{\mathrm{alg}}(\mathfrak{a}, \omega^i)$ for this value.*

(2) *Let $\mathbf{T}(\mathfrak{a}_1)$ and $\mathbf{T}(\mathfrak{a}_2)$ be two branches of the Hida family of $\bar{\rho}$. Then*

$$\lambda^{\mathrm{alg}}(\mathfrak{a}_1, \omega^i) - \lambda^{\mathrm{alg}}(\mathfrak{a}_2, \omega^i) = \sum_{v \neq v_p} \delta_v(\mathfrak{a}_1, \omega^i) - \delta_v(\mathfrak{a}_2, \omega^i).$$

*Proof.* The first statement is immediate from Corollary 4.2.5 and the definition of $\delta_v(\mathfrak{a}, \omega^i)$. The second statement follows from the first. □

We remark that in Section 5 we will see that the formulas of the preceding theorem compare well with the formulas of Theorem 3.7.7.

## 5. Applications to the main conjecture and examples

5.1. **The main conjecture.** Let $\mathcal{O}$ be the ring of integers of a finite extension $K$ of $\mathbf{Q}_p$ and let $f \in \mathcal{O}[[q]]$ be a $p$-ordinary eigenform such that the residual representation $\bar{\rho}_f$ is irreducible. Let $L_p^{\mathrm{alg}}(f, \omega^i) \in \Lambda_{\mathcal{O}}$ denote a generator of the characteristic power series of the $\Lambda_{\mathcal{O}}$-dual of the Selmer group $\mathrm{Sel}(\mathbf{Q}_\infty, A_{f,i})$ of Section 4.1. Let $L_p^{\mathrm{an}}(f, \omega^i) \in \Lambda_{\mathcal{O}}$ denote the usual $p$-adic $L$-function of $f \otimes \omega^i$ (computed with respect to some canonical period). The main conjecture of Iwasawa theory in this context is the following; it is independent of the particular choice of coefficient field $K$ over which $f$ is defined.

**Conjecture 5.1.1.** *There is a unit $u \in \Lambda_{\mathcal{O}}^{\times}$ such that*

$$L_p^{\mathrm{alg}}(f, \omega^i) \cdot u = L_p^{\mathrm{an}}(f, \omega^i).$$

We remark that the Selmer group in the above conjecture is not the Bloch–Kato Selmer group in which the local condition at $p$ is defined via crystalline periods, but instead is the Greenberg Selmer group in which the local condition at $p$ is defined via the ordinary filtration as in Section 4.1. The former group is always contained in the latter group and the quotient is trivial unless the analytic $p$-adic $L$-function has a trivial zero, in which case it has corank one (see [10, pp. 108-109]). It is for this reason that in our statement of the main conjecture we do not need to consider separately the case of trivial zeroes.

The following deep theorem of Kato [20] establishes one divisibility of the main conjecture.

**Theorem 5.1.2** (Kato)**.** *There is a $u \in \Lambda_{\mathcal{O}} \otimes \mathbf{Q}_p$ such that*

$$L_p^{\mathrm{alg}}(f, \omega^i) \cdot u = L_p^{\mathrm{an}}(f, \omega^i).$$

*In particular, to verify the main conjecture for $f \otimes \omega^i$ it suffices to check that*

$$\mu^{\mathrm{alg}}(f, \omega^i) = \mu^{\mathrm{an}}(f, \omega^i) \text{ and } \lambda^{\mathrm{alg}}(f, \omega^i) = \lambda^{\mathrm{an}}(f, \omega^i).$$

Our results in Sections 3 and 4 yield the following result.

**Theorem 5.1.3.** *Let $k$ be a finite field of characteristic $p$ and let $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ be an irreducible, modular, $p$-ordinary and $p$-distinguished representation; as always we fix a choice of $p$-stabilization. Suppose that*

$$\mu^{\mathrm{alg}}(f_0, \omega^i) = \mu^{\mathrm{an}}(f_0, \omega^i) = 0 \text{ and } \lambda^{\mathrm{alg}}(f_0, \omega^i) = \lambda^{\mathrm{an}}(f_0, \omega^i)$$

*for some $f_0$ in the Hida family attached to $\bar{\rho}$ and some $i$. Then*

$$\mu^{\mathrm{alg}}(f, \omega^i) = \mu^{\mathrm{an}}(f, \omega^i) = 0 \text{ and } \lambda^{\mathrm{alg}}(f, \omega^i) = \lambda^{\mathrm{an}}(f, \omega^i)$$

*for every $f$ in the Hida family attached to $\bar{\rho}$.*

Before giving a proof, we first state an immediate corollary of Theorem 5.1.3 and Kato's result.

**Corollary 5.1.4.** *Let $\bar{\rho}$ be as above and suppose that $\mu^{\mathrm{alg}}(\bar{\rho}, \omega^i) = \mu^{\mathrm{an}}(\bar{\rho}, \omega^i) = 0$ for some $i$. If the main conjecture holds for $f_0 \otimes \omega^i$ for one form $f_0$ in the Hida family of $\bar{\rho}$, then the main conjecture holds for $f \otimes \omega^i$ for every form $f$ in the Hida family of $\bar{\rho}$.*

Since every Hida family contains a form of weight two, this corollary in particular reduces the main conjecture to the case of weight two and to the conjecture on the vanishing of the $\mu$-invariants.

The proof of Theorem 5.1.3 is based on the following lemma which relates the invariants $e_\ell(\mathfrak{a}, \omega^i)$ and $\delta_v(\mathfrak{a}, \omega^i)$ of Theorem 3.7.7 and Section 4.3.

**Lemma 5.1.5.** *Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be minimal primes of $\mathbf{T}_\Sigma$. For any prime $\ell \neq p$*

$$\sum_{v | \ell} \delta_v(\mathfrak{a}_1, \omega^i) - \delta_v(\mathfrak{a}_2, \omega^i) = e_\ell(\mathfrak{a}_2, \omega^i) - e_\ell(\mathfrak{a}_1, \omega^i)$$

*where the sum is taken over all primes $v$ of $\mathbf{Q}_\infty$ over $\ell$.*

*Proof.* To prove the lemma it suffices to see that for any minimal prime $\mathfrak{a}$ of $\mathbf{T}_\Sigma$, the sum
$$e_\ell(\mathfrak{a},\omega^i)+\sum_{v|\ell}\delta_v(\mathfrak{a},\omega^i)$$
is independent of $\mathfrak{a}$. For this, fix a classical newform $f$ on the branch $\mathbf{T}(\mathfrak{a})$. Consider first the group $H^1(\mathbf{Q}_{\infty,v},A_{f,i})$. Since $H^1(\mathbf{Q}_{\infty,v},A_{f,i})$ is divisible (as $G_v$ has $p$-cohomological dimension one) we have
$$\dim_k H^1(\mathbf{Q}_{\infty,v},A_{f,i})[\pi]=\lambda\left(\mathrm{char}_{\Lambda_{\mathcal{O}}}(H^1(\mathbf{Q}_{\infty,v},A_{f,i})^\vee)\right)$$
where $M^\vee=\mathrm{Hom}(M,\mathbf{Q}_p/\mathbf{Z}_p)$.

By [15, Proposition 2.4] we have
$$\mathrm{char}_{\Lambda_{\mathcal{O}}}\left(\oplus_{v|\ell}H^1(\mathbf{Q}_{\infty,v},A_{f,i})^\vee\right)=E_\ell(f,\langle\ell\rangle_{p,i}^{-1})\cdot\Lambda_{\mathcal{O}}.$$
Since $E_\ell(f,\langle\ell\rangle_{p,i}^{-1})$ is simply $E_\ell(\mathfrak{a},\langle\ell\rangle_{p,i}^{-1})$ mod $\wp_f$, we conclude that
$$\sum_{v|\ell}\dim_k H^1(\mathbf{Q}_{\infty,v},A_{f,i})[\pi]=e_\ell(\mathfrak{a},\omega^i). \tag{5.1}$$

Consider now the exact sequence
$$0\to A_{f,i}^{G_v}/\pi\to H^1(\mathbf{Q}_{\infty,v},\bar{A}\otimes\omega^i)\to H^1(\mathbf{Q}_{\infty,v},A_{f,i})[\pi]\to 0.$$
Since the first term has $k$-dimension $\delta_v(\mathfrak{a},\omega^i)$ and the second term is certainly independent of $\mathfrak{a}$, the lemma now follows from (5.1). □

*Proof of Theorem 5.1.3.* Let $f$ be a form in the Hida family of $\bar\rho$. The vanishing of the $\mu$-invariants of $f$ is immediate from that for $f_0$ and Theorems 4.3.3 and 3.7.5.

By Lemma 5.1.5 and Theorems 4.3.4 and 3.7.7 we see also that
$$\lambda^{\mathrm{alg}}(f,\omega^i)-\lambda^{\mathrm{alg}}(f_0,\omega^i)=\lambda^{\mathrm{an}}(f,\omega^i)-\lambda^{\mathrm{an}}(f_0,\omega^i).$$
Since we are assuming the main conjecture for $f_0\otimes\omega^i$, we have that
$$\lambda^{\mathrm{alg}}(f_0,\omega^i)=\lambda^{\mathrm{an}}(f_0,\omega^i),$$
so that it follows that
$$\lambda^{\mathrm{alg}}(f,\omega^i)=\lambda^{\mathrm{an}}(f,\omega^i)$$
as desired. □

5.2. **Raising the level.** We conclude our general discussion with some results on branches of Hida families. It is well-known (see [8, 9]) which levels can occur among forms in the Hida family of $\bar\rho$. In the next proposition we list the cases in which the invariant $\delta_v(\cdot,\omega^i)$ increases.

**Proposition 5.2.1.** *Let $\bar\rho: G_{\mathbf{Q}}\to\mathrm{GL}_2(k)$ be an irreducible, $p$-ordinary and $p$-distinguished modular Galois representation; assume further that $\bar\rho$ is ramified at $p$. Let $\Sigma$ be some finite set of primes not containing $p$ and let $\ell\neq p$ be a prime at which $\bar\rho$ is unramified. Set $\Sigma'=\Sigma\cup\{\ell\}$ and let $\mathbf{T}_\Sigma$, $\mathbf{T}_{\Sigma'}$ be the Hida algebras associated to $\bar\rho$. Let $a_\ell=\mathrm{Trace}\,\bar\rho(\mathrm{Frob}_\ell)$ and $c_\ell=\det\bar\rho(\mathrm{Frob}_\ell)$, both viewed as elements of $k$. Then for every branch $\mathbf{T}(\mathfrak{a})$ of $\mathbf{T}_\Sigma$:*

1. *If there is an $i$ such that $a_\ell=\ell^{-i}+\ell^{1-i}$ and $c_\ell=\ell^{1-2i}$, then there is a branch $\mathbf{T}(\mathfrak{b})$ of $\mathbf{T}_{\Sigma'}$ such that*
$$N(\mathfrak{b})=N(\mathfrak{a})\ell \text{ and } \delta_v(\mathfrak{b},\omega^i)=1$$
*for all places $v$ dividing $\ell$.*

(2) *If $\ell \equiv 1 \pmod{p}$ and $a_\ell = c_\ell + 1$, then there is a branch $\mathbf{T}(\mathfrak{b})$ of $\mathbf{T}_{\Sigma'}$ (distinct from the branch provided by (1)) such that*

$$N(\mathfrak{b}) = N(\mathfrak{a})\ell \text{ and } \delta_v(\mathfrak{b}, \omega^i) = 1$$

*for all places $v$ dividing $\ell$ and all $i$.*

(3) *If $\ell \equiv 1 \pmod{p}$, $a_\ell = 2$ and $c_\ell = 1$, then there are distinct branches $\mathbf{T}(\mathfrak{b}_1)$ and $\mathbf{T}(\mathfrak{b}_2)$ of $\mathbf{T}_{\Sigma'}$ such that*

$$N(\mathfrak{b}_1) = N(\mathfrak{b}_2) = N(\mathfrak{a})\ell^2 \text{ and } \delta_v(\mathfrak{b}_1, \omega^i) = \delta_v(\mathfrak{b}_2, \omega^i) = 2$$

*for all places $v$ dividing $\ell$ and all $i$.*

(4) *If $\ell \equiv -1 \pmod{p}$, $a_\ell = 0$ and $c_\ell = -1$, then there is a branch $\mathbf{T}(\mathfrak{b})$ of $\mathbf{T}_{\Sigma'}$ such that*

$$N(\mathfrak{b}) = N(\mathfrak{a})\ell^2 \text{ and } \delta_v(\mathfrak{b}, \omega^i) = 1$$

*for all places $v$ dividing $\ell$ and all $i$.*

Note in particular that if (3) holds, then (1) and (2) also hold, so that the proposition exhibits four distinct non-minimal branches of $\mathbf{T}'_\Sigma$ for each branch of $\mathbf{T}_\Sigma$.

*Proof.* If (1) holds, then one sees easily that the eigenvalues of $\mathrm{Frob}_\ell$ on $\bar{\rho}$ equal $\ell^{1-i}$ and $\ell^{-i}$. It follows that there exists a ramified representation

$$\tau_\ell : G_\ell \to \mathrm{GL}_2(\mathbf{Z}_p)$$

such that $\bar{\tau}_\ell \otimes k \cong \bar{\rho}|_{G_\ell}$ and

$$\tau_\ell = \begin{pmatrix} \omega^{1-i} & * \\ 0 & \omega^{-i} \end{pmatrix}.$$

By [9, Theorem 1] (which applies since $\bar{\rho}$ is modular of weight 2 by [5, Theorem 1.1]) there exists a newform $g$ of tame level $N\ell$ in the Hida family of $\bar{\rho}$ such that $\rho_g|_{I_\ell} \cong \tau_\ell|_{I_\ell}$. Since $\bar{\rho} \otimes \omega^i$ is unramified at $\ell$ with Frobenius eigenvalues $\ell$ and 1, one computes easily that

$$\delta_v(A_g, \omega^i) = 1$$

for any place $v$ dividing $\ell$. In particular, the branch $\mathfrak{a}_1$ on which $g$ lies satisfies the requirements of the proposition.

The proofs for (2) (principal series), (3) (either special or principal series) and (4) (supercuspidal) are entirely similar; the conditions on $\ell$, $a_\ell$ and $c_\ell$ are simply those obtained by combining the conditions of [8, p. 435] for ramified lifts of the appropriate type to exist with the requirement that $\bar{\rho}(\mathrm{Frob}_\ell)$ has a trivial eigenvalue. Note also that $\omega|_{G_\ell}$ is trivial (resp. quadratic) in (2) and (3) (resp. (4)), so that the choice of $i$ is irrelevant. Finally, the branches in (1) and (2) (resp. in (3)) are distinct since one is special at $\ell$ and the other is principal series at $\ell$. □

It would be interesting to determine when the branches of the preceding proposition intersect. The following proposition uses $\Lambda$-adic level raising to give some insight into this question.

**Proposition 5.2.2.** *We maintain the hypotheses of the previous proposition. Let $\mathfrak{a}$ be a minimal prime of $\mathbf{T}_\Sigma$ such that*

$$(T_\ell \textit{ mod } \mathfrak{a}')^2 - \ell^{-2}\langle \ell \rangle_{N(\mathfrak{a})p}(\ell+1)^2$$

*is not a unit in* $\mathbf{T}(\mathfrak{a})^\circ$. *(Recall that* $\mathfrak{a}'$ *is a minimal prime of* $\mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}$ *sitting over* $\mathfrak{a}$ *and that* $\mathbf{T}(\mathfrak{a})^\circ \cong \mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}'$*.) Then there exists some minimal prime* $\mathfrak{b}$ *of* $\mathbf{T}_{\Sigma'}$ *with* $N(\mathfrak{b}) = N(\mathfrak{a})\ell$ *and a height one prime* $\wp$ *of* $\mathbf{T}_{\Sigma'}$ *such that* $\mathbf{T}(\mathfrak{a})$ *and* $\mathbf{T}(\mathfrak{b})$ *cross at* $\wp$*. (That is, both* $\mathfrak{b}$ *and the pre-image of* $\mathfrak{a}$ *under the natural map* $\mathbf{T}_{\Sigma'} \to \mathbf{T}_\Sigma$ *are contained in* $\wp$*.)*

*Proof.* Attached to the minimal prime $\mathfrak{a}$ we have the $\Lambda$-adic modular form

$$f(\mathfrak{a}, q) = \sum_{n \geq 1} (T_n \bmod \mathfrak{a}') q^n.$$

By [4, Theorem 6C], there exists a finite extension $R$ of $\mathbf{T}(\mathfrak{a})^\circ$, a $\Lambda$-adic modular form $g(q) = \sum_{n\geq 1} b_n q^n \in R[[q]]$ new of level $N(\mathfrak{a})\ell$, and a height one prime $\wp'$ of $R$ such that

$$(T_n \bmod \mathfrak{a}') \equiv b_n \pmod{\wp'} \tag{5.2}$$

for each $n$ relatively prime to $N(\mathfrak{a})\ell$.

The $\Lambda$-adic form $g(q)$ corresponds to some minimal prime ideal of $\mathbf{T}_\Sigma$. Indeed, there is a map $\mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})\ell} \to R$ that sends $T_n$ to $b_n$. Let $\mathfrak{b}' \subseteq \mathbf{T}^{\mathrm{new}}_{N(\mathfrak{a})\ell}$ denote the kernel of this map; it is necessarily a minimal prime since both the source and the target are finite extensions of $\Lambda$. The preimage of $\mathfrak{b}'$ in $\prod_{M|N(\Sigma')} \mathbf{T}^{\mathrm{new}}_M$ is a minimal prime and thus, by Proposition 2.3.2, corresponds to some minimal prime of $\mathbf{T}'_{N(\Sigma')}$. Since the residual representation attached to $g(q)$ equals $\bar{\rho}$, this minimal prime yields a minimal prime of $\mathbf{T}_{\Sigma'}$ which we denote by $\mathfrak{b}$. Note that by construction $N(\mathfrak{b}) = N(\mathfrak{a})\ell$.

We thus have a map $\mathbf{T}_{\Sigma'} \to R$ (sending $T_n$ to $b_n$) with kernel $\mathfrak{b}$. If let $\tilde{\mathfrak{a}}$ denotes the pre-image of $\mathfrak{a}$ in $\mathbf{T}_{\Sigma'}$, then we also have a map $\mathbf{T}_{\Sigma'} \to R$ with kernel $\tilde{\mathfrak{a}}$ (given by reducing mod $\tilde{\mathfrak{a}}$ and then embedding the image $\mathbf{T}(\mathfrak{a})$ into $R$). By (5.2), the reduction of these two maps modulo $\wp'$ are the same. If we let $\wp$ denote the kernel of either of these maps modulo $\wp'$, then $\wp$ is a height one prime containing both of $\mathfrak{b}$ and $\tilde{\mathfrak{a}}$, as desired. □

The ideal $\wp$ of the previous proposition could potentially be a prime ideal lying over the principal ideal $(p)$. It would be interesting to determine whether or not these branches actually meet at a prime ideal of residue characteristic zero.

## 5.3. **Examples.**

**Example 5.3.1.** Set $p = 11$ and let $f$ denote the weight 2 newform associated to the elliptic curve $X_0(11)$. It follows from the fact that $f$ is the only newform of weight 2 and level dividing 11 that $\mathbf{T} := \mathbf{T}_\Sigma(\bar{\rho}_f) \cong \Lambda$ for $\Sigma = \emptyset$. Thus for $k \geq 2$, there is a unique newform $f_k$ of weight $k$ and level dividing 11 that is congruent to $f$ modulo 11. For example, $f_{12}$ is the 11-ordinary, 11-stabilized oldform of level 11 attached to the Ramanujan $\Delta$-function.

The $p$-adic $L$-function of this family was studied in detail in [14]. Also, in [11], the congruence between $X_0(11)$ and $\Delta$ was exploited to gain information about the Selmer groups of both forms. We review below what is known in these examples and then go on to study the other branches of this Hida family with tame conductor greater than one.

We first verify the main conjecture for $f$. Since $X_0(11)$ has split multiplicative reduction at $p = 11$, the $p$-adic $L$-function $L_p^{\mathrm{an}}(f, T)$ has a trivial zero at $T = 0$.

By the Mazur, Tate and Teitelbaum conjecture (proved by Greenberg and Stevens [14]) we have an explicit formula for the derivative of $L_p^{\mathrm{an}}(f,T)$ at 0; that is,

$$\frac{d}{dT}L_p^{\mathrm{an}}(f,T)\Big|_{T=0} = \frac{\mathcal{L}_p(f)}{\log_p(\gamma)} \cdot \frac{L(f,1)}{\Omega_f}.$$

(Here, $\gamma$ is a topological generator of $1+p\mathbf{Z}_p$ that is implicitly chosen by writing the $p$-adic $L$-function in the $T$-variable; the appearance of the extra factor of $\log_p(\gamma)$ above is accounted for by the fact that we have written this formula in the $T$-variable rather than the more standard $s$-variable.) One checks that $\mathcal{L}_p(f)$ and $\log_p(\gamma)$ are exactly divisible by 11 while $L(f,1)/\Omega_f$ is an 11-adic unit. Thus $L_p^{\mathrm{an}}(f,T)$ is a unit multiple of $T$ so that $\lambda^{\mathrm{an}}(f) = 1$ and $\mu^{\mathrm{an}}(f) = 0$.

Since $L(f,1)/\Omega_f$ is an 11-adic unit, by an Euler system argument of Kolyvagin, the classical 11-adic Selmer group $\mathrm{Sel}_p(X_0(11)/\mathbf{Q})$ of the elliptic curve $X_0(11)$ vanishes. It thus follows from a refined control theorem of Greenberg ([13, Proposition 3.7]) that $\mathrm{Sel}_p(X_0(11)/\mathbf{Q}_\infty)$ vanishes as well. The latter group agrees with the Bloch–Kato Selmer group of $f$; as $f$ has a trivial zero, it follows that the Greenberg Selmer group $\mathrm{Sel}_p(\mathbf{Q}_\infty, A_f)$ is simply $\mathbf{Q}_p/\mathbf{Z}_p$ with trivial Galois action. (This example is worked out in detail in [11, Example 3].) Thus $L_p^{\mathrm{alg}}(f,T)$ is a unit multiple of $T$, so that $\lambda^{\mathrm{alg}}(f) = 1$ and $\mu^{\mathrm{alg}}(f) = 0$. In particular, this verifies the main conjecture for $X_0(11)$ at $p = 11$.

Corollary 5.1.4 now shows that the main conjecture holds at $p = 11$ for each form $f_k$ with $k \geq 2$, and Theorem 5.1.3 shows that

$$\lambda^{\mathrm{an}}(f_k) = \lambda^{\mathrm{alg}}(f_k) = 1 \text{ for } k \geq 2.$$

Note that the $p$-adic $L$-function corresponding to $X_0(11)$ is the only $p$-adic $L$-function in the Hida family with a trivial zero. Nonetheless, the existence of this trivial zero forces every $p$-adic $L$-function in the family to have at least one zero.

For $k \equiv 2 \pmod{10}$, the unique zero of $L_p^{\mathrm{an}}(f_k)$ can be explained by the functional equation of the $p$-adic $L$-function. Namely, the sign of this functional equation is $-1$ for such $k$ and thus $L_p^{\mathrm{an}}(f_k)$ vanishes at the character that sends $x$ to $x^{\frac{k-2}{2}}$. Using this observation, we can determine the two-variable $p$-adic $L$-function attached to $\bar{\rho}_f$ (which we denote by $L(\bar{\rho}_f)$). By Theorem 3.7.7, we have that $\lambda(L(\bar{\rho}_f)) = 1$ and hence, under the identification of (3.9),

$$L(\bar{\rho}_f) = (T - a_0) \cdot u(T)$$

with $a_0 \in \mathbf{T}$ and $u(T) \in \mathbf{T}[[T]]$ a unit. Since $L(\bar{\rho}_f)$ specializes at weight $k$ to $L_p(f_k)$ (Proposition 3.4.3), we know that $a_0$ specializes at weight $k$ to the unique zero of $L_p(f_k)$. In particular, for $k \equiv 2 \pmod{10}$, we have that $a_0$ specializes to $\gamma^{\frac{k-2}{2}}$. Thus, $a_0 = \gamma^{-1}\langle\gamma\rangle_p^{\frac{1}{2}}$ since this element has the correct specialization for infinitely many $k$.

So far we have only applied our results to minimal lifts of $\bar{\rho}_f$; however, Corollary 5.1.4 applies to every modular form lifting this representation. For instance, we can use Proposition 5.2.1 to explicitly find primes $\ell$ to add to the level that will produce modular forms that have higher $\lambda$-invariants and for which we still know the main conjecture.

The first prime that satisfies condition (3) (and thus (1) and (2)) of Proposition 5.2.1 is $\ell = 1321$. If we set $\Sigma = \{1321\}$, then $\mathbf{T}_\Sigma$ will have four distinct branches $\mathbf{T}(\mathfrak{a}_1), \ldots, \mathbf{T}(\mathfrak{a}_4)$ with $\lambda(\mathfrak{a}_1) = \lambda(\mathfrak{a}_2) = 2$ and $\lambda(\mathfrak{a}_3) = \lambda(\mathfrak{a}_4) = 3$. (Here and in what follows, we do not distinguish between the analytic and algebraic $\lambda$-invariants in cases where the main conjecture is known to be true.) Note that 1321 is inert in the cyclotomic $\mathbf{Z}_{11}$-extension of $\mathbf{Q}$ and so there is a unique prime $v$ of $\mathbf{Q}_\infty$ sitting over 1321.

The prime $\ell' = 2113$ is also inert in $\mathbf{Q}_\infty/\mathbf{Q}$ and satisfies condition (3) of Proposition 5.2.1. If $\Sigma' = \{1321, 2113\}$, we may apply Proposition 5.2.1 to each of the five branches we have exhibited in $\mathbf{T}_\Sigma$ to see that $\mathbf{T}_{\Sigma'}$ possesses at least 20 additional branches with $\lambda$-invariants ranging between 3 and 5.

**Example 5.3.2.** We conclude by considering an example of Greenberg and Vatsal in [15]. Consider the elliptic curves

$$E_1 : y^2 = x^3 + x - 10 \qquad\qquad E_2 : y^2 = x^3 - 584x + 5444,$$

of conductors 52 and 364 respectively, which are ordinary and congruent mod 5. One computes that $\lambda^{\mathrm{an}}(E_1) = \mu^{\mathrm{an}}(E_1) = 0$ so that Kato's divisibility (Theorem 5.1.2) yields the main conjecture for $E_1$ at $p = 5$. The main conjecture for $E_2$ at $p = 5$ follows and in this case we have $\lambda(E_2) = 5$. (This example is discussed in detail in [15, pp. 22, 44].)

We now examine this congruence from the point of view of Hida theory. Let $f_i$ denote the newform of weight 2 associated to $E_i$. One checks that $f_1$ is not congruent modulo 5 to any other modular forms of weight two and level dividing $2^2 \cdot 5 \cdot 13^2$ (using [32], for example). It follows that we have $\mathbf{T}_\Sigma \cong \Lambda$ for $\Sigma = \{2, 13\}$. For consistency of notation, let us denote the unique irreducible component of this space as $\mathbf{T}(\mathfrak{a}_1)$. Moreover, since $\mu(f_1) = \lambda(f_1) = 0$, the two-variable $p$-adic $L$-function attached to $\mathbf{T}_\Sigma$ is a unit and $\lambda(\mathfrak{a}_1) = 0$.

The prime $\ell = 7$ satisfies condition (1) of Proposition 5.2.1 (with $i = 0$). Thus, if $\Sigma' = \{2, 7, 13\}$, then $\mathbf{T}_{\Sigma'}$ contains an irreducible component $\mathbf{T}(\mathfrak{a}_2)$ not contained in $\mathbf{T}_\Sigma$. Moreover, since $f_2$ is the unique normalized newform of weight 2 congruent modulo 5 to $f_1$ with level dividing $2^2 \cdot 5 \cdot 7^2 \cdot 13^2$, it follows that $\mathbf{T}_{\Sigma'}$ has rank 2 over $\Lambda$. Hence, $\mathbf{T}(\mathfrak{a}_2)$ is the only branch of $\mathbf{T}_{\Sigma'}$ not coming from $\mathbf{T}_\Sigma$. By Theorems 3.7.7 and 4.3.4, we have that $\lambda(\mathfrak{a}_2) = 5$. Since $f_2$ sits on the branch $\mathbf{T}(\mathfrak{a}_2)$, it follows that $\lambda(f_2) = 5$.

We close with some questions. Proposition 5.2.2 establishes that the branches $\mathbf{T}(\mathfrak{a}_1)$ and $\mathbf{T}(\mathfrak{a}_2)$ must cross at some (non-classical) height one prime, but does not exclude the possibility that they cross at the prime $(p)$. Do they in fact cross at a prime of residue characteristic zero and if so could one compute $p$-adic approximations of this prime? How many such crossing points do these two branches share? It appears at present that little is known about the shape of these Hida families when multiple branches appear (even in any particular case).

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