Draft: October 10, 2007

JACQUET MODULES OF LOCALLY ANALYTIC REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS II. THE RELATION TO PARABOLIC INDUCTION

Matthew Emerton

Northwestern University

To Nicholas

Contents

Let L be a finite extension of $\mathbb{Q}_p$ (for some prime p), let G be (the L-valued points of) a connected reductive linear algebraic group over L, and let P be a parabolic subgroup of G, with unipotent radical N and Levi factor M. We also fix an extension K of L, complete with respect to a discrete valuation extending that of L. All representations will be on K-vector spaces (whether or not this is explicitly stated).

In the paper [7] we defined a Jacquet module functor J_P that takes locally analytic G-representations to locally analytic M-representations, which when restricted to the category of admissible smooth representations of G, coincides with the Jacquet module functor of [5] (i.e. the functor of N-coinvariants). This functor plays an important role in the study of certain global questions related to the p-adic interpolation of automorphic forms [8], and its applications in that context provided the original motivation for its introduction and study. However, it is natural to ask whether this functor also has an intrinsic representation theoretic significance. In considering this question, we are guided by the fact that in the theory of admissible smooth G-representations, the Jacquet module functor is both left, and, for more subtle reasons – namely the Casselman Duality Theorem – right adjoint to parabolic induction. More precisely, one has the two formulas

$$\mathrm{Hom}_G(V, (\mathrm{Ind}_P^G U)_{\mathrm{sm}}) \xrightarrow{\sim} \mathrm{Hom}_M(J_P(V), U) \tag{0.1}$$

and

$$\mathrm{Hom}_G((\mathrm{Ind}_{\overline{P}}^G U)_{\mathrm{sm}}, V) \xrightarrow{\sim} \mathrm{Hom}_M(U(\delta), J_P(V)), \tag{0.2}$$

The author would like to acknowledge the support of the National Science Foundation (award number DMS-0070711)

where U and V are arbitrary admissible smooth representations of M and G respectively, $\overline{P}$ denotes an opposite parabolic to P, $U(\delta)$ denotes U with its M-action twisted by the modulus character δ of P (thought of as a character of M), and $(\mathrm{Ind}_P^G U)_{\mathrm{sm}}$ and $(\mathrm{Ind}_{\overline{P}}^G U)_{\mathrm{sm}}$ denote the smooth induction of U to G from P and $\overline{P}$ respectively. (See [5, §4].)

In this paper we investigate the relation between the locally analytic Jacquet module functor J_P and parabolic induction, with the goal of extending the adjointness formula (0.2) to the locally analytic context. (The formula (0.1) cannot extend to the locally analytic context, since J_P is not right exact, as Example 5.1.9 below shows.) The formula that we will establish not only serves to establish the representation theoretic significance of the functor J_P, but also has important global applications – for example, to the construction of p-adic L-functions attached to automorphic forms [8, § 4.5].

The first result that we establish in the direction of our goal is the following lemma, proved in Subsection 2.8. (We refer to the *Notations and conventions* below for the definition of the various categories of locally analytic representations that appear in the statement of this and subsequent results.)

Lemma 0.3. *If U is any object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then there is a closed M-equivariant embedding $U(\delta) \to J_P(\mathrm{Ind}_{\overline{P}}^G U)$. (Here $\mathrm{Ind}_{\overline{P}}^G U$ denotes the locally analytic parabolic induction of U from $\overline{P}$ to G.)*

For any object U of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, this lemma allows us to regard U (with its M-action twisted by δ) as a subspace of $J_P(\mathrm{Ind}_{\overline{P}}^G U)$. We define $I_{\overline{P}}^G(U)$ to be the closed G-subrepresentation of $\mathrm{Ind}_{\overline{P}}^G U$ generated by the image of U under the canonical lifting of $J_P(\mathrm{Ind}_{\overline{P}}^G U)$ to $\mathrm{Ind}_{\overline{P}}^G U$. (Recall from [7, (0.9)] that the canonical lift depends for its definition on a choice of compact open subgroup N_0 of N, as well as a lift of M to a Levi factor of P. Here and below we fix such a choice of N_0, while we take the Levi factor to be $M = P \bigcap \overline{P}$. In fact $I_{\overline{P}}^G(U)$ is well-defined independent of the choice of N_0; see Lemma 2.8.3 below.) The subrepresentation $I_{\overline{P}}^G(U)$ is the part of $\mathrm{Ind}_{\overline{P}}^G U$ that can be "detected" by its Jacquet module. (See Corollary 5.1.4 below for a precise statement in the case when G is quasi-split and P is a Borel subgroup.) For example, if U is an admissible smooth M-representation, then $I_{\overline{P}}^G(U) = (\mathrm{Ind}_{\overline{P}}^G U)_{\mathrm{sm}}$. Passing to Jacquet modules thus yields, for any $U \in \mathrm{Rep}^z_{\mathrm{la.c}}(M)$ and $V \in \mathrm{Rep}_{\mathrm{la.c}}(G)$, an injection

$$\mathcal{L}_G(I_{\overline{P}}^G(U), V) \to \mathcal{L}_M(U(\delta), J_P(V)). \tag{0.4}$$

It is not the case in general that (0.4) is surjective, for the following reason. If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$, then the canonical lifting $J_P(V) \to V$ is a $\mathrm{U}(\mathfrak{p})$-equivariant map (where $\mathrm{U}(\mathfrak{p})$ acts on $J_P(V)$ through its quotient $\mathrm{U}(\mathfrak{m})$), and so induces a $\mathrm{U}(\mathfrak{g})$-equivariant map

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} J_P(V) \to V. \tag{0.5}$$

(Here we are using gothic letters to denote Lie algebras, and $\mathrm{U}(-)$ to denote universal enveloping algebras.) Taking $V = \mathrm{Ind}_{\overline{P}}^G(U)$ in (0.5) (for any object U of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$), and taking into account Lemma 0.3, we obtain in particular a $\mathrm{U}(\mathfrak{g})$-equivariant map

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U \to I_{\overline{P}}^G(U). \tag{0.6}$$

(We have dropped the twist by δ, since δ is a smooth character, and so induces the trivial character of $\mathfrak{p}$.) Any map in the source of (0.4) induces a commutative diagram

$$\begin{array}{ccc} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U & \longrightarrow & \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} J_P(V) \\ \downarrow{\scriptstyle (0.6)} & & \downarrow{\scriptstyle (0.5)} \\ I_P^G(U) & \longrightarrow & V. \end{array} \tag{0.7}$$

Definition 0.8. We say that a map in the target of (0.4) is *balanced* if the kernel of the induced map

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U \longrightarrow \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} J_P(V) \xrightarrow{(0.5)} V$$

contains the kernel of (0.6). We let $\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$ denote the subspace of $\mathcal{L}_M(U(\delta), J_P(V))$ consisting of balanced maps. (Note that this space depends not just on $J_P(V)$ as an M-representation, but on its realization as the Jacquet module of V. On the other hand, the kernel of (0.6), and hence the condition that a map be balanced, is independent of the choice of N_0 used to determine the canonical lifting; see Lemma 0.18 below.)

The commutativity of (0.7) shows that $\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}$ contains the image of (0.4).

The present paper aims to study the following question.

Question 0.9. Is the injection

$$\mathcal{L}_G(I_P^G(U), V) \to \mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}} \tag{0.10}$$

induced by (0.4) an isomorphism?

A positive answer (perhaps with some hypotheses on U and V) would yield a generalization of (0.2) to the locally analytic context. (Note that the condition of maps being balanced does not arise in the context of smooth representations, since the Lie algebra actions on smooth representations are trivial.)

Before stating our main result, we make two further definitions.

Definition 0.11. Let U be an object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$, and let A denote the space of linear M-equivariant endomorphisms of U. We say that U is allowable if for any pair of finite dimensional algebraic M-representations W_1 and W_2, each element of $\mathcal{L}_{A[M]}(U \otimes_K W_1, U \otimes_K W_2)$ is strict (i.e. has closed image). (Here each $U \otimes_K W_i$ is regarded as an $A[M]$-module via the A action on the left-hand factor along with the diagonal M-action.)

Any essentially admissible locally analytic representation of M is allowable, in the sense of this definition [6, Prop. 6.4.11, Prop. 6.4.16].

Definition 0.12. We say that an admissible locally analytic G-representation V is very strongly admissible if there is a continuous G-equivariant K-linear injection $V \to W$, where W is an admissible continuous representation of G on a K-Banach space.

Any very strongly admissible locally analytic G-representation is strongly admissible (see [6, Prop. 6.2.4] and [16, Thm. 7.1]), and conversely when G is compact [6, Prop. 6.5.9]. The author does not know an example of a strongly admissible locally analytic representation that is not very strongly admissible.

We may now state our main result (proved in Subsection 4.3).

Theorem 0.13. *If U is an allowable object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, and if V is a very strongly admissible G-representation, then the map (0.10) is an isomorphism.*

When combined with an explicit computation of Jacquet modules of the locally analytic principal series (see Proposition 5.1.5 and Remark 5.1.8 below), Theorem 0.13 has the following corollaries (proved in Subsection 5.3).

Corollary 0.14. *Suppose that G is quasi-split, and that P is a Borel subgroup of G. If V is a topologically irreducible very strongly admissible locally analytic representation of G for which $J_P(V) \neq 0$ then V is a quotient of $I^G_{\overline{P}}(U)$ for some finite dimensional locally analytic representation U of the maximal torus M of G.*

Corollary 0.15. *Suppose that G is quasi-split, and that P is a Borel subgroup of G. If V is an admissible locally analytic representation of G of finite length, whose composition factors are very strongly admissible, then $J_P(V)$ is a finite dimensional M-representation.*

As we remarked above, in addition to these purely representation theoretic applications, Theorem 0.13 has applications in the global context. For example, in the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, it can be applied to the construction of two-variable p-adic L-functions attached to overconvergent p-adic modular forms [8, §4.5].

A sketch of the proof of Theorem 0.13. For any object U of $\mathrm{Rep}_{\mathrm{la.c}}(M)$, denote by $\mathcal{C}^{\mathrm{sm}}_c(N, U)$ the space of locally constant compactly supported U-valued functions on N; see Subsection 2.2 below for a description of the topology of $\mathcal{C}^{\mathrm{sm}}_c(N, U)$, and of the natural P-action on this space. Let $I^G_{\overline{P}}(U)(N)$ denote the $(\mathfrak{g}, P)$-invariant closed subspace of $I^G_{\overline{P}}(U)$ consisting of elements supported on N. (Here we regard N as an open subset of $\overline{P}\backslash G$; we refer to Subsection 2.3 for a definition of the support of an element of $\mathrm{Ind}^G_{\overline{P}} U$ as a subset of $\overline{P}\backslash G$. Note that Lemma 2.3.5 implies that $I^G_{\overline{P}}(U)(N)$ is closed in $\mathrm{Ind}^G_{\overline{P}} U$.) The open immersion $N \to \overline{P}\backslash G$ induces a closed P-equivariant embedding $\mathcal{C}^{\mathrm{sm}}_c(N, U) \to \mathrm{Ind}^G_{\overline{P}} U$ which factors through $I^G_{\overline{P}}(U)(N)$, and hence induces a $(\mathfrak{g}, P)$-equivariant map $\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}^{\mathrm{sm}}_c(N, U) \to \mathrm{Ind}^G_{\overline{P}} U$. If $I^G_{\overline{P}}(U)^{\mathrm{lp}}(N)$ denotes the coimage of this map (i.e. the quotient of $\mathrm{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} \mathcal{C}^{\mathrm{sm}}_c(N, U)$ by the kernel of this map), then there is induced a continuous $(\mathfrak{g}, P)$-equivariant injection

$$I^G_{\overline{P}}(U)^{\mathrm{lp}}(N) \to I^G_{\overline{P}}(U)(N). \tag{0.16}$$

If U is an allowable object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then $I^G_{\overline{P}}(U)$ is a local closed G-subrepresentation of $\mathrm{Ind}^G_{\overline{P}} U$, polynomially generated by degree zero (using terminology to be introduced in Section 2), and $I^G_{\overline{P}}(U)^{\mathrm{lp}}(N)$ consists of the locally polynomial elements of $I^G_{\overline{P}}(U)(N)$; see Subsection 2.8. The results of Section 2 imply that $I^G_{\overline{P}}(U)$ is generated as a G-representation by $I^G_{\overline{P}}(U)(N)$, and that (0.16) has dense image.

We can now outline the proof of Theorem 0.13. Let U be an allowable object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, and V be a very strongly admissible locally analytic G-representation. The adjointness result of [7, Thm. 3.5.6] yields an isomorphism

$$\mathcal{L}_M(U(\delta), J_P(V)) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}^{\mathrm{sm}}_c(N,U), V). \tag{0.17}$$

The following result follows essentially by definition.

Lemma 0.18. *The isomorphism (0.17) restricts to an isomorphism*

$$\mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}} \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(I_P^G(U)^{\mathrm{lp}}(N), V).$$

Now restriction from $I_P^G(U)$ to $I_P^G(U)(N)$ induces a map

$$\mathcal{L}_G(I_P^G(U), V) \to \mathcal{L}_{(\overline{\mathfrak{g}},P)}(I_P^G(U)(N), V), \tag{0.19}$$

while (0.16) induces a map

$$\mathcal{L}_{(\mathfrak{g},P)}(I_P^G(U)(N), V) \to \mathcal{L}_{(\mathfrak{g},P)}(I_P^G(U)^{\mathrm{lp}}(N), V). \tag{0.20}$$

Taking into account Lemma 0.18, we see that Theorem 0.13 will follow if we can show that each of these maps is an isomorphism.

The proof that (0.19) is an isomorphism is the subject of Subsection 4.1. It relies crucially on the fact that $I_P^G(U)$ is generated by $I_P^G(U)(N)$ (and in fact, on the slightly stronger result of Lemma 2.4.13, which also addresses the issue of topologies), and on the assumption that the G-action on V is locally analytic. The proof that (0.20) is an isomorphism is the subject of Subsection 4.2. Since the map (0.16) is injective, with dense image (U being allowable), the key point is to show that any element of the source of (0.20) remains continuous when $I_P^G(U)^{\mathrm{lp}}(N)$ is equipped with the topology obtained by regarding it as a subspace of $I_P^G(U)$ via (0.16). This is done using a variant of the method of Amice-Vélu and Vishik [1, 18]; it is here that we use the assumption that V is very strongly admissible.

The arrangement of the paper. Section 1 develops some necessary preliminary results. In Subsection 1.1 we establish a useful structural result for objects of $\mathrm{Rep}^z_{\mathrm{la.c}}(G)$, for any locally L-analytic group G. In Subsection 1.2 we prove some basic results on what we call locally integrable $\mathfrak{g}$-representations; these are continuous representations of the Lie algebra $\mathfrak{g}$ of a locally L-analytic group G for which the action on any vector may be integrated to an action of some compact open subgroup of G. In Subsection 1.3 we prove a certain technical lemma. In Subsection 1.4 we describe some simple functional analysis related to spaces of compactly supported functions, and also to spaces of germs of locally analytic functions, on locally L-analytic spaces. In Subsection 1.5 we recall a result from highest weight theory.

In Section 2 we return to the setting of the introduction. After an initial discussion of locally analytic parabolic induction in Subsection 2.1, and after establishing some notation in Subsection 2.2, in Subsections 2.3 through 2.7 we develop the structure theory of what we call local closed G-subrepresentations of $\mathrm{Ind}_P^G U$. In Subsection 2.8 we apply these results to the representation $I_P^G(U)$.

As above, let N_0 denote a compact open subgroup of N. Let Z_M denote the centre of M, and define $Z_M^+ = \{z \in Z_M \mid zN_0z^{-1} \subset N_0\}$. In Section 3 we fix a K-Banach space U equipped with a continuous Z_M-action (note that there is then a natural action of the semidirect product $N_0 Z_M^+$ on $\mathcal{C}^{\mathrm{la}}(N_0, U)$), a pair of $N_0 Z_M^+$-invariant closed subspaces $S \subset T \subset \mathcal{C}^{\mathrm{la}}(N_0, U)$, and a K-Banach space W equipped with a continuous $N_0 Z_M^+$-action. We then give a criterion for the restriction map $\mathcal{L}_{N_0 Z_M^+}(T, W) \to \mathcal{L}_{N_0 Z_M^+}(S, W)$ to be an isomorphism. (Actually, we replace S and T by their ultrabornologicalizations; this is harmless from the point of view of our intended applications.) This result provides the variant of the results of Amice-Vélu and Vishik on tempered distributions that was alluded to above. The criterion that we give is described in terms of certain extra structure whose existence we assume. In Subsection 3.1 we describe this extra structure, and state our isomorphism criterion in terms of it, while the proof of the isomorphism criterion is given in Subsection 3.2.

In Section 4 we present the proof of Theorem 0.13. In fact we prove a more general result, in which $I_{\overline{P}}^G(U)$ is replaced by any local closed G-subrepresentation X of $\mathrm{Ind}_{\overline{P}}^G U$ that is polynomially generated by bounded degrees (as defined in Subsection 2.7); this result will be useful in future applications. Subsection 4.1 (respectively Subsection 4.2) proves that (the analogue for arbitrary X of) the map (0.19) (respectively the map (0.20)) is an isomorphism. In Subsection 4.3 we bring these results together to prove our main result, which then has Theorem 0.13 as a consequence.

In Section 5 we give some complements to and applications of our results, focusing on the case when G is quasi-split and P is a Borel subgroup. In Subsection 5.1 we present some results concerning the Jacquet modules of parabolically induced representations. In Subsection 5.2 we establish a certain technical result (Theorem 5.2.18). In Subsection 5.3 we prove Corollaries 0.14 and 0.15. In an appendix we establish some general results about admissible locally analytic representations that are required in Subsection 5.2.

A special case of the main theorem. A proof of the main theorem of Section 4, namely Theorem 4.3.2, in the special case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, the parabolic P is a Borel subgroup, the representation U being induced is a locally analytic character, and the local closed G-subrepresentation X of $\mathrm{Ind}_{\overline{P}}^G U$ is taken to be $\mathrm{Ind}_{\overline{P}}^G U$ itself, is given in the appendix to [3]. The argument in this special case follows the same lines as in the general case, but with many simplifications, both in the group theory and in the functional analysis. The reader interested in following the details of our argument in the general case may find it helpful to first study the argument in this simplified setting.

Notation and conventions. As in the introduction, we fix a prime p, a finite extension L of $\mathbb{Q}_p$, and an extension K of L that is complete with respect to a discrete valuation that extends the discrete valuation on L. All vector spaces that we consider will have coefficients in K (whether or not this is explicitly mentioned).

Throughout this paper, we will systematically identify linear algebraic groups over L with their groups of points; this should cause no confusion, since for a linear algebraic group, the set of L-valued points is Zariski dense in the corresponding algebraic group.

If V is a topological vector space, then by a topological action of a group (or semigroup, ring, Lie algebra, etc.) on V we mean an action on V via continuous endomorphisms. (We use this term in distinction to a continuous action on V, meaning an action for which the action map is jointly continuous, with respect to the topology on V and whatever topology is under consideration on the object that is acting.)

We refer to [6, §1] for the notion of a BH-space, and of a BH-subspace of a convex K-space. If V is a BH-space, then we let $\overline{V}$ denote the latent Banach space underlying V. (See [6, Def. 1.1.1, Prop. 1.1.2].) More generally, if V is an arbitrary Hausdorff convex K-vector space, then we let $\overline{V}$ denote the ultrabornologicalization of V; that is, $\overline{V} := \varinjlim_{W} \overline{W}$, where W runs over all BH-subspaces of V. (It follows from [6, Prop. 1.1.2] that the transition maps are continuous, and we endow $\overline{V}$ with the locally convex inductive limit topology.) Since $\overline{V}$ is a locally convex inductive limit of Banach spaces, it is ultrabornological [2, ex. 20, p. III.46]. The continuous injections $\overline{W} \to V$ induce a continuous bijection $\overline{V} \to V$, which is an isomorphism if and only if V is itself ultrabornological.

We refer to [6, §1] for the notion of a space of LB-type, and of an LB-space, and to [2, Cor., p. II.34] for the version of the Open Mapping Theorem that applies to maps between LB-spaces.

All locally L-analytic spaces appearing are assumed to be paracompact (and hence strictly paracompact in the sense of [13, p. 446], by [12, Satz 8.6]).

For any locally L-analytic group G, we let $\mathrm{Rep}_{\mathrm{top.c}}(G)$ denote the category whose objects are Hausdorff locally convex K-vector spaces of compact type, equipped with a topological action of G, and whose morphisms are continuous G-equivariant K-linear maps. We let $\mathrm{Rep}_{\mathrm{la.c}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{top.c}}(G)$ consisting of locally analytic representations of G on convex K-vector spaces of compact type. (The notion of a locally analytic representation of G is defined in [13, p. 12]; see also [6, Def. 3.6.9].)

If the centre Z_G of G is topologically finitely generated then we let $\mathrm{Rep}_{\mathrm{es}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{la.c}}(G)$ consisting of essentially admissible locally analytic representations of G (as defined in [6, Def. 6.4.9]). We let $\mathrm{Rep}_{\mathrm{ad}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{es}}(G)$ consisting of admissible locally analytic representations of G (as defined in [16]; see also [6, Def. 6.1.1]).

There is another full subcategory of $\mathrm{Rep}_{\mathrm{la.c}}(G)$ that we will consider, to be denoted by $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$. The objects of this category are convex K-spaces V of compact type, equipped with a locally analytic G-representation, that may be written as the union of an increasing sequence of Z_G-invariant BH-subspaces. Recall that by definition, any object of $\mathrm{Rep}_{\mathrm{es}}(G)$ is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$.

We refer to [7, §3.1] for a summary of the basic results regarding these categories of G-representations.

We briefly recall the notion of an analytic open subgroup of an L-analytic group, referring to [6, §3.5, §5.2] for a detailed discussion. If G is an L-analytic group, then an analytic open subgroup of G actually consists of the following data: an open subgroup H of G, a rigid L-analytic group $\mathbb{H}$ that is isomorphic as a rigid analytic space to a closed ball, and a locally L-analytic isomorphism $H \xrightarrow{\sim} \mathbb{H}(L)$. Although all three pieces of data are required to specify an analytic open subgroup, typically we will refer simply to an analytic open subgroup H of G. We adopt the convention that if an analytic open subgroup is denoted by H (possibly decorated with some subscripts or superscripts), then the underlying rigid analytic group will always be denoted by $\mathbb{H}$ (decorated with the same superscripts or subscripts). Given analytic open subgroups H_1 and H_2 of G, and an inclusion $H_2 \subset H_1$, we say that this is an inclusion of analytic open subgroups if it arises from a rigid analytic open immersion $\mathbb{H}_2 \to \mathbb{H}_1$ by passing to L-valued points.

If H is an analytic open subgroup of G, and $g \in G$, then we will write $g\mathbb{H}$ to denote a copy of the rigid analytic space $\mathbb{H}$, with the understanding that the locally analytic space of L-valued points of $g\mathbb{H}$ is to be naturally identified with the coset $gH \subset G$ via the locally L-analytic isomorphism $(g\mathbb{H})(L) = \mathbb{H}(L) \xrightarrow{\sim} H \xrightarrow{\sim} gH$ (the final isomorphism being provided by left multiplication by g). For example, if $\{g_i\}_{i\in I}$ is a collection of right coset representatives of H in G, then $\coprod_{i\in I} g_i\mathbb{H}$ is a rigid analytic space whose set of L-valued points is naturally isomorphic to $G = \coprod_{i\in I} g_i H$ as a locally L-analytic space. We will apply similar notation with regard to the left cosets Hg of H in G.

Finally, we recall from [6] that we call an analytic open subgroup H of G *good* if there exists an $\mathcal{O}_L$-Lie sublattice $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G (i.e. $\mathfrak{h}$ is a finitely generated $\mathcal{O}_L$-submodule of $\mathfrak{g}$ which spans $\mathfrak{g}$ over L, and which is closed under the Lie bracket) on which the Lie bracket may be exponentiated to form a rigid analytic group $\mathbb{H}$, whose underlying group of L-valued points may be identified with H (in a manner compatible with the inclusion $\mathfrak{h} \subset \mathfrak{g}$). Good analytic open subgroups of G exist, and are cofinal among all analytic open subgroups of G [17, LG 5.35, Cor. 2].

If H is a good analytic open subgroup H of G, and $r \in (0,1) \bigcap |L^\times|$, then the polydisk of radius r around the origin of $\mathbb{H}$ is an affinoid rigid analytic group, which we denote by $\mathbb{H}_r$. We let $H_r := \mathbb{H}_r(L)$ denote the corresponding locally analytic groups of L-valued points. We also write $\mathbb{H}^\circ := \bigcup_{r<1} \mathbb{H}_r$, and write $H^\circ := \mathbb{H}^\circ(L)$. (Note that $H^\circ = H_r$ for r close enough to 1.) See [6, §5.2] for more details of these constructions.

1. Preliminaries

(1.1) Let G be a locally L-analytic group, and let H be a compact open subgroup of G. Note that HZ_G is then an open subgroup of G.

Lemma 1.1.1. *We may find a a cofinal sequence $\{H_i\}_{i\geq 0}$ of analytic open subgroups of H, chosen so that H normalizes each $\mathbb{H}_i$. (Recall that this means that H normalizes the group of L-rational points H_i of $\mathbb{H}_i$, and that the resulting action of H on H_i via conjugation is induced by a corresponding action of H on $\mathbb{H}_i$ via rigid analytic automorphisms.)*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G (and hence also of H), and fix an $\mathcal{O}_L$-Lie sublattice $\mathfrak{h}$ of $\mathfrak{g}$. We may find an $\mathcal{O}_L$-Lie sublattice $\mathfrak{h}_0$ of $\mathfrak{h}$ which is invariant under the adjoint action of H. Replacing $\mathfrak{h}_0$ by $p^n\mathfrak{h}_0$ for sufficiently large n if necessary, we may furthermore assume that $\mathfrak{h}_0$ exponentiates to a good analytic open subgroup H_0 of H. If we let H_i be the good analytic open subgroup attached to $p^i\mathfrak{h}_0$ for each $i \geq 0$, then $\{H_i\}_{i\geq 0}$ satisfies the requirements of the lemma. □

We now prove a useful structural result regarding objects of $\mathrm{Rep}^z_{\mathrm{la.c}}(G)$.

Lemma 1.1.2. *If $\{H_i\}_{i\geq 0}$ is any cofinal sequence of analytic open subgroups of H satisfying the conclusion of Lemma 1.1.1, and if $U \in \mathrm{Rep}^z_{\mathrm{la.c}}(G)$, then we may write $U \xrightarrow{\sim} \varinjlim_i U_i$, where each U_i is a K-Banach space equipped with an $\mathbb{H}_i$-analytic HZ_G-action, and the transition maps $U_i \to U_{i+1}$ are compact, injective, and HZ_G-equivariant.*

Proof. Since U is a locally analytic representation of G, and hence of H, on a compact type space (and so in particular on an LB-space), we may write $U = \bigcup_{i\geq 0} U_{1,i}$, where $\{U_{1,i}\}_{i\geq 0}$ is an increasing sequence of H-invariant BH-subspaces of U, such that for each $i \geq 0$, the induced H-representation on $\overline{U}_{1,i}$ is $\mathbb{H}_i$-analytic. (This follows from [6, Prop. 3.2.15, Thm. 3.6.12].) Since U is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(G)$, we may also write $U = \bigcup_{i\geq 0} U_{2,i}$, where $\{U_{2,i}\}_{i\geq 0}$ is an increasing sequence of Z_G-invariant BH-subspaces of U.

For any $i \geq 0$, the unit ball $U^{\circ}_{2,i}$ of $\overline{U}_{2,i}$ (with respect to some norm defining the Banach space structure on $\overline{U}_{2,i}$) is bounded in U, and thus so is the closure of $HU^{\circ}_{2,i}$ in U (since H is compact, and so acts equicontinuously on U, by [6, Lem. 3.1.4]). The K-span of $HU^{\circ}_{2,i}$ is thus an HZ_G-invariant BH-subspace of U (since U is complete; see [2, Cor., p. III.8]) that contains $U_{2,i}$. Replacing $U_{2,i}$ by this BH-subspace, we may assume that each $U_{2,i}$ is in fact HZ_G-invariant.

By passing to appropriate subsequences of the two sequences $\{U_{1,i}\}_{i\geq 0}$ and $\{U_{2,i}\}_{i\geq 0}$ of BH-subspaces of U, and relabelling as necessary, we may furthermore assume that $U_{1,i} \subset U_{2,i}$ for all $i \geq 0$ [2, Prop. 1, p. I.20]. Passing to latent Banach space structures (taking into account [6, Prop. 1.1.2 (ii)]) and then to $\mathbb{H}_i$-analytic vectors (taking into account [6, Thm. 3.6.3]) this inclusion induces continuous injections

$$\overline{U}_{1,i} = (\overline{U}_{1,i})_{\mathbb{H}_i-\mathrm{an}} \to (\overline{U}_{2,i})_{\mathbb{H}_i-\mathrm{an}} \to \overline{U}_{2,i}.$$

We deduce that $U \xrightarrow{\sim} \varinjlim_{i} (\overline{U}_{2,i})_{\mathbb{H}_i-\mathrm{an}}$.

By [6, Prop. 1.1.2 (ii)] and the functoriality of passing to $\mathbb{H}_i$-analytic vectors, we see that the HZ_G-action on $U_{2,i}$ induces a HZ_G-action on $(\overline{U}_{2,n})_{\mathbb{H}_i}$ for each $i \geq 0$. Since U is of compact type, by passing to a subsequence of $\{(\overline{U}_{2,i})_{\mathbb{H}_i-\mathrm{an}}\}_{i\geq 0}$ and then relabelling we may furthermore assume that each of the transition maps $(\overline{U}_{2,i})_{\mathbb{H}_i-\mathrm{an}} \to (\overline{U}_{2,i+1})_{\mathbb{H}_{i+1}-\mathrm{an}}$ is compact. Taking $U_i = (\overline{U}_{2,i})_{\mathbb{H}_i-\mathrm{an}}$ gives the lemma. □

(1.2) Let G be a locally L-analytic group with Lie algebra $\mathfrak{g}$. In this subsection we develop some simple definitions and results related to $\mathfrak{g}$-representations that are locally integrable, in the sense that the $\mathfrak{g}$-action on any vector may be integrated to an action of some compact open subgroup of G.

Let H be an analytic open subgroup of G. If V is a K-Banach space, then $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V) := \mathcal{C}^{\mathrm{an}}(\mathbb{H}, K) \otimes_K V$ is again a K-Banach space, equipped with mutually commuting actions of H by each of the left and right regular actions. By [6, Prop. 3.3.4], these actions are $\mathbb{H}$-analytic in the sense of [6, Def. 3.6.1]. If V is furthermore equipped with a topological $\mathfrak{g}$-action (which is then necessarily continuous, since V is Banach, and so barrelled), then we obtain a topological action of $\mathfrak{g}$ on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$, namely the action induced by the action on V (the "pointwise action"), which commutes with each of the regular H-actions.

We define a $\mathfrak{g} \times \mathfrak{g}$-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ by having the first factor act through the pointwise action, and the second factor act through the derivative of the left regular H-action; this $\mathfrak{g} \times \mathfrak{g}$-action commutes with the right regular H-action. If $\Delta : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ denotes the diagonal embedding, then composing the $\mathfrak{g} \times \mathfrak{g}$-action with Δ induces yet another $\mathfrak{g}$-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$; we denote this copy of $\mathfrak{g}$ acting on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ by $\Delta(\mathfrak{g})$, to avoid confusion.

Definition 1.2.1. We write $V_{\mathbb{H}-\mathrm{int}} := \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta(\mathfrak{g})}$, the closed subspace of invariants in $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ under the action of $\Delta(\mathfrak{g})$, and refer to $V_{\mathbb{H}-\mathrm{int}}$ as the space of $\mathbb{H}$-integrable vectors in V. Note that since the right regular H-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ commutes with the $\Delta(\mathfrak{g})$-action, this action restricts to an H-action on $V_{\mathbb{H}-\mathrm{int}}$, and we regard $V_{\mathbb{H}-\mathrm{int}}$ as an H-representation in this way.

Since $V_{\mathbb{H}-\mathrm{int}}$ is a closed subspace of the Banach space $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$, it is again a Banach space. The formation of $V_{\mathbb{H}-\mathrm{int}}$ is evidently functorial in V. Evaluation at the identity of H induces a natural continuous map

$$V_{\mathbb{H}-\mathrm{int}} \to V. \tag{1.2.2}$$

Lemma 1.2.3. *(i) The H-action on $V_{\mathbb{H}-\mathrm{int}}$ is $\mathbb{H}$-analytic.*

(ii) The map (1.2.2) is injective and $\mathfrak{g}$-equivariant, if we endow $V_{\mathbb{H}-\mathrm{int}}$ with the $\mathfrak{g}$-action obtained by differentiating the H-action. (This make sense, by (i).)

Proof. Claim (i) follows from [6, Prop. 3.6.2], since $V_{\mathbb{H}-\mathrm{int}}$ is a closed subrepresentation of the $\mathbb{H}$-analytic H-representation $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$.

If we differentiate the H-action on $V_{\mathbb{H}-\mathrm{int}}$, we obtain a $\mathfrak{g}$-action on $V_{\mathbb{H}-\mathrm{int}}$, with respect to which the map (1.2.2) (which we recall is induced by evaluating elements of $V_{\mathbb{H}-\mathrm{int}} \subset \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ at the identity element of H) is immediately checked to be $\mathfrak{g}$-equivariant. The kernel of (1.2.2) is thus $\mathfrak{g}$-invariant, and so an element in the kernel of (1.2.2) is a rigid analytic V-valued function on $\mathbb{H}$ whose power series expansion at the identity vanishes identically. Consequently (1.2.2) has trivial kernel, and (ii) is proved. □

Definition 1.2.4. We say that a topological $\mathfrak{g}$-action on a K-Banach space V is $\mathbb{H}$-integrable if the map (1.2.2) is a bijection (in which case the Open Mapping Theorem shows that it is a topological isomorphism, since the source and domain are both Banach spaces).

If the Banach space V is equipped with an $\mathbb{H}$-integrable action of $\mathfrak{g}$, then we may use the isomorphism (1.2.2) to transport the H-action on $V_{\mathbb{H}-\mathrm{int}}$ to an H-action on V. We will say that this H-action is obtained by "integrating the $\mathfrak{g}$-action".

Lemma 1.2.5. *If V is a K-Banach space equipped with an $\mathbb{H}$-analytic action of H, then the $\mathfrak{g}$-action on V obtained by differentiating the H-action is $\mathbb{H}$-integrable, and the map (1.2.2) is H-equivariant.*

Proof. If we define the $\Delta_{1,2}(H)$-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ as in the discussion of [6, §3.3], then its derivative gives the $\Delta(\mathfrak{g})$-action described above, and so we have continuous H-equivariant injections

$$\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta_{1,2}(H)} \longrightarrow \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)^{\Delta(\mathfrak{g})} \overset{(1.2.2)}{\longrightarrow} V.$$

By definition (see [6, Def. 3.3.1, Def. 3.6.1]), our assumption implies that the composite of these injections is a bijection, and thus so is the second, that is, the map (1.2.2). □

Proposition 1.2.6. *If V is a K-Banach space equipped with a topological $\mathfrak{g}$-action, and if W is a closed $\mathfrak{g}$-invariant subspace of V, then the natural commutative diagram*

$$\begin{array}{ccc} W_{\mathbb{H}-\mathrm{int}} & \longrightarrow & V_{\mathbb{H}-\mathrm{int}} \\ \downarrow & & \downarrow \\ W & \longrightarrow & V \end{array}$$

(in which the vertical arrows are provided by (1.2.2), the bottom horizontal arrow is the inclusion, and the top horizontal arrow arises from the bottom one by applying the functor "pass to $\mathbb{H}$-integrable vectors") is Cartesian (in the category of K-Banach spaces).

Proof. By [6, Prop. 2.1.23], the given closed embedding of W into V induces a closed embedding $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, W) \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$. Passing to $\Delta(\mathfrak{g})$-invariants yields the proposition. □

Corollary 1.2.7. *If V is a K-Banach space equipped with an $\mathbb{H}$-integrable action of $\mathfrak{g}$, and if W is a closed $\mathfrak{g}$-invariant subspace of V, then the $\mathfrak{g}$-action on W is again $\mathbb{H}$-integrable, and the inclusion $W \subset V$ is H-equivariant, if we endow W and V with the H-actions provided by integrating their respective $\mathfrak{g}$-actions.*

Proof. This follows directly from Proposition 1.2.6. □

Proposition 1.2.8. *Let $H_2 \subset H_1$ be an inclusion of analytic open subgroups of G. If V is a K-Banach space equipped with a topological $\mathfrak{g}$-action, then there is a natural $\mathfrak{g}$-equivariant continuous injection $V_{\mathbb{H}_1-\mathrm{int}} \to V_{\mathbb{H}_2-\mathrm{int}}$, uniquely determined by the requirement that the diagram*

$$\begin{array}{ccc} V_{\mathbb{H}_1-\mathrm{int}} & \longrightarrow & V_{\mathbb{H}_2-\mathrm{int}} \\ & \searrow & \downarrow \\ & & V, \end{array}$$

in which the diagonal and vertical arrows are provided by (1.2.2), commutes.

Proof. Since $H_2 \subset H_1$ is an inclusion of analytic open subgroups, by definition it arises from a rigid analytic open map $\mathbb{H}_2 \to \mathbb{H}_1$ by passing to L-valued points. The required map is obtained from the pull-back map $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, V) \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_2, V)$ by passing to $\Delta(\mathfrak{g})$-invariants. □

Let V be a convex K-space equipped with a topological $\mathfrak{g}$-action. If W is a $\mathfrak{g}$-invariant BH-subspace of V, then [6, Prop. 1.1.2 (ii)] shows that the $\mathfrak{g}$-action on W lifts to a topological $\mathfrak{g}$-action on the latent Banach space $\overline{W}$; and so for any analytic open subgroup H of G, we may define $\overline{W}_{\mathbb{H}-\mathrm{int}}$ following Definition 1.2.1. If $W_1 \subset W_2$ is an inclusion of $\mathfrak{g}$-invariant BH-subspaces of V, and $H_2 \subset H_1$ an inclusion of analytic open subgroups of G, then we obtain a commutative diagram of continuous $\mathfrak{g}$-equivariant injections

$$\begin{array}{ccc} (\overline{W}_1)_{\mathbb{H}_1-\mathrm{int}} & \longrightarrow & (\overline{W}_2)_{\mathbb{H}_1-\mathrm{int}} \\ \downarrow & & \downarrow \\ (\overline{W}_1)_{\mathbb{H}_2-\mathrm{int}} & \longrightarrow & (\overline{W}_2)_{\mathbb{H}_2-\mathrm{int}}, \end{array}$$

in which the bottom (respectively top) arrow is induced by the functoriality of the formation of $\mathbb{H}_1$-integrable (respectively $\mathbb{H}_2$-integrable) vectors, while the two vertical arrows are induced by Proposition 1.2.8. (The commutativity of this diagram follows from the naturality of the maps given by Proposition 1.2.8.) Composing either the left-hand vertical and bottom arrows or the top and right-hand vertical arrows of this diagram, we obtain a continuous $\mathfrak{g}$-equivariant injection

$$(\overline{W}_1)_{\mathbb{H}_1-\mathrm{int}} \to (\overline{W}_2)_{\mathbb{H}_2-\mathrm{int}}. \tag{1.2.9}$$

As W ranges over the directed set of all BH-subspaces of V, and H ranges over the directed set of all analytic open subgroups of G, the maps (1.2.9) make a directed set out of the collection of $\mathfrak{g}$-representations $\overline{W}_{\mathbb{H}-\mathrm{int}}$.

Definition 1.2.10. If V is a convex K-space equipped with a topological $\mathfrak{g}$-action, then we define the space of locally integrable vectors, V_{lint}, in V to be the locally convex inductive limit $V_{\mathrm{lint}} := \varinjlim_{H,W} \overline{W}_{\mathbb{H}-\mathrm{int}}$, where H runs over the directed set of analytic open subgroups of G, W runs over the directed set of $\mathfrak{g}$-invariant BH-subspaces of V, and the transition maps are defined via (1.2.9).

Since each of the space $\overline{W}_{\mathbb{H}-\mathrm{int}}$ appearing in the inductive limit that defines V_{lint} is equipped in a natural way with a continuous $\mathfrak{g}$-action, and since the transition maps (1.2.9) are $\mathfrak{g}$-equivariant, we see that V_{lint} is equipped in a natural way with a continuous $\mathfrak{g}$-action. (One sees immediately that the $\mathfrak{g}$-action on V_{lint} is topological; however it is then necessarily continuous, since V_{lint} is barrelled, being defined as the locally convex inductive limit of Banach spaces). The formation of V_{lint} (with its $\mathfrak{g}$-action) is obviously functorial in V, and the maps (1.2.2) (with V replaced by the spaces $\overline{W}$) induce a natural continuous $\mathfrak{g}$-equivariant injection

$$V_{\mathrm{lint}} \to V. \tag{1.2.11}$$

Proposition 1.2.12. *If V is a Hausdorff convex K-vector space equipped with a topological $\mathfrak{g}$-action, then the natural map $(V_{\mathrm{lint}})_{\mathrm{lint}} \to V_{\mathrm{lint}}$ (obtained by substituting V_{lint} for V in (1.2.11)) is a topological isomorphism.*

Proof. Let W be a $\mathfrak{g}$-invariant BH-subspace of V, and let H be an analytic open subgroup of V. Then by definition of V_{lint} (and taking into account Lemma 1.2.3) the space $\overline{W}_{\mathbb{H}-\mathrm{int}}$ is identified with a $\mathfrak{g}$-invariant BH-subspace of V_{lint}. It is also equipped with an $\mathbb{H}$-analytic representation, by Lemma 1.2.3, and so Lemma 1.2.5 shows that the map $(\overline{W}_{\mathbb{H}-\mathrm{int}})_{\mathbb{H}-\mathrm{int}} \to \overline{W}_{\mathbb{H}-\mathrm{int}}$ (obtained by taking V to be $\overline{W}_{\mathbb{H}-\mathrm{int}}$ in (1.2.2)) is a topological isomorphism. Passing to the locally convex inductive limit over all such W and H, we obtain the isomorphism that forms the top row of the commutative diagram

$$\begin{array}{ccc} \varinjlim_{H,W} (\overline{W}_{\mathbb{H}-\mathrm{int}})_{\mathbb{H}-\mathrm{int}} & \xrightarrow{\sim} & \varinjlim_{H,W} \overline{W}_{\mathbb{H}-\mathrm{int}} \\ \downarrow & & \downarrow \wr \\ (V_{\mathrm{lint}})_{\mathrm{lint}} & \longrightarrow & V_{\mathrm{lint}}, \end{array} \tag{1.2.13}$$

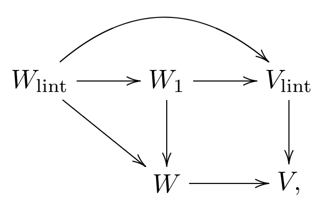
in which the left vertical arrow is the continuous injection induced by definition of $(V_{\mathrm{lint}})_{\mathrm{lint}}$, the right vertical arrow is the isomorphism induced by definition of V_{lint}, and the bottom arrow is induced by replacing V by V_{lint} in (1.2.11). Since the top and right-hand arrows of (1.2.13) are topological isomorphisms, while the other two arrows are continuous injections, we see that these other two arrows (in particular, the bottom arrow) are also topological isomorphisms, as required. □

Proposition 1.2.14. *For any Hausdorff convex K-vector space V of LB-type equipped with a topological $\mathfrak{g}$-action, V_{lint} is an LB-space.*

Proof. Since V is the union of a countable collection of BH-spaces, and since G admits a countable neighbourhood basis of the identity consisting of analytic open subgroups, we may replace the directed set appearing in the inductive limit of Definition 1.2.10 by a cofinal directed set that is countable. Thus V_{lint} may be written as the locally convex inductive limit of a sequence of Banach spaces, and so by definition is an LB-space. □

Proposition 1.2.15. *Let V be a Hausdorff convex K-vector space equipped with a topological $\mathfrak{g}$-action, and let W be a $\mathfrak{g}$-invariant closed subspace of V. Let W_1 denote the preimage of W in V_{lint} under (1.2.11), which is a $\mathfrak{g}$-invariant closed subspace of V_{lint}. Then the map $W_{\mathrm{lint}} \to V_{\mathrm{lint}}$ (induced by functoriality of the formation of locally integrable vectors) induces a continuous bijection of W_{lint} onto W_1.*

Proof. Consider the diagram of continuous $\mathfrak{g}$-equivariant injections



in which the outer part of the diagram (obtained by omitting W_1) arises from the natural transformation (1.2.11) and the functoriality of the formation of locally integrable vectors, while the square is a Cartesian diagram of topological K-vector spaces. We must show that any vector $w \in W_1$ lies in the image of the map $W_{\mathrm{lint}} \to W_1$. From the definition of V_{lint} and the construction of W_1, we see that we may find a $\mathfrak{g}$-invariant BH-subspace X of V, and an analytic open subgroup H of G, so that under the map $V_{\mathrm{lint}} \to V$, the image of w lies in $W \bigcap X$, while w lies in $\overline{X}_{\mathbb{H}-\mathrm{int}}$. Since W is closed in V, we see that $W \bigcap X$ is a BH-subspace of W, and that $\overline{W \bigcap X}$ is a closed $\mathfrak{g}$-invariant subspace of $\overline{X}$. Proposition 1.2.6 thus shows that w lies in $(\overline{W \bigcap X})_{\mathbb{H}-\mathrm{int}}$. From the definition of W_{lint}, we see that w does lie in the image of the map $W_{\mathrm{lint}} \to W_1$, as required. □

Definition 1.2.16. We say that a topological $\mathfrak{g}$-action on a convex K-space V is locally integrable if V is barrelled, and if the map (1.2.11) is a bijection.

Proposition 1.2.17. *If V is an LB-space equipped with a locally integrable $\mathfrak{g}$-action, then the continuous bijection (1.2.11) is in fact a topological isomorphism.*

Proof. Proposition 1.2.14 implies that V_{lint} is again an LB-space. Thus (1.2.11) is a continuous bijection between LB-spaces, and so by the Open Mapping Theorem necessarily is a topological isomorphism. □

Proposition 1.2.18. *If V is a barrelled convex K-space equipped with a locally analytic G-action, then the resulting $\mathfrak{g}$-action on V makes V a locally integrable $\mathfrak{g}$-representation.*

Proof. By assumption the map $V_{\mathrm{la}} := \varinjlim_{H} \varinjlim_{W} \overline{W}_{\mathbb{H}-\mathrm{an}}$ is an isomorphism (the inductive limit being taken over the directed set of analytic open subgroups H of G, and (H being fixed), the directed set of H-invariant BH-subspaces W of V). Each of the Banach spaces $\overline{W}_{\mathbb{H}-\mathrm{an}}$ is an $\mathbb{H}$-analytic representation of H, and so by Lemma 1.2.5 the continuous injection $(\overline{W}_{\mathbb{H}-\mathrm{an}})_{\mathbb{H}-\mathrm{int}} \to \overline{W}_{\mathbb{H}-\mathrm{an}}$ is an isomorphism. Since the images of the spaces $(\overline{W}_{\mathbb{H}-\mathrm{an}})$ in V (with respect to the maps $\overline{W}_{\mathbb{H}-\mathrm{an}} \to \overline{W} \to W \subset V$) have union equal to all of V by assumption, we see that indeed the map (1.2.11) is a bijection. □

Proposition 1.2.19. *Let V be a Hausdorff convex K-vector space equipped with a locally integrable representation of $\mathfrak{g}$, and let W be a closed G-invariant subspace of V. If W is barrelled, then W is also a locally integrable $\mathfrak{g}$-representation.*

Proof. Proposition 1.2.15 implies that the natural map $W_{\mathrm{lint}} \to W$ is a bijection. □

Proposition 1.2.20. *If V is a Hausdorff convex K-vector space equipped with a topological $\mathfrak{g}$-action, and if there is a $\mathfrak{g}$-equivariant isomorphism $\varinjlim_{i\in I} V_i \xrightarrow{\sim} V$, where $\{V_i\}_{i\in I}$ is a $\mathfrak{g}$-equivariant inductive system of Hausdorff K-vector spaces, each equipped with a locally integrable action of $\mathfrak{g}$, then the $\mathfrak{g}$-action on V is again locally integrable.*

Proof. Since V is the locally convex inductive limit of the barrrelled spaces V_i, it is barrelled. Functoriality of the formation of locally integrable vectors yields the commutative diagram

$$\begin{array}{ccc} \varinjlim_{i\in I} (V_i)_{\mathrm{lint}} & \longrightarrow & V_{\mathrm{lint}} \\ \downarrow & & \downarrow \\ \varinjlim_{i\in I} V_i & \longrightarrow & V. \end{array}$$

The left-hand vertical arrow and lower horizontal arrow are both continuous bijections, by assumption, and thus so is the right-hand vertical arrow (since it is *a priori* a continuous injection). □

Proposition 1.2.21. *Let H be an analytic open subgroup of G, let W be a Banach space equipped with an $\mathbb{H}$-integrable action of $\mathfrak{g}$, let Y be a dense $\mathfrak{g}$-invariant subspace of W, and let V be an LB-space equipped with a topological G-action. Suppose that $\phi : W \to V$ is a continuous K-linear map, with the property that $\phi_{|Y}$ factors through a $\mathfrak{g}$-equivariant map $Y \to V_{\mathbb{H}-\mathrm{an}}$. Then ϕ itself factors through a continuous H-equivariant map $W \to V_{\mathbb{H}-\mathrm{an}}$.*

Proof. Since $\mathfrak{g}$ acts in an $\mathbb{H}$-integrable fashion on W, Lemma 1.2.3 provides an $\mathbb{H}$-analytic H-action on W. Consider the continuous map $\Phi : H \times W \to V$ defined by $\Phi(h, w) = \phi(hw) - h\phi(w)$. If we fix $w \in Y$, then the resulting map is an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}, V)$ (since $\phi_{|Y}$ factors through $V_{\mathbb{H}-\mathrm{an}}$), and all its derivatives vanish (since $\phi_{|Y}$ is $\mathfrak{g}$-equivariant). Thus $\Phi_{|H\times Y}$ vanishes, and since Y is dense in W, we conclude that

Φ vanishes, and thus that ϕ is H-equivariant. Since H acts $\mathbb{H}$-analytically on W, we conclude that ϕ factors through a continuous H-equivariant map $W \to V_{\mathbb{H}-\mathrm{an}}$, as claimed. □

Corollary 1.2.22. *Let H be an analytic open subgroup of G, let W be a Banach space equipped with an $\mathbb{H}$-integrable action of $\mathfrak{g}$, and let V be an LB-space equipped with a locally analytic G-action. If $\phi : W \to V$ is a $\mathfrak{g}$-equivariant continuous K-linear map, then there is an analytic open subgroup $H' \subset H$ such that ϕ factors through a continuous H'-equivariant map $W \to V_{\mathbb{H}'-\mathrm{an}}$.*

Proof. By assumption $V_{\mathrm{la}} \to V$ is a bijection. Since V is an LB-space and W is a Banach space, it follows from the definition of V_{la} [6, Def. 3.5.3] and [2, Prop. 1, p. I.20] that we may find an inclusion of analytic open subgroups $H' \subset H$ so that ϕ factors through a continuous map $W \to \overline{U}$, where U is an H'-invariant BH-subspace of V for which the H'-action on the underlying latent Banach space $\overline{U}$ is $\mathbb{H}'$-analytic. The corollary thus follows from Proposition 1.2.21 (taking $Y = W$). □

Proposition 1.2.23. *Let W be an LB-space over K equipped with a locally integrable $\mathfrak{g}$-action, and write $W \xrightarrow{\sim} \varinjlim_{i\geq 0} \overline{W}_i$, where each W_i is a $\mathfrak{g}$-invariant BH-subspace of W on which the $\mathfrak{g}$-action is again locally integrable. Fix a $\mathfrak{g}$-invariant subspace Y of W, and for each $i \geq 0$ let Y_i denote the preimage of Y under the continuous injection $\overline{W}_i \to W$. Suppose that:*

(i) Y_i is dense in $\overline{W}_i$ for each $i \geq 0$;

(ii) Y_i contains a Banach subspace of $\overline{W}_i$ that generates Y_i as $\mathfrak{g}$-module.

In addition, let V be a convex K-space of LB-type equipped with a topological G-action, and let $\phi : W \to V$ be a continuous K-linear map with the property that $\phi_{|Y}$ factors through a $\mathfrak{g}$-equivariant map $Y \to V_{\mathrm{la}}$. Then ϕ itself factors through a continuous $\mathfrak{g}$-equivariant map $W \to V_{\mathrm{la}}$.

Proof. Fix a value of $i \geq 0$, and let $B_i \subset Y_i$ be a Banach subspace of Y_i that generates Y_i as a $\mathfrak{g}$-module. (Such a subspace exists, by assumption (ii).) Since $B_i \subset Y$, we have that $\phi(B_i) \subset V_{\mathrm{la}}$ by assumption. Thus $\phi(B_i) \subset V_{\mathbb{H}-\mathrm{an}}$ for some analytic open subgroup H of V (since B_i is a Banach space; cf. the proof of the preceding corollary). Since $\overline{W}_i$ is a Banach space on which the $\mathfrak{g}$-action is locally integrable, it follows from [2, Prop. 1, p. I.20] that by shrinking H if necessary, we may arrange for the $\mathfrak{g}$-action on $\overline{W}_i$ to be $\mathbb{H}$-integrable. Since $\phi_{|Y}$ is $\mathfrak{g}$ equivariant, since $\mathrm{U}(\mathfrak{g})B_i = Y_i$, and since $V_{\mathbb{H}-\mathrm{an}}$ is a $\mathfrak{g}$-invariant subspace of V_{la}, we see that $\phi_{|Y_i}$ is $\mathfrak{g}$-equivariant, with image lying in $V_{\mathbb{H}-\mathrm{an}}$. Proposition 1.2.21 and assumption (i) thus imply that $\phi_{|\overline{W}_i}$ factors through a continuous $\mathfrak{g}$-equivariant map $\overline{W}_i \to V_{\mathbb{H}-\mathrm{an}}$, and so in particular through a continuous $\mathfrak{g}$-equivariant map $\overline{W}_i \to V_{\mathrm{la}}$. Letting $i \to \infty$ yields the proposition. □

(1.3) Let $H_2 \subset H_1$ be an inclusion of analytic open subgroups of a locally L-analytic group G, and let U be a K-Banach space. We endow $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, U)$ (respectively $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_2, U)$) with an H_1-action (respectively an H_2-action) via the right regular representation. As was recalled in Subsection 1.2, these actions are $\mathbb{H}_1$-analytic and $\mathbb{H}_2$-analytic respectively and differentiating them induces a $\mathfrak{g}$-action on each of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, U)$ and $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_2, U)$. (As usual $\mathfrak{g}$ denotes the Lie algebra of G.) Restricting functions from H_1 to H_2 yields a $\mathfrak{g}$-equivariant map $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, U) \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_2, U)$.

Lemma 1.3.1. *Let X be a closed $\mathfrak{g}$-invariant subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_2, U)$. If $f \in \mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, U)$ is such that $f_{|H_2} \in X$, then for each $h \in H_1$ the function $(hf)_{|H_2}$ again lies in X.*

Proof. Since the H_1-action on $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, U)$ is $\mathbb{H}_1$-analytic, the map $h \mapsto hf$ is a rigid analytic map from $\mathbb{H}_1$ to $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_1, U)$, and thus

$$h \mapsto (hf)_{|H_2} \tag{1.3.2}$$

is a rigid analytic map from $\mathbb{H}_1$ to $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_2, U)$. Since the map $f \mapsto f_{|H_2}$ is $\mathfrak{g}$-equivariant, since X is $\mathfrak{g}$-invariant, and since $f_{|H_2}$ lies in X, one furthermore computes that all derivatives of arbitrary order of (1.3.2) at e lie in X. Thus (since X is closed) we see that (1.3.2) is in fact an X-valued rigid analytic function on $\mathbb{H}_1$. This proves the lemma. □

(1.4) For any locally L-analytic space X and any Hausdorff convex K-vector space U, we let $\mathcal{C}^{\mathrm{la}}_c(X, U)$ denote the vector space of compactly supported locally analytic U-valued functions on X. If $\Omega \subset \Omega'$ is an inclusion of compact open subsets of X, then extension by zero defines an injective map

$$\mathcal{C}^{\mathrm{la}}(\Omega, U) \to \mathcal{C}^{\mathrm{la}}(\Omega', U). \tag{1.4.1}$$

Extension by zero also defines an injective map

$$\mathcal{C}^{\mathrm{la}}(\Omega, U) \to \mathcal{C}^{\mathrm{la}}_c(X, U), \tag{1.4.2}$$

for each compact open subset Ω of X. Clearly the maps (1.4.2) are compatible with the maps (1.4.1), and so we obtain a map

$$\varinjlim_{\Omega} \mathcal{C}^{\mathrm{la}}(\Omega, U) \to \mathcal{C}^{\mathrm{la}}_c(X, U), \tag{1.4.3}$$

in which the inductive limit is taken over all compact open subsets of X.

Lemma 1.4.4. *The map (1.4.3) is an isomorphism of abstract K-vector spaces.*

Proof. This follows from the very definition of a function having compact support. □

Lemma 1.4.5. *If we equip source and target with their usual topology (as recalled in* [6, Def. 2.1.25] *or* [13, p. 5]*), then (1.4.1) is a closed embedding.*

Proof. If we write $\Omega'' := \Omega' \setminus \Omega$, then the equation $\Omega' = \Omega \coprod \Omega''$ expresses Ω' as the disjoint union of two compact open subsets. Thus there is a natural isomorphism $\mathcal{C}^{\mathrm{la}}(\Omega', U) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(\Omega, U) \oplus \mathcal{C}^{\mathrm{la}}(\Omega'', U)$. The map (1.4.1) corresponds to the inclusion of the first direct summand, and so is indeed a closed embedding. □

In particular, this lemma shows that the transition maps in the inductive limit of (1.4.3) are continuous, and so we may endow it with its locally convex inductive limit topology.

Definition 1.4.6. We topologize $\mathcal{C}^{\mathrm{la}}_c(X,U)$ by equipping $\varinjlim_{\Omega} \mathcal{C}^{\mathrm{la}}(\Omega,U)$ with its locally convex inductive limit topology, and declaring (1.4.3) to be a topological isomorphism.

The formation of $\mathcal{C}^{\mathrm{la}}_c(X,U)$ is evidently covariantly functorial in U, and contravariantly functorial with respect to proper maps in X.

The obvious natural map $\mathcal{C}^{\mathrm{la}}_c(X,U) \to \mathcal{C}^{\mathrm{la}}(X,U)$ is a continuous injection. Since $\mathcal{C}^{\mathrm{la}}(X,U)$ is Hausdorff (see the remark following [6, Prop. 2.1.26]) we see that $\mathcal{C}^{\mathrm{la}}_c(X,U)$ is also Hausdorff. Alternatively, this follows from the fact that the transition maps in (1.4.3) are closed embeddings (by Lemma 1.4.5), since a strict inductive limit of Hausdorff convex spaces is Hausdorff [2, Prop. 9, p. II.32].

Lemma 1.4.7. *If X is σ-compact and U is of compact type, then $\mathcal{C}^{\mathrm{la}}_c(X,U)$ is of compact type.*

Proof. Since X is σ-compact, we may find a cofinal subsequence of compact open subsets of X. By [6, Prop. 2.1.28] the inductive limit (1.4.3) is thus isomorphic to the inductive limit with injective transition maps of a sequence of compact type spaces, and so is again of compact type. □

Lemma 1.4.8. *If G is a locally L-analytic group and U is a Hausdorff locally convex K-space, then the right regular action of G on $\mathcal{C}^{\mathrm{la}}_c(G,U)$ is locally analytic.*

Proof. We fix a compact open subgroup G_0 of G. We may then restrict the compact open subsets Ω in the inductive limit (1.4.3) to consist of finite unions of right cosets of G_0 in G. The right regular action of G_0 on such a coset gG_0 induces a locally analytic action of G on $\mathcal{C}^{\mathrm{la}}(gG_0,U)$, by the discussion following [6, Def. 3.6.9], and hence on $\mathcal{C}^{\mathrm{la}}(\Omega,U)$ for any finite union Ω of such cosets. It follows from [6, Prop. 3.6.17] that $\mathcal{C}^{\mathrm{la}}_c(G,U)$ is locally analytic as a G_0-representation, and so also as a G-representation. □

Definition 1.4.9. If x is a point of a locally L-analytic space X, and U is a Hausdorff locally convex K-vector space, then we let $\mathcal{C}^{\omega}(X,U)_x$ denote the space of germs at the identity x of locally analytic U-valued functions on X; that is

$$\mathcal{C}^{\omega}(X,U)_x = \varinjlim_{\Omega \ni x} \mathcal{C}^{\mathrm{la}}(\Omega,U),$$

where the locally convex inductive limit is taken over the directed set of open neighbourhoods Ω of x in X, and the transition maps are given by restricting functions.

Lemma 1.4.10. *If the V is a convex K-space of compact type, then for any point x on a locally L-analytic space X, the space of germs $\mathcal{C}^{\omega}(X,V)_x$ is again of compact type.*

Proof. Let X_0 be a chart around x in X, so that there exists a closed rigid analytic ball $\mathbb{X}_0$ such that $X_0 \xrightarrow{\sim} \mathbb{X}_0(L)$. Let $\{\mathbb{X}_i\}_{i\geq 0}$ be a cofinal sequence of admissible open affinoid balls containing x in $\mathbb{X}_0$, chosen so that the inclusion $\mathbb{X}_{i+1} \subset \mathbb{X}_i$ is relatively compact. Write $V \xrightarrow{\sim} \varinjlim_i V_i$, where each transition map $V_i \to V_{i+1}$ is a compact injection between K-Banach spaces. There is then a natural topological isomorphism $\varinjlim_i \mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, V_i) \xrightarrow{\sim} \mathcal{C}^{\omega}(X,K)_x$, where the transition maps defining the inductive limit are obtained by restricting from $\mathbb{X}_i$ to $\mathbb{X}_{i+1}$, together with mapping from V_i into V_{i+1}. We may rewrite the transition map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, V_i) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_{i+1}, V_{i+1})$ in the form

$$\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \mathop{\hat{\otimes}}_K V_i \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_{i+1}, K) \mathop{\hat{\otimes}}_K V_{i+1}, \tag{1.4.11}$$

where the arrow is obtained as the completed tensor product of the restriction map $\mathcal{C}^{\mathrm{an}}(\mathbb{X}_i, K) \to \mathcal{C}^{\mathrm{an}}(\mathbb{X}_{i+1}, K)$ with the transition map $V_i \to V_{i+1}$. The former map is compact since the inclusion $\mathbb{X}_{i+1} \subset \mathbb{X}_i$ is relatively compact, while the latter map is compact by assumption. Thus it follows from [6, Cor. 1.1.27] and [11, Lem. 18.12] that (1.4.11) is injective and compact, and hence that $\mathcal{C}^{\omega}(X, K)_x$ is of compact type, as claimed. □

(1.5) In this subsection we recall a standard lemma from highest weight theory. Not knowing a reference, we include a proof.

We first establish the necessary notation. Let $\mathfrak{g}$ be a split reductive Lie algebra over K, let $\mathfrak{p}$ be a fixed Borel subalgebra of $\mathfrak{g}$, with nilpotent radical $\mathfrak{n}$, let $\mathfrak{m}$ be a Cartan subalgebra of $\mathfrak{p}$ (so that $\mathfrak{p} = \mathfrak{m} \bigoplus \mathfrak{n}$), let $\overline{\mathfrak{p}}$ be an opposite Borel to $\mathfrak{p}$, chosen so that $\mathfrak{p} \bigcap \overline{\mathfrak{p}} = \mathfrak{m}$, and let $\overline{\mathfrak{n}}$ be the nilpotent radical of $\overline{\mathfrak{p}}$ (so that $\overline{\mathfrak{p}} = \mathfrak{m} \bigoplus \overline{\mathfrak{n}}$). We let Δ denote the set of roots of $\mathfrak{m}$ acting on $\mathfrak{g}$. We write $\Delta = \Delta_+ \coprod \Delta_-$ as the disjoint union of the positive roots (i.e. the roots appearing in $\mathfrak{n}$) and the negative roots (i.e. the roots appearing in $\overline{\mathfrak{n}}$). If $\mathfrak{r} \in \Delta_+$ is a positive root, then we fix a basis element $X_{\mathfrak{r}}$ of the $\mathfrak{r}$-root space. We also fix a basis element $X_{-\mathfrak{r}}$ of the $-\mathfrak{r}$-root space, chosen so that if $H_{\mathfrak{r}} := [X_{\mathfrak{r}}, X_{-\mathfrak{r}}]$, then $[H_{\mathfrak{r}}, X_{\pm\mathfrak{r}}] = \pm 2 X_{\pm\mathfrak{r}}$. The elements $X_{\mathfrak{r}}$ (respectively $X_{-\mathfrak{r}}$) for $\mathfrak{r} \in \Delta_+$ form a basis for $\mathfrak{n}$ (respectively $\overline{\mathfrak{n}}$) as a vector space.

The Weyl group W of $\mathfrak{g}$ with respect to $\mathfrak{m}$ acts simply transitively on the set of Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{m}$, and also acts on Δ, and these actions are compatible, in the following sense: if $w(\mathfrak{p})$ is the transform of $\mathfrak{p}$ under some $w \in W$, with nilpotent radical $w(\mathfrak{n})$, then $w(\Delta_+)$ is the set of roots which appear in $w(\mathfrak{n})$. (Equivalently, the elements $X_{w(\mathfrak{r})}$, for $\mathfrak{r} \in \Delta_+$, form a basis for $w(\mathfrak{n})$.)

If $w \in W$, let $\Delta_w := \Delta_+ \bigcap w(\Delta_-)$, and let $I_w := \{(k_{\mathfrak{r}})_{\mathfrak{r} \in \Delta_w} \mid k_{\mathfrak{r}} \in \mathbb{Z}_{\geq 0}\}$. Thus Δ_w denotes the set of roots that $\mathfrak{n}$ and $w(\overline{\mathfrak{n}})$ have in common, while I_w denotes the set of non-negative integral multi-indices indexed by the set Δ_w. We will denote a typical element of I_w by $\underline{k}$, and a typical component of $\underline{k}$ by $k_{\mathfrak{r}}$ ($\mathfrak{r} \in \Delta_w$).

If $\mathfrak{r} \in \Delta_+$, we let $s_{\mathfrak{r}} \in W$ denote the reflection associated to $\mathfrak{r}$. If $w \in W$, then the length of w is the minimal number of factors appearing in a factorization of w into a product of simple reflections (i.e. elements $s_{\mathfrak{r}}$ attached to simple roots $\mathfrak{r} \in \Delta_+$). If w has length l, then the order of Δ_w is also equal to l, and for an appropriately chosen ordering of the elements $\mathfrak{r}_1, \ldots, \mathfrak{r}_l$ of Δ_w, we have the factorization $w = s_{\mathfrak{r}_1} \cdots s_{\mathfrak{r}_l}$.

If V is a $\mathfrak{g}$-representation, then for any integer $k \geq 0$ let $V^{\mathfrak{n}^k}$ denote the $\mathfrak{p}$-subrepresentation of V consisting of elements annihilated by $\mathfrak{n}^k$. We write $V^{\mathfrak{n}^\infty} := \bigcup_{k \geq 0} V^{\mathfrak{n}^k}$. (Note that this is a $\mathfrak{g}$-subrepresentation of V.) If $w \in W$ and $\underline{k} \in I_w$, we write

$$(V^{\mathfrak{n}})_{\underline{k}} := \{v \in V^{\mathfrak{n}} \mid (H_{\mathfrak{r}} - k_{\mathfrak{r}})v = 0 \text{ for all } \mathfrak{r} \in \Delta_w\}.$$

Lemma 1.5.1. *Let V be a $\mathfrak{g}$-representation, and suppose that $V = V^{\mathfrak{n}^\infty}$. If $w \in W$ has length l, and if $\mathfrak{r}_1, \ldots, \mathfrak{r}_l$ is an ordering of the elements of Δ_w, chosen so that*

$w = s_{\mathfrak{r}_1} \dots s_{\mathfrak{r}_l}$, then the space $V^{w(\mathfrak{n})}$ of $w(\mathfrak{n})$-invariants in V is contained in the subspace $\sum_{\underline{k} \in I_w} (X_{-\mathfrak{r}_1})^{k_{\mathfrak{r}_1}} \cdots (X_{-\mathfrak{r}_l})^{k_{\mathfrak{r}_l}} (V^{\mathfrak{n}})_{\underline{k}}$ of V.

Proof. We prove the lemma by induction on the length l of the element $w \in W$. If w is the identity element (i.e. $l = 0$), then Δ_w is empty, and the assertion to be proved reduces to the tautology $V^{\mathfrak{n}} = V^{\mathfrak{n}}$. Suppose now w is non-trivial (so that Δ_w is non-empty). Note that $\mathfrak{r}_1$ lies in Δ_w, and that $-\mathfrak{r}_1$ is a simple root of $w(\mathfrak{n})$.

For each $k \geq 0$, let $(V^{w(\mathfrak{n})})_k := \{v \in V^{w(\mathfrak{n})} \mid (X_{\mathfrak{r}_1})^k v = 0\}$. Clearly $V^{w(\mathfrak{n})} = \bigcup_{k \geq 0} (V^{w(\mathfrak{n})})_k$. We first prove that $(V^{w(\mathfrak{n})})_k = (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}(H_{\mathfrak{r}_1}+1)\cdots(H_{\mathfrak{r}_1}+k-1)}$ $(=: \{v \in V^{w(\mathfrak{n})} \mid H_{\mathfrak{r}_1}(H_{\mathfrak{r}_1}+1)\cdots(H_{\mathfrak{r}_1}+k-1)v = 0\})$. If $k = 0$, then there is nothing to prove, so we may assume that $k \geq 1$. Now let $v \in (V^{w(\mathfrak{n})})_k$. By assumption $X_{-\mathfrak{r}_1} v = (X_{\mathfrak{r}_1}^k) v = 0$. Thus we compute that

$$0 = (X_{\mathfrak{r}_1})^k X_{-\mathfrak{r}_1} v = (X_{-\mathfrak{r}_1}(X_{\mathfrak{r}_1})^k + (X_{\mathfrak{r}_1})^{k-1}(H_{\mathfrak{r}_1} + k - 1))v = (X_{\mathfrak{r}_1})^{k-1}(H_{\mathfrak{r}_1} + k - 1)v.$$

Hence $(V^{w(\mathfrak{n})})_{k-1} \subset (H_{\mathfrak{r}_1} + k - 1)(V^{w(\mathfrak{n})})_k \subset (V^{w(\mathfrak{n})})_{k-1}$ (the first inclusion holding from the fact that $(H_{\mathfrak{r}_1} + k - 1)$ is coprime to $H_{\mathfrak{r}_1}(H_{\mathfrak{r}_1}+1)\cdots(H_{\mathfrak{r}_1}+k-2)$, which annihilates $(V^{w(\mathfrak{n})})_{k-1}$, by induction on k), and so in fact $(V^{w(\mathfrak{n})})_{k-1} = (H_{\mathfrak{r}_1} + k - 1)(V^{w(\mathfrak{n})})_k$. Suppose now that $v \in V^{w(\mathfrak{n})}$ is such that $(H_{\mathfrak{r}_1} + k - 1)v \in (V^{w(\mathfrak{n})})_{k-1}$; we wish to show that $v \in V_k^{w(\mathfrak{n})}$. By what we have just shown, we may find $v' \in (V^{w(\mathfrak{n})})_k$ such that $(H_{\mathfrak{r}_1} + k - 1)v = (H_{\mathfrak{r}_1} + k - 1)v$, and so replacing v by $v - v'$, we may assume that $(H_{\mathfrak{r}_1} + k - 1)v = 0$. If we choose l minimally so that $v \in (V^{w(\mathfrak{n})})_l$, then, by what we have shown, $(V^{w(\mathfrak{n})})_{l-1} \ni (H_{\mathfrak{r}_1} + l - 1)v = (l-k)v$. Since l is minimal, we find that $l = k$, and so $v \in (V^{w(\mathfrak{n})})_k$. Thus we have shown that $(V^{w(\mathfrak{n})})_k = \{v \in V^{w(\mathfrak{n})} \mid (H_{\mathfrak{r}_1} + k - 1)v \in (V^{w(\mathfrak{n})})_{k-1}\}$. By induction on k, we conclude that $(V^{w(\mathfrak{n})})_k = (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}(H_{\mathfrak{r}_1}+1)\cdots(H_{\mathfrak{r}_1}+k-1)}$. Consequently, we may decompose $V^{w(\mathfrak{n})}$ as the direct sum

$$V^{w(\mathfrak{n})} = \bigoplus_{k \geq 0} (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1} = -k}, \tag{1.5.2}$$

with the non-zero elements of the kth direct summand being annihilated by $X_{\mathfrak{r}_1}^{k+1}$, but by no smaller power of $X_{\mathfrak{r}_1}$.

If we write $w' := s_{\mathfrak{r}_1} w = s_{\mathfrak{r}_2} \cdots s_{\mathfrak{r}_l} \in W$, then $\Delta_{w'} = \Delta_w \setminus \{\mathfrak{r}_1\} = \{\mathfrak{r}_2, \dots, \mathfrak{r}_l\}$. By induction we may assume that

$$V^{w'(\mathfrak{n})} \subset \sum_{\underline{k}' \in \Delta_{w'}} (X_{-\mathfrak{r}_2})^{k_{\mathfrak{r}_2}} \cdots (X_{-\mathfrak{r}_l})^{k_{\mathfrak{r}_l}} (V^{\mathfrak{n}})_{\underline{k}'}. \tag{1.5.3}$$

We claim that

$$(X_{\mathfrak{r}_1})^k (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1} = -k} \subset (V^{w'(\mathfrak{n})})^{H_{\mathfrak{r}_1} = k}. \tag{1.5.4}$$

Note that $w'(\Delta_+) = (w(\Delta_+) \setminus \{-\mathfrak{r}_1\}) \bigcup \{\mathfrak{r}_1\}$. Since $(X_{\mathfrak{r}_1})^{k+1}$ annihilates the space $(V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1} = -k}$, the left-hand side of (1.5.4) is annihilated by $X_{\mathfrak{r}_1}$. On the other hand, if $\mathfrak{r} \in w(\Delta_+) \setminus \{-\mathfrak{r}_1\}$, then any iterated commutator $[X_{\mathfrak{r}_1}, [X_{\mathfrak{r}_1}, \dots, [X_{\mathfrak{r}_1}, X_{\mathfrak{r}}]]]$

either lies in $w(\mathfrak{n})$ or else vanishes. Thus the left-hand side of (1.5.4) is annihilated by $X_{\mathfrak{r}}$. This shows that

$$(X_{\mathfrak{r}_1})^k (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}=-k} \subset V^{w'(\mathfrak{n})}. \tag{1.5.5}$$

On the other hand, it follows immediately from the formula $[X_{\mathfrak{r}_1}, X_{-\mathfrak{r}_1}] = H_{\mathfrak{r}_1}$ that

$$(X_{\mathfrak{r}_1})^k (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}=-k} \subset V^{H_{\mathfrak{r}_1}=k}. \tag{1.5.6}$$

The inclusion (1.5.4) follows from (1.5.5) and (1.5.6).

One easily deduces from (1.5.3) that

$$\begin{aligned} (V^{w'(\mathfrak{n})})^{H_{\mathfrak{r}_1}=k} &\subset \sum_{\underline{k}' \in \Delta_{w'}} (X_{-\mathfrak{r}_2})^{k_{\mathfrak{r}_2}} \cdots (X_{-\mathfrak{r}_l})^{k_{\mathfrak{r}_l}} ((V^{\mathfrak{n}})_{\underline{k}'})^{H_{\mathfrak{r}_1}=k} \\ &= \sum_{\underline{k} \in \Delta_w \text{ s.t. } k_{\mathfrak{r}_1}=k} (X_{-\mathfrak{r}_2})^{k_{\mathfrak{r}_2}} \cdots (X_{-\mathfrak{r}_l})^{k_{\mathfrak{r}_l}} (V^{\mathfrak{n}})_{\underline{k}}. \end{aligned} \tag{1.5.7}$$

Finally, a simple commutator calculation shows that the actions of $(X_{-\mathfrak{r}_1})^k (X_{\mathfrak{r}_1})^k$ and $(-1)^k H_{\mathfrak{r}_1}(H_{\mathfrak{r}_1}+1)\cdots(H_{\mathfrak{r}_1}+k-1)$ on $V^{w(\mathfrak{n})}$ coincide. Thus $(X_{-\mathfrak{r}_1})^k (X_{\mathfrak{r}_1})^k$ acts on $(V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}=-k}$ as multiplication by $k!$, and hence (combining (1.5.4) and (1.5.7))

$$\begin{aligned} (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}=-k} &= (X_{-\mathfrak{r}_1})^k (X_{\mathfrak{r}_1})^k (V^{w(\mathfrak{n})})^{H_{\mathfrak{r}_1}=-k} \\ &\subset \sum_{\underline{k} \in \Delta_w \text{ s.t. } k_{\mathfrak{r}_1}=k} (X_{-\mathfrak{r}_1})^{k} \cdots (X_{-\mathfrak{r}_l})^{k_{\mathfrak{r}_l}} (V^{\mathfrak{n}})_{\underline{k}}. \end{aligned}$$

Taken together with (1.5.2), this proves the lemma. □

2. Parabolically induced representations

(2.1) Let G be a connected reductive linear algebraic group over L. Fix a parabolic subgroup P of G, with Levi quotient M. If U is an object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$, then we define

$$\mathrm{Ind}_P^G U := \{f \in \mathcal{C}^{\mathrm{la}}(G, U) \mid f(pg) = pf(g) \text{ for all } p \in P\},$$

regarded as a closed subspace of $\mathcal{C}^{\mathrm{la}}(G, U)$, and equipped with the right regular action of G. (Here we regard P as acting on U through its Levi quotient M.)

We may find a compact open subgroup G_0 of G such that $G = PG_0$. (For example, G_0 could be taken to be a special, good, maximal compact subgroup of G [4, p. 140].) If we write $P_0 = G_0 \bigcap P$, then as a G_0-representation, we may identify $\mathrm{Ind}_P^G U$ with the closed subspace

$$\mathrm{Ind}_{P_0}^{G_0} U = \{f \in \mathcal{C}^{\mathrm{la}}(G_0, U) \mid f(pg) = pf(g) \text{ for all } p \in P_0\}$$

of the space $\mathcal{C}^{\mathrm{la}}(G_0, U)$. This latter space is of compact type, by [6, Prop. 2.2.28], and thus $\mathrm{Ind}_P^G U$ is a space of compact type. It follows from [6, Prop. 3.5.11, Lem. 3.6.14] that $\mathrm{Ind}_P^G U$ is an object of $\mathrm{Rep}_{\mathrm{la.c}}(G)$.

Evaluation at the identity of G induces a continuous P-equivariant map

$$\mathrm{Ind}_P^G U \to U.$$

It follows from [6, Prop. 5.1.1 (iii)] that if V is an object of $\mathrm{Rep}_{\mathrm{la.c}} G$, then this map induces a "Frobenius reciprocity" isomorphism

$$\mathcal{L}_G(V, \mathrm{Ind}_P^G U) \xrightarrow{\sim} \mathcal{L}_P(V, U).$$

Let us remark that a thorough discussion of the basic properties of locally analytic induction, including this Frobenius reciprocity isomorphism, appeared in the 1997 dissertation of Feaux de Lacroix [10, §4].

Proposition 2.1.1. *If U is an object of* $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$*, then* $\mathrm{Ind}_P^G U$ *is an object of* $\mathrm{Rep}^z_{\mathrm{la.c}}(G)$.

Proof. Apply Lemma 1.1.1 to find a cofinal sequence $\{\mathbb{H}_i\}_{i\geq 1}$ of analytic open subgroups of G_0, each normalized by G_0. For each $i \geq 1$, let $\mathbb{P}_i$ denote the intersection of $\mathbb{H}_i$ with P (thinking of P as a closed rigid analytic subspace of G), and let $\mathbb{M}_i$ denote the image of $\mathbb{P}_i$ under the projection $P \to M$. Then $\{\mathbb{M}_i\}_{i\geq 1}$ forms a cofinal sequence of analytic open subgroups of M, each of which is normalized by the image M_0 of P_0 in M.

Apply Lemma 1.1.2 to write $U \xrightarrow{\sim} \varinjlim_{i\geq 1} U_i$, where each U_i is a K-Banach space equipped with a locally analytic $M_0 Z_M$-action that is $\mathbb{M}_i$-analytic. Then we may write

$$\begin{aligned}\mathrm{Ind}_P^G U &\xrightarrow{\sim} \mathrm{Ind}_{P_0}^{G_0} U \\ &\xrightarrow{\sim} \varinjlim_i \{f \in \mathcal{C}^{\mathrm{la}}(G_0, U_i)_{\mathbb{H}_i-\mathrm{an}} \mid f(pg) = pf(g) \text{ for all } p \in P_0, g \in G_0\}.\end{aligned}$$

Each term in this inductive limit is a Banach space whose image in $\mathrm{Ind}_P^G U$ is invariant under Z_G (since $Z_G \subset Z_M$), and thus we have exhibited $\mathrm{Ind}_P^G U$ as an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(G)$. □

Proposition 2.1.2. *If U is an admissible (respectively strongly admissible, respectively very strongly admissible) locally analytic representation of M, then $\mathrm{Ind}_P^G U$ is an admissible (respectively strongly admissible, respectively very strongly admissible) locally analytic representation of G.*

Proof. As above, we exploit the isomorphism $\mathrm{Ind}_P^G U \xrightarrow{\sim} \mathrm{Ind}_{P_0}^{G_0} U$. Fix a choice of lifting of M to a Levi factor of P, and write $M_0 := G_0 \bigcap M$. By definition, U is an admissible (respectively strongly admissible) locally analytic representation of M if and only if it is such a representation of M_0.

Let $\overline{P}$ denote an opposite parabolic to P, chosen so that $P \bigcap \overline{P} = M$, our fixed choice of Levi factor of P. Applying [7, Prop. 4.1.6] (after fixing a minimal parabolic of G contained in P) we choose a cofinal sequence $\{H_i\}_{i\geq 0}$ of analytic open subgroups of G_0, whose underlying rigid analytic subgroups $\mathbb{H}_i$ admit a rigid analytic Iwahori decomposition $\overline{\mathbb{N}}_i \times \mathbb{M}_i \times \mathbb{N}_i \to \mathbb{H}_i$ with respect to P and $\overline{P}$. Note that the subgroups $M_i := \mathbb{M}_i(L)$ then form a cofinal sequence of analytic open subgroups of M_0.

Let U be an admissible locally analytic M_0-representation. Fix $i \geq 0$. If $g \in G_0$, let U^g denote U regarded as a locally analytic representation of $g^{-1}Pg$, via $(g^{-1}pg) \cdot u = pu$, and write $P_i^g = H_i \bigcap g^{-1}Pg$. Choose an analytic open subgroup L_i^g of the locally analytic group P_i^g, for which the closed embedding $L_i^g \subset H_i$ arises from a rigid analytic closed embedding $\mathbb{L}_i^g \subset \mathbb{H}_i$. Since U is an admissible locally analytic M_0-representation, and so in particular an admissible locally analytic P_0-representation, U^g is an admissible locally analytic P_i^g-representation. In particular, the space $(U^g)_{\mathbb{L}_i^g-\mathrm{an}}$ of $\mathbb{L}_i^g$-analytic vectors in U^g is a Banach space admitting a L_i^g-equivariant closed embedding $(U^g)_{\mathbb{L}_i^g-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{L}_k^g, K)^r$, for some $r \geq 0$. (Here and below we are using the characterization of admissible locally analytic representations provided by [6, Def. 6.1.1], and the discussion following that definition.)

The decomposition of G_0 into right H_i-cosets induces a natural H_i-equivariant closed embedding

$$\operatorname{Ind}_{P_0}^{G_0} U \to \bigoplus_{g \in G_0/H_i} \operatorname{Ind}_{P_i^g}^{H_i} U^g \to \bigoplus_{g \in G_0/H_i} \operatorname{Ind}_{L_i^g}^{H_i} U^g,$$

and hence [6, Prop. 3.3.23] a closed embedding of H_i-representations

$$(\operatorname{Ind}_{P_0}^{G_0} U)_{\mathbb{H}_i-\mathrm{an}} \to \bigoplus_{g \in G_0/H_i} (\operatorname{Ind}_{L_i^g}^{H_i} U^g)_{\mathbb{H}_i-\mathrm{an}}.$$

We will show that each $(\operatorname{Ind}_{L_i^g}^{H_i} U^g)_{\mathbb{H}_i-\mathrm{an}}$ admits an H_i-equivariant closed embedding into $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, K)^r$, for some $r \geq 0$. and thus conclude that $(\operatorname{Ind}_{P_0}^{G_0} U)_{\mathbb{H}_i-\mathrm{an}}$ admits a closed H_i-equivariant embedding into $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, K)^s$, for some $s \geq 0$. Since $i \geq 0$ was arbitrary, we will have established that $\operatorname{Ind}_{P_0}^{G_0} U$ is an admissible locally analytic representation.

It follows from [6, props. 3.3.23, 3.3.24] that $(\operatorname{Ind}_{L_i^g}^{H_i} U^g)_{\mathbb{H}_i-\mathrm{an}}$ is equal to the preimage of $\operatorname{Ind}_{L_i^g}^{H_i} U^g$ under the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, U^g) \to \mathcal{C}^{\mathrm{la}}(H_i, U^g)$. Any element in $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, U^g)$ whose image in $\mathcal{C}^{\mathrm{la}}(H_i, U^g)$ lies in $\operatorname{Ind}_{L_i^g}^{H_i} U^g$ must actually lie in $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, (U^g)_{\mathbb{L}_i^g-\mathrm{an}})$, as follows immediately from the closed embedding $\mathbb{L}_i^g \subset \mathbb{H}_i$ and the definition of $\operatorname{Ind}_{L_i^g}^{H_i} U^g$ as a subspace of $\mathcal{C}^{\mathrm{la}}(H_i, U^g)$. Thus $(\operatorname{Ind}_{L_i}^{H_i} U^g)_{\mathbb{H}_i-\mathrm{an}}$ is the preimage of $\operatorname{Ind}_{L_i^g}^{H_i} U^g$ under the natural map $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, (U^g)_{\mathbb{L}_i^g-\mathrm{an}}) \to \mathcal{C}^{\mathrm{la}}(H_i, U^g)$, and so in particular embeds as a closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, (U^g)_{\mathbb{L}_i^g-\mathrm{an}})$. The closed embedding $(U^g)_{\mathbb{L}_i^g-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{L}_i^g, K)^r$ thus induces a closed embedding

$$(\operatorname{Ind}_{L_i^g}^{H_i} U^g)_{\mathbb{H}_i-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, \mathcal{C}^{\mathrm{an}}(\mathbb{L}_i^g, K))^r \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{L}_i^g \times \mathbb{H}_i, K)^r.$$

Again, by the definition of $\operatorname{Ind}_{L_i^g}^{H_i} U^g$, this closed embedding lies in the closed subspace of $\mathcal{C}^{\mathrm{an}}(\mathbb{L}_i^g \times \mathbb{H}_i, K)^r$ consisting of functions f satisfying the condition $f(p^g, h) = f(1, p^g h)$ for any $p^g \in \mathbb{L}_i^g$, $h \in \mathbb{H}_i$. Restricting to the closed subvariety $1 \times \mathbb{H}_i$ of $\mathbb{L}_i^g \times \mathbb{H}_i$ induces an isomorphism between this subspace and $\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, K)^r$. Thus we obtain the required closed embedding $(\operatorname{Ind}_{L_i^g}^{H_i} U^g)_{\mathbb{H}_i-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, K)^r$.

If we suppose that U is a strongly admissible locally analytic representation of M_0, then a similar, but more straightforward, argument shows that $\operatorname{Ind}_{P_0}^{G_0} U$ is a strongly admissible locally analytic representation of G_0. Indeed, since U is strongly admissible, it admits a closed M_0-equivariant embedding $U \to \mathcal{C}^{\mathrm{la}}(M_0, K)^r$, for some $r \geq 0$. (See the remark following [6, Def. 6.2.1].) Thus $\mathcal{C}^{\mathrm{la}}(G_0, U)$ admits a closed embedding into

$$\mathcal{C}^{\mathrm{la}}(G_0, \mathcal{C}^{\mathrm{la}}(M_0, K))^r \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(M_0 \times G_0, K)^r.$$

The closed subspace $\operatorname{Ind}_{P_0}^{G_0} U$ of $\mathcal{C}^{\mathrm{la}}(G_0, U)$ lands in the subspace of functions f for which $f(m, g) = f(1, mg)$ (for $m \in M_0$, $g \in G_0$). Restricting such functions to $1 \times G_0 \subset M_0 \times G_0$ induces an isomorphism of this subspace with $\mathcal{C}^{\mathrm{la}}(G_0, K)^r$. Thus we obtain a closed embedding $\operatorname{Ind}_{P_0}^{G_0} U \to \mathcal{C}^{\mathrm{la}}(G_0, K)^r$, as required.

Finally, suppose that U is very strongly admissible, and choose an admissible Banach space representation W of M for which U admits a continuous M-equivariant injection $U \to W$. There is then a continuous G-equivariant injection

$$\operatorname{Ind}_P^G U \to \{f \in \mathcal{C}(G,W) \mid f(pg) = pf(g) \text{ for all } p \in P\}. \tag{2.1.3}$$

(The target of this map is the continuous induction of W from M to G.) The preceding argument shows that the source of (2.1.3) is a strongly admissible G-representation, and a similar argument, using the fact that W is an admissible Banach space representation of M, shows that the target of (2.1.3) is an admissible Banach space representation of G. By definition, we conclude that the source of (2.1.3) is a very strongly admissible locally analytic G-representation. □

(2.2) Let G be a connected reductive linear algebraic group over L. Fix a parabolic subgroup P of G, a Levi factor M of P, and let $\overline{P}$ denote the opposite parabolic to P, chosen so that $P \bigcap \overline{P} = M$. Let N and $\overline{N}$ denote the unipotent radicals of P and $\overline{P}$ respectively, so that $P = MN$ and $\overline{P} = M\overline{N}$. Fix an object U of $\mathrm{Rep}_{\mathrm{la.c}}(M)$.

Recall the definitions of the convex spaces $\mathcal{C}_c^{\mathrm{la}}(N,U)$ and $\mathcal{C}^{\omega}(N,U)_e$ from Subsection 1.4. These are of compact type, by Lemma 1.4.7 and 1.4.10 respectively.

We endow $\mathcal{C}_c^{\mathrm{la}}(N,U)$ with a locally analytic P-action as follows: If $f \in \mathcal{C}_c^{\mathrm{la}}(N,U)$ and $p \in P$, then factor $p = mn$ with $m \in M$ and $n \in N$, and define the function $pf \in \mathcal{C}_c^{\mathrm{la}}(N,U)$ via the formula $(pf)(n') = mf(m^{-1}n'mn)$ for all $n' \in N$; here m acts on the element $f(m^{-1}n'mn)$ of U via the given locally analytic action of M on U. Lemma 1.4.8 shows that the N-action is locally analytic. It is easily verified that this is also true of the M-action (and hence of the P-action). We will not give a direct proof here, however, since the local analyticity of the P-action follows from Lemma 2.3.6 below and the fact that G, and so P, acts locally analytically on $\operatorname{Ind}_{\overline{P}}^G U$.

Let $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ denote the space of compactly supported locally constant functions on N. Note that $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ coincides with the closed subspace of $\mathfrak{n}$-invariants in $\mathcal{C}_c^{\mathrm{la}}(N,U)$. Thus $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ is a closed P-invariant subspace of $\mathcal{C}_c^{\mathrm{la}}(N,U)$, and so in particular is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$. (By construction this locally analytic P-action on $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ agrees with that defined in [7, §3.5].)

We define a locally analytic M-action on $\mathcal{C}^{\omega}(N,U)_e$ as follows: If $f \in \mathcal{C}^{\omega}(N,U)_e$, and $m \in M$, then let the germ $mf \in \mathcal{C}^{\omega}(N,U)_e$ be defined via the formula $(mf)(n) = mf(m^{-1}nm)$ for $n \in N$ sufficiently close to e (i.e. close enough for $f(m^{-1}nm)$ to be defined); here m acts on the element $f(m^{-1}nm)$ of U via the given locally analytic action of M on U. Again, we omit the (easy) direct proof that this action is locally analytic, since local analyticity of the action follows from Corollary 2.3.4 below and the fact that the $\overline{P}$ (and hence the M-action) on $(\operatorname{Ind}_{\overline{P}}^G U)_e$ is locally analytic (by Lemma 2.3.5 (i)).

We let Z_M denote the centre of M, let A denote the maximal split torus in Z_M, and fix a coweight $\alpha : L^{\times} \to A$ that pairs strictly negatively with each restricted root of A acting on N. Set $z_0 = \alpha(p)^{-1}$. In the remainder of this subsection we introduce some auxiliary constructions that will be useful for what follows. We apply [7, Prop. 4.1.6] (after fixing a minimal parabolic of G contained in P) to find a cofinal sequence $\{\mathbb{H}_i\}_{i \geq 0}$ of good analytic open subgroups of G that admit a rigid analytic Iwahori decomposition with respect to P and $\overline{P}$, such that $\mathbb{H}_i$ is normal in

H_0 for each $i \geq 0$, such that each inclusion $\mathbb{H}_{i+1} \subset \mathbb{H}_i$ is relatively compact, and such that $\mathbb{N}_i := z_0^i \mathbb{N}_0 z_0^{-i}$ for all $i \geq 0$. Let $\mathbb{H}_i \xrightarrow{\sim} \overline{\mathbb{N}}_i \mathbb{M}_i \mathbb{N}_i$ be the rigid analytic Iwahori decomposition of $\mathbb{H}_i$. Also, fix an increasing sequence $\{N^i\}_{i \geq 0}$ of compact open subsets of N, chosen so that $N^0 \supset N_0$, and so that $N = \bigcup_{i \geq 0} N^i$.

We now define two submonoids of the centre Z_M of M that will play an important role in what follows.

Definition 2.2.1. Write $Z_M^+ = \{z \in Z_M \mid z N_0 z^{-1} \subset N_0\}$ (respectively $Z_M^- = \{z \in Z_M \mid N_0 \subset z N_0 z^{-1}\}$). Note that each of Z_M^+ and Z_M^- is a submonoid of Z_M.

Write $U \xrightarrow{\sim} \varinjlim_n U_i$, where U_i is a K-Banach space equipped with an $\mathbb{M}_i$-analytic action of M_0, and the transition maps are compact, injective, and M_0-equivariant. In the case when U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, we furthermore choose the U_i to be Z_M-invariant, and the transition maps to be Z_M-equivariant (as we may, by Lemma 1.1.2).

For each pair of integers $i, j \geq 0$, we define the K-Banach spaces of functions $A_{i,j} := \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i, U_j)$ and $B_{i,j} := \bigoplus_{n \in N_i \backslash N_0} \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n, U_j)$ (where the direct sum is over a set of left coset representatives of N_i in N_0). These are spaces of U_j-valued rigid analytic functions on the spaces $\mathbb{N}_i$ and $\coprod_{n \in N_i \backslash N_0} \mathbb{N}_i n$ respectively.

If $i' \geq i \geq 0$ and $j' \geq j \geq 0$, then there are continuous injections

$$A_{i,j} \to A_{i',j'} \tag{2.2.2}$$

and

$$B_{i,j} \to B_{i',j'} \tag{2.2.3}$$

induced by the continuous injection $U_j \to U_{j'}$, together with restricting functions from $\mathbb{N}_i$ to $\mathbb{N}_{i'}$, or from $\coprod_{n \in N_i \backslash N_0} \mathbb{N}_i n$ to $\coprod_{n \in N_{i'} \backslash N_0} \mathbb{N}_{i'} n$.

Taking the germ of an element of $A_{i,j}$ at e yields a continuous injection

$$A_{i,j} \to \mathcal{C}^{\omega}(N, U)_e. \tag{2.2.4}$$

Passing to the inductive limit in i and j, we obtain an isomorphism

$$\varinjlim_{i,j} A_{i,j} \xrightarrow{\sim} \mathcal{C}^{\omega}(N, U)_e. \tag{2.2.5}$$

(The transition maps in the inductive limit are provided by the maps (2.2.2).) Also, extension by zero yields a natural continuous injection

$$A_{i,j} \to \mathcal{C}_c^{\mathrm{la}}(N, U). \tag{2.2.6}$$

Composing (2.2.6) with the natural map $\mathcal{C}_c^{\mathrm{la}}(N, U) \to \mathcal{C}^{\omega}(N, U)_e$ given by passing to the germ of a locally analytic function at the identity yields the map (2.2.4).

Regarding an element of $B_{i,j}$ as a U_j-valued function on N_0 we obtain a continuous injection

$$B_{i,j} \to \mathcal{C}^{\mathrm{la}}(N_0, U_j). \tag{2.2.7}$$

Fixing j, and passing to the inductive limit in i, we obtain an isomorphism

$$\varinjlim_{i} B_{i,j} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(N_0, U_j). \tag{2.2.8}$$

The continuous injection $U_j \to U$ induces a continuous injection

$$\mathcal{C}^{\mathrm{la}}(N_0, U_j) \to \mathcal{C}^{\mathrm{la}}(N_0, U), \tag{2.2.9}$$

which when composed with (2.2.7) yields a continuous injection

$$B_{i,j} \to \mathcal{C}^{\mathrm{la}}(N_0, U). \tag{2.2.10}$$

Passing to the inductive limit in i and j yields an isomorphism

$$\varinjlim_{i,j} B_{i,j} \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(N_0, U), \tag{2.2.11}$$

which may also be obtained by taking the inductive limit with respect to j of the isomorphisms (2.2.8). (The transition maps in the inductive limits (2.2.8) and (2.2.11) are provided by the maps (2.2.3).)

(**2.3)** We maintain the notation of Subsection 2.2. In this subsection, and those that follow, we will consider parabolic induction from $\overline{P}$ to G (rather than from P to G).

Unless the M-action on our fixed object U of $\mathrm{Rep}_{\mathrm{la.c}}(M)$ is trivial, the elements of $\mathrm{Ind}_{\overline{P}}^{G} U$ are not U-valued functions on $\overline{P}\backslash G$. (Rather, they are sections of a bundle on $\overline{P}\backslash G$, with fibres isomorphic to U.) Nevertheless, any element f of $\mathrm{Ind}_{\overline{P}}^{G} U$ has a well-defined support (that is, region over which its germ is non-zero) on $\overline{P}\backslash G$, which is a compact open subset of this quotient. (The support of f as a function on G is an open and closed subset of G, invariant under left multiplication by $\overline{P}$, and so corresponds to a compact open subset of $\overline{P}\backslash G$.)

If $f \in \mathrm{Ind}_{\overline{P}}^{G} U$ and Ω is a compact open subset of $\overline{P}\backslash G$, then we write $f_{|\Omega}$ to denote the function

$$f_{|\Omega}(g) = \begin{cases} f(g) \text{ if } \overline{P}g \in \Omega \\ 0 \text{ otherwise.} \end{cases}$$

The function $f_{|\Omega}$ is clearly again an element of $\mathrm{Ind}_{\overline{P}}^{G} U$.

Definition 2.3.1. (i) If Ω is any open subset of $\overline{P}\backslash G$, we let $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega)$ denote the subspace of elements whose support is contained in Ω.

(ii) If $x \in \overline{P}\backslash G$, then we define the stalk of $\mathrm{Ind}_{\overline{P}}^{G} U$ at x to be

$$(\mathrm{Ind}_{\overline{P}}^{G} U)_x := \varinjlim_{\Omega \ni x} (\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega),$$

where the locally convex inductive limit is taken over the directed set of compact open neighbourhoods of x, with the transition map $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega) \to (\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega')$ (for $\Omega \supset \Omega'$ an inclusion of neighbourhood of x) being given by $f \mapsto f_{|\Omega'}$. If Ω is an open subset of $\overline{P}\backslash G$ containing x, then we denote by st_x (for "stalk at x") the continuous map $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega) \to (\mathrm{Ind}_{\overline{P}}^{G} U)_x$, arising from the description of the target as an inductive limit.

Since $N \bigcap \overline{P} = \{e\}$, the map $n \mapsto \overline{P}n$ is an open immersion

$$N \to \overline{P}\backslash G, \tag{2.3.2}$$

and we thus regard N as an open subset of $\overline{P}\backslash G$.

Lemma 2.3.3. *If Ω is a compact open subset of N, regarded also as a compact open subset of $\overline{P}\backslash G$ via (2.3.2), then there is a natural topological isomorphism $\mathcal{C}^{\mathrm{la}}(\Omega, U) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega)$.*

Proof. The preimage of Ω (regarded as a subset of $\overline{P}\backslash G$) in G is equal to $\overline{P}\Omega$. We may write $G = \overline{P}\Omega \sqcup (G \setminus (\overline{P}\Omega))$; this is a partition of G into two open sets invariant under the left action of $\overline{P}$ on G. Thus we obtain a topological isomorphism

$$(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega) \xrightarrow{\sim} \{f \in \mathcal{C}^{\mathrm{la}}(\overline{P}\Omega, U) \mid f(\overline{p}g) = \overline{p}f(g) \text{ for all } g \in \overline{P}\Omega\}.$$

Since $N \bigcap \overline{P} = \{e\}$, multiplication induces a locally analytic isomorphism $\overline{P} \times \Omega \to \overline{P}\Omega$. Thus restricting to Ω induces an isomorphism $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(\Omega, U)$. That we do indeed obtain an isomorphism follows from the fact that $\overline{P}$ acts (through M) in a locally analytic fashion on U, so that we may extend any element of $\mathcal{C}^{\mathrm{la}}(\Omega, U)$ in a unique manner to an element of $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega)$ (using the natural orbit map $U \to \mathcal{C}^{\mathrm{la}}(\overline{P}, U)$ provided by [6, Thm. 3.5.7, Thm. 3.6.12]). □

Corollary 2.3.4. *The open immersion (2.3.2) induces an M-equivariant topological isomorphism $\mathcal{C}^{\omega}(N, K)_e \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^{G} U)_e$.*

Proof. This follows immediately from Lemma 2.3.3, by passing to the inductive limit over the compact open neighbourhoods Ω of e. (The claimed M-equivariance follows immediately from a consideration of the M-action on $\mathcal{C}^{\omega}(N, K)_e$ as defined at the end of Subsection 2.2, the definition of the induction $\mathrm{Ind}_{\overline{P}}^{G} U$, and the construction of the isomorphism.) □

As usual, for any $x \in \overline{P}\backslash G$, we write G_x to denote the stabilizer of x in G (with respect to the action of G on $\overline{P}\backslash G$ given by right multiplication). If $x = \overline{P}g$, then of course $G_x = g^{-1}\overline{P}g$.

Lemma 2.3.5. *(i) For each $x \in \overline{P}\backslash G$, the stalk $(\mathrm{Ind}_{\overline{P}}^{G} U)_x$ is a compact type space, equipped in a natural way with a $(\mathfrak{g}, G_x)$-action. Furthermore, the $\mathfrak{g}$-action is locally integrable, and the G_x-action is locally analytic.*

(ii) If Ω is an open subset of $\overline{P}\backslash G$, then $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega)$ is a closed $\mathfrak{g}$-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$. Furthermore, the $\mathfrak{g}$-action is locally integrable.

(iii) For any $g \in G$ and $x \in \overline{P}\backslash G$, the action of g induces a topological isomorphism $g\cdot : (\mathrm{Ind}_{\overline{P}}^{G} U)_{gx} \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^{G} U)_x$, compatible with the $(\mathfrak{g}, G_{gx})$-action on the source and the $(\mathfrak{g}, G_x)$-action on the target (in the sense that $g(Xf) = \mathrm{Ad}_g(X)gf$ and $g(g'f) = (gg'g^{-1})gf$ for any $g \in G$, $X \in \mathfrak{g}$, $g' \in G_{gx}$ (note that $gg'g^{-1}$ is then an element of G_x), and $f \in (\mathrm{Ind}_{\overline{P}}^{G} U)_{gx}$).

(iv) For each $x \in \overline{P}\backslash G$, the map $\mathrm{st}_x : \mathrm{Ind}_{\overline{P}}^{G} U \to (\mathrm{Ind}_{\overline{P}}^{G} U)_x$ is $(\mathfrak{g}, G_x)$-equivariant.

Proof. Since $\mathfrak{g}$ acts on $\mathrm{Ind}_{\overline{P}}^{G} U$ through differential operators, it is obvious that $(\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega)$ is a $\mathfrak{g}$-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$ for any open subset Ω of $\overline{P}\backslash G$. Thus for any $x \in \overline{P}\backslash G$, the stalk $(\mathrm{Ind}_{\overline{P}}^{G} U)_x$ is described as the locally convex inductive limit of $\mathfrak{g}$-representations, with $\mathfrak{g}$-equivariant transition maps, and so is naturally a topological $\mathfrak{g}$-representation. Incidentally, we see that the claim of (iv) regarding $\mathfrak{g}$-equivariance is true by definition of the $\mathfrak{g}$-action on $(\mathrm{Ind}_{\overline{P}}^{G} U)_x$.

It is clear that for any $g \in G$ and $x \in \overline{P}\backslash G$, the action of g induces a topological isomorphism $(\mathrm{Ind}_{\overline{P}}^{G} U)_{gx} \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^{G} U)_x$, such that $g(Xf) = \mathrm{Ad}_g(X)gf$ for any

$X \in \mathfrak{g}$ and $f \in (\mathrm{Ind}_{\overline{P}}^G U)_{gx}$. In particular, if $g \in G_x$ then g induces a topological automorphism of $(\mathrm{Ind}_{\overline{P}}^G U)_x$, and so we find that $(\mathrm{Ind}_{\overline{P}}^G U)_x$ is a topological $(\mathfrak{g}, G_x)$-representation. This establishes (iii) (since the remaining claimed compatibility "$g(g'f) = (gg'g^{-1})gf$" follows directly from the definitions of the actions involved) and a part of (i). By construction of the G_x-action on $(\mathrm{Ind}_{\overline{P}}^G U)_x$, we see that the claim of (iv) regarding G_x-equivariance holds (and this completes the proof of (iv)).

Corollary 2.3.4 and Lemma 1.4.10 show that $(\mathrm{Ind}_{\overline{P}}^G U)_e$ is of compact type. Since acting by $g^{-1} \in G$ gives a topological isomorphism $(\mathrm{Ind}_{\overline{P}}^G U)_e \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^G U)_x$ for any $x = \overline{P}g \in \overline{P}\backslash G$, we see that every $(\mathrm{Ind}_{\overline{P}}^G U)_x$ is of compact type, establishing another claim of (i).

By definition we see that $(\mathrm{Ind}_{\overline{P}}^G U)(\Omega) = \bigcap_{x\in\Omega} \ker(\mathrm{st}_x)$. The target of each map st_x is of compact type, and so Hausdorff, and hence $\ker(\mathrm{st}_x)$ is closed for all $x \in \overline{P}\backslash G$. Consequently $(\mathrm{Ind}_{\overline{P}}^G U)(\Omega)$ is closed. The $\mathfrak{g}$-action on $\mathrm{Ind}_{\overline{P}}^G U$ is locally integrable, by Proposition 1.2.18, and so Proposition 1.2.19 shows that the same is true of the the $\mathfrak{g}$-action on $(\mathrm{Ind}_{\overline{P}}^G U)(\Omega)$. This completes the proof of (ii).

To complete the proof of (i), and hence of the lemma, we must show that the $\mathfrak{g}$-action and G_x-action on $(\mathrm{Ind}_{\overline{P}}^G U)_x$ are locally integrable and locally analytic respectively. By the result of the preceding paragraph, we see that $(\mathrm{Ind}_{\overline{P}}^G U)_x$ is the $\mathfrak{g}$-equivariant locally convex inductive limit of a sequence of locally integrable $\mathfrak{g}$-representations, and so by Proposition 1.2.20 it is a locally integrable $\mathfrak{g}$-representation. Since the G_x-action on $\mathrm{Ind}_{\overline{P}}^G U$ is locally analytic, and since $st_x : \mathrm{Ind}_{\overline{P}}^G U \to (\mathrm{Ind}_{\overline{P}}^G U)_x$ is surjective and G_x-equivariant, it follows from [6, Prop. 3.6.14] that the G_x-action on $(\mathrm{Ind}_{\overline{P}}^G U)_x$ is also locally analytic. □

Lemma 2.3.6. *The open immersion (2.3.2) induces a P-equivariant topological isomorphism* $\mathcal{C}_c^{\mathrm{la}}(N, U) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^G U)(N)$.

Proof. Since any element of $(\mathrm{Ind}_{\overline{P}}^G U)(N)$ has compact support, we see that the isomorphisms of Lemma 2.3.3 give rise to a continuous bijection

$$\mathcal{C}_c^{\mathrm{la}}(N, U) \to (\mathrm{Ind}_{\overline{P}}^G U)(N). \tag{2.3.7}$$

The source of (2.3.7) is of compact type, by Lemma 1.4.7, and so is the target, since by Lemma 2.3.5 (ii) it is a closed subspace of the compact type space $\mathrm{Ind}_{\overline{P}}^G U$. The Open Mapping Theorem thus implies that (2.3.7) is an isomorphism. As in the proof of Corollary 2.3.4, the P-equivariance of this isomorphism is immediately checked. □

We transport the $(\mathfrak{g}, \overline{P})$-action on $(\mathrm{Ind}_{\overline{P}}^G)_e$ given by Lemma 2.3.5 (iii) to a $(\mathfrak{g}, \overline{P})$-action on $\mathcal{C}^\omega(N, U)_e$ via the isomorphism of Corollary 2.3.4, and hence regard $\mathcal{C}^\omega(N, U)_e$ as a $(\mathfrak{g}, \overline{P})$-representation. Similarly, we transport the $\mathfrak{g}$-action on $(\mathrm{Ind}_{\overline{P}}^G U)(N)$ to a $\mathfrak{g}$-action on $\mathcal{C}_c^{\mathrm{la}}(N, U)$ via the isomorphism of Lemma 2.3.6, and hence regard $\mathcal{C}_c^{\mathrm{la}}(N, U)$ as a $(\mathfrak{g}, P)$-representation.

Composing the closed embedding $(\mathrm{Ind}_{\overline{P}}^G U)(N) \to \mathrm{Ind}_{\overline{P}}^G U$ with the isomorphism of Lemma 2.3.6 and the continuous injection (2.2.6) yields for each $i \geq 0$ a continuous injection

$$A_{i,i} \to \mathrm{Ind}_{\overline{P}}^G U, \tag{2.3.8}$$

which we use to regard $A_{i,i}$ as the latent Banach space underlying a BH-subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$. Similarly, composing the isomorphism of Corollary 2.3.4 with (2.2.4) yields for each $i \geq 0$ a continuous injection

$$A_{i,i} \to (\mathrm{Ind}_{\overline{P}}^{G} U)_e, \tag{2.3.9}$$

via which we regard $A_{i,i}$ as the latent Banach space underlying a BH-subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$.

Proposition 2.3.10. *(i) For any $i \geq 0$ the image of $A_{i,i}$ in $\mathrm{Ind}_{\overline{P}}^{G} U$ under (2.3.8) is H_i-invariant. The resulting H_i-action on $A_{i,i}$ (given by* [6, Prop. 1.1.2 (ii)]*) is furthermore $\mathbb{H}_i$-analytic.*

(ii) For any $i \geq 0$, the image of $A_{i,i}$ in $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$ under (2.3.9) is $\mathfrak{g}$-invariant. If U is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$, then it is furthermore Z_M^{-}-invariant.

Proof. Since $\mathbb{H}_i$ is a rigid analytic group, the group operation on $\mathbb{H}_i$ induces a rigid analytic map

$$\mathbb{N}_i \times \mathbb{H}_i \to \mathbb{H}_i \xrightarrow{\sim} \overline{\mathbb{N}}_i \times \mathbb{M}_i \times \mathbb{N}_i \tag{2.3.11}$$

(where the second arrow is the rigid analytic Iwahori decomposition of $\mathbb{H}_i$). Set $\overline{P}_i := H_i \bigcap \overline{P} = \overline{N}_i M_i$, and write

$$(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}} := \{ f \in \mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, U_i) \mid f(\overline{p}g) = \overline{p}f(g) \text{ for all } \overline{p} \in \overline{P}_i, g \in G_i \}$$

(where $\overline{P}_i$ acts on U_i through its quotient M_i). Taking into account (2.3.11) and the fact that the M_i-action on U_i is $\mathbb{M}_i$-analytic, we see that restricting functions to N_i induces an isomorphism

$$(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i, U_i) = A_{i,i}. \tag{2.3.12}$$

On the one hand, right translation by elements of H_i equips the source of (2.3.12) with an $\mathbb{H}_i$-analytic H_i-action. On the other hand, since H_i admits an Iwahori decomposition, any element of $(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}}$ evidently admits a unique extension by zero to an element of $\mathrm{Ind}_{\overline{P}}^{G} U$, and we obtain a continuous H_i-equivariant injection

$$(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}} \to \mathrm{Ind}_{\overline{P}}^{G} U, \tag{2.3.13}$$

which fits into the commutative diagram

$$\begin{array}{ccc} (\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}} & \xrightarrow{(2.3.12)} & A_{i,i} \\ & \searrow & \downarrow {\scriptstyle (2.3.8)} \\ & & \mathrm{Ind}_{\overline{P}}^{G} U. \end{array}$$

Part (i) follows from the fact that the diagonal arrow is H_i-equivariant, while the horizontal arrow is an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccc} A_{i,i} & \xrightarrow{(2.3.8)} & \operatorname{Ind}_{\overline{P}}^{G} U \\ & \searrow^{(2.3.9)} & \downarrow \mathrm{st}_e \\ & & (\operatorname{Ind}_{\overline{P}}^{G} U)_e. \end{array}$$

Part (i) shows that the image of the horizontal arrow is $\mathfrak{g}$-invariant, and since the vertical arrow is $\mathfrak{g}$-equivariant, by Lemma 2.3.5 (iv), we see that the image of the diagonal arrow is also $\mathfrak{g}$-invariant. To complete the proof of (ii), we must show that when U is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$ this image is also Z_M^--invariant.

Since we have chosen U_i to be Z_M-invariant when U is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$, we find that for any $z \in Z_M$ and f in the image of (2.3.9), the translate zf is a U_i-valued rigid analytic function on $z\mathbb{N}_i z^{-1}$. If $z \in Z_M^-$, then $N_i \subset zN_i z^{-1}$, and (since N_i and zN_iz^{-1} are good analytic open subgroups of G) this inclusion extends to a rigid analytic embedding $\mathbb{N}_i \subset z\mathbb{N}_i z^{-1}$. Thus zf in particular restricts to a U_i-valued rigid analytic function on $\mathbb{N}_i$. Hence the image of (2.3.9) is Z_M^--invariant, as claimed. □

We use [6, Prop. 1.1.2 (ii)] to lift the $(\mathfrak{g}, Z_M^-)$-action on the image of (2.3.9) given by part (ii) of the preceding lemma to a continuous $(\mathfrak{g}, Z_M^-)$-action on $A_{i,i}$. Part (i) of the proposition (together with Lemma 1.2.5) shows that the $\mathfrak{g}$-action on $A_{i,i}$ is then $\mathbb{H}_i$-integrable, and that the resulting H_i-action on $A_{i,i}$ is compatible with the map (2.3.8).

Lemma 2.3.5 shows that $(\operatorname{Ind}_{\overline{P}}^{G} U)(N_0)$ is a closed $\mathfrak{g}$-invariant subspace of $\operatorname{Ind}_{\overline{P}}^{G} U$, and so $\mathcal{C}^{\mathrm{la}}(N_0, U)$ (which the isomorphism of Lemma 2.3.6 maps isomorphically to $(\operatorname{Ind}_{\overline{P}}^{G} U)(N_0)$) is a closed $\mathfrak{g}$-invariant subspace of $\mathcal{C}^{\mathrm{la}}_c(N, U)$.

Lemma 2.3.14. *For each $i \geq 0$, the image of $B_{i,i}$ under the continuous injection (2.2.10) is a $\mathfrak{g}$-invariant subspace of $\mathcal{C}^{\mathrm{la}}(N_0, U)$. The induced $\mathfrak{g}$-action on $B_{i,i}$ is locally integrable.*

Proof. There is a natural isomorphism

$$B_{i,i} \xrightarrow{\sim} \bigoplus_{n \in N_i \backslash N_0} nA_{i,i}. \tag{2.3.15}$$

Proposition 2.3.10 (i) shows that the image of $A_{i,i}$ in $\mathcal{C}^{\mathrm{la}}(N_0, U)$ is $\mathfrak{g}$-invariant, and (together with Lemma 1.2.5) that the $\mathfrak{g}$-action on $A_{i,i}$ is locally integrable. Since translation by n intertwines the action of $X \in \mathfrak{g}$ on $A_{i,i}$ with the action of $\mathrm{Ad}_n(X)$ on $nA_{i,i}$, and since Ad_n is an automorphism of $\mathfrak{g}$, we see that the image of $nA_{i,i}$ is also $\mathfrak{g}$-invariant, and that the induced $\mathfrak{g}$-action on $nA_{i,i}$ is again locally integrable, for each $n \in N_i \backslash N_0$. Taking into account the isomorphism (2.3.15), the lemma is proved. □

We close this subsection with a result that strengthens part (i) of Lemma 2.3.5. For the remainder of this subsection, let us break with our conventions and write $\overline{\mathbb{N}}$ to denote the algebraic group underlying $\overline{N}$. Since $\overline{\mathbb{N}}$ is unipotent, it is σ-affinoid in the sense of [6, §3.4], i.e. it may be written as an increasing union $\overline{\mathbb{N}} = \bigcup_{j=1}^{\infty} \overline{\mathbb{N}}^j$, where $\overline{\mathbb{N}}^j$ is an admissible affinoid open subgroup of $\overline{\mathbb{N}}$. We write $\overline{N}^j := \overline{\mathbb{N}}^j(L)$ to denote the locally analytic group of L-valued points of $\overline{\mathbb{N}}^j$ (which is a compact open subgroup of $\overline{N}$). We may and do choose the affinoid groups $\overline{\mathbb{N}}^j$ so that $\overline{N}^j$ is Zariski dense in $\overline{\mathbb{N}}^j$ for each $j \geq 1$.

Lemma 2.3.16. *The stalk* $(\mathrm{Ind}_{\overline{P}}^G U)_e$*, which by Lemma 2.3.5 (i) is a locally analytic* $\overline{P}$*-representation, is* $\overline{\mathbb{N}}$*-analytic as an* $\overline{N}$*-representation, in the sense of* [6, Def. 3.6.1].

Proof. By [6, Thm. 3.3.16], it suffices to show that for each $j \geq 1$, and for each $f \in (\mathrm{Ind}_{\overline{P}}^G U)_e$, the orbit map $o_f : N^j \to (\mathrm{Ind}_{\overline{P}}^G U)_e$ is induced by a rigid analytic function on $\overline{N}^j$.

We may choose $i \geq 0$ such that f lies in the image of (2.3.9), and hence (via the isomorphism (2.3.12)) may regard f as an element of $(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}}$, and hence also (via the embedding (2.3.13)) as an element of $\mathrm{Ind}_{\overline{P}}^G U$ (extended by zero from $\overline{P}H_i$ to G). Again breaking with our conventions, write $\mathbb{G}$ to denote the algebraic group underlying G. There is an algebraic, and hence rigid analytic, action of $\overline{N}$ on G via conjugation. Explicitly, we consider the action

$$\overline{\mathbb{N}} \times \mathbb{G} \to \mathbb{G} \tag{2.3.17}$$

given by the formula $(\overline{n}, g) \mapsto \overline{n}^{-1} g \overline{n}$ on the level of L-valued points. Evidently $e \in G$ is fixed by this action. Since $\overline{\mathbb{N}}^j$ is affinoid, and since the collection of affinoid open subgroups $\{\mathbb{H}_{i'}\}$ forms a cofinal sequence of open affinoid neighbourhoods of e in $\mathbb{G}$, we may find $i' \geq 0$ such that (2.3.17) restricts to a rigid analytic map

$$\overline{\mathbb{N}}^j \times \mathbb{H}_{i'} \to \mathbb{H}_i,$$

and so in particular to a rigid analytic map

$$\overline{\mathbb{N}}^j \times \mathbb{N}_{i'} \to \mathbb{H}_i.$$

This map induces a corresponding map on spaces of U_i-valued rigid analytic functions:

$$\mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, U_i) \to \mathcal{C}^{\mathrm{an}}(\overline{\mathbb{N}}^j \times \mathbb{N}_{i'}, U_i) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\overline{\mathbb{N}}^j, \mathcal{C}^{\mathrm{an}}(\mathbb{N}_{i'}, U_i)).$$

Precomposing this with the closed embedding $(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{H}_i, U_i)$, and postcomposing it with the map $\mathcal{C}^{\mathrm{an}}(\overline{\mathbb{N}}^j, \mathcal{C}^{\mathrm{an}}(\mathbb{N}_{i'}, U_i)) \to \mathcal{C}^{\mathrm{an}}(\overline{\mathbb{N}}^j, (\mathrm{Ind}_{\overline{P}}^G U)_e)$ induced by (2.2.4) and the isomorphism of Corollary 2.3.4, we obtain a continuous map

$$(\mathrm{Ind}_{\overline{P}_i}^{H_i} U_i)_{\mathbb{H}_i-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\overline{\mathbb{N}}^j, (\mathrm{Ind}_{\overline{P}}^G U)_e).$$

Evaluating this map on the element f of its source, we obtain a rigid analytic map

$$\overline{\mathbb{N}}^j \to (\mathrm{Ind}_{\overline{P}}^G U)_e. \tag{2.3.18}$$

A consideration of the formula $f(n\overline{n}) = f(\overline{n}^{-1} n \overline{n})$ for $n \in N_{i'}$ and $\overline{n} \in \overline{N}^j$ shows that the restriction of (2.3.18) to the group $\overline{N}^j$ of L-valued points of $\overline{\mathbb{N}}^j$ coincides with o_f. Thus o_f is indeed a rigid analytic map, and the lemma is proved. □

(2.4) We maintain the notation of the preceding subsections. In this subsection we introduce certain G-subrepresentations of $\mathrm{Ind}_{\overline{P}}^G U$ (namely, the local closed subrepresentations) that play a crucial role in our later argument.

Definition 2.4.1. We say that a vector subspace X of $\mathrm{Ind}_{\overline{P}}^{G} U$ is local if for any $f \in X$ and compact open subset Ω of $\overline{P}\backslash G$ the function $f_{|\Omega}$ again lies in X.

Definition 2.4.2. Let X be a local subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$.

(i) For any open subset Ω of $\overline{P}\backslash G$, we write $X(\Omega) = X \bigcap (\mathrm{Ind}_{\overline{P}}^{G} U)(\Omega)$.

(ii) If $x \in P\backslash G$, then we define the stalk of X at x to be

$$X_x := \varinjlim_{\Omega \ni x} X(\Omega),$$

where the locally convex inductive limit is taken over the directed set of compact open neighbourhoods of x, with the transition map $X(\Omega) \to X(\Omega')$ (for $\Omega \supset \Omega'$ an inclusion of neighbourhood of x) being given by $f \mapsto f_{|\Omega'}$.

We begin by establishing some basic lemmas regarding local subspaces of $\mathrm{Ind}_{\overline{P}}^{G} U$.

Lemma 2.4.3. *Let X be a local subspace of* $\mathrm{Ind}_{\overline{P}}^{G} U$*. We have the following characterization of X, namely*

$$X = \{f \in \mathrm{Ind}_{\overline{P}}^{G} U \,|\, \mathrm{st}_x(f) \in X_x \textit{ for all } x \in \overline{P}\backslash G\}.$$

Proof. Let us write $Y = \{f \in \mathrm{Ind}_{\overline{P}}^{G} U \,|\, \mathrm{st}_x(f) \in X_x \text{ for all } x \in \overline{P}\backslash G\}$. It is clear that $X \subset Y$, and we must show that this is an equality. Let f be an element of Y. For each element x of $\overline{P}\backslash G$, choose an element F_x of X whose stalk at x coincides with that of f. We may find an analytic chart Y_x containing x such that both f and F_x are rigid analytic on the underlying rigid analytic ball $\mathbb{Y}_x$, and thus coincide on Y_x. Since $\overline{P}\backslash G$ is compact, we may partition it into a disjoint union of open subsets $\Omega_x \subset Y_x$, as x ranges over some finite set S of points of $\overline{P}\backslash G$. If we set $F := \sum_{x \in S} F_{x|\Omega_x}$, then F lies in X (since X is local), and coincides with f. □

Lemma 2.4.4. *If X is a local subspace of* $\mathrm{Ind}_{\overline{P}}^{G} U$*, then so is its closure Y in* $\mathrm{Ind}_{\overline{P}}^{G} U$*. Furthermore, for any open subset Ω of $\overline{P}\backslash G$, the closure of $X(\Omega)$ is equal to $Y(\Omega)$.*

Proof. Since the map

$$f \mapsto f_{|\Omega} \tag{2.4.5}$$

(for any compact open subset Ω of $\overline{P}\backslash G$) is a continuous endomorphism of $\mathrm{Ind}_{\overline{P}}^{G} U$, we see that if X is local, then so its closure Y. If we fix an open subset Ω of $\overline{P}\backslash G$, then (taking into account Lemma 2.3.5 (ii) and the fact that Y is closed) we see that $Y(\Omega)$ is closed, and so certainly contains the closure of $X(\Omega)$. Conversely, if $F \in Y(\Omega)$, we will show that F lies in the closure of $X(\Omega)$. Replacing Ω by the support of F, we may furthermore assume that Ω is compact. Since $F = F_{|\Omega}$, and F does lie in the closure of X by assumption, this again follows from the continuity of (2.4.5). □

Lemma 2.4.6. *If X is a local closed N-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$, then the stalk X_e is a closed subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$.*

Proof. We will use the notation introduced in Subsection 2.2. Via Corollary 2.3.4 we identify $(\mathrm{Ind}_{\overline{P}}^{G} U)(N)_e$ with $\mathcal{C}^{\omega}(N,U)_e$, and so regard X_e as a subspace of $\mathcal{C}^{\omega}(N,U)_e$, and for each $i \geq 0$ we let X_i denote the closed subspace of $A_{i,i}$ obtained as the preimage of X under the continuous map (2.3.8).

The natural map $\varinjlim_i X_i \to X_e$ is evidently an isomorphism of abstract K-vector spaces. We claim that for each $i \geq 0$, the preimage of X_{i+1} in $A_{i,i}$ under the transition map $A_{i,i} \to A_{i+1,i+1}$ provided by (2.2.2) is equal to X_i. Given this, it follows from [6, Prop. 1.1.41], Lemma 1.4.10, and the isomorphism (2.2.5) that $\varinjlim_i X_i$ embeds as a closed subspace of $\mathcal{C}^{\omega}(N,U)_e$. This will complete the proof of part (i).

To prove the claim, suppose that $f \in A_{i,i}$ is such that $f_{|N_{i+1}}$ lies in X_{i+1}. Lemma 1.3.1 implies that for each $n \in N_i$, the function $(nf)_{|N_{i+1}} = n(f_{|N_{i+1}n})$ lies in X_{i+1}. Since X is N-invariant, we find that $f_{|N_{i+1}n}$ lies in X for each $n \in N_i$. Letting n run over a set of left coset representatives of N_{i+1} in N_i, we find that $f = \sum_n f_{|N_{i+1}n}$ lies in X, and so in X_i. $\square$

The next set of results is aimed at providing a useful description of local closed G-invariant subspaces of $\mathrm{Ind}_{\overline{P}}^{G} U$ in terms of closed $(\mathfrak{g}, \overline{P})$-invariant subspaces of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$.

Proposition 2.4.7. *Let X be a local closed G-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$.*

(i) X_e is a closed $(\mathfrak{g}, \overline{P})$-invariant subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$.

(ii) If $x = \overline{P}g \in \overline{P}\backslash G$, then the isomorphism of Lemma 2.3.5 (iii) induced by multiplication by g induces an isomorphism $X_x \xrightarrow{\sim} X_e$.

(iii) We have the following characterization of X, namely

$$X = \{f \in \mathrm{Ind}_{\overline{P}}^{G} U \mid g\mathrm{st}_x(f) \in X_e \text{ for all } x = \overline{P}g \in \overline{P}\backslash G\}.$$

Proof. It is clear that X_e is a $(\mathfrak{g}, \overline{P})$-invariant subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$. The remainder of part (i) follows from Lemma 2.4.6, part (ii) is immediate, and part (iii) is a restatement of Lemma 2.4.3 in the situation of the proposition (taking into account the statement of part (ii)). $\square$

We will establish a converse to the preceding Proposition. We begin by introducing some notation. Suppose that $\mathcal{X}$ is a subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$. For each $i \geq 0$, let $\mathcal{X}_i$ denote the preimage of $\mathcal{X}$ under the map (2.3.9). Recall from Proposition 2.3.10 (i) that for each $i \geq 0$, the space $A_{i,i}$ is equipped with an $\mathbb{H}_i$-analytic H_i-action.

Lemma 2.4.8. *If $\mathcal{X}$ is a closed $\mathfrak{g}$-invariant subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$, then for each $i \geq 0$ the subspace $\mathcal{X}_i$ of $A_{i,i}$ is closed and H_i-invariant.*

Proof. Since (2.3.9) is continuous and $\mathfrak{g}$-invariant, we see that $\mathcal{X}_i$ is a closed $\mathfrak{g}$-invariant subspace of $A_{i,i}$. The lemma then follows from Lemma 1.2.5 and Corollary 1.2.7. $\square$

We now prove the converse to Proposition 2.4.7.

Proposition 2.4.9. *If $\mathcal{X}$ is a closed $(\mathfrak{g}, \overline{P})$-invariant subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$, and if we define*

$$X = \{f \in \mathrm{Ind}_{\overline{P}}^{G} U \mid g\mathrm{st}_x(f) \in \mathcal{X} \textit{ for all } x = \overline{P}g \in \overline{P}\backslash G\}$$

(where $g \in G$ acts on $\mathrm{st}_x(f)$ via the isomorphism of Lemma 2.3.5 (iii)), then X is a closed local G-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$, and the natural map $X_e \to \mathcal{X}$ induced by st_e is an isomorphism.

Proof. Note that X is well-defined, since we have assumed that $\mathcal{X}$ is $\overline{P}$-invariant, and X is G-invariant, local, and closed in $\mathrm{Ind}_{\overline{P}}^{G} U$ by construction. Furthermore, the map $X_e \to \mathcal{X}$ is certainly an injection, and we must show that it is also surjective. Let $f \in \mathrm{Ind}_{\overline{P}}^{G} U$ be an element for which $\mathrm{st}_e(f)$ is equal to some given element of $\mathcal{X}$. We will find a neighbourhood Ω of e such that $f_{|\Omega}$ lies in X, and thus show that $\mathrm{st}_e(f)$ lies in X_e.

Choose a value of $i \geq 0$ so that $f_{|N_i}$ lies in the image of $A_{i,i}$ under the continuous injection (2.3.8). Since $f_{|N_i}$ lies in $\mathcal{X}_i$ (using the notation introduced above), it follows from Lemma 2.4.8 that $nf_{|N_i}$ lies in $\mathcal{X}_i$ for all $n \in N_i$. We conclude that $\mathrm{st}_e(nf) = \mathrm{st}_n(f)$ lies in $\mathcal{X}$ for all $n \in N_i$, and thus that $f_{|N_i}$ lies in X, as required. □

Propositions 2.4.7 and 2.4.9 together show that closed local G-invariant subspaces of $\mathrm{Ind}_{\overline{P}}^{G} U$ are determined by their stalks at the base point e of $\overline{P}\backslash G$, and that the association $X \mapsto X_e$ provides a bijection between the lattice of local closed G-invariant subspaces of $\mathrm{Ind}_{\overline{P}}^{G} U$, and the closed $(\mathfrak{g}, \overline{P})$-invariant subspaces of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$. The following result will be useful for detecting such subspaces of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$.

Lemma 2.4.10. *A closed $\mathfrak{g}$-invariant subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$ is $\overline{P}$-invariant if and only if it is M-invariant.*

Proof. The only if direction is obvious. The if direction follows from Lemma 2.3.16, which shows that an $\overline{\mathfrak{n}}$-invariant closed subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$ is automatically $\overline{N}$-invariant. □

If X is a local G-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$ and Ω is an open subset of $\overline{P}\backslash G$, then the inclusion $X(\Omega) \subset X$ together with the G-action on X induces a map

$$K[G] \otimes_K X(\Omega) \to X, \tag{2.4.12}$$

which is continuous if we endow $K[G]$ with its finest convex topology and the tensor product with the inductive tensor product topology. (This tensor product may be described less canonically as a direct sum of copies of $X(\Omega)$, indexed by the elements of G.)

Lemma 2.4.13. *If X is a local closed G-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$ and Ω is an open subset of $\overline{P}\backslash G$, then (2.4.12) is a strict surjection.*

Proof. Since $\overline{P}\backslash G$ is compact, we may find a finite set of elements $\{g_1, \ldots, g_n\} \subset G$ such that $\overline{P}\backslash G = \bigcup_{i=1}^{n} \Omega g_i$, and thus (since X is local) such that the map

$$\bigoplus_{i=1}^{n} X(\Omega) \to X \tag{2.4.14}$$

defined by $(f_i) \mapsto \sum_i g_i f_i$ is surjective. Since the source and target of (2.4.14) are of compact type, it is furthermore a strict surjection. Since it factors as the composite of (2.4.12) and the continuous map $\bigoplus_{i=1}^n X(\Omega) \to K[G] \otimes_K X$ defined by $(f_i) \mapsto \sum_i g_i \otimes f_i$, it is immediate that (2.4.12) is also a strict surjection, as claimed. $\square$

(2.5) We maintain the notation of the preceding subsections. We let $\mathcal{C}^{\mathrm{pol}}(N, K)$ denote the ring of algebraic K-valued functions on the affine algebraic group N; i.e. if $\mathbb{N}$ denotes the linear algebraic group underlying N, then $\mathcal{C}^{\mathrm{pol}}(N, K)$ is the subspace of $\mathcal{C}^{\mathrm{la}}(N, K)$ obtained by restricting the global sections of the structure sheaf of $\mathbb{N}$ over K to the set N of L-valued points of $\mathbb{N}$. Since N is Zariski dense in $\mathbb{N}$, the ring of functions $\mathcal{C}^{\mathrm{pol}}(N, K)$ is naturally identified with the affine ring of $\mathbb{N}$ over K.

Since $\mathbb{N}$ is unipotent, we may naturally identify it as an algebraic variety with its Lie algebra $\mathfrak{n}$ (regarded as an affine space over L); we denote this identification as usual via $X \mapsto \exp(X)$. The group law on $\mathbb{N}$ is then determined by the Lie bracket on $\mathfrak{n}$ together with the Baker-Cambpell-Hausdorff formula:

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \cdots).$$

This identification induces a natural isomorphism between $\mathcal{C}^{\mathrm{pol}}(N, K)$ and the symmetric algebra over K of the dual to $\mathfrak{n}$:

$$\mathcal{C}^{\mathrm{pol}}(N, K) \xrightarrow{\sim} \mathrm{Sym}^{\bullet}_K \check{\mathfrak{n}}, \tag{2.5.1}$$

where $\check{\mathfrak{n}} := \mathrm{Hom}_L(\mathfrak{n}, K)$ is the K-dual space to $\mathfrak{n}$.

Since the group structure on N is induced by the algebraic group law of $\mathbb{N}$, the space of functions $\mathcal{C}^{\mathrm{pol}}(N, K)$ is invariant under the right regular action of N. We may extend the right regular N-action on $\mathcal{C}^{\mathrm{pol}}(N, K)$ to a P-action as follows. If $m \in M$, define $(mf)(n) = f(m^{-1}nm)$ for any $m \in M, f \in \mathcal{C}^{\mathrm{pol}}(N, K)$, and $n \in N$. One immediately checks that mf again lies in $\mathcal{C}^{\mathrm{pol}}(N, K)$. Using the equality $P = MN$, we combine the M and N-action on $\mathcal{C}^{\mathrm{pol}}(N, K)$ into a P-action. The resulting P-action on $\mathcal{C}^{\mathrm{pol}}(N, K)$ is algebraic, in the sense that $\mathcal{C}^{\mathrm{pol}}(N, K)$ may be written as the union of an increasing sequence of finite dimensional P-invariant subspaces, on each of which P acts through an algebraic representation. Differentiating this P-action we make $\mathcal{C}^{\mathrm{pol}}(N, K)$ a $\mathfrak{p}$-module (where $\mathfrak{p}$ denotes the Lie algebra of P). Note that since P is a connected algebraic group, this $\mathfrak{p}$-action uniquely determines the P-action that gives rise to it.

We define a bilinear pairing $\mathcal{C}^{\mathrm{pol}}(N, K) \times \mathrm{U}(\mathfrak{n}) \to K$ via

$$(f, X) \mapsto (Xf)(e) \tag{2.5.2}$$

(where as usual e denotes the identify element of N). We may use this pairing to give another useful description of $\mathcal{C}^{\mathrm{pol}}(N, K)$.

Lemma 2.5.3. *The pairing (2.5.2) induces an $\mathfrak{n}$-equivariant isomorphism*

$$\mathcal{C}^{\mathrm{pol}}(N, K) \xrightarrow{\sim} \mathrm{Hom}_K(\mathrm{U}(\mathfrak{n}), K)^{\mathfrak{n}^{\infty}}.$$

(Here $\mathrm{Hom}_K(\mathrm{U}(\mathfrak{n}), K)$ *is equipped with its right regular* $\mathrm{U}(\mathfrak{n})$*-action, and the target is the* $\mathrm{U}(\mathfrak{n})$*-submodule of* $\mathrm{Hom}_K(\mathrm{U}(\mathfrak{g}), K)$ *consisting of elements that are annihilated by* $\mathfrak{n}^k$ *for some* $k \geq 0$*.)*

Proof. This is well-known and straightforward. □

Note that since the $\mathfrak{n}$-action on $\mathrm{Hom}_K(\mathrm{U}(\mathfrak{n}), K)^{\mathfrak{n}^\infty}$ is locally finite by definition, we may integrate the $\mathfrak{n}$-action to obtain an N-action. The isomorphism of Lemma 2.5.3 is then N-equivariant (since we observed above that the N-action on $\mathcal{C}^{\mathrm{pol}}(N, K)$ is also obtained by integrating the corresponding $\mathfrak{n}$-action).

We endow $\mathcal{C}^{\mathrm{pol}}(N, K)$ with its finest convex topology; since $\mathcal{C}^{\mathrm{pol}}(N, K)$ is of countable dimension, this topology makes it a compact type space. The preceding description of the P-action on $\mathcal{C}^{\mathrm{pol}}(N, K)$ shows that it is $\mathbb{P}_0$-analytic for any analytic open subgroup P_0 of P, and so in particular is locally analytic.

If W is any Hausdorff convex K-space, then we write $\mathcal{C}^{\mathrm{pol}}(N, W) := W \otimes_K \mathcal{C}^{\mathrm{pol}}(N, K)$ to denote the space of polynomial functions on N with coefficients in W. We endow $\mathcal{C}^{\mathrm{pol}}(N, W)$ with the inductive tensor product topology. Since $\mathcal{C}^{\mathrm{pol}}(N, K)$ is endowed with its finest convex topology, we see that $\mathcal{C}^{\mathrm{pol}}(N, W)$ is topologically isomorphic to a direct sum of copies of W (indexed by the elements of a basis of $\mathcal{C}^{\mathrm{pol}}(N, K)$). We will apply this notation with W taken either to be U (our fixed object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$), or one of the BH-subspaces U_i of U (in the notation of Subsection 2.2).

We let P act on U through its quotient M, and define a P-action on $\mathcal{C}^{\mathrm{pol}}(N, U)$ by taking the tensor product of the P-action on U and the P-action discussed above on $\mathcal{C}^{\mathrm{pol}}(N, K)$. If we regard $\mathcal{C}^{\mathrm{pol}}(N, U)$ as a space of U-valued functions on N, then this action is given by the formula

$$(mnf)(n') = mf(m^{-1}n'mn) \tag{2.5.4}$$

for $m \in M, n, n' \in N$, and $f \in \mathcal{C}^{\mathrm{pol}}(N, U)$. Since the M-action on U and the P-action on $\mathcal{C}^{\mathrm{pol}}(N, K)$ are both locally analytic, the same is true of the P-action on $\mathcal{C}^{\mathrm{pol}}(N, U)$. Differentiating this action makes $\mathcal{C}^{\mathrm{pol}}(N, U)$ a $\mathrm{U}(\mathfrak{p})$-module.

We will now explain how the $\mathfrak{p}$-action on $\mathcal{C}^{\mathrm{pol}}(N, U)$ may be extended in a natural manner to a $\mathfrak{g}$-action, making $\mathcal{C}^{\mathrm{pol}}(N, U)$ a $(\mathfrak{g}, P)$-module. We will do this by giving another description of $\mathcal{C}^{\mathrm{pol}}(N, U)$, which will naturally exhibit the $(\mathfrak{g}, P)$-action on this space.

We first note that the isomorphism of Lemma 2.5.3 induces an isomorphism

$$\mathcal{C}^{\mathrm{pol}}(N, U) \xrightarrow{\sim} \mathrm{Hom}_K(\mathrm{U}(\mathfrak{n}), U)^{\mathfrak{n}^\infty}. \tag{2.5.5}$$

If we regard U as a $\overline{\mathfrak{p}}$-representation, via the quotient map $\overline{\mathfrak{p}} \to \mathfrak{m}$, then the isomorphism $\mathrm{U}(\overline{\mathfrak{p}}) \otimes_K \mathrm{U}(\mathfrak{n}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g})$ induced by multiplication in $\mathrm{U}(\mathfrak{g})$ gives rise to a natural isomorphism

$$\mathrm{Hom}_{\mathrm{U}(\overline{\mathfrak{p}})}(\mathrm{U}(\mathfrak{g}), U)^{\mathfrak{n}^\infty} \xrightarrow{\sim} \mathrm{Hom}_K(\mathrm{U}(\mathfrak{n}), U)^{\mathfrak{n}^\infty} \tag{2.5.6}$$

(where the source is defined by regarding $\mathrm{U}(\mathfrak{g})$ as a $\mathrm{U}(\overline{\mathfrak{p}})$-module via left multiplication). Combining the isomorphisms (2.5.5) and (2.5.6) yields an isomorphism

$$\mathcal{C}^{\mathrm{pol}}(N, U) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{U}(\overline{\mathfrak{p}})}(\mathrm{U}(\mathfrak{g}), U)^{\mathfrak{n}^\infty}. \tag{2.5.7}$$

We may use this isomorphism to transport the $\mathrm{U}(\mathfrak{g})$-action on the target induced by right multiplication to a $\mathrm{U}(\mathfrak{g})$-action on the source.

Lemma 2.5.8. *The $\mathfrak{g}$-action on $\mathcal{C}^{\mathrm{pol}}(N,U)$ arising from the isomorphism (2.5.7) is continuous, and makes $\mathcal{C}^{\mathrm{pol}}(N,U)$ a $(\mathfrak{g},P)$-representation.*

Proof. This is easily checked by the reader. □

We will need to consider certain gradings on the spaces $\mathcal{C}^{\mathrm{pol}}(N,K)$. Recall that we have fixed a coweight α of the maximal split torus in Z_M that pairs strictly negatively with all the restricted roots of N. The adjoint action of α on $\mathfrak{g}$ and the coadjoint action of α on $\check{\mathfrak{n}}$ induce gradings on $\mathfrak{g}$ and on $\mathrm{Sym}^{\bullet}_K \check{\mathfrak{n}}$. (Note that the grading on the latter space typically does not coincide with the grading by symmetric powers).

We use the isomorphism (2.5.1) to transport the grading on $\mathrm{Sym}^{\bullet}_K \check{\mathfrak{n}}$ to a grading of $\mathcal{C}^{\mathrm{pol}}(N,K)$. We let $\mathcal{C}^{\mathrm{pol},d}(N,K)$ denote the dth graded piece of $\mathcal{C}^{\mathrm{pol}}(N,K)$ with respect to this grading. Our assumption on α implies that $\mathcal{C}^{\mathrm{pol},d}(N,K)$ is finite dimensional for each d, and vanishes if $d<0$. Since the grading on $\mathcal{C}^{\mathrm{pol}}(N,K)$ is defined using a coweight of the centre of M, we see that each of the graded pieces $\mathcal{C}^{\mathrm{pol},d}(N,K)$ is an M-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N,K)$.

We write

$$\mathcal{C}^{\mathrm{pol},\leq d}(N,K) := \bigoplus_{d'\leq d} \mathcal{C}^{\mathrm{pol},d'}(N,K)$$

and

$$\mathcal{C}^{\mathrm{pol},>d}(N,K) := \bigoplus_{d'>d} \mathcal{C}^{\mathrm{pol},d'}(N,K).$$

Since our choice of α ensures that $\mathfrak{n}$ is graded in negative degrees, we see that each of the spaces $\mathcal{C}^{\mathrm{pol},\leq d}(N,K)$ is $\mathfrak{n}$-invariant, and hence forms a P-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N,K)$.

For any Hausdorff convex K-vector space W we induce a grading on the tensor product $\mathcal{C}^{\mathrm{pol}}(N,W) = W\otimes_K \mathcal{C}^{\mathrm{pol}}(N,K)$ by defining

$$\mathcal{C}^{\mathrm{pol},d}(N,W) := W\otimes_K \mathcal{C}^{\mathrm{pol},d}(N,K)$$

for all $d\geq 0$. We also write

$$\mathcal{C}^{\mathrm{pol},\leq d}(N,W) = W\otimes_K \mathcal{C}^{\mathrm{pol},\leq d}(N,K) = \bigoplus_{d'\leq d} \mathcal{C}^{\mathrm{pol},d'}(N,W)$$

and

$$\mathcal{C}^{\mathrm{pol},>d}(N,W) = W\otimes_K \mathcal{C}^{\mathrm{pol},>d}(N,K) = \bigoplus_{d'>d} \mathcal{C}^{\mathrm{pol},d'}(N,W)$$

for all $d\geq 0$.

Taking $W=U$, each of the spaces $\mathcal{C}^{\mathrm{pol},d}(N,U)$ (respectively $\mathcal{C}^{\mathrm{pol},\leq d}(N,U)$) is a closed M-invariant (respectively closed P-invariant) subspace of $\mathcal{C}^{\mathrm{pol}}(N,U)$.

Lemma 2.5.9. *The $\mathfrak{g}$-action on $\mathcal{C}^{\mathrm{pol}}(N,U)$ arising from the isomorphism (2.5.7) is a graded action, if we endow each of $\mathfrak{g}$ and $\mathcal{C}^{\mathrm{pol}}(N,U)$ with the grading induced by α described above.*

Proof. This is clear. □

Suppose now that $\mathbb{Y}$ is an open affinoid rigid analytic polydisk contained in $\mathbb{N}$. Since the space of polynomial functions on N is dense in the space of rigid analytic functions on $\mathbb{Y}$, we obtain a continuous injection

$$\bigoplus_{d\geq 0} \mathcal{C}^{\mathrm{pol},d}(N,U_j) \xrightarrow{\sim} \mathcal{C}^{\mathrm{pol}}(N,U_j) \to \mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j) \tag{2.5.10}$$

with dense image. Thus we may regard $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$ as a completion of $\mathcal{C}^{\mathrm{pol}}(N,U_j)$ with respect to a certain topology. We now describe this completion more explicitly.

We begin by introducing some notation.

Definition 2.5.11. Let W^d (for $d \geq 0$) denote a sequence of Banach spaces, each equipped with a fixed norm $||-||^d$ that defines the topology on W^d. Let $\hat{\bigoplus}_{d\geq 0} W^d$ denote the completion of the direct sum $\bigoplus_{d\geq 0} W^d$ with respect to the norm $||\,(w^d)\,|| := \max_{d\geq 0} ||\,w^d\,||^d$ (where (w^d) denotes an element of $\bigoplus_{d\geq 0} W^d$).

Note that the construction of $\hat{\bigoplus}_{d\geq 0} W^d$ depends on the particular choice of norms $||-||^d$, and not just on the topological vector space structures of the Banach spaces W^d.

Let us now fix a norm on U_j that defines the topology on U_j; the induced sup norm on elements of $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$ then defines the topology on $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$. Note that for any fixed degree d, the continuous injection (2.5.10) restricts to a closed embedding

$$\mathcal{C}^{\mathrm{pol},d}(N,U_j) \to \mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j). \tag{2.5.12}$$

If we write $||-||^d_{\mathbb{Y},j}$ to denote the pull-back of the sup norm on $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$ to the Banach space $\mathcal{C}^{\mathrm{pol},d}(N,U_j)$ via (2.5.12), then $||-||^d_{\mathbb{Y},j}$ defines the topology on $\mathcal{C}^{\mathrm{pol},d}(N,U_j)$.

Lemma 2.5.13. *The map (2.5.10) induces a topological isomorphism*

$$\hat{\bigoplus_{d\geq 0}} \mathcal{C}^{\mathrm{pol},d}(N,U_j) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j),$$

the source of the isomorphism being defined via Definition 2.5.11, using the norms $||-||^d_{\mathbb{Y},j}$.

Proof. This follows directly from definition of the space of rigid analytic functions $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$. $\square$

Recall from Subsection 2.2 that $A_{i,j} := \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i,U_j)$, for any $i,j \geq 0$. The previous discussion applies in particular when we take $\mathbb{Y} := \mathbb{N}_i$. In this case (2.5.10) induces a continuous injection

$$\bigoplus_{d\geq 0} \mathcal{C}^{\mathrm{pol},d}(N,U_j) \xrightarrow{\sim} \mathcal{C}^{\mathrm{pol}}(N,U_j) \to A_{i,j} \tag{2.5.14}$$

with dense image. Writing $||-||^d_{i,j} := ||-||^d_{\mathbb{N}_i,j}$, we obtain the following special case of Lemma 2.5.13.

Lemma 2.5.15. *The map (2.5.14) induces a topological isomorphism*

$$\hat{\bigoplus_{d\geq 0}} \mathcal{C}^{\mathrm{pol},d}(N, U_j) \xrightarrow{\sim} A_{i,j},$$

the source of the isomorphism being defined via Definition 2.5.11, using the norms $||-||^d_{i,j}$.

Note that the Banach spaces appearing as direct summands in the source of the isomorphism of Lemma 2.5.15 are independent of i; however, the norms $||-||^d_{i,j}$ do depend on the choice of i (as they must, since $A_{i,j}$ certainly depends on i as well as j). The next result quantifies the dependence of these norms on i.

Lemma 2.5.16. *For any* $i \geq 0$, *the norms* $||-||^d_{0,j}$ *and* $||-||^d_{i,j}$ *on* $\mathcal{C}^{\mathrm{pol},d}(N, U_j)$ *are related by the formula*

$$||-||^d_{i,j} = |p|^{di}||-||^d_{0,j}.$$

Proof. Since $z_0 = \alpha(p)^{-1}$ and $\mathbb{N}_i = z_0^i \mathbb{N}_0 z_0^{-i}$, this follows from the fact that the grading on $\mathcal{C}^{\mathrm{pol}}(N, U_j)$ is defined via the coweight α. $\square$

The isomorphism $\varinjlim_i U_j \xrightarrow{\sim} U$ of Subsection 2.2 induces an isomorphism

$$\varinjlim_j \mathcal{C}^{\mathrm{pol}}(N, U_j) \xrightarrow{\sim} \mathcal{C}^{\mathrm{pol}}(N, U). \tag{2.5.17}$$

Passing to the inductive limit of the maps (2.5.14), and taking into account the isomorphisms (2.5.17) and (2.2.5), we obtain a map

$$\mathcal{C}^{\mathrm{pol}}(N, U) \to \mathcal{C}^{\omega}(N, U)_e. \tag{2.5.18}$$

Lemma 2.5.19. *The maps (2.5.18) is a continuous* $(\mathfrak{g}, M)$*-equivariant injection with dense image.*

Proof. Since for each $i, j \geq 0$ the morphism (2.5.14) is a continuous injection, with dense image, the same is true of (2.5.18). The claimed equivariance is easily checked by the reader. $\square$

Composing (2.5.18) with the isomorphism of Corollary 2.3.4 then yields a continuous $(\mathfrak{g}, M)$-equivariant injection

$$\mathcal{C}^{\mathrm{pol}}(N, U) \to (\mathrm{Ind}_P^G U)_e \tag{2.5.20}$$

with dense image.

In the remainder of this subsection we introduce some terminology and basic results related to spaces of locally polynomial functions.

Definition 2.5.21. Let $\mathcal{C}^{\mathrm{lp}}_c(N, U)$ denote the tensor product

$$\mathcal{C}^{\mathrm{pol}}(N, U) \otimes_K \mathcal{C}^{\mathrm{sm}}_c(N, K) \quad (\xrightarrow{\sim} U \otimes_K \mathcal{C}^{\mathrm{pol}}(N, K) \otimes_K \mathcal{C}^{\mathrm{sm}}_c(N, K) \xrightarrow{\sim} \mathcal{C}^{\mathrm{pol}}(N, K) \otimes_K \mathcal{C}^{\mathrm{sm}}_c(N, U)).$$

equipped with the inductive (or equivalently, by [6, Prop. 1.1.31], projective) tensor product topology. (The superscript "lp" stands for "locally polynomial".) It is a compact type convex K-space, which we endow with a continuous $(\mathfrak{g}, P)$-representation by having $\mathrm{U}(\mathfrak{g})$ act via its action on the first factor $\mathcal{C}^{\mathrm{pol}}(N, U)$, and having P act diagonally.

Lemma 2.5.22. *The P-action on $\mathcal{C}_c^{\mathrm{lp}}(N,U)$ is locally analytic.*

Proof. This follows from [6, 3.6.18] and the fact that the P-action on each of $\mathcal{C}^{\mathrm{pol}}(N,U)$ and $\mathcal{C}_c^{\mathrm{sm}}(N,K)$ is locally analytic. $\square$

Multiplication of polynomials by smooth functions induces a map

$$\mathcal{C}_c^{\mathrm{lp}}(N,U) \to \mathcal{C}_c^{\mathrm{la}}(N,U). \tag{2.5.23}$$

Lemma 2.5.24. *The morphism (2.5.23) is a continuous $(\mathfrak{g},P)$-equivariant injection, whose image is equal to the subspace of $\mathcal{C}_c^{\mathrm{la}}(N,U)$ consisting of functions f that are locally polynomial, in the sense that in a neighbourhood of each point of N, the function f is defined by an element of $\mathcal{C}^{\mathrm{pol}}(N,U)$.*

Proof. This is easily confirmed by the reader. $\square$

Definition 2.5.25. For any open subset Ω of N, define $\mathcal{C}_c^{\mathrm{lp}}(\Omega,U) := \mathcal{C}^{\mathrm{pol}}(N,U)\otimes_K \mathcal{C}_c^{\mathrm{sm}}(\Omega,K)$. For $d \geq 0$, let $\mathcal{C}_c^{\mathrm{lp},\leq d}(\Omega,U)$ denote the closed subspace $\mathcal{C}^{\mathrm{pol},\leq d}(N,U)\otimes_K \mathcal{C}_c^{\mathrm{sm}}(\Omega,K)$ of $\mathcal{C}^{\mathrm{lp}}(\Omega,U)$.

The closed embedding $\mathcal{C}_c^{\mathrm{sm}}(\Omega,K) \to \mathcal{C}_c^{\mathrm{sm}}(N,K)$ given by extending functions by zero induces a closed embedding $\mathcal{C}_c^{\mathrm{lp}}(\Omega,U) \to \mathcal{C}_c^{\mathrm{lp}}(N,U)$, via which we regard $\mathcal{C}_c^{\mathrm{lp}}(\Omega,U)$ as a $\mathfrak{g}$-invariant closed subspace of $\mathcal{C}_c^{\mathrm{lp}}(N,U)$.

Lemma 2.5.26. *The closed subspace $\mathcal{C}_c^{\mathrm{lp}}(\Omega,U)$ of $\mathcal{C}_c^{\mathrm{lp}}(N,U)$ is equal to the preimage of the closed subspace $\mathcal{C}_c^{\mathrm{la}}(\Omega,U)$ of $\mathcal{C}_c^{\mathrm{la}}(N,U)$ under the map (2.5.23).*

Proof. This is immediate. $\square$

The composite of the map (2.5.23) with the isomorphism of Lemma 2.3.6 restricts to a continuous injection

$$\mathcal{C}_c^{\mathrm{lp}}(\Omega,U) \to (\mathrm{Ind}_P^G U)(\Omega). \tag{2.5.27}$$

for any open subset Ω of N. The description of the image of (2.5.23) provided by Lemma 2.5.24 shows that the image of (2.5.27) is a local subspace of $\mathrm{Ind}_P^G U$ (in the sense of Definition 2.4.1), whose stalk at e is equal to the image of (2.5.20).

(**2.6**) In this section we develop some further results related to representations on spaces of polynomial functions. Although these results are not required for the proof of the main theorem in Subsection 4.3, they will be useful in applications. (See for example Propositions 2.7.16 and 5.2.11 below.)

Fix a chart Y around $e \in N$ and integers $d, j \geq 0$. Let $\Pi_j^d : \mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j) \to \mathcal{C}^{\mathrm{pol},d}(N,U_j)$ denote the projection onto the dth direct summand in the isomorphism of Lemma 2.5.13.

Lemma 2.6.1. *If I denotes the identity operator from $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$ to itself, then there is an equality $I = \sum_{d=1}^{\infty} \Pi_j^d$ in $\mathcal{L}(\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j),\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j))_s$ (i.e. the series on the right converges in weak topology to the identity operator).*

Proof. This is clear from the isomorphism of Lemma 2.5.13. $\square$

Let d/dt denote the standard non-zero tangent vector at the identity of the torus $L^{\times}$, and write ∂_α to denote the pushforward of d/dt with respect to α (so ∂_α is a non-zero element in the Lie algebra of Z_M). The action of Z_M on $\mathcal{C}^{\omega}(N,U)$ being locally analytic, it induces an action of ∂_α on $\mathcal{C}^\omega(N,U)_e$, as well as on $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)_e$ for each open affinoid polydisk $\mathbb{Y}$ contained in $\mathbb{N}$ and each $j \geq 0$. These actions induce maps

$$K[\partial_\alpha] \to \mathcal{L}(\mathcal{C}^\omega(N,U)_e, \mathcal{C}^\omega(N,U)_e)_s \tag{2.6.2}$$

and

$$K[\partial_\alpha] \to \mathcal{L}(\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j), \mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j))_s. \tag{2.6.3}$$

(Here $K[\partial_\alpha]$ denotes the polynomial ring over K in ∂_α, or equivalently, the universal enveloping algebra of the lie algebra of the one dimensional torus $L^\times$. Note also that the targets of these two maps are equipped with their weak topologies.)

The following result is a special case of the theory of diagonalizable modules developed in [10, §1.3.1]. We give a complete proof, since [10] remains unpublished, and the argument is in any case straightforward.

Proposition 2.6.4. *If ∂_α acts on the locally analytic M-representation U via a scalar, then for each $d, j \geq 0$, the projection Π_j^d lies in the closure of the image of (2.6.3).*

Proof. Recall that $\mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j) = \mathcal{C}^{\mathrm{an}}(\mathbb{Y},K)\,\hat{\otimes}_K\, U_j$, and that the ∂_α-action on this tensor product is simply the tensor product action of the ∂_α-action on each factor. By assumption ∂_α acts on U, and hence U_j, through a scalar λ.

Fix $d \geq 0$. For any $r \geq 0$ such that $0 \leq d < p^r$, let $\phi_{d,r} : \mathbb{Z}_p \to \{0,1\}$ denote the characteristic function of the coset $d + p^r\mathbb{Z}_p \subset \mathbb{Z}_p$. It is a continuous function on $\mathbb{Z}_p$, and so admits a Mahler expansion $\phi_{d,r}(x) := \sum_{k=0}^\infty a_{d,r,k}\binom{x}{k}$, with $a_{d,r,k} \in \mathbb{Z}_p$. Let $P_{d,r} := \sum_{k=0}^{p^r-1} a_{d,r,k}\binom{x}{k} \in \mathbb{Z}_p[x]$. Note that then $P_{d,r}(x) = \phi_{d,r}(x)$ if $0 \leq x < p^r$; more concretely,

$$P_{d,r}(x) = \begin{cases} 1 \text{ if } x = d \\ 0 \text{ if } 0 \leq x < p^r \text{ and } x \neq d. \end{cases} \tag{2.6.5}$$

Let $f \in \mathcal{C}^{\mathrm{an}}(\mathbb{Y},U_j)$, and via the isomorphism of Lemma 2.5.13 write $f = \sum_{d'=1}^\infty f_{d'}$, where $f_{d'} \in \mathcal{C}^{\mathrm{pol},d'}(N,U_j) = \mathcal{C}^{\mathrm{pol},d'}(N,K) \otimes_K U_i$ for each $d' \geq 0$, and $||f_{d'}||_{\mathbb{Y},j}^{d'} \to 0$ as $d' \to \infty$. Taking into account (2.6.5), one computes that

$$P_{d,r}(\partial_\alpha - \lambda)f = \sum_{d'=0}^\infty P_{d,r}(d')f_{d'} = f_d + \sum_{d' \geq p^r} P_{d,r}(d')f_{d'}. \tag{2.6.6}$$

Since $P_{d,r}(d') \in \mathbb{Z}_p$ for all $d' \geq 0$, we find that

$$\lim_{d' \to \infty} ||P_{d,r}(d')f_{d'}||_{i,j}^{d'} \leq \lim_{d' \to \infty} ||f_{d'}||_{i,j}^{d'} = 0,$$

which when combined with (2.6.6) shows that

$$\lim_{r \to \infty} P_{d,r}(\partial_\alpha - \lambda)f = f_d.$$

Thus $P_{d,r}(\partial_\alpha - \lambda)$ converges weakly to Π_j^d as $r \to \infty$, and so Π_j^d does lie in the closure of the image of (2.6.3). □

The preceding discussion applies in particular taking $\mathbb{Y} := \mathbb{N}_i$ for some $i \geq 0$. We then write

$$\Pi^d_{i,j} : A_{i,j} = \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i, U_j) \to \mathcal{C}^{\mathrm{pol},d}(N, U_j)$$

for the projection denoted above by Π^d_j. The projections $\Pi^d_{i,j}$ are compatible with the transition maps of (2.2.5) and (2.5.17), and so passing to the corresponding inductive limits, they yield maps

$$\Pi^d : \mathcal{C}^{\omega}(N, U)_e \to \mathcal{C}^{\mathrm{pol},d}(N, U).$$

Corollary 2.6.7. *(i) If I denotes the identity operator from $\mathcal{C}^{\omega}(N, U)_e$ to itself, then there is an equality $I = \sum_{d=1}^{\infty} \Pi^d$ in $\mathcal{L}(\mathcal{C}^{\omega}(N, U)_e, \mathcal{C}^{\omega}(N, U)_e)_s$ (i.e. the series on the right converges in weak topology to the identity operator).*

(ii) If ∂_α acts on the locally analytic M-representation U via a scalar, then for each $d, j \geq 0$, the projection Π^d_j lies in the closure of the image of (2.6.2).

Proof. The claims follow from Lemma 2.6.1 and Proposition 2.6.4, together with the isomorphism (2.2.5). □

(2.7) We maintain the notation of the preceding subsections.

Definition 2.7.1. If $\mathcal{X}$ is a subspace of $(\mathrm{Ind}_P^G U)_e$, then we let $\mathcal{X}^{\mathrm{pol}}$ denote the preimage of $\mathcal{X}$ under the map (2.5.20).

The map (2.5.20) induces a continuous injection

$$\mathcal{X}^{\mathrm{pol}} \to \mathcal{X}. \tag{2.7.2}$$

Lemma 2.7.3. *If $\mathcal{X}$ is a closed $(\mathfrak{g}, M)$-invariant subspace of $(\mathrm{Ind}_P^G U)_e$, then $\mathcal{X}^{\mathrm{pol}}$ is closed $(\mathfrak{g}, P)$-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$, and (2.7.2) is a $(\mathfrak{g}, M)$-equivariant map.*

Proof. Evidently $\mathcal{X}^{\mathrm{pol}}$ is a closed $(\mathfrak{g}, M)$-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$, since the map (2.5.20) is continuous and $(\mathfrak{g}, M)$-equivariant. Furthermore (2.7.2) is $(\mathfrak{g}, M)$-equivariant, being a restriction of (2.5.20). Since the N-action on $\mathcal{C}^{\mathrm{pol}}(N, U)$ is obtained by integrating the $\mathfrak{n}$-action, we see that $\mathcal{X}^{\mathrm{pol}}$, being both M and $\mathfrak{n}$-invariant, is in fact $P = MN$-invariant. □

The following lemma gives a result in the converse direction to that just proved.

Lemma 2.7.4. *If $\mathbf{X}$ is a closed $(\mathfrak{g}, M)$-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$, and if $\hat{\mathbf{X}}$ denotes the closure in $(\mathrm{Ind}_P^G U)_e$ of the image of $\mathbf{X}$ under (2.5.20), then $\hat{\mathbf{X}}$ is a $(\mathfrak{g}, \overline{P})$-invariant subspace of $(\mathrm{Ind}_P^G U)_e$.*

Proof. Since $\mathbf{X}$ is assumed to be $(\mathfrak{g}, M)$-invariant, the same is true of its image under (2.5.20), and thus of the closure $\hat{\mathbf{X}}$ of this image. Lemma 2.4.10 shows that if U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(G)$ then $\mathcal{X}$ is in fact $(\mathfrak{g}, \overline{P})$-invariant. □

Definition 2.7.5. If X is a local closed G-invariant subspace of $\mathrm{Ind}_P^G U$, then for any open subset Ω of N we let $X^{\mathrm{lp}}(\Omega)$ denote the preimage of $X(\Omega)$ under the continuous injection (2.5.27). For any $d \geq 0$, we let $X^{\mathrm{lp}, \leq d}(\Omega)$ denote the intersection of $X^{\mathrm{lp}}(\Omega)$ and $\mathcal{C}_c^{\mathrm{lp}, \leq d}(\Omega, U)$.

For any open subset Ω of N, the map (2.5.27) evidently restricts to a continuous injection

$$X^{\text{lp}}(\Omega) \to X(\Omega). \tag{2.7.6}$$

In the case when $\Omega = N$, Lemma 2.5.24 implies that (2.5.27) is $(\mathfrak{g}, P)$-equivariant. Since $X(N)$ is a closed $(\mathfrak{g}, P)$-invariant subspace of $(\text{Ind}_P^G U)(N)$, we thus see that $X^{\text{lp}}(N)$ is a closed $(\mathfrak{g}, P)$-invariant subspace of $\mathcal{C}_c^{\text{lp}}(N, U)$.

Since X_e is a closed subspace of $(\text{Ind}_P^G U)_e$ (by Proposition 2.4.7 (i)), Lemma 2.7.3 shows that $(X_e)^{\text{pol}}$ is a closed $(\mathfrak{g}, P)$-invariant subspace of $\mathcal{C}^{\text{pol}}(N, U)$. By definition we have an isomorphism $\mathcal{C}^{\text{pol}}(N, U) \otimes_K \mathcal{C}^{\text{sm}}(N, K) \xrightarrow{\sim} \mathcal{C}_c^{\text{lp}}(N, U)$ (see Definition 2.5.21), which restricts to a closed embedding

$$(X_e)^{\text{pol}} \otimes_K \mathcal{C}^{\text{sm}}(N, K) \to \mathcal{C}_c^{\text{lp}}(N, U). \tag{2.7.7}$$

Lemma 2.7.8. *The map (2.7.7) induces an isomorphism*

$$(X_e)^{\text{pol}} \otimes_K \mathcal{C}^{\text{sm}}(N, K) \xrightarrow{\sim} X^{\text{lp}}(N).$$

Proof. Let Y denote the image of (2.7.7). To show that $Y \subset X^{\text{lp}}(N)$, it suffices (since X is local) to show that for every element $F \in (X_e)^{\text{pol}}$, and for every sufficiently small compact open subset Ω of N, the function $F_{|\Omega}$ lies in X. (Here $F_{|\Omega}$ denotes the locally polynomial function defined by F on Ω, and by zero elsewhere.) If $n \in \Omega$, then $F_{|\Omega} = n^{-1}(nF)_{|\Omega n^{-1}}$. Since X is an N-invariant subspace of $\text{Ind}_P^G U$, while $(X_e)^{\text{pol}}$ is an N-invariant subspace of $\mathcal{C}^{\text{pol}}(N, U)$ (by Lemma 2.7.3), we see that it suffices to treat the case when Ω is a sufficiently small neighbourhood of e. It then follows from the very definition of $(X_e)^{\text{pol}}$ that $F_{|\Omega}$ lies in X for sufficiently small neighbourhoods Ω of e.

To prove that $X^{\text{lp}}(N) \subset Y$, it suffices to show that if a locally polynomial function $f \in \mathcal{C}_c^{\text{la}}(N, U)$ lies in X, then for any sufficiently small compact open subset Ω of N we have $f_{|\Omega} = F_{|\Omega}$ for some $F \in (X_e)^{\text{pol}}$. If $n \in \Omega$, then $f_{|\Omega} = n^{-1}(nf)_{|\Omega n^{-1}}$. Again using the N-invariance of X and of $(X_e)^{\text{pol}}$, we see that it suffices to treat the case when Ω is a sufficiently small neighbourhood of e. Since f is locally polynomial, we see that $F := \text{st}_e(f)$ is a polynomial and that $f_{|\Omega} = F_{|\Omega}$ for any sufficiently small neighbourhood Ω of e. Since $f \in X$, it follows that F lies in $(X_e)^{\text{pol}}$. This completes the proof of the lemma. □

Proposition 2.7.9. *If X is a local closed G-invariant subspace of $\text{Ind}_P^G U$, then the following are equivalent.*

(i) The image of $(X_e)^{\text{pol}}$ in X_e under the map (2.7.2) (with $\mathcal{X}$ taken to be X_e) is dense in X_e.

(ii) The image of (2.7.6) is dense in $X(\Omega)$ for some (or equivalently, for every) non-empty open subset Ω of N.

Furthermore, X is then equal to the closed G-subrepresentation of $\text{Ind}_P^G U$ generated by the image of (2.7.6) for any non-empty open subset Ω of N.

Proof. Suppose that (i) holds. The image of (2.7.6) for $\Omega = N$ is a local $(\mathfrak{g}, P)$-invariant subspace of $X(N)$. If we let Y denote the closure of this image, then Y is $(\mathfrak{g}, P)$-invariant, and Lemma 2.4.4 shows that Y is again local. We claim that there is an equality of stalks $Y_n = X_n$ for all $n \in N$. Granting this, it follows from Lemma 2.4.3 that $Y = X(N)$. Since, by Lemma 2.4.4, $Y(\Omega)$ coincides with the closure of the image of (2.7.6) for any open subset Ω of N, we conclude that (ii) holds.

If we consider stalks at e, then it is clear that Y_e contains the image of (2.7.2) (applied to $(X_e)^{\mathrm{pol}}$), and is contained in X_e. Since Y_e is closed in $(\mathrm{Ind}_{\overline{P}}^G U)_e$, by Lemma 2.4.6, our hypothesis implies that $\mathrm{st}_e(Y) = X_e$. Since both Y and $X(N)$ are N-invariant, it follows that indeed $Y_n = X_n$ for all $n \in N$, as claimed.

Conversely, suppose that (ii) holds, i.e. that (2.7.6) has dense image for some open subset Ω of N. Since the map st_e is continuous, the image of the composite

$$X^{\mathrm{lp}}(\Omega) \xrightarrow{(2.7.6)} X(\Omega) \xrightarrow{\mathrm{st}_e} (\mathrm{Ind}_{\overline{P}}^G U)_e \tag{2.7.10}$$

is dense in $X_e = \mathrm{st}_e(X)$. The isomorphism of Lemma 2.7.8 restricts to an isomorphism

$$(X_e)^{\mathrm{pol}} \otimes_K \mathcal{C}^{\mathrm{sm}}(\Omega, K) \xrightarrow{\sim} X^{\mathrm{lp}}(\Omega).$$

Thus we see that the image of (2.7.10) coincides with the image of (2.7.2), and so (i) holds.

Finally, if condition (ii) holds then $X(\Omega)$ is by assumption the closure of the image of (2.7.6). Since (2.4.12) is a surjection (X being local) we see that X is indeed the closed G-subrepresentation of $\mathrm{Ind}_{\overline{P}}^G U$ generated by the image of (2.7.6). □

Definition 2.7.11. We say that a subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$ is graded if it is graded with respect to the grading induced by α on $\mathcal{C}^{\mathrm{pol}}(N, U)$ that was discussed in Subsection 2.5.

Suppose now that $\mathbf{X}$ is a closed graded $(\mathfrak{g}, M)$-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$. The isomorphism (2.5.17) then induces isomorphisms

$$\mathbf{X} \xrightarrow{\sim} \varinjlim_j \mathbf{X}_j \xrightarrow{\sim} \varinjlim_j \bigoplus_{d \geq 0} \mathbf{X}_j^d, \tag{2.7.12}$$

where $\mathbf{X}_j$ denotes the preimage of $\mathbf{X}$ under the natural injection $\mathcal{C}^{\mathrm{pol}}(N, U_j) \to \mathcal{C}^{\mathrm{pol}}(N, U)$, and $\mathbf{X}_j^d := \mathbf{X}_j \bigcap \mathcal{C}^{\mathrm{pol},d}(N, U_j)$ (where the intersection takes place in $\mathcal{C}^{\mathrm{pol}}(N, U_j)$). If we let $\hat{\mathbf{X}}_{i,j}$ denote the closure in $A_{i,j}$ of the image of $\mathbf{X}_j$ under (2.5.14), then the isomorphism of Lemma 2.5.15 induces an isomorphism

$$\hat{\mathbf{X}}_{i,j} \xrightarrow{\sim} \hat{\bigoplus_{d \geq 0}} \mathbf{X}_j^d,$$

where the target is defined via Definition 2.5.11, using the restriction of the norms $||-||_{i,j}^d$ to $\mathbf{X}_{i,j}^d$.

From this description of $\hat{\mathbf{X}}_{i,j}$ it is clear that $\hat{\mathbf{X}}_{i,j}$ coincides with the preimage of $\hat{\mathbf{X}}_{i',j'}$ under the transition map (2.2.2), for any $i' \geq i, j' \geq j$. It follows from [6, Prop. 1.1.41] together with the isomorphism (2.2.5) and the isomorphism of Corollary 2.3.4 that we have an isomorphism

$$\varinjlim_{i,j} \hat{\mathbf{X}}_{i,j} \xrightarrow{\sim} \hat{\mathbf{X}} \tag{2.7.13}$$

(where as in the statement of Lemma 2.7.4, we let $\hat{\mathbf{X}}$ denote the closure of in $(\mathrm{Ind}_{\overline{P}}^G U)_e$ of the image of $\mathbf{X}$ under (2.5.20)).

Corollary 2.7.14. *The evident inclusion* $\mathbf{X} \subset \hat{\mathbf{X}}^{\mathrm{pol}}$ *is in fact an equality.*

Proof. If follows directly from the explicit description of $\hat{\mathbf{X}}$ provided by (2.7.12) and (2.7.13) that the preimage of $\hat{\mathbf{X}}$ under the injection

$$\mathcal{C}^{\mathrm{pol}}(N, U_j) \longrightarrow \mathcal{C}^{\mathrm{pol}}(N, U) \stackrel{(2.5.20)}{\longrightarrow} (\mathrm{Ind}_{\overline{P}}^{G} U)_e$$

coincides with $\mathbf{X}_j$, for each $j \geq 0$, and thus that the preimage of $\hat{\mathbf{X}}$ under (2.5.20) coincides with $\mathbf{X}$, as claimed. □

Definition 2.7.15. If X is a local closed G-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$, then we say that X is polynomially generated if $(X_e)^{\mathrm{pol}}$ is a graded (in the sense of Definition 2.7.11) subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$, and if the image of $(X_e)^{\mathrm{pol}}$ under (2.5.20) is dense in X_e.

We say that X is polynomially generated by bounded degrees if furthermore $(X_e)^{\mathrm{pol}}$ is generated as a $\mathfrak{g}$-module by $(X_e)^{\mathrm{pol}, \leq d} := (X_e)^{\mathrm{pol}} \bigcap \mathcal{C}^{\mathrm{pol}, \leq d}(N, U)$, for some $d \geq 0$.

If U is an object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$, then the results of this Subsection and of Subsection 2.4 show that the polynomially generated local closed G-invariant subspaces of $\mathrm{Ind}_{\overline{P}}^{G} U$ are in natural bijection with the closed graded $(\mathfrak{g}, M)$-invariant subspaces of $\mathcal{C}^{\mathrm{pol}}(N, U)$, the bijection being provided by $X \mapsto (X_e)^{\mathrm{pol}}$.

We close this subsection with a result which helps illustrate the scope of Definition 2.7.15.

Proposition 2.7.16. *If the M-representation U admits a central character, then every local closed G-subrepresentation of $\mathrm{Ind}_{\overline{P}}^{G} U$ is polynomially generated. If U is furthermore finite dimensional, then any such G-subrepresentation is in fact generated by bounded degrees.*

Proof. If X is a local closed G-subrepresentation of $\mathrm{Ind}_{\overline{P}}^{G} U$, then it follows from Proposition 2.4.7 (i) that X_e is a $(\mathfrak{g}, \overline{P})$-invariant, and so in particular ∂_α-invariant, closed subspace of $(\mathrm{Ind}_{\overline{P}}^{G} U)_e$. Corollary 2.6.7 then implies that $(X_e)^{\mathrm{pol}}$ is graded, and that its image under (2.5.20) is dense in X_e. Thus X is polynomially generated, as claimed.

If U is finite dimensional, then $\mathcal{C}^{\mathrm{pol}}(N, U)$ is a finitely generated $\mathrm{U}(\mathfrak{g})$-module. Since $\mathrm{U}(\mathfrak{g})$ is Noetherian, $(X_e)^{\mathrm{pol}}$ is then also finitely generated, and so is generated over $\mathfrak{g}$ by $(X_e)^{\mathrm{pol}, \leq d}$ for some sufficiently large value of d. This shows that X is generated by bounded degrees. □

(2.8) We maintain the notation of the previous subsections. We begin by proving Lemma 0.3.

Proof of Lemma 0.3. Composing the closed embedding

$$\mathcal{C}^{\mathrm{sm}}_c(N, U) \to \mathcal{C}^{\mathrm{la}}_c(N, U) \tag{2.8.1}$$

with the isomorphism of Lemma 2.3.6 yields a closed embedding

$$\mathcal{C}^{\mathrm{sm}}_c(N, U) \to \mathrm{Ind}_{\overline{P}}^{G} U. \tag{2.8.2}$$

Passing to Jacquet modules, and taking into account [7, Lem. 3.5.2], we obtain a closed embedding $U(\delta) \to J_P(\mathrm{Ind}_P^G U)$, as required. □

Recall from the introduction that if U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then $I_P^G(U)$ is defined to be the closed G-subrepresentation of $\mathrm{Ind}_P^G U$ generated by the image of U under the canonical lifting of $J_P(\mathrm{Ind}_P^G U)$ to $\mathrm{Ind}_P^G U$. Recall from the discussion of [7, §3.4] that the canonical lifting depends on a choice of compact open subgroup N_0 of N (and also a choice of Levi factor of P; however, we have already fixed such a choice). The following lemma gives another description of $I_P^G(U)$, which shows that it is well-defined independently of the choice of N_0.

Lemma 2.8.3. *If U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then $I_P^G(U)$ coincides with the closed G-subrepresentation of $\mathrm{Ind}_P^G U$ generated by the image of the closed embedding (2.8.2).*

Proof. Fix a compact open subgroup N_0 of N, and for $u \in U$, let $u_{|N_0} \in \mathcal{C}_c^{\mathrm{sm}}(N, U)$ denote the function that is identically equal to u on N_0, and that is zero on the complement of N_0 in N. The canonical lifting $U \to \mathrm{Ind}_P^G U$ is defined by sending an element $u \in U$ to the image of $u_{|N_0}$ under (2.8.1). (See the discussion following the proof of [7, Thm. 3.5.6].) Since $\{u_{|N_0} \,|\, u \in U\}$ generates $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ as a P-representation, we see that the image of U under the canonical lift generates the image of (2.8.2) as a P-representation. Consequently, the closed G-representation generated by the image of U under the canonical lift coincides with the closed G-representation generated by the image of (2.8.2). □

Regarding an element of U as a constant U-valued function on N induces a closed P-equivariant embedding

$$U \to \mathcal{C}^{\mathrm{pol}}(N, U). \tag{2.8.4}$$

(Here P acts on U through its quotient M.) Tensoring (2.8.4) with $\mathcal{C}_c^{\mathrm{sm}}(N, K)$ over K induces a closed P-equivariant embedding

$$\mathcal{C}_c^{\mathrm{sm}}(N, U) \to \mathcal{C}_c^{\mathrm{lp}}(N, U). \tag{2.8.5}$$

Note that the composite of (2.8.5) and (2.5.23) coincides with (2.8.1).

The P-equivariant maps (2.8.4) and (2.8.5) induce $(\mathfrak{g}, P)$-equivariant maps

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U \to \mathcal{C}^{\mathrm{pol}}(N, U) \tag{2.8.6}$$

and

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N, U) \to \mathcal{C}_c^{\mathrm{lp}}(N, U). \tag{2.8.7}$$

respectively. We may regard (2.8.7) as being obtained from (2.8.6) by tensoring with $\mathcal{C}_c^{\mathrm{sm}}(N, K)$ over K.

Lemma 2.8.8. *If U is an allowable object of $\mathrm{Rep}_{\mathrm{la.c}}(M)$, then the image of (2.8.6) is a closed $(\mathfrak{g}, P)$-invariant subspace of $\mathcal{C}^{\mathrm{pol}}(N, U)$.*

Proof. Multiplication in $\mathrm{U}(\mathfrak{g})$ induces an isomorphism $\mathrm{U}(\overline{\mathfrak{n}}) \otimes_K \mathrm{U}(\mathfrak{p}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g})$. Thus we may rewrite (2.8.6) as a map $\mathrm{U}(\overline{\mathfrak{n}}) \otimes_K U \to \mathcal{C}^{\mathrm{pol}}(N, U)$. We may further write this map as the direct sum of maps

$$\mathrm{U}^d(\overline{\mathfrak{n}}) \otimes_K U \to \mathcal{C}^{\mathrm{pol},d}(N, U) = \mathcal{C}^{\mathrm{pol},d}(N, K) \otimes_K U, \tag{2.8.9}$$

as d ranges over the non-negative integers. (Here $\mathrm{U}^d(\overline{\mathfrak{n}})$ denotes the degree d graded piece of $\mathrm{U}(\overline{\mathfrak{n}})$ with respect to the grading induced by α.)

Since the formation of the maps (2.8.9) is functorial in U, they are $\mathrm{End}_M(U)$-equivariant, and hence have closed image (since each of $\mathrm{U}^d(\overline{\mathfrak{n}})$ and $\mathcal{C}^{\mathrm{pol},d}(N,K)$ is a finite dimensional algebraic M-representation, and U is assumed allowable). It follows from [2, Cor. 1, p. II.31] that (2.8.6) has closed image, as asserted. □

Proposition 2.8.10. *If U is an allowable object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then $I^G_{\overline{P}}(U)$ is a local closed polynomially generated G-invariant subspace of $\mathrm{Ind}^G_{\overline{P}}U$, and $I^G_{\overline{P}}(U)^{\mathrm{lp}}(N)$ (as given by Definition 2.7.5) coincides with the image of (2.8.7).*

Proof. Let $\mathbf{X}$ denote the image of (2.8.6), which by Lemma 2.8.8 is a closed subspace of $\mathcal{C}^{\mathrm{pol}}(N,U)$. Since (2.8.4) identifies U with $\mathcal{C}^{\mathrm{pol},0}(N,U)$, Lemma 2.5.9 shows that $\mathbf{X}$ is furthermore graded, in the sense of Definition 2.7.11. If $\hat{\mathbf{X}}$ denotes the closure of the image of $\mathbf{X}$ under the map (2.5.20), then Lemma 2.7.4 shows that $\hat{\mathbf{X}}$ is a $(\mathfrak{g},\overline{P})$-invariant subspace of $(\mathrm{Ind}^G_{\overline{P}}U)_e$, which corresponds to a local closed subrepresentation X of $\mathrm{Ind}^G_{\overline{P}}U$ via the construction of Proposition 2.4.9. Corollary 2.7.14 implies that $\mathbf{X} = \hat{\mathbf{X}}^{\mathrm{pol}}$, and thus that X is polynomially generated in the sense of Definition 2.7.15.

Proposition 2.7.9 implies that X coincides with the closed G-representation generated by $X^{\mathrm{lp}}(N)$, which Lemma 2.7.8 shows to be isomorphic to $\mathbf{X}\otimes_K \mathcal{C}^{\mathrm{sm}}_c(N,K)$. The definition of $\mathbf{X}$ then shows that $X^{\mathrm{lp}}(N)$ is equal to the image of (2.8.7). Since this image maps under (2.5.27) (taking Ω to be N) to the $\mathfrak{g}$-subrepresentation of $\mathrm{Ind}^G_{\overline{P}}U$ generated by the image of (2.8.2), we conclude from Lemma 2.8.3 that X coincides with $I^G_{\overline{P}}(U)$. This establishes the proposition. □

In the context of the preceding proposition, note that by construction $I^G_{\overline{P}}(U)$ is in fact polynomially generated in degree zero. Note also that the proof of this proposition shows that $I^G_{\overline{P}}(U)^{\mathrm{lp}}(N)$ coincides with the space denoted in the same manner in the introduction.

3. A variant of the method of Amice-Vélu and Vishik

(3.1) We maintain the notation introduced in Subsection 2.2, and fix in addition a K-Banach space V equipped with a continuous action of Z_M. We recall from Subsection 1.4 the definition of $\mathcal{C}^{\mathrm{la}}_c(N,V)$ as a convex K-vector space. We endow $\mathcal{C}^{\mathrm{la}}_c(N,V)$ with the right regular action of N, and with the Z_M-action defined by $(zf)(n) = zf(z^{-1}nz)$. These actions together give a locally analytic action of NZ_M on $\mathcal{C}^{\mathrm{la}}_c(N,V)$. If we embed $\mathcal{C}^{\mathrm{la}}(N_0,V)$ as the closed subspace of $\mathcal{C}^{\mathrm{la}}_c(N,V)$ consisting of functions supported on N_0, then $\mathcal{C}^{\mathrm{la}}(N_0,V)$ is an $N_0Z_M^+$-invariant subspace of $\mathcal{C}^{\mathrm{la}}_c(N,V)$.

For each $i\geq 0$, consider the space $B_i := \bigoplus_{n\in N_i\backslash N_0}\mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n,V)$ (the direct sum ranging over a set of left coset representatives of N_i in N_0). This may be naturally identified with the space of $\mathbb{N}_i$-analytic vectors in $\mathcal{C}^{\mathrm{la}}(N_0,V)$ with respect to the left regular action of N_0, and so is equipped with a natural map

$$B_i \to \mathcal{C}^{\mathrm{la}}(N_0,V). \tag{3.1.1}$$

Since any $\mathbb{N}_i$-analytic vector is also $\mathbb{N}_{i+1}$-analytic, there is a natural map

$$B_i \to B_{i+1}, \tag{3.1.2}$$

compatible with (3.1.1) for i and $i+1$. We may form the inductive limit $\varinjlim_i B_i$ with respect to the maps (3.1.2), and the maps (3.1.1) induce an isomorphism

$$\varinjlim_i B_i \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(N_0, V); \tag{3.1.3}$$

this gives a concrete realization of $\mathcal{C}^{\mathrm{la}}(N_0, V)$ as an inductive limit of Banach spaces.

The right regular N_0-action on $\mathcal{C}^{\mathrm{la}}(N_0, V)$ induces an N_0-action on each B_i, and the maps (3.1.1), (3.1.2), and (3.1.3) are all N_0-equivariant. The action of z_0 on $\mathcal{C}^{\mathrm{la}}(N_0, V)$ induces an isomorphism $z_0^i B_0 \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i, V)$. Since the action of any element $n \in N_0$ induces an isomorphism $\mathcal{C}^{\mathrm{an}}(\mathbb{N}_i, V) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n^{-1}, V)$, we obtain an isomorphism

$$\bigoplus_{n \in N_i \backslash N_0} n z_0^i B_0 \xrightarrow{\sim} B_i. \tag{3.1.4}$$

Suppose that $S \subset T$ are closed $N_0 Z_M^+$-invariant subspaces of $\mathcal{C}^{\mathrm{la}}(N_0, V)$. For each $i \geq 0$, let T_i (respectively S_i) denote the preimage of T (respectively S) under the map (3.1.1). The closed embedding $S \subset T$ induces a closed embedding

$$S_i \subset T_i \tag{3.1.5}$$

for each $i \geq 0$, while the isomorphism (3.1.4) induces closed embeddings

$$\bigoplus_{n \in N_i \backslash N_0} n z_0^i T_0 \to T_i \tag{3.1.6}$$

and

$$\bigoplus_{n \in N_i \backslash N_0} n z_0^i S_0 \to S_i. \tag{3.1.7}$$

As discussed in the *Notations and conventions*, let $\overline{T}$ and $\overline{S}$ denote the ultrabornologicalizations of S and T respectively. It follows from [2, Prop. 1, p. I.20] that there are natural isomorphisms $\varinjlim_i T_i \xrightarrow{\sim} \overline{T}$ and $\varinjlim_i S_i \xrightarrow{\sim} \overline{S}$ (the transition maps for these locally convex inductive limits being induced by the maps (3.1.2)). Recall that we have continuous bijections

$$\overline{T} \to T \qquad \text{and} \qquad \overline{S} \to S. \tag{3.1.8}$$

The closed embeddings (3.1.5) induce a continuous injection

$$\overline{S} \to \overline{T} \tag{3.1.9}$$

in a manner compatible with the bijections (3.1.8).

The reason for considering $\overline{T}$ and $\overline{S}$ is that T and S, while being of LB-type – that is, they are a countable union of BH-subspaces, being closed subspaces of the LB-space $\mathcal{C}^{\mathrm{la}}(N_0, V)$ – may not be LB-spaces. The space $\overline{T}$ (respectively $\overline{S}$) may be regarded as being the same underlying abstract vector space as T (respectively S), endowed with the coarsest locally convex topology that both makes it an LB-space and is finer than the given topology on T (respectively S).

Since the formation of ultrabornologicalizations is functorial, the $N_0Z_M^+$-action on T and S induces an $N_0Z_M^+$-action on each of the space $\overline{T}$ and $\overline{S}$, uniquely determined by the requirement that the bijections (3.1.8) be $N_0Z_M^+$-equivariant. (These actions are *a priori* separately continuous, but since $\overline{T}$ and $\overline{S}$ are LB-, and so barrelled, spaces, they are in fact continuous.) The map (3.1.9) is then also necessarily $N_0Z_M^+$-equivariant.

Let W be a K-Banach space equipped with a continuous action of $N_0Z_M^+$. Our goal in the remainder of this subsection is to state a theorem giving conditions under which the map

$$\mathcal{L}_{N_0Z_M^+}(\overline{T}, W) \to \mathcal{L}_{N_0Z_M^+}(\overline{S}, W) \tag{3.1.10}$$

induced by (3.1.9) is an isomorphism. The proof of this theorem will be given in the following subsection.

In order to state our theorem, we must introduce further notation. We fix a norm $||-||_V$ on V that determines its Banach space structure. Since $B_i := \bigoplus_{n\in N_i\backslash N_0}\mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n, V) = \mathcal{C}^{\mathrm{an}}(\coprod_{n\in N_i\backslash N_0}\mathbb{N}_i n, V)$ is equal to the space of rigid analytic V-valued functions on the affinoid $\coprod_{n\in N_i\backslash N_0}\mathbb{N}_i n$, the choice of the norm $||-||_V$ on V determines a norm on B_i, namely the usual rigid analytic sup norm. We denote this norm by $||-||_i$; it determines the Banach space structure on B_i. We also fix a norm $||-||_W$ on W that determines its Banach space structure.

Since N_0 is compact and acts continuously on W, we may and do assume that $||-||_W$ is N_0-invariant (by [6, Lem. 6.5.3] for example). Since z_0 acts as a continuous endomorphism of W, we may find $C_1 > 0$ such that $||\,z_0w\,||_W \leq C_1||\,w\,||_W$ for all $w \in W$. Since z_0 induces a continuous automorphism of V, we may find $C_2 > 0$ such that $||\,z_0(u)\,||_V > C_2||\,u\,||$ for all $u \in V$. Write $C = C_1/C_2$.

For each $i \geq 0$, fix a topological complement $S_i^\perp$ to S_i in T_i (as we may, by [11, Prop. 10.5]), and let π_i (respectively $\pi_i^\perp$) denote the projection $T_i \to S_i$ with kernel $S_i^\perp$ (respectively the projection $T_i \to S_i^\perp$ with kernel S_i).

Theorem 3.1.11. *Suppose that the closed embeddings (3.1.6) and (3.1.7) are isomorphisms, and that there exists $\epsilon > 0$ such that $\epsilon C < 1$, and such that for all $i' \geq i \geq 0$, one has*

$$||\,\pi_{i'}^\perp(f)\,||_{i'} \leq \epsilon^{i'-i}||\,f\,||_i \tag{3.1.12}$$

for all $f \in T_i$. Then (3.1.10) is an isomorphism.

As remarked above, we present the proof of this result in Subsection 3.2. The following lemma gives a criterion for verifying the first hypothesis of Theorem 3.1.11.

Lemma 3.1.13. *Let T be a closed subspace of $\mathcal{C}^{\mathrm{la}}(N_0, V)$ such that:*

(i) For all $f \in T$, the element $z_0^{-1}(f_{|N_1})$ of $\mathcal{C}^{\mathrm{la}}(N_0, V)$ again lies in T.

(ii) For any $f \in T$ and any compact open subspace $\Omega \subset N_0$, the restriction $f_{|\Omega}$ (regarded as an element of $\mathcal{C}^{\mathrm{la}}(N_0, V)$ via extending by zero) again lies in T.

Then the closed embedding (3.1.6) is an isomorphism.

Proof. Fix $i \geq 0$ and $n \in N_0$, and let $f \in T_i \bigcap \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n, V)$. Since T_i is N_0-invariant, we see that $nf \in T_i \bigcap \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i, V)$, and so by (i) (applied i times) we find that $z_0^{-i} n f \in T_0$. Thus $T_i \bigcap \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n, V) \subset n^{-1} z_0^i T_0$. The reverse inclusion is clear (T being $N_0 Z_M^+$-invariant), and so in fact $T_i \bigcap \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n, V) = n^{-1} z_0^i T_0$. Since (ii) implies that $T_i = \bigoplus_{n \in N_i \backslash N_0} T_i \bigcap \mathcal{C}^{\mathrm{an}}(\mathbb{N}_i n, V)$, we see that (3.1.6) is an isomorphism, as claimed. $\square$

(3.2) In this subsection we prove Theorem 3.1.11. Thus we assume that the closed embeddings (3.1.6) and (3.1.7) are in fact isomorphisms, and that (3.1.12) holds for all $i' \geq i \geq 0$ and some ϵ (which we fix) satisfying $0 < \epsilon < C^{-1}$. We begin by establishing some estimates.

Lemma 3.2.1. *If $i \geq 0$ and $f \in T_i$, then $|| f ||_{i+1} \leq || f ||_i$.*

Proof. Recall that $|| f ||_{i+1}$ is computed as the sup norm of f over the rigid analytic space $\coprod_{n \in N_{i+1} \backslash N_0} \mathbb{N}_{i+1} n$, while $|| f ||_i$ is computed as the sup norm of f over the rigid analytic space $\coprod_{n \in N_i \backslash N_0} \mathbb{N}_i n$. Since the former rigid analytic space is an open subset of the latter, the claimed inequality obviously holds. $\square$

Lemma 3.2.2. *If $i \geq 0$ and $f \in T_i$, so that $z_0 f \in T_{i+1}$, then $|| z_0 f ||_{i+1} \geq C_2 || f ||_i$.*

Proof. By assumption f is a rigid analytic function on $\coprod_{n \in N_i \backslash N_0} \mathbb{N}_i n$, and so $z_0 f$ is the rigid analytic function on $\coprod_{n \in N_{i+1} \backslash N_1} \mathbb{N}_{i+1} n$ defined by

$$(z_0 f)(x) = z_0(f(z_0^{-1} x z_0))$$

(and extended by zero to $\coprod_{n \in N_{i+1} \backslash N_0} \mathbb{N}_{i+1} n$). We compute that

$$\begin{aligned} || z_0 f ||_{i+1} &= \sup\{|| (z_0 f)(x) ||_V \mid x \in \coprod_{n \in N_{i+1} \backslash N_1} \mathbb{N}_{i+1} n\} \\ &= \sup\{|| z_0(f(z_0^{-1} x z_0)) ||_V \mid x \in \coprod_{n \in N_{i+1} \backslash N_1} \mathbb{N}_{i+1} n\} \\ &\geq \sup\{C_2 || f(z_0^{-1} x z_0) ||_V \mid x \in \coprod_{n \in N_{i+1} \backslash N_1} \mathbb{N}_{i+1} n\} \\ &= C_2 \sup\{|| f(x) ||_V \mid x \in \coprod_{n \in N_i \backslash N_0} \mathbb{N}_i n\} = C_2 || f ||_i, \end{aligned}$$

as claimed. $\square$

Lemma 3.2.3. *Let t be a continuous semi-norm on $\overline{T}$ that satisfies the following conditions:*

(i) The semi-norm t is N_0-invariant.

(ii) We have $t(z_0 f) \leq C_1 t(f)$ for all $f \in \overline{T}$.

Then there exists $A > 0$ such that $t(f) \leq A C^i || f ||_i$ for all $i \geq 0$ and $f \in T_i$.

Proof. Since t is continuous, there is a constant $A > 0$ such that $t(f) \leq A || f ||_0$ for all $f \in T_0$. Now let $f \in T_i$. Since (3.1.6) is an isomorphism, we may write $f = \sum_{n \in N_i \backslash N_0} n z_0^i g_n$ for some $g_n \in T_0$. We then compute (taking into account conditions (i) and (ii)) that

$$t(f) \leq \max\{t(z_0^i g_n)\} \leq C_1^i \max\{t(g_n)\} \leq A C_1^i \max\{|| g_n ||_0\}.$$

On the other hand, Lemma 3.2.2 shows that

$$|| f ||_i = \max\{|| z_0^i g_n ||_i\} \geq C_2^i \max || g_n ||_0.$$

Combining these two inequalities (and recalling that $C = C_1/C_2$) yields the inequality of the lemma. $\square$

Lemma 3.2.4. *If t is any continuous semi-norm on $\overline{T}$ that satisfies conditions (i) and (ii) of Lemma 3.2.3, then for all $i' \geq i \geq 0$ and $f \in T_i$ we have $t(\pi_{i'}^{\perp}(f)) \leq AC^i(\epsilon C)^{i'-i} || f ||_i$.*

Proof. This follows from Lemma 3.2.3 (applied to $\pi_{i'}^{\perp}(f) \in T_{i'}$) and (3.1.12). □

Lemma 3.2.5. *If t is any continuous semi-norm on $\overline{T}$ that satisfies conditions (i) and (ii) of Lemma 3.2.3, then the image of (3.1.9) is dense in $\overline{T}$ with respect to the topology induced by t.*

Proof. If $f \in T_i$, then it follows from Lemma 3.2.4 and the assumption that $\epsilon C < 1$ that $\lim_{i \leq i' \to \infty} t(f - \pi_{i'}(f)) = \lim_{i \leq i' \to \infty} t(\pi_{i'}^{\perp}(f)) = 0$. □

We can now prove that (3.1.10) is injective. Let $F : \overline{T} \to W$ be an $N_0 Z_M^+$-equivariant continuous K-linear map $\overline{T} \to W$, and let t denote the continuous semi-norm on $\overline{T}$ defined by $t(f) = || F(f) ||_W$ for all $f \in \overline{T}$. It follows directly from the corresponding properties of $|| - ||_W$ that t satisfies conditions (i) and (ii) of Lemma 3.2.3. If the restriction of F to the image of (3.1.9) vanishes, then the restriction of t to the image of (3.1.9) also vanishes. Lemma 3.2.5 then implies that t vanishes identically on $\overline{T}$, and thus that F vanishes. Consequently (3.1.10) is injective.

The proof that (3.1.10) is surjective is similar. We begin by proving the following result.

Proposition 3.2.6. *Let s be a continuous semi-norm on $\overline{S}$ that satisfies conditions (i) and (ii) of Lemma 3.2.3, with t and $\overline{T}$ replaced by s and $\overline{S}$. Then there is a uniquely determined continuous semi-norm t on $\overline{T}$ that satisfies conditions (i) and (ii) of Lemma 3.2.3, and such that s is equal to the composite of t with (3.1.9).*

Proof. The uniqueness is a direct consequence of Lemma 3.2.5. As for the existence, we have the following explicit definition of t: If $f \in T_i$, then $t(f) = \lim_{i \leq i' \to \infty} s(\pi_{i'}(f))$. We will show that this gives a well-defined continuous semi-norm on $\overline{T}$ that satisfies conditions (i) and (ii) of Lemma 3.2.3. Note that if $f \in S_i$ then $\pi_{i'}(f) = f$ for all $i' \geq i$, and so it is clear that s is equal to the composite of t and (3.1.9).

To show that $t(f)$ is well-defined, we must show that the limit exists. For this, it suffices to show that $\lim_{i \leq i' \to \infty} s(\pi_{i'+1}(f) - \pi_{i'}(f))$ exists and equals zero. We compute that (for some $A > 0$)

$$\begin{aligned} & s(\pi_{i'+1}(f) - \pi_{i'}(f)) \\ \leq\ & AC^{i'+1} || \pi_{i'+1}(f) - \pi_{i'}(f) ||_{i'+1} \\ =\ & AC^{i'+1} || \pi_{i'+1}^{\perp}(f) - \pi_{i'}^{\perp}(f) ||_{i'+1} \\ \leq\ & AC^{i'+1} \max\{|| \pi_{i'+1}^{\perp}(f) ||_{i'+1}, || \pi_{i'}^{\perp}(f) ||_{i'+1}\} \\ \leq\ & AC^{i'+1} \max\{|| \pi_{i'+1}^{\perp}(f) ||_{i'+1}, || \pi_{i'}^{\perp}(f) ||_{i'}\} \\ \leq\ & AC^{i'+1} \epsilon^{i'-i} \max\{\epsilon, 1\} || f ||_i \\ =\ & AC^{i+1} \max\{\epsilon, 1\} (\epsilon C)^{i'-i} || f ||_i \end{aligned} \tag{3.2.7}$$

(where the first inequality follows from Lemma 3.2.3, applied with s and $S_{i'+1}$ in place of t and T_i, the subsequent equality follows from the equation $\pi_{i'+1}(f) -$

$\pi_{i'}(f) = f - \pi^{\perp}_{i'+1}(f) - (f - \pi^{\perp}_{i'}(f)) = \pi^{\perp}_{i'}(f) - \pi^{\perp}_{i'+1}(f)$, the second inequality is an application of the non-archimedean triangle inequality, the third inequality follows from Lemma 3.2.1, and the fourth inequality follows from (3.1.12).) Since $\epsilon C < 1$ by assumption, we conclude that indeed $\lim_{i \leq i' \to \infty} s(\pi_{i'+1}(f) - \pi_{i'}(f)) = 0$, and so t is well-defined. It is clear that t is a semi-norm, since s is a semi-norm and the projections $\pi_{i'}$ are linear.

The inequality (3.2.7) immediately generalizes: If $i \leq i' \leq i''$, and $f \in T_i$ then

$$\begin{aligned} s(\pi_{i''}(f) - \pi_{i'}(f)) &\leq \max\{s(\pi_{i'+1}(f) - \pi_{i'}(f)), \dots, s(\pi_{i''}(f) - \pi_{i''-1}(f))\} \\ &\leq AC^{i+1}\max\{\epsilon, 1\}(\epsilon C)^{i'-i} || \, f \, ||_i, \end{aligned} \tag{3.2.8}$$

using (3.2.7) and the fact that $\epsilon C < 1$.

Now $|| \, \pi_i(f) \, ||_i \leq \max\{|| \, \pi^{\perp}_i(f) \, ||_i, || \, f \, ||_i\} = || \, f \, ||_i$ (applying (3.1.12) with $i' = i$), which together with Lemma 3.2.3 (putting $\pi_i(f)$, s and S_i in place of f, t and T_i) implies that $s(\pi_i(f)) \leq AC^i || \, f \, ||_i$. Combining this and (3.2.8), we find that

$$s(\pi_{i'}(f)) \leq \max\{s(\pi_i(f)), s(\pi_{i'}(f) - \pi_i(f))\} \leq AC^i \max\{1, C\max\{\epsilon, 1\}\} || \, f \, ||_i.$$

Passing to the limit as $i' \to \infty$, we deduce that

$$t(f) \leq AC^i \max\{1, C\max\{\epsilon, 1\}\} || \, f \, ||_i.$$

This implies that t is continuous on T_i. Since $\overline{T} \xrightarrow{\sim} \varinjlim_i T_i$, we conclude that t is continuous on $\overline{T}$.

If $i \geq 0$ and $f \in T_i$, then for any $i' \geq i$, we compute that

$$\begin{aligned} t(f - \pi_{i'}(f)) &= \lim_{i' \leq i'' \to \infty} s(\pi_{i''}(f - \pi_{i'}(f))) \\ &= \lim_{i' \leq i'' \to \infty} s(\pi_{i''}(f) - \pi_{i'}(f)) \leq AC^{i+1}\max\{\epsilon, 1\}(\epsilon C)^{i'-i} || \, f \, ||_i \end{aligned}$$

(where the inequality follows from (3.2.8)). Passing to the limit as $i' \to \infty$, and remembering that $\epsilon C < 1$, we find that $\lim_{i \leq i' \to \infty} t(f - \pi_{i'}(f)) = 0$, and thus that $\pi_{i'}(f) \to f$ in the t-topology on $\overline{T}$ as $i' \to \infty$. In particular, we conclude that the image of (3.1.9) is dense in $\overline{T}$ with respect to the t-topology. Since conditions (i) and (ii) of Lemma 3.2.3 hold for s on $\overline{S}$ by assumption, we conclude that they also hold for t on $\overline{T}$. □

We can now prove that (3.1.10) is surjective. Let $F_1 : \overline{S} \to W$ be an $N_0 Z_M^+$-equivariant continuous K-linear map, and let s denote the continuous semi-norm on $\overline{S}$ defined by $s(f) = || \, F_1(f) \, ||_W$. The semi-norm s satisfies conditions (i) and (ii) of Lemma 3.2.3, and so we may extend s to a semi-norm t on $\overline{T}$ as in Proposition 3.2.6. Since t induces the s-topology on the image of $\overline{S}$ in $\overline{T}$ under the injection (3.1.9), since this image is dense in the t-topology on $\overline{T}$, since F_1 is continuous with respect to the s-topology on $\overline{S}$ (by the very definition of s), and since W is complete, we conclude that there is a t-continuous map $F : \overline{T} \to W$ whose composition with (3.1.9) coincides with F_1. Furthermore, since the action of $N_0 Z_M^+$ is continuous with respect to the t-topology on $\overline{T}$, and since F_1 is $N_0 Z_M^+$-equivariant, we conclude that F is $N_0 Z_M^+$-equivariant. Since t is continuous with respect to the inductive limit topology on $\overline{T}$, we see that F is continuous with respect to the inductive limit topology on $\overline{T}$. Thus F is the required element of $\mathcal{L}_{N_0 Z_M^+}(\overline{T}, W)$ that maps to F_1 under (3.1.10).

4. Proof of Theorem 0.13

(4.1) We maintain the notation introduced in Subsection 2.2. In particular, we fix an object U of $\mathrm{Rep}_{\mathrm{la.c}}(M)$, which in fact we assume to lie in $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$. We also fix a local closed G-invariant subspace X of $\mathrm{Ind}_{\overline{P}}^{G} U$. If V is a locally analytic G-representation, then restriction of morphisms from X to $X(N)$ induces a morphism

$$\mathcal{L}_G(X,V) \to \mathcal{L}_{(\mathfrak{g},P)}(X(N),V). \tag{4.1.1}$$

The following result show that an element in the target of (4.1.1) is as G-equivariant as it can be.

Lemma 4.1.2. *Suppose that V is an LB-space equipped with a locally analytic G-representation, and let ϕ be an element in the target of (4.1.1). If $f \in X(N)$ and $g \in G$ are chosen so that gf also lies in $X(N)$, then $\phi(gf) = g\phi(f)$.*

Proof. Let $\Omega \subset N$ be the support of f. Since Ω is compact, we may choose i sufficiently large so that for any $n \in \Omega$ we have $N_i n \subset \Omega$, and $(nf)_{|N_i}$ is $\mathbb{N}_i$-analytic, taking values in U_i. If we fix this value of i, and let X_i denote the preimage under (2.3.8) of $X(N)$ (or equivalently, the preimage under (2.3.9) of X_e), then we see (taking into account Proposition 2.3.10 and the fact that X is a G-invariant subspace of $\mathrm{Ind}_{\overline{P}}^{G} U$) that X_i is an (H_i, Z_M^-)-invariant closed subspace of $A_{i,i}$, which by assumption contains $(nf)_{|N_i}$ for any $n \in \Omega$. Since ϕ is a continuous $\mathfrak{g}$-equivariant map between the LB-spaces $X(N)$ and V, and since the G-action on V is locally analytic, we deduce from [2, Prop. 1, p. I.20] that we may find a BH-subspace W of V that is H_j-invariant for some $j \geq i$, such that the H_j action on $\overline{W}$ obtained by [6, Prop. 1.1.1 (ii)] is $\mathbb{H}_j$-analytic, and such that ϕ induces a $\mathfrak{g}$-equivariant continuous map $X_i \to \overline{W}$.

By assumption the compact open subset Ωg^{-1} of $\overline{P}\backslash G$ (which is the support of gf) is contained in N (regarded as an open subset of $\overline{P}\backslash G$ via the open immersion (2.3.2)). Equivalently, the product Ωg^{-1}, now computed in G (by regarding Ω as a compact subset of G via the closed embedding $N \subset G$), is contained in $\overline{P}N = \overline{N}P$. If we let $\overline{\Omega}$ denote the projection of Ωg^{-1} onto $\overline{N}$, then $\overline{\Omega}$ is a compact subset of $\overline{N}$. Thus we may find $z \in Z_M^-$ such that $z\overline{\Omega}z^{-1} \subset \overline{N}_j$.

Fix $n \in \Omega$. Since $\Omega g^{-1} \subset \overline{\Omega}P$, we may write

$$gn^{-1} = p\overline{n} \tag{4.1.3}$$

for some $\overline{n} \in \overline{\Omega}^{-1}$ and $p \in P$. We thus compute that

$$\begin{aligned} \phi(g(f_{|z^{-1}N_i zn})) &= \phi(gn^{-1}z^{-1}(znf)_{|N_i}) = \phi(p\overline{n}z^{-1}(znf)_{|N_i}) \\ &= pz^{-1}\phi(z\overline{n}z^{-1}(znf)_{|N_i}) = pz^{-1}(z\overline{n}z^{-1})\phi((znf)_{|N_i}) \\ &= p\overline{n}n\phi(n^{-1}z^{-1}(znf)_{|N_i}) = g\phi(n^{-1}z^{-1}(znf)_{|N_i}) = g\phi(f_{|z^{-1}N_i zn}). \end{aligned} \tag{4.1.4}$$

Here the first and last equalities follow from the formula

$$f_{|z^{-1}N_i zn} = n^{-1}z^{-1}(znf)_{|N_i},$$

the second and second-last equalities follow from (4.1.3), and the third and third-last equalities follow from the P-equivariance of ϕ. The middle equality of (4.1.4) follows from the $\mathfrak{g}$-equivariance of ϕ, the fact that $(znf)_{|N_i}$ (which lies in X_i, since $(nf)_{|N_i}$ does, and since X_i is Z_M^--invariant) is $\mathbb{H}_i$-analytic, and so also $\mathbb{H}_j$-analytic, the fact that all the elements of $\phi(X_i)$ are $\mathbb{H}_j$-analytic, and the containment $z\overline{n}z^{-1} \in \overline{N}_j \subset H_j$ (since $\overline{n} \in \overline{\Omega}^{-1}$). We now write Ω as a disjoint union of left $z^{-1}N_iz$-cosets, say $\Omega = \coprod_{l=1}^m z^{-1}N_izn_l$, and compute that

$$\begin{aligned}\phi(gf) = \phi(g\sum f_{|z^{-1}N_izn_l}) &= \sum \phi(gf_{|z^{-1}N_izn_l}) \\ &= \sum g\phi(f_{|z^{-1}N_izn_l}) = g\phi(\sum f_{|z^{-1}N_izn_l}) = g\phi(f)\end{aligned}$$

(the third equality following from (4.1.4)), as required. □

Theorem 4.1.5. *If V is an LB-space equipped with a locally analytic action of G, then (4.1.1) is an isomorphism.*

Proof. Since $X(N)$ generates X as a G-representation, we see that the map (4.1.1) is an injection. We must show that is is also surjective.

Let ϕ be an element of the target of (4.1.1). Since V is endowed with a G-action, we see that ϕ induces a natural map

$$K[G] \otimes_K X(N) \to V. \tag{4.1.6}$$

We will show that (4.1.6) factors through the map (2.4.12) (with Ω taken to be N). Since (2.4.12) is a strict surjection, by Lemma 2.4.13, this will imply that (4.1.6) factors through a continuous G-equivariant map $X \to V$, which will then be an element in the source of (4.1.1) that maps under (4.1.1) to ϕ.

Thus we must prove the following claim: if $g_1, \dots, g_l$ is a finite sequence of elements of G, and $f_1, \dots, f_l$ a finite sequence of elements of $X(N)$, such that

$$\sum_{i=1}^{l} g_i f_i = 0 \text{ in } X, \tag{4.1.7}$$

then

$$\sum_{i=1}^{l} g_i \phi(f_i) \stackrel{?}{=} 0 \text{ in } V. \tag{4.1.8}$$

Since X is local, it suffices to show that each point x of $\overline{P}\backslash G$ has a compact open neighbourhood Ω_x such that for any neighbourhood $\Omega'_x \subset \Omega_x$ of x we have

$$\sum_{i=1}^{l} g_i \phi(f_{i|\Omega'_x g_i}) \stackrel{?}{=} 0. \tag{4.1.9}$$

Indeed, we can then partition $\overline{P}\backslash G$ into a finite disjoint union of such neighbourhoods, say $\overline{P}\backslash G = \coprod_{j=1}^s \Omega'_{x_j}$, and writing

$$g_i f_i = \sum_{j=1}^{s} (g_i f_i)_{|\Omega'_{x_j}} = \sum_{j=1}^{s} g_i f_{i|\Omega'_{x_j} g_i},$$

so that

$$f_i = \sum_{j=1}^{s} f_{i|\Omega'_{x_j} g_i},$$

we conclude that if (4.1.9) holds, then so does (4.1.8).

If $x = \overline{P}g$ for some $g \in G$, then replacing $(g_1, \ldots, g_l)$ by $(gg_1, \ldots, gg_l)$, we see that it suffices to treat the case when x equals the identity coset, which we identify with the identity e of N under the open immersion (2.3.2). Let Ω_e be a compact open neighbourhood of e in N, chosen so that $\Omega_e g_i \subset N$ (as open subsets of $\overline{P}\backslash G$) for all i for which $g_i \in \overline{P}N$, and so that $\Omega_e g_i$ (regarded as an open subset of $\overline{P}\backslash G$) is disjoint from the support of f_i for all other i. It follows from (4.1.7) that

$$\sum_{i=1}^{l} (g_i f_i)_{|\Omega_e} = \sum_{i=1}^{l} g_i f_{i|\Omega_e g_i} = 0 \text{ in } X(N). \tag{4.1.10}$$

Applying the map ϕ to the second equality of (4.1.10) (which is an equation involving elements of $X(N)$, since $f_{|\Omega_e g_i} = 0$ if $g_i \notin N$, by virtue of our choice of Ω_e), and taking into account Lemma 4.1.2, we deduce that indeed $\sum_{i=1}^{l} g_i \phi(f_{i|\Omega_e g_i}) = 0$, and we are done. □

(4.2) We maintain the notation and assumptions of Subsection 4.1. We furthermore assume that the given local closed G-invariant subspace X of $\mathrm{Ind}_{\overline{P}}^{G} U$ is polynomially generated, in the sense of Definition 2.7.15.

Taking Ω to be N and N_0 in turn in (2.7.6) yields continuous injections

$$X^{\mathrm{lp}}(N) \to X(N) \tag{4.2.1}$$

and

$$X^{\mathrm{lp}}(N_0) \to X(N_0) \tag{4.2.2}$$

which sit in the Cartesian diagram

$$\begin{array}{ccc} X^{\mathrm{lp}}(N_0) & \longrightarrow & X(N_0) \\ \downarrow & & \downarrow \\ X^{\mathrm{lp}}(N) & \longrightarrow & X(N), \end{array} \tag{4.2.3}$$

whose vertical arrows are given by extension by zero. The injection (4.2.1) is $(\mathfrak{g}, P)$-equivariant. If P^+ denotes the submonoid of P generated by N_0 and M^+, then one immediately checks that $X(N_0)$ is a $(\mathfrak{g}, P^+)$-invariant subspace of $X(N)$. Thus $X^{\mathrm{lp}}(N_0)$ is a $(\mathfrak{g}, P^+)$-invariant subspace of $X^{\mathrm{lp}}(N)$, and the map (4.2.2) is $(\mathfrak{g}, P^+)$-equivariant.

Note that the N-actions on $X(N)$ and on $X^{\mathrm{lp}}(N)$ induce isomorphisms

$$K[N] \otimes_{K[N_0]} X(N_0) \xrightarrow{\sim} X(N) \tag{4.2.4}$$

and

$$K[N] \otimes_{K[N_0]} X^{\mathrm{lp}}(N_0) \xrightarrow{\sim} X^{\mathrm{lp}}(N). \tag{4.2.5}$$

Fix $i, j \geq 0$. Let T_j denote the preimage of $X(N_0)$ under (2.2.9), and let $T_{i,j}$ denote the preimage of T_j under (2.2.7) (or equivalently, the preimage of $X(N_0)$ under (2.2.10)). The isomorphisms (2.2.8) and (2.2.11) give rise to a continuous bijection

$$\varinjlim_i T_{i,j} \xrightarrow{\sim} T_j \tag{4.2.6}$$

and an isomorphism

$$\varinjlim_{i,j} T_{i,j} \xrightarrow{\sim} X(N_0) \tag{4.2.7}$$

respectively. If $\overline{T}_j$ is the ultrabornologicalization of T_j, then the continuous bijection (4.2.6) induces an isomorphism

$$\varinjlim_i T_{i,j} \xrightarrow{\sim} \overline{T}_j.$$

For any $d, i, j \geq 0$, let S_j^d denote the preimage of $X^{\mathrm{lp}, \leq d}(N_0)$ under (2.2.9), and let $S_{i,j}^d$ denote the preimage of S_j^d under (2.2.7) (or equivalently, the preimage of $X^{\mathrm{lp}, \leq d}(N_0)$ under (2.2.10). The isomorphisms (2.2.8) and (2.2.11) give rise to a continuous bijection

$$\varinjlim_i S_{i,j}^d \xrightarrow{\sim} S_j^d \tag{4.2.8}$$

and an isomorphism

$$\varinjlim_{i,j} S_{i,j}^d \xrightarrow{\sim} X^{\mathrm{lp}, \leq d}(N_0) \tag{4.2.9}$$

respectively. If $\overline{S}_j^d$ is the ultrabornologicalization of S_j^d, then the continuous bijection (4.2.8) induces an isomorphism

$$\varinjlim_i S_{i,j} \xrightarrow{\sim} \overline{S}_j^d.$$

Since the Z_M-action on U induces a Z_M-action on each U_j, the $N_0 Z_M^+$-action on $\mathcal{C}^{\mathrm{la}}(N_0, U)$ induces an $N_0 Z_M^+$-action on $\mathcal{C}^{\mathrm{la}}(N_0, U_j)$, and each of the subspaces T_j and S_j^d is $N_0 Z_M^+$-invariant.

Since $X^{\mathrm{lp}, \leq d}(N_0)$ is a closed subspace of $X(N_0)$, we find that S_j^d is a closed subspace of T_j for each $d, j \geq 0$. As in Subsection 3.1, the closed embedding of S_j^d in T_j induces a $N_0 Z_M^+$-equivariant continuous injection

$$\overline{S}_j^d \to \overline{T}_j. \tag{4.2.10}$$

The isomorphisms (4.2.7) and (4.2.9) give rise to $N_0 Z_M^+$-equivariant isomorphisms

$$\varinjlim_j \overline{T}_j \xrightarrow{\sim} X(N_0), \tag{4.2.11}$$

$$\varinjlim_j \overline{S}_j^d \xrightarrow{\sim} X^{\mathrm{lp}, \leq d}(N_0), \tag{4.2.12}$$

and (as a consequence of (4.2.12))

$$\varinjlim_{j,d} \overline{S}_j^d \xrightarrow{\sim} X^{\mathrm{lp}}(N_0). \tag{4.2.13}$$

Lemma 4.2.14. *If W is a K-Banach space equipped with a continuous $N_0 Z_M^+$-action, and if we fix $j \geq 0$, then the map*

$$\mathcal{L}_{N_0 Z_M^+}(\overline{T}_j, W) \xrightarrow{\sim} \mathcal{L}_{N_0 Z_M^+}(\overline{S}_j^d, W) \tag{4.2.15}$$

given by pulling back via (4.2.10) is a natural isomorphism provided d is sufficiently large (in a manner possibly depending on j).

Proof. We will prove the lemma via an application of Theorem 3.1.11. We fix j, and take the Banach V of Subsection 3.1 to be U_j; thus the spaces B_i of Subsection 3.1 correspond to the spaces $B_{i,j}$ of Subsection 2.2. As in Subsection 3.1, fix norms $||-||_{U_j}$ and $||-||_W$ defining the topologies of U_j and W respectively.

We take the spaces S and T of Subsection 3.1 to be S_j^d and T_j. It is clear from the definition of S_j^d and T_j that the conditions of Lemma 3.1.13 hold for both of them, and thus that the closed embeddings (3.1.6) and (3.1.7) are isomorphisms.

We write $\mathbf{X} := (X_e)^{\mathrm{pol}}$, and use the notation introduced in Subsection 2.7, following Definition 2.7.11. Note that our assumption that X is polynomially generated implies in particular that $X_e = \hat{\mathbf{X}}$. In terms of that notation, we may write

$$T_{i,j} = \bigoplus_{n \in N_i \backslash N_0} n \hat{\mathbf{X}}_{i,j}$$

and

$$S_{i,j}^d = \bigoplus_{n \in N_i \backslash N_0} n \mathbf{X}_{i,j}^{\leq d} = \bigoplus_{n \in N_i \backslash N_0} \bigoplus_{d' \leq d} n \mathbf{X}_{i,j}^{d'}.$$

We define

$$S_{i,j}^{d,\perp} = \bigoplus_{n \in N_i \backslash N_0} n \mathbf{X}_{i,j}^{> d} = \bigoplus_{n \in N_i \backslash N_0} \hat{\bigoplus_{d' > d}} n \mathbf{X}_{i,j}^{d'}$$

(where the completed direct sum is taken with respect to the norms $||-||_{i,j}^{d'}$ on the Banach spaces $\mathbf{X}_{i,j}^{d'}$). Let $\pi_{i,j}^{d,\perp}$ denote the projection onto the second factor of the isomorphism

$$T_{i,j} \xrightarrow{\sim} S_{i,j}^d \bigoplus S_{i,j}^{\perp,d}.$$

If $f \in T_{i,j}$, write $f = \sum_{n \in N_i \backslash N_0} n f_n$, where $f_n = \sum_{d'=0}^{\infty} f_{n,d'} \in \hat{\mathbf{X}}_{i,j}$ with $f_{n,d'} \in \mathbf{X}_{i,j}^d$. Then

$$\begin{aligned} ||\, \pi_{i',j}^{d,\perp}(f)\, ||_{i',j} &= \max_{n \in N_i \backslash N_0} \sup_{d' > d} ||\, f_{n,d'}\, ||_{i',j}^{d'} \\ &= \max_{n \in N_i \backslash N_0} \sup_{d' > d} |p|^{d'(i'-i)} ||\, f_{n,d'}\, ||_{i,j}^{d'} \\ &\leq \max_{n \in N_i \backslash N_0} \sup_{d' \geq 0} |p|^{(d+1)(i'-i)} ||\, f_{n,d'}\, ||_{i,j}^{d'} \\ &= |p|^{(d+1)(i'-i)} ||\, f\, ||_{i,j}. \end{aligned}$$

(The second equality follows from the formula of Lemma 2.5.16.) Since $|p|^{d+1} \to 0$ as $d \to \infty$, we see that if we choose d large enough, then condition (3.1.12) will be satisfied if we take $\epsilon = |p|^{d+1}$. It now follows from Theorem 3.1.11 that pulling back along the map $\overline{S}_j^d \to \overline{T}_j$ induces an isomorphism

$$\mathcal{L}_{N_0 Z_M^+}(\overline{T}_j, W) \xrightarrow{\sim} \mathcal{L}_{N_0 Z_M^+}(\overline{S}_j^d, W),$$

as claimed. □

Theorem 4.2.16. *If W is any K-Banach space equipped with a continuous P^+-action, then pulling back via (4.2.2) induces a natural isomorphism*

$$\mathcal{L}_{P^+}(X(N_0), W) \xrightarrow{\sim} \mathcal{L}_{P^+}(X^{\mathrm{lp}}(N_0), W).$$

Proof. Passing to the projective limit over d and j of the maps (4.2.15), and taking into account the statement of Lemma 4.2.14 and the isomorphisms (4.2.11) and (4.2.13), we find that the map

$$\mathcal{L}_{N_0 Z_M^+}(X(N_0), W) \to \mathcal{L}_{N_0 Z_M^+}(X^{\mathrm{lp}}(N_0), W) \tag{4.2.17}$$

given by pulling back via (4.2.2) is also an isomorphism. Since (4.2.2) has dense image, we find that a map in the source of (4.2.17) is M^+-equivariant if and only if this is true of its pull-back via (4.2.2). Thus (4.2.17) restricts to the isomorphism in the statement of the theorem. □

Corollary 4.2.18. *If X is polynomially generated by bounded degrees, then for any K-Banach space W equipped with a continuous G-action, pulling back via (4.2.1) induces a natural isomorphism*

$$\mathcal{L}_{(\mathfrak{g},P)}(X(N), W_{\mathrm{la}}) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(X^{\mathrm{lp}}(N), W_{\mathrm{la}}).$$

Proof. Since P is generated by P^+ as a group, and since we have the isomorphisms (4.2.4) and (4.2.5), it suffices to prove that the map

$$\mathcal{L}_{(\mathfrak{g},P^+)}(X(N_0), W_{\mathrm{la}}) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P^+)}(X^{\mathrm{lp}}(N_0), W_{\mathrm{la}}) \tag{4.2.19}$$

given by pulling back via (4.2.2) is an isomorphism. It follows from Theorem 4.2.16 that (4.2.19) is an injection, and that any map in the target may be extended to a continuous P^+-equivariant map $X(N_0) \to W$. Thus to prove the corollary, it suffices to show that any continuous map $X(N_0) \to W$ with the property that its composite with (4.2.2) factors through a continuous $\mathfrak{g}$-equivariant map $X^{\mathrm{lp}}(N_0) \to W_{\mathrm{la}}$ itself necessarily factors through a continuous $\mathfrak{g}$-equivariant map $X(N_0) \to W_{\mathrm{la}}$.

The isomorphism (2.2.11) induces an isomorphism

$$\varinjlim_i T_{i,i} \xrightarrow{\sim} X(N_0).$$

If Y denotes the image of (4.2.2), then the preimage Y_i of Y in $T_{i,i}$ is dense in $T_{i,i}$. Furthermore, this preimage contains the closed Banach subspace $S^d_{i,i}$ of $T_{i,i}$ for all $d \geq 0$, and by assumption, if d is sufficiently large this subspace generates Y_i as a $\mathfrak{g}$-representation. The corollary now follows from Proposition 1.2.23, once we note that the $\mathfrak{g}$-action on $T_{i,i}$ is locally integrable (by Proposition 1.2.19 and Lemma 2.3.14, since $T_{i,i}$ is a closed $\mathfrak{g}$-invariant subspace of $B_{i,i}$). □

(4.3) We maintain the notation introduced in Subsection 4.1. The closed embedding of $X(N)$ into X, when composed with (4.2.1), yields a map

$$X^{\mathrm{lp}}(N) \to X. \tag{4.3.1}$$

Theorem 4.3.2. *Suppose that X is polynomially generated by bounded degrees. If W is a K-Banach space equipped with a continuous G-representation, then (4.3.1) induces an isomorphism*

$$\mathcal{L}_G(X, W_{\mathrm{la}}) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(X^{\mathrm{lp}}(N), W_{\mathrm{la}}).$$

Proof. Theorem 4.1.5 and Corollary 4.2.18 show that we have isomorphisms

$$\mathcal{L}_G(X, W_{\mathrm{la}}) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(X(N), W_{\mathrm{la}}) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(X^{\mathrm{lp}}(N), W_{\mathrm{la}}),$$

proving the theorem. □

Corollary 4.3.3. *Suppose that X is polynomially generated by bounded degrees. If V is a very strongly admissible locally analytic G-representation (in the sense of Definition 0.12), then (4.3.1) induces an isomorphism*

$$\mathcal{L}_G(X, V) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(X^{\mathrm{lp}}(N), V).$$

Proof. By assumption we may find a continuous G-equivariant injection $V \to W$, where W is a Banach space equipped with a continuous admissible G-action. Since the G-action on V is locally analytic, this continuous injection factors through a continuous map $V \to W_{\mathrm{la}}$. Since V and W_{la} are both admissible locally analytic G-representations (the latter by [6, Prop. 6.2.4] or [16, Thm. 7.1 (ii)]), this map is furthermore a closed embedding [16, Prop. 6.4 (ii)].

Theorem 4.3.2 yields an isomorphism

$$\mathcal{L}_G(X, W_{\mathrm{la}}) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(X^{\mathrm{lp}}(N), W_{\mathrm{la}}). \tag{4.3.4}$$

Since X is topologically generated as a G-representation by the image of (4.3.1), and since V is closed in W_{la}, we see that a map in the source of (4.3.4) takes values in V if and only the same is true of the corresponding map in the target. □

We are now ready to prove Theorem 0.13.

Proof of Theorem 0.13. Let U be an allowable object of $\mathrm{Rep}^{z}_{\mathrm{la.c}}(M)$. Proposition 2.8.10 shows that $I_{\overline{P}}^{G}(U)$ is a local closed G-subrepresentation that is polynomially graded and generated by degree zero (and so in particular by bounded degrees). Corollary 4.3.3 thus yields an isomorphism

$$\mathcal{L}_G(I_{\overline{P}}^{G}(U), V) \xrightarrow{\sim} \mathcal{L}_{(\mathfrak{g},P)}(I_{\overline{P}}^{G}(U)^{\mathrm{lp}}(N), V),$$

while by Lemma 0.18, passing to Jacquet modules induces an isomorphism

$$\mathcal{L}_{(\mathfrak{g},P)}(I_{\overline{P}}^{G}(U)^{\mathrm{lp}}(N), V) \xrightarrow{\sim} \mathcal{L}_M(U(\delta), J_P(V))^{\mathrm{bal}}.$$

This proves Theorem 0.13. □

5. Examples, complements, and applications

(5.1) We maintain the notation introduced in Subsection 2.2. We furthermore suppose that G is quasi-split, and we take P to be a Borel subgroup of G. Thus $\overline{P}$ is an opposite Borel to P, and the Levi factor $M = P \bigcap \overline{P}$ is a maximal torus in G.

Proposition 5.1.1. *If U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then the inclusion $I_{\overline{P}}^G(U) \subset \mathrm{Ind}_{\overline{P}}^G U$ induces an equality of $\mathfrak{n}$-invariants $I_{\overline{P}}^G(U)^{\mathfrak{n}} = (\mathrm{Ind}_{\overline{P}}^G U)^{\mathfrak{n}}$.*

Proof. Fix an element $f \in (\mathrm{Ind}_{\overline{P}}^G U)^{\mathfrak{n}}$. We may find a partition $\overline{P}\backslash G = \coprod_i X_i$ of $\overline{P}\backslash G$ into a finite disjoint union of charts, where each chart X_i is of the form $Y_i g_i$, for some chart Y_i of N and some $g_i \in G$, and where f is rigid analytic on the rigid analytic polydisk $\mathbb{X}_i$ underlying X_i. Since $f = \sum_i f_{|X_i}$, it suffices to prove that

$$f_{|X_i} \overset{?}{\in} I_{\overline{P}}^G(U)^{\mathfrak{n}} \tag{5.1.2}$$

for each value of i.

Let W be the Weyl group of M in G. We recall the Bruhat decomposition

$$\overline{P}\backslash G = \coprod_{w \in W} wN. \tag{5.1.3}$$

(Here, for each $w \in W$, we use the same letter to denote the image of w in $\overline{P}\backslash G$.) Each chart X_i must have non-empty intersection with at least one of the strata of the Bruhat decomposition, and thus we may choose $n_i \in N$ and $w_i \in W$ such that $w_i n_i \in X_i$. We may also write $w_i n_i = n_i' g_i$ for some $n_i' \in Y_i$. Translating the putative inclusion (5.1.2) by $w_i n_i$, and noting that $X_i(w_i n_i)^{-1} = Y_i(n_i')^{-1}$, and also that $\mathrm{Ad}_{w_i n_i}(\mathfrak{n}) = \mathrm{Ad}_{w_i}(\mathfrak{n})$, we find that it suffices to prove the containment

$$w_i n_i f_{|X_i} \in I_{\overline{P}}^G(U)(Y_i(n_i')^{-1})^{\mathrm{Ad}_{w_i}(\mathfrak{n})}$$

for each value of i. Note that $Y_i(n_i')^{-1}$ is a chart of N containing e, and that $w_i n_i f_{|X_i}$ is rigid analytic on the polydisk underlying $Y_i(n_i')^{-1}$. Thus, to simply notation, we now let Y denote an arbitrary chart of N containing e, and let f denote an element of $(\mathrm{Ind}_{\overline{P}}^G U)(Y)^{\mathrm{Ad}_w(\mathfrak{n})}$ which is rigid analytic on the polydisk $\mathbb{Y}$ underlying Y. We will prove that $f \in I_{\overline{P}}^G(U)(Y)^{\mathrm{Ad}_w(\mathfrak{n})}$ (which by the preceding discussion suffices to establish the proposition).

Choose $j \geq 0$ such that f takes values in the BH-subspace U_j of U. We may regard f as an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{Y}, U_j)$, and then via the isomorphism of Lemma 2.5.13, write $f = \sum_{d=0}^{\infty} f_d$, where $f_d \in \mathcal{C}^{\mathrm{pol},d}(N, U_j)$. The action of $\mathrm{Ad}_w(\mathfrak{n})$ on $\mathcal{C}^{\mathrm{pol}}(N, U_j)$ is a graded action, and thus if f is annihilated by $\mathrm{Ad}_w(\mathfrak{n})$, so are each of its components f_d. Since $I_{\overline{P}}^G(U)(Y)$ is closed, it suffices to show that $f_d \in I_{\overline{P}}^G(U)(Y)^{\mathrm{Ad}_w(\mathfrak{n})}$ for each d. In other words, we may assume that f, when thought of as an element of $\mathcal{C}^{\mathrm{an}}(\mathbb{Y}, U_j)$, in fact lies in $\mathcal{C}^{\mathrm{pol}}(N, U_j)$. The isomorphism of (2.5.7), together with Lemma 1.5.1 (which we may apply after making an extension of scalars so as to split G), then shows that f lies in the $\mathfrak{g}$-subrepresentation of $\mathcal{C}^{\mathrm{pol}}(N, U)$ generated by $\mathcal{C}^{\mathrm{pol}}(N, U)^{\mathfrak{n}}$, and thus in $I_{\overline{P}}^G(U)$, as required. $\square$

Corollary 5.1.4. *If U is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(M)$, then the inclusion $I^G_{\overline{P}}(U) \subset \mathrm{Ind}^G_{\overline{P}} U$ includes an isomorphism $J_P(I^G_{\overline{P}}(U)) \xrightarrow{\sim} J_P(\mathrm{Ind}^G_{\overline{P}} U)$.*

Proof. This follows from the very definition of the functor J_P [7, Def. 3.4.5], together with Proposition 5.1.1. □

Proposition 5.1.5. *If U is a finite dimensional locally analytic representation of M, then $J_P(\mathrm{Ind}^G_{\overline{P}} U)$ is a finite dimensional M-representation.*

Proof. Extending scalars, we may assume that U is filtered by one dimensional representations of M. Since the formation of $\mathrm{Ind}^G_{\overline{P}} U$ is exact, while J_P is left exact, we may thus suppose that U is equal to some locally analytic character ψ of M. Extending scalars again, if necessary, we may and do assume that G splits over K.

Let $\hat{M}$ denote the rigid analytic space over K that parameterizes the locally analytic characters of M (as constructed in [6, §6.4]). Proposition 2.1.2 shows that the locally analytic induction $\mathrm{Ind}^G_{\overline{P}} \psi$ is a strongly admissible locally analytic G-representation, and so by [7, Thm. 0.5], the Jacquet module $J_P(\mathrm{Ind}^G_{\overline{P}} \psi)$ is an essentially admissible M-representation. Hence its dual space is identified with the space of sections of a rigid analytic coherent sheaf $\mathcal{F}$ on $\hat{M}$. The claim of the proposition is equivalent to showing that $\mathcal{F}$ is supported on a finite subset of $\hat{M}$. If $\chi \in \hat{M}(\overline{K})$ (where $\overline{K}$ denotes an algebraic closure of K), with field of definition E_χ (a finite extension of K), then the fibre of $\mathcal{F}$ at χ is dual to the χ-eigenspace $J^\chi_P(E_\chi \otimes_K \mathrm{Ind}^G_{\overline{P}} \psi)$ of $J_P(E_\chi \otimes_K \mathrm{Ind}^G_{\overline{P}} \psi)$ $(= E_\chi \otimes_K J_P(\mathrm{Ind}^G_{\overline{P}} \psi))$. Thus to prove the proposition, it suffices to prove that $J^\chi_P(E_\chi \otimes_K \mathrm{Ind}^G_{\overline{P}} \psi)$ is non-zero for only a finite number of $\chi \in \hat{M}(\overline{K})$.

Thus we fix an element $\chi \in \hat{M}(\overline{K})$, and suppose that $J^\chi_P(E_\chi \otimes_K \mathrm{Ind}^G_{\overline{P}} \psi) \neq 0$. To simplify our notation, we extend scalars if necessary so as to assume that $E_\chi = K$. We will show that χ belongs to a certain finite set of characters that depends only on ψ.

As in the proof of Proposition 5.1.1, we let W denote the Weyl group of M in G. Choose a labelling $\{w_i \mid 1 \leq i \leq |W|\}$ of the elements of W, with the property that if $1 \leq i \leq j \leq |W|$ then the length of w_i is less than or equal to that of w_j. For any $1 \leq i \leq |W|$, we write $B_i := \bigcup_{j \leq i} w_j N \subset \overline{P}\backslash G$. From the theory of the Bruhat decomposition (5.1.3), it is known that each B_i is an open subset of $\overline{P}\backslash G$, and that $B_{|W|} = \overline{P}\backslash G$. If we write $V_i = (\mathrm{Ind}^G_{\overline{P}} \psi)(B_i)$, then

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{|W|} = \mathrm{Ind}^G_{\overline{P}} \psi$$

is an increasing and exhaustive filtration of $\mathrm{Ind}^G_{\overline{P}} \psi$ by closed P-invariant subspaces, inducing a corresponding filtration

$$0 = J^\chi_P(V_0) \subset J^\chi_P(V_1) \subset J^\chi_P(V_2) \subset \cdots \subset J^\chi_P(V_{|W|}) = J^\chi_P(\mathrm{Ind}^G_{\overline{P}} \psi)$$

of $J^\chi_P(\mathrm{Ind}^G_{\overline{P}} \psi)$. By [7, Prop. 3.4.9], we may (and do) identify $J^\chi_P(V_i)$ (for each $i \in \{1, \ldots, |W|\}$) with the subspace of $V_i^{N_0}$ on which the Hecke operators $\pi_{N_0,m}$ (for $m \in M^+$) act via the eigenvalues $\chi(m)$.

Claim 5.1.6: For each $i \in \{1, \ldots, |W|\}$, the map $f \mapsto \mathrm{st}_{w_i} f \in (\mathrm{Ind}^G_{\overline{P}} \psi)_{w_i}$ induces an injection

$$J^\chi_P(V_i)/J^\chi_P(V_{i-1}) \to ((\mathrm{Ind}^G_{\overline{P}} \psi)_{w_i})^{\mathfrak{n}}. \tag{5.1.7}$$

To see this, let $f \in J_P(V_i)$, and let $\Omega \subset B_i$ denote the support of f. Since $\pi_{N_0,m} f = \chi(m) f$ for all $m \in M^+$, one easily deduces (taking into account the fact that $w_i N = B_i \setminus B_{i-1}$ is a closed subset of B_i, so that $\Omega \bigcap w_i N$ is compact) either that $\Omega \bigcap w_i N = \emptyset$ (in which case f is supported in B_{i-1}, and so lies in $J_P(V_{i-1})$) or else that $\Omega \bigcap w_i N = w_i N_0$. In any case, the germ of f at any point of $w_i N_0$ is evidently determined by the germ of f at w_i (since f is N_0-invariant), establishing the claim.

Before drawing conclusions from the claim, we require some computations. We first compute the action of M on the target of (5.1.7). Translation by w_i induces the first isomorphism in the sequence of isomorphisms

$$(\mathrm{Ind}_P^G \psi)_{w_i} \xrightarrow{\sim} (\mathrm{Ind}_P^G \psi)_e \xrightarrow{\sim} \mathcal{C}^\omega(N, \psi)_e,$$

the second being provided by Corollary 2.3.4. Let $d\psi : \mathfrak{m} \to K$ denote the derivative of the character ψ. Note that M acts on $(\mathcal{C}^\omega(N,\psi)_e)^{\mathfrak{n}}$ via ψ, and thus that $\mathfrak{m}$ acts on this space via $d\psi$. Lemma 1.5.1 shows that $(\mathcal{C}^\omega(N,\psi)_e)^{w_i(\mathfrak{n})}$ vanishes unless $d\psi(H_{\mathfrak{r}})$ is a non-negative integer for each $\mathfrak{r} \in \Delta_{w_i}$. If this condition holds, write $k_{\mathfrak{r}} := d\psi(H_{\mathfrak{r}})$. Lemma 1.5.1 then implies that

$$(\mathcal{C}^\omega(N,\psi)_e)^{w_i(\mathfrak{n})} \subset X_{-\mathfrak{r}_1}^{k_{\mathfrak{r}_1}} \cdots X_{-\mathfrak{r}_l}^{k_{\mathfrak{r}_l}} (\mathcal{C}^\omega(N,\psi)_e)^{\mathfrak{n}}$$

for an appropriately chosen ordering $\mathfrak{r}_1, \ldots, \mathfrak{r}_l$ of the elements of Δ_{w_i}. Thus M acts on $(\mathcal{C}^\omega(N,\psi)_e)^{w_i(\mathfrak{n})}$ via the character $\mathfrak{r}_1^{-k_1} \cdots \mathfrak{r}_l^{-k_l} \psi$. (In this last expression we are regarding the roots as characters of M rather than $\mathfrak{m}$, and hence use multiplicative rather than additive notation.) Applying the isomorphism of Corollary 2.3.4 and then translating by w_i^{-1}, we find that M acts on $((\mathrm{Ind}_P^G \psi)_{w_i})^{\mathfrak{n}}$ via the character

$$\psi_i := \mathfrak{r}_1^{k_1} \cdots \mathfrak{r}_l^{k_l} \psi^{w_i}$$

(where $\psi^{w_i}(m) := \psi(w_i m w_i^{-1})$).

We next compute $\mathrm{st}_{w_i}(\pi_{N_0,m} f)$ in terms of $\mathrm{st}_{w_i}(f)$, for $f \in J_P^\chi(V_i)$ and $m \in M^+$. Recall that

$$\pi_{N_0,m} f = m \pi_{m^{-1} N_0 m} f = m\delta(m) \sum_{n' \in m^{-1} N_0 m / N_0} n' f,$$

where $\delta(m) = [N_0 : m N_0 m^{-1}]^{-1}$. We also define a variant of the modulus function for $m \in M$, namely

$$\delta_i(m) := \frac{[N_0 \bigcap w_i^{-1} \overline{N} w_i : m N_0 m^{-1} \bigcap w_i^{-1} \overline{N} w_i]}{[N_0 : m N_0 m^{-1}]}.$$

We now compute:

$$\begin{aligned}
\mathrm{st}_{w_i}(\pi_{N_0,m} f) &= \mathrm{st}_{w_i}(m\delta(m) \sum_{n' \in m^{-1} N_0 m / N_0} n' f) \\
&= \psi_i(m)\delta(m) \sum_{n' \in m^{-1} N_0 m / N_0} \mathrm{st}_{w_i}(n' f) \\
&= \psi_i(m)\delta(m) \sum_{n' \in m^{-1} N_0 m / N_0} n' \mathrm{st}_{w_i(n')^{-1}}(f) \\
&= \psi_i(m)\delta(m) \sum_{n' \in (m^{-1} N_0 m \bigcap w_i^{-1} \overline{N} w_i)/(N_0 \bigcap w_i^{-1} \overline{N} w_i)} \mathrm{st}_{w_i}(f) \\
&= \psi_i(m)\delta_i(m) \mathrm{st}_{w_i}(f)
\end{aligned}$$

(where the second equality follows from the result of the preceding paragraph, while the second to last equality follows from the following observations: the stabilizer in G of (the $\overline{P}$-coset of) w_i is $w_i^{-1}\overline{P}w_i$, the intersection of the support of f and w_iN is contained in w_iN_0, and $\mathrm{st}_{w_i}(f)$ is fixed by $\mathfrak{n}$, and hence by the elements of $N\bigcap w_i^{-1}\overline{N}w_i$, since the action of $w_i^{-1}\overline{N}w_i$ on $(\mathrm{Ind}_{\overline{P}}^{G}\psi)_{w_i}$ is $w_i^{-1}\overline{\mathbb{N}}w_i$-analytic, by Lemmas 2.3.5 (iii) and 2.3.16).

Combining this computation with Claim 5.1.6, we find that $J_P^{\chi}(V_i)/J_P^{\chi}(V_{i-1})$ vanishes unless $\chi = \psi_i\delta_i$. Consequently $J_P^{\chi}(\mathrm{Ind}_{\overline{P}}^{G}\psi)$ vanishes unless $\chi = \psi_i\delta_i$ for some $i \in \{1, \ldots, |W|\}$. This proves the proposition. $\square$

Remark 5.1.8. Suppose that ψ is a locally dominant algebraic character of M, i.e. that $\psi = \theta\lambda$ where θ is smooth (i.e. locally constant) and λ is an algebraic character of M that is dominant (with respect to P). Let W be the irreducible algebraic representation of G of highest (with respect to P) weight λ. The reader can easily check that there is an isomorphism

$$I_{\overline{P}}^{G}(\psi) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^{G}\theta)_{\mathrm{sm}} \otimes_K W$$

(where the first factor denotes the smooth parabolic induction of θ). Corollary 5.1.4 and [7, Prop. 4.3.6] taken together then show that

$$J_P(\mathrm{Ind}_{\overline{P}}\psi) \xrightarrow{\sim} J_P(I_{\overline{P}}^{G}(\psi)) \xrightarrow{\sim} ((\mathrm{Ind}_{\overline{P}}^{G}\theta)_{\mathrm{sm}})_N \otimes_K \lambda.$$

Combining this with [5, Thm. 6.3.5] (applied with $P_\Theta = P_\Omega = P$) yields a precise description of the semi-simplification of the finite dimensional M-representation $J_P(\mathrm{Ind}_{\overline{P}}\psi)$.

One can obtain an analogous description of (the semi-simplification of) the finite dimensional M-representation $J_P(\mathrm{Ind}_{\overline{P}}\psi)$ in general. To see this, note first that Claim 5.1.6 can be strengthened slightly: it evidently holds true if we replace $J_P^{\chi}(V_i)/J_P^{\chi}(V_{i-1})$ by $J_P^{\mathrm{gen}-\chi}(V_i)/J_P^{\mathrm{gen}-\chi}(V_{i-1})$, where the notation "gen $-\,\chi$" denotes the generalized χ-eigenspace. The argument in the proof of the proposition then shows that $J_P^{\mathrm{gen}-\chi}(V_i)/J_P^{\mathrm{gen}-\chi}(V_{i-1})$ vanishes except under the following conditions: $k_{\mathfrak{r}} := d\psi(H_{\mathfrak{r}}) \in \mathbb{Z}_{\geq 0}$ for each root $\mathfrak{r}$ that appears in $\mathrm{Ad}_{w_i}(\overline{\mathfrak{n}})\bigcap\mathfrak{n}$, and $\chi = \mathfrak{r}_1^{k_1}\cdots\mathfrak{r}_l^{k_l}\psi^{w_i}\delta_i$, in which case its dimension is at most one. In fact, in this latter case, one can show that $J_P^{\mathrm{gen}-\chi}(V_i)/J_P^{\mathrm{gen}-\chi}(V_{i-1})$ has dimension exactly one. Thus (for any given choice of character ψ), we can compute the dimension of $J_P^{\mathrm{gen}-\chi}(\mathrm{Ind}_{\overline{P}}^{G}\psi)$ precisely for each of the finitely many characters χ which may appear in $J_P(\mathrm{Ind}_{\overline{P}}^{G}\psi)$, and hence determine precisely the semi-simplification of $J_P(\mathrm{Ind}_{\overline{P}}^{G}\psi)$. (We leave it to the reader to check that this general description of $J_P(\mathrm{Ind}_{\overline{P}}^{G}\psi)$ is consistent with the description already given in the case when ψ is locally dominant algebraic.)

Example 5.1.9. Suppose that $L = \mathbb{Q}_p$ and that $G = \mathrm{SL}_2(\mathbb{Q}_p)$. Take P (respectively $\overline{P}$) to be the Borel subgroup of upper triangular (respectively lower triangular matrices), so that $M = P\bigcap\overline{P}$ is the subgroup of diagonal matrices. The coroot $\alpha : a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$ yields an isomorphism $\mathbb{Q}_p^{\times} \xrightarrow{\sim} M$, and hence yields an isomorphism

$$K \xrightarrow{\sim} \mathfrak{m} \tag{5.1.10}$$

via $k \mapsto k\partial_\alpha$ (where as in Subsection 2.6, we denote by ∂_α the pushforward under α of the element dt/t in the Lie algebra of $\mathbb{Q}_p^\times$). We let $\mathfrak{w} : \mathfrak{m} \to K$ denote the inverse of the isomorphism (5.1.10); thus $\mathfrak{w}$ generates the weight lattice of M. If $\mathfrak{r}$ denotes the (unique) root of M acting on $\mathfrak{n}$, then $\mathfrak{r} = \mathfrak{w}^2$, while α coincides with the coroot $\mathfrak{r}^\vee$ associated to $\mathfrak{r}$ (and so also ∂_α and $H_\mathfrak{r}$ coincide). The Weyl group $W = \{1, w\}$ has order two (one can take $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$), and the Bruhat decomposition decomposes $\overline{P}\backslash G$ (which is isomorphic to the $\mathbb{Q}_p$-points of the projective line) into the union of N (the affine line) and a point (the $\overline{P}$-coset of w, or the point at infinity).

Let $\psi : M \to K^\times$ be a locally analytic character, and write $d\psi = k\mathfrak{w}$. Using the notation of the proof of Proposition 5.1.5, we see that $V_1 = (\mathrm{Ind}_{\overline{P}}^G \psi)(N) \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{la}}(N, \psi)$ (the isomorphism being given by Lemma 2.3.6), that $V_2 = \mathrm{Ind}_{\overline{P}}^G \psi$, and that taking stalks at w induces an isomorphism

$$V_2/V_1 \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^G \psi)_e.$$

The proof of Proposition 5.1.5 shows that $J_P(V_2)/J_P(V_1) = 0$ unless $k \in \mathbb{Z}_{\geq 0}$, that is, unless ψ is locally dominant algebraic.

Thus if ψ is not locally dominant algebraic, we find that

$$J_P(\mathrm{Ind}_{\overline{P}}^G \psi) = J_P(\mathcal{C}_c^{\mathrm{la}}(N, \psi)) = \psi\delta.$$

Note also that in this case $I_{\overline{P}}^G(\psi) = \mathrm{Ind}_{\overline{P}}^G \psi$, as follows from Proposition 2.8.10 and the fact that $\mathcal{C}^{\mathrm{pol}}(N, \psi)$ is an irreducible $\mathfrak{g}$-representation when $d\psi$ is not dominant integral. (Alternatively, one might note $\mathrm{Ind}_{\overline{P}}^G \psi$ itself is topologically irreducible [13, Thm. 6.1].)

Suppose on the other hand that ψ is locally dominant algebraic, and write $\psi = \theta\mathfrak{w}^k$, with θ smooth. As was noted in the preceding remark,

$$I_{\overline{P}}^G(\psi) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^G \theta)_{\mathrm{sm}} \otimes_K (\mathrm{Sym}^k K^2)$$

(recall that $\mathrm{Sym}^k K^2$ has highest weight equal to $\mathfrak{w}^k$) and so

$$J_P(\mathrm{Ind}_{\overline{P}}^G \psi) = J_P(I_{\overline{P}}^G(\psi)) = J_P((\mathrm{Ind}_{\overline{P}}^G \theta)_{\mathrm{sm}}) \otimes \mathfrak{w}^k.$$

Applying [5, Thm. 6.3.5], we find that $J_P((\mathrm{Ind}_{\overline{P}}^G \theta)_{\mathrm{sm}})$ is a two dimensional M-representation, whose Jordan-Hölder constituents are $\theta\delta$ and θ^w. Thus $J_P(\mathrm{Ind}_{\overline{P}}^G \psi)$ is a two dimensional M-representation, with Jordan-Hölder factors $\psi\delta$ and $\psi^w\mathfrak{r}^k$.

Finally, we note that there is an is a short exact sequence

$$0 \to I_{\overline{P}}^G(\psi) \to \mathrm{Ind}_{\overline{P}}^G \psi \to \mathrm{Ind}_{\overline{P}}^G \psi\mathfrak{r}^{-(k+1)} \to 0.$$

(This is the short exact sequence $(*)$ on p. 123 of [14].) Taking Jacquet modules yields the exact sequence

$$0 \longrightarrow J_P(I_{\overline{P}}^G(\psi)) \xrightarrow{\sim} J_P(\mathrm{Ind}_{\overline{P}}^G \psi) \longrightarrow J_P(\mathrm{Ind}_{\overline{P}}^G \psi\mathfrak{r}^{-(k+1)}).$$

Since the final object in this sequence is non-zero, we see that the sequence is not exact on the right, and thus that J_P is not an exact functor.

(5.2) As in the preceding section we suppose that G is quasi-split, and we take P to be a Borel subgroup of G. In general, if $V \to W$ is a G-equivariant continuous surjection of admissible locally analytic G-representations, then the induced map $J_P(V) \to J_P(W)$ need not be surjective. This is the case even when V is of the form $\mathrm{Ind}_{\overline{P}}^G \psi$ for some locally analytic character ψ of M, as Example 5.1.9 shows. The main result of this section is Theorem 5.2.18, which shows by contrast that when $V = I_{\overline{P}}^G(\psi)$, any surjection $V \to W$ does induce a surjection on the corresponding Jacquet modules.

Definition 5.2.1. We say that a $\mathfrak{g}$-equivariant surjection of $\mathfrak{g}$-modules $V \to W$ is locally split if for any finitely generated $\mathfrak{g}$-subrepresentation W_1 of W, with preimage V_1 in V, the induced surjection $V_1 \to W_1$ is $\mathfrak{g}$-equivariantly split.

Lemma 5.2.2. *If $V \to W$ is a continuous G-equivariant surjection of admissible locally analytic G-representations that is locally split as a surjection of $\mathfrak{g}$-modules, then the induced map $J_P(V) \to J_P(W)$ is surjective.*

Proof. As in the proof of Proposition 5.1.5, let $\hat{M}$ denote the rigid analytic space over K that parameterizes the locally analytic characters of M. For any $\chi \in \hat{M}(K)$, let $I_{d\chi}$ denote the kernel of the composite homomorphism $\mathrm{U}(\mathfrak{p}) \to \mathrm{U}(\mathfrak{m}) \to K$, in which the first arrow is induced by the projection $\mathfrak{p} \to \mathfrak{m}$, and the second arrow is induced by the character $d\chi : \mathfrak{m} \to K$ (the derivative of χ). Similarly, for each $i \geq 0$ we let $I_{\chi,i}$ denote the kernel of the composite $K[P_i] \to K[M_i] \to K$, in which $K[P_i]$ and $K[M_i]$ denote the group rings over K of P_i and M_i respectively, the first arrow is induced by the projection $P_i \to M_i$, and the second arrow is induced by $\chi|M_i$. For any $i, r \geq 0$, we let $V^{(I_{d\chi})^r,\mathfrak{n}}$ denote the P-invariant closed subspace of V annihilated by both the ideal $(I_{d\chi})^r \subset \mathrm{U}(\mathfrak{p})$ and the lie algebra $\mathfrak{n}$, and let $V^{(I_{\chi,i})^r,N_i}$ denote the P_i-invariant closed subspace of V annihilated by both the ideal $(I_{\chi,i})^r \subset K[P_i]$ and the group N_i. We employ similar notation with W in place of V.

The assumption that the surjection $V \to W$ is locally split as a map of $\mathfrak{g}$-modules implies that the induced map

$$V^{(I_{d\chi})^r,\mathfrak{n}} \to W^{(I_{d\chi})^r,\mathfrak{n}} \tag{5.2.3}$$

is also surjective. Since the P-actions on V and W are locally analytic, we have an equality $V^{(I_{d\chi})^r,\mathfrak{n}} = \bigcup_{i \geq 0} V^{(I_{\chi,i})^r,N_i}$, and similarly with W in place of V. Averaging over right P_i-cosets in P_0 induces projections $V^{(I_{\chi,i})^r,N_i} \to V^{(I_{\chi,0})^r,N_0}$ and $W^{(I_{\chi,i})^r,N_i} \to W^{(I_{\chi,0})^r,N_0}$ for any $i, r \geq 0$. Applying these projection to the source and target of (5.2.3), we find that the induced map

$$V^{(I_{\chi,0})^r,N_0} \to W^{(I_{\chi,0})^r,N_0} \tag{5.2.4}$$

is surjective.

Since $M = Z_M$, we write $M^+ := Z_M^+$. Any element $m \in M^+$ induces a Hecke operator $\pi_{N_0,m}$ on V^{N_0} (respectively W^{N_0}) [7, Def. 3.4.2]. These operators induce an action of the monoid ring $K[M^+]$ on V^{N_0} and W^{N_0} [7, Lem. 3.4.4]. The space

$V^{(I_\chi,0)^r,N_0}$ (respectively $W^{(I_\chi,0)^r,N_0}$) is a $K[M^+]$-submodule of V^{N_0} (respectively W^{N_0}), and the map (5.2.4) is M^+-equivariant. We let I_χ denote the kernel in $K[M^+]$ of the homomorphism $K[M^+] \to K$ induced by $\chi|M^+$, and let V^{N_0,I_χ^∞} (respectively $V^{(I_\chi,0)^r,N_0,I_\chi^\infty}$, respectively $W^{(I_\chi,0)^r,N_0,I_\chi^\infty}$) denote the subspace of V^{N_0} (respectively $V^{(I_\chi,0)^r,N_0}$, respectively $W^{(I_\chi,0)^r,N_0}$) consisting of vectors annihilated by some power of I_χ.

It was shown in the proof of [7, Prop. 4.2.33] that the Hecke operator π_{N_0,z_0} acts as a compact operator on each of the π_{N_0,z_0}-invariant subspaces $V^{(I_\chi,0)^r,N_0}$ and $W^{(I_\chi,0)^r,N_0}$. It follows from the theory of compact operators that the generalized $\chi(z_0)$-eigenspace of π_{N_0,z_0} on each of $V^{(I_\chi,0)^r,N_0}$ and $W^{(I_\chi,0)^r,N_0}$ is finite dimensional, and that the surjection (5.2.4) induces a surjection on generalized $\chi(z_0)$-eigenspaces. Consequently each of the spaces $V^{(I_\chi,0)^r,N_0,I_\chi^\infty}$ and $W^{(I_\chi,0)^r,N_0,I_\chi^\infty}$ are finite dimensional, and the surjection (5.2.4) induces a surjection $V^{(I_\chi,0)^r,N_0,I_\chi^\infty} \to W^{(I_\chi,0)^r,N_0,I_\chi^\infty}$. Letting r tend to infinity, we obtain a surjection

(5.2.5) $$V^{N_0,I_\chi^\infty} \to W^{N_0,I_\chi^\infty}.$$

The Jacquet module $J_P(V)$ is an essentially admissible locally analytic M-representation [7, Thm. 0.5]. Thus the dual space $J_P(V)'$ is naturally isomorphic to the space of global sections of a coherent sheaf $\mathcal{E}$ on $\hat{M}$. Similarly, $J_P(W)'$ is naturally isomorphic to the space of global sections of a coherent sheaf $\mathcal{F}$ on $\hat{M}$, and the induced map $J_P(V) \to J_P(W)$ corresponds to a map of coherent sheaves

(5.2.6) $$\mathcal{F} \to \mathcal{E}.$$

If we complete this morphism at the point χ of $\hat{M}$, we obtain a morphism

(5.2.7) $$\hat{\mathcal{F}}_\chi \to \hat{\mathcal{E}}_\chi$$

which is naturally identified with the topological dual to (5.2.5) (as follows from the definition of the Jacquet module [7, Def. 3.2.5]; compare [7, Prop. 3.4.9]). Since (5.2.5) is surjective, we see that (5.2.7) is injective.

Applying the above reasoning to the base-change of the map $V \to W$ over arbitrary finite extensions of K, we find that the morphism (5.2.6) becomes injective after being completed at arbitrary $\overline{K}$-valued points of $\hat{M}$ (where $\overline{K}$ denotes an algebraic closure of K). Thus (5.2.6) is itself injective. Passing to global sections and dualizing again, we find that the map $J_P(V) \to J_P(W)$ is surjective, as claimed. □

The following result allows one to verify the hypothesis of the preceding lemma.

Lemma 5.2.8. *Let $V \to W$ be a G-equivariant continuous surjection of admissible locally analytic G-representations. If there is a cofinal sequence $\{H_i\}$ of good analytic open subgroups of G such that, for each value of i, the space $V_{\mathbb{H}_i^\circ-\mathrm{an}}$ is topologically isomorphic, as an H_i°-representation, to a finite direct sum of topologically irreducible H_i°-representations, then $V \to W$ is locally split as a surjection of $\mathfrak{g}$-modules.*

Proof. Since any finitely generated $\mathfrak{g}$-subrepresentation of W embeds into $W_{\mathbb{H}_i^\circ-\mathrm{an}}$ for sufficiently large i, it suffices to show that the induced map $V_{\mathbb{H}_i^\circ-\mathrm{an}} \to W_{\mathbb{H}_i^\circ-\mathrm{an}}$ is a split surjection of H_i°-representations for all i. That this map is a surjection follows from Corollary A.13 of the appendix. Dualizing, we find that the induced map

$$(W_{\mathbb{H}_i^\circ-\mathrm{an}})'_b \to (V_{\mathbb{H}_i^\circ-\mathrm{an}})'_b \tag{5.2.9}$$

is a closed embedding of $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_i^\circ, K)$-modules, these modules being finitely presented, by Corollary A.14 of the appendix. It suffices to show that the embedding (5.2.9) is split.

Our assumption on $V_{\mathbb{H}_i^\circ-\mathrm{an}}$ implies that $(V_{\mathbb{H}_i^\circ-\mathrm{an}})'_b$ is topologically semi-simple as a $\mathcal{D}^{\mathrm{an}}(\mathbb{H}_i^\circ, K)$-module (i.e. is a direct sum of topologically irreducible submodules). That the embedding (5.2.9) is split now follows from standard properties of semi-simple objects in abelian categories, together with Proposition A.10 of the appendix. $\square$

As usual, we regard N as an open subset of $\overline{P}\backslash G$ via the open immersion (2.3.2). In particular, we regard the identity $e \in N$ as a point of $\overline{P}\backslash G$.

Lemma 5.2.10. *Let H be a good analytic open subgroup of G for which the locally analytic open immersion $H \to G$ underlies a rigid analytic open immersion $\mathbb{H} \to \mathbb{G}$. Let $r \in (0,1) \bigcap |\overline{L}^\times|$, and write $\overline{P}_r := H_r \bigcap \overline{P}$. Let $\overline{\mathbb{P}}_r$ denote the rigid analytic Zariski closure of $\overline{P}_r$ in $\mathbb{H}_r$. If the orbit of H on e (via the right action of H on $\overline{P}\backslash G$) is contained in N, then $\overline{\mathbb{P}}_r\backslash\mathbb{H}_r$ is an affinoid polydisk, and the locally analytic open immersion $\overline{P}_r\backslash H_r \xrightarrow{\sim} e \cdot H_r \subset N$ underlies a rigid analytic open immersion $\overline{\mathbb{P}}_r\backslash\mathbb{H}_r \to \mathbb{N}$.*

Proof. In the course of the proof it will be necessary to distinguish between various rigid analytic spaces, and their underlying locally analytic manifolds of L-valued points. Thus we begin by introducing some additional notation for this purpose.

Let $\underline{\mathfrak{g}}$ (respectively $\underline{\mathfrak{n}}$) denote the rigid analytic affine space whose underlying locally analytic set of L-valued points is equal to the Lie algebra $\mathfrak{g}$ (respectively $\mathfrak{n}$). As usual, let $\mathfrak{h}$ denote the $\mathcal{O}_L$-Lie sublattice of $\mathfrak{g}$ from which H is obtained by exponentiation. This sublattice defines a norm on $\mathfrak{g}$, the gauge of $\mathfrak{h}$, and we let $\underline{\mathfrak{h}_r}$ denote the rigid analytic closed polydisk in $\underline{\mathfrak{g}}$ of radius r with respect to this gauge. We let $\mathfrak{h}_r$ denote the set of L-valued points of $\underline{\mathfrak{h}_r}$. The exponential map defines a rigid analytic isomorphism $\exp : \underline{\mathfrak{h}_r} \xrightarrow{\sim} \mathbb{H}_r$. Also, since $\mathfrak{n}$ is a nilpotent Lie algebra, the exponential map defines a rigid analytic isomorphism $\exp : \underline{\mathfrak{n}} \xrightarrow{\sim} \mathbb{N}$. Of course, these isomorphisms induce isomorphisms of the underlying locally analytic manifolds of L-valued points. Finally, we note that since the H-orbit of e in $\overline{P}\backslash G$ is contained in N, we have the inclusion $H \subset \overline{P}N$ as open subsets of G, and hence a rigid analytic open immersion $\mathbb{H} \subset \overline{\mathbb{P}}\mathbb{N}$.

Consider the diagram

$$\begin{array}{ccccc} \underline{\mathfrak{h}_r} & \longrightarrow & \underline{\mathfrak{g}} & \longrightarrow & \underline{\mathfrak{n}} \\ \sim\downarrow \exp & & & & \sim\downarrow \exp \\ \mathbb{H}_r & \longrightarrow & \overline{\mathbb{P}}\mathbb{N} & \longrightarrow & \mathbb{N} \end{array}$$

of maps of rigid analytic spaces (in which the upper right horizontal arrow is induced by the projection $\mathfrak{g} \to \overline{\mathfrak{p}}\backslash\mathfrak{g} \xrightarrow{\sim} \mathfrak{n}$, and the lower right horizontal arrow is induced by the projection $\overline{\mathbb{P}}\mathbb{N} \to \mathbb{N}$), as well as the corresponding diagram

$$\begin{array}{ccccc} \mathfrak{h}_r & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{n} \\ \sim \downarrow \exp & & & & \sim \downarrow \exp \\ H_r & \longrightarrow & \overline{P}N & \longrightarrow & N, \end{array}$$

obtained from it by passing to L-valued points. One sees that the first diagram is commutative (by rigid analytic continuation, since it is evidently so if we replace $\mathfrak{h}_r$ by a sufficiently small $\mathcal{O}_L$-Lie sublattice), and thus so is the second diagram. This establishes the lemma. $\square$

Proposition 5.2.11. *Let H be a good analytic open subgroup of G, and let Ω denote the H°-orbit of $e \in \overline{P}\backslash G$. If Ω is contained in N, then for any locally analytic K-valued character ψ of M, the H°-representation $I_{\overline{P}}^G(\psi)(\Omega)_{\mathbb{H}^\circ-\mathrm{an}}$ is topologically irreducible.*

Proof. For each $r \in (0,1) \bigcap |\overline{L}^\times|$, let Ω_r denote the H_r-orbit of e, so that in the notation of the preceding lemma, $\Omega_r := \overline{P}_r\backslash H_r$. We have an isomorphism

$$I_{\overline{P}}^G(\psi)(\Omega)_{\mathbb{H}^\circ-\mathrm{an}} \xrightarrow{\sim} \varprojlim_{1>r} I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}}.$$

Thus if V is a closed subspace of $I_{\overline{P}}^G(\psi)(\Omega)_{\mathbb{H}^\circ-\mathrm{an}}$, and if V_r denotes the closure of V in $I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}}$ for each $1 > r > 0$, then the induced map $V \to \varprojlim_{1>r} V_r$ is an isomorphism. Hence it suffices to show that $I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}}$ is a topologically irreducible H_r-representation, or equivalently, a topologically irreducible $\mathfrak{g}$-representation, for each $1 > r > 0$.

The preceding lemma yields a closed embedding $I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(\mathbb{Y}, \psi)$ for some open affinoid polydisk $\mathbb{Y}$ in $\mathbb{N}$. If $(I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}})^{\mathrm{pol}}$ denotes the preimage under this isomorphism of $\mathcal{C}^{\mathrm{pol}}(N, \psi) \subset \mathcal{C}^{\mathrm{an}}(\mathbb{Y}, \psi)$, then $(I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}})^{\mathrm{pol}}$ is a $\mathfrak{g}$-invariant subspace of $I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}}$. In order to show that this latter space is topologically irreducible as a $\mathfrak{g}$-representation, it suffices, by Lemma 2.6.1 and Proposition 2.6.4, to show that $(I_{\overline{P}}^G(\psi)(\Omega_r)_{\mathbb{H}_r-\mathrm{an}})^{\mathrm{pol}}$ is irreducible as a $\mathfrak{g}$-representation. From the construction of $I_{\overline{P}}^G(\psi)$, one sees that this space coincides with the image of the natural $\mathfrak{g}$-equivariant map

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} K(\psi) \to \mathrm{Hom}_{\mathrm{U}(\overline{\mathfrak{p}})}(\mathrm{U}(\mathfrak{g}), K(\psi)).$$

(Here $K(\psi)$ denotes the one dimensional M-representation given by the character ψ, and the map is obtained by composing (2.8.6) with the isomorphism (2.5.7), taking U to be $K(\psi)$.) This image is well-known from the theory of Verma modules to be an irreducible $\mathfrak{g}$-representation. $\square$

Lemma 5.2.12. *Let H be a subgroup of G, let x be a point of $\overline{P}\backslash G$, and let $\Omega = xH$ denote the H-orbit of x. If there exists $g \in G$ such that $x = \overline{P}g$ and such that $\Omega \subset Ng$ (where Ng denotes the translate of $N \subset \overline{P}\backslash G$ by g), then for every point $x' \in \Omega$, we may find $g' \in G$ such that $x' = \overline{P}g'$ and such that $\Omega \subset Ng'$.*

Proof. We may write $x' = \overline{P}gh$ for some $h \in H$, and take $g' = gh$. Since $\Omega \subset Ng$, translating on the right by h, and using the fact that $\Omega = xH$ is an H-orbit, we find that also $\Omega \subset Ngh = Ng'$, as required. $\square$

Lemma 5.2.13. *If $\{H_i\}$ is a cofinal sequence of open subgroups of G, then we may find a natural number i_0, such that for any fixed $i \geq i_0$ there exists a finite subset $\{g_1, \ldots, g_m\} \in G$ with the property that, if we write Ω_j to denote the H_i-orbit of $\overline{P}g_j$ in $\overline{P}\backslash G$, then $\overline{P}\backslash G$ is the disjoint union of the H_i-orbits Ω_j, and there is an inclusion $\Omega_j \subset Ng_j$ for each $j = 1, \ldots, m$.*

Proof. Since N is an open subset of $\overline{P}\backslash G$, and since $\overline{P}g$ lies in Ng for each $g \in G$, we may choose for each $g \in G$ an index i_g such that the H_{i_g}-orbit of $\overline{P}g$ lies in Ng. Since $\overline{P}\backslash G$ is compact, we may cover $\overline{P}\backslash G$ by a finite number of these orbits, associated to the elements $g_1, \ldots, g_l$ say. Let i_0 be the maximum of the indices i_{g_j} for $1 \leq j \leq l$.

Fix $i \geq i_0$, and choose $\{x_1, \ldots, x_m\} \subset \overline{P}\backslash G$ such that $\overline{P}\backslash G$ is the disjoint union of the H_i-orbits $\Omega_{j'} := x_{j'}H_i$ for $1 \leq j' \leq m$. (This is possible, since $\overline{P}\backslash G$ is compact.) For each $1 \leq j' \leq m$, we may find $1 \leq j \leq l$ such that $x_{j'}$ lies in the $H_{i_{g_j}}$-orbit of $\overline{P}g_j$. By the choice of i_{g_j}, this orbit is contained in Ng_j. Lemma 5.2.12 allows us to find $g'_{j'}$ such that $x_{j'} = \overline{P}g'_{j'}$, and such that this $H_{i_{g_j}}$-orbit is contained in $N'g_{j'}$. Since $i \geq i_0 \geq i_{g_j}$, we see that $H_i \subset H_{i_{g_j}}$, and thus that $\Omega_{j'} \subset Ng_{j'}$. Relabelling $\{g'_1, \ldots, g'_m\}$ as $\{g_1, \ldots, g_m\}$ completes the proof of the lemma. □

Corollary 5.2.14. *We may find a cofinal sequence of good analytic open subgroups $\{H_i\}$ of G so that for any locally analytic character ψ of M, and any value of i, the H_i°-representation $I_{\overline{P}}^G(\psi)_{\mathbb{H}_i^\circ-\mathrm{an}}$ is topologically isomorphic to a finite direct sum of topologically irreducible H_i°-representations.*

Proof. Fix a cofinal sequence $\{H_i\}$ of good analytic open subgroups of G. Apply Lemma 5.2.13 to the cofinal sequence $\{H_i^\circ\}$. If $i \geq i_0$, then we may find a finite subset $\{g_1, \ldots, g_m\} \subset G$ such that $\overline{P}\backslash G$ is the disjoint union of the H_i°-orbits Ω_j of the points $\overline{P}g_j$, and such that

$$\Omega_j \subset Ng_j \tag{5.2.15}$$

for each $1 \leq j \leq m$. We then have a topological isomorphism of H_i°-representations

$$I_{\overline{P}}^G(\psi) \xrightarrow{\sim} \bigoplus_{j=1}^m I_{\overline{P}}^G(\psi)(\Omega_j),$$

which induces an isomorphism

$$I_{\overline{P}}^G(\psi)_{\mathbb{H}_i^\circ-\mathrm{an}} \xrightarrow{\sim} \bigoplus_{j=1}^m I_{\overline{P}}^G(\psi)(\Omega_j)_{\mathbb{H}_i^\circ-\mathrm{an}} \tag{5.2.16}$$

upon passing to $\mathbb{H}_i^\circ$-analytic vectors. Translating by g_j induces a topological isomorphism

$$I_{\overline{P}}^G(\psi)(\Omega_j)_{\mathbb{H}_i^\circ-\mathrm{an}} \xrightarrow{\sim} I_{\overline{P}}^G(\psi)(\Omega_j g_j^{-1})_{g_j\mathbb{H}_i^\circ g_j^{-1}-\mathrm{an}} \tag{5.2.17}$$

for each $1 \leq j \leq m$, which is compatible with the H_i°-action on the source, the $g_jH_i^\circ g_j^{-1}$-action on the target, and the isomorphism $H_i^\circ \xrightarrow{\sim} g_jH_i^\circ g_j^{-1}$ induced by conjugation by g_j.

Now $\Omega_j g_j^{-1}$ is the $g_j H^\circ g_j^{-1}$-orbit of $e \in N$, and (5.2.15) implies that $\Omega_j g_j^{-1} \subset N$. It follows from Proposition 5.2.11 (applied to the good analytic open subgroup gH_ig^{-1}) and from the isomorphism (5.2.17) that $I_P^G(\psi)(\Omega_j)_{\mathbb{H}_i^\circ-\mathrm{an}}$ is a topologically irreducible H_i°-representation. Taking into account the isomorphism (5.2.16), we conclude that $I_P^G(\psi)_{\mathbb{H}_i^\circ-\mathrm{an}}$ is topologically isomorphic as an H_i°-representation to a finite direct sum of topologically irreducible representations. The proposition follows upon taking the sequence of good analytic open subgroups to be $\{H_i\}_{i\geq i_0}$. □

Theorem 5.2.18. *If U is a finite dimensional locally analytic representation of M, then any G-equivariant surjection $I_P^G(U) \to W$ of admissible locally analytic G-representations induces a surjection of Jacquet modules $J_P(I_P^G(U)) \to J_P(W)$.*

Proof. We may verify the assertion of the theorem after making a finite extension of scalars, and thus may assume that U is a direct sum of locally analytic characters. The theorem thus follows from Lemmas 5.2.2 and 5.2.8 and Corollary 5.2.14. □

(5.3) In this subsection we prove Corollaries 0.14 and 0.15.

Proof of Corollary 0.14. Since $J_P(V)$ is a non-zero object of $\mathrm{Rep}_{\mathrm{es}}(M)$, we may find a character $\chi \in \hat{M}(E)$ for some finite extension E of K for which the χ-eigenspace of $J_P(V \otimes_K E)$ is non-zero. Taking U to be $\chi\delta^{-1}$, we let W denote the image of the map $\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathcal{C}_c^{\mathrm{sm}}(N, U) \to V \otimes_K E$ corresponding via (0.17) to the inclusion of $U(\delta)$ into $J_P(V \otimes_K E)$. If $d\chi$ denotes the derivative of χ (regarded as a weight of the Lie algebra $\mathfrak{m}$ of M) then $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ is isomorphic to a direct sum of copies of $d\chi$ as a $\mathrm{U}(\mathfrak{p})$-module, and so W is a direct sum of copies of a quotient of the Verma module $\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} d\chi$.

The space of $\mathfrak{n}$-invariants $W^{\mathfrak{n}}$ decomposes as a direct sum of weights of $\mathfrak{m}$. Furthermore, for every weight λ of $\mathfrak{m}$ that appears, there is a corresponding character $\tilde{\chi}$ appearing in $J_P(V \otimes_K E)$ for which $d\tilde{\chi} = \lambda$. (Compare the proof of [7, Prop. 4.4.4].) The theory of Verma modules shows that we may find a weight λ of $\mathfrak{m}$ appearing in $W^{\mathfrak{n}}$ such that $\lambda - \mu$ does not appear in $W^{\mathfrak{n}}$ for any element μ in the positive cone of the root lattice of $\mathfrak{m}$. Let $\tilde{\chi}$ be a character of M appearing in $J_P(V \otimes_K E)$ for which $\lambda = d\tilde{\chi}$, and set $\tilde{U} = \tilde{\chi}\delta^{-1}$. Our choice of λ ensures that the resulting inclusion $\tilde{U}(\delta) \to J_P(V \otimes_K E)$ is balanced, and so Theorem 0.13 yields a non-zero E-linear and G-equivariant map $I_P^G(\tilde{U}) \to V \otimes_K E$. Forgetting the E-linear structure, and regarding the source and target of this maps simply as K-linear representations of G, we see that the target is isomorphic to a finite direct sum of copies of the topologically irreducible representation V. Since the map is non-zero, the projection onto at least one of these direct summands must again be non-zero, and so we obtain a K-linear and G-equivariant map $I_P^G(\tilde{U}) \to V$, as required. □

Proof of Corollary 0.15. The functor J_P is left exact (see [7, Thm. 4.2.32]), and so it suffices to prove the result when V is a topologically irreducible very strongly admissible locally analytic G-representation. If $J_P(V)$ is non-zero then Corollary 0.14 yields a surjection $I_P^G(U) \to V$ for some finite dimensional locally analytic representation of M. Theorem 5.2.18 shows that the induced map $J_P(I_P^G(U)) \to J_P(V)$ is surjective. Proposition 5.1.5 shows that the source of this map is finite dimensional, and thus so is its target. □

Appendix

In this appendix we derive some results concerning admissible locally analytic representations, based on the results of [6, §5].

Proposition A.1. *If V is an admissible locally analytic representation of G, and if H is a good analytic open subgroup of G, then there is a natural isomorphism*

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b \underset{\mathcal{D}^{\mathrm{la}}(H^{\circ},K)_b}{\hat{\otimes}} V'_b \xrightarrow{\sim} (V_{\mathbb{H}^{\circ}-\mathrm{an}})'_b.$$

Proof. We will use the notation introduced in the proof of [6, Prop. 5.3.1]. Choose a real number $0 < r < 1$ such that $r \in |\overline{L}^{\times}|$. There are isomorphisms

$$\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)_b \xrightarrow{\sim} \varprojlim_n D(\mathbb{H}^{\circ}_{r^n}, H^{\circ}) \tag{A.2}$$

and

$$V'_b \xrightarrow{\sim} \varprojlim_n (V_{\mathbb{H}^{\circ}_{r^n}-\mathrm{an}})'_b. \tag{A.3}$$

Since furthermore the natural map

$$\mathcal{D}^{\mathrm{la}}(H^{\circ}, K)_b \to D(\mathbb{H}^{\circ}_{r^n}, H^{\circ}) \tag{A.4}$$

induced by (A.2) has dense image, tensoring (A.2) and (A.3) induces the first of the sequence of isomorphisms

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b \underset{\mathcal{D}^{\mathrm{la}}(H^{\circ},K)_b}{\hat{\otimes}} V'_b \xrightarrow{\sim} \varprojlim_n \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b \underset{D(\mathbb{H}^{\circ}_{r^n},H^{\circ})}{\hat{\otimes}} (V_{\mathbb{H}^{\circ}_{r^n}-\mathrm{an}})'_b$$
$$\xrightarrow{\sim} \varprojlim_n (V_{\mathbb{H}^{\circ}-\mathrm{an}})'_b \xrightarrow{\sim} (V_{\mathbb{H}^{\circ}-\mathrm{an}})'_b.$$

The second isomorphism is provided by [6, (6.1.15)] (which [6, Cor. 6.1.19] shows to be indeed an isomorphism), while the third isomorphism is tautological, since its source is the projective limit of a constant projective system. Composing these isomorphisms gives the isomorphism of the proposition. □

In order to apply the preceding result, it will be useful to establish some facts concerning topological modules over $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b$. The key property of $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b$ is the isomorphism of topological algebras

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b \xrightarrow{\sim} \varinjlim_m \mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b^{(m)} \tag{A.5}$$

provided by [6, Prop. 5.2.3], which shows that $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^{\circ}, K)_b$ is coherent, and of compact type.

Lemma A.6. *Suppose that $\{A_m\}$ is an inductive sequence of Noetherian Banach algebras, for which the transition maps $A_m \to A_{m+1}$ are injective, compact (as maps between Banach spaces), and flat homomorphisms of K-algebras, and let $A := \varinjlim_m A_m$ be the inductive limit algebra (which is thus of compact type). If M_m is a finitely generated A_m-Banach module, for some value of m, then $A \otimes_{A_m} M_m$ is of compact type (in particular it is Hausdorff and complete), and is a finitely presented A-module.*

Proof. For each $m' \geq m$, write $M_{m'} := A_{m'} \otimes_{A_m} M_m$, so that $M_{m'}$ is a finitely generated $A_{m'}$-Banach module (by [6, Prop. 1.2.5] and the remark following its proof). The natural map $A_{m'} \to A$ induces a map

$$M_{m'} \to A \otimes_{A_m} M_m \tag{A.7}$$

for each $m' \geq m$.

Our convention is that $A \otimes_{A_m} M_m$ is equipped with the quotient topology obtained by regarding it as a quotient of $A \otimes_{K,\pi} M_m$. Since A is of compact type, and M_m is a Banach space, [6, Prop. 1.1.31] shows that this topology coincides with the topology obtained by regarding $A \otimes_{A_m} M_m$ as a quotient of $A \otimes_{K,\imath} M_m$. A similar remark applies to each of the tensor products $M_{m'} = A_{m'} \otimes_{A_m} M_m$ (for $m' \geq m$), since $A_{m'}$ and M_m are both Banach spaces. From these remarks, together with [6, Lem. 1.1.30], one easily deduces that the natural map

$$\varinjlim_{m' \geq m} M_{m'} \to A \otimes_{A_m} M \tag{A.8}$$

induced by the maps (A.7) is an isomorphism.

For $m' \geq m$, let $N_{m'}$ be the quotient of $M_{m'}$ by the kernel of the map (A.7). We may then rewrite the isomorphism (A.8) as an isomorphism

$$\varinjlim_{m' \geq m} N_{m'} \xrightarrow{\sim} A \otimes_{A_m} M, \tag{A.9}$$

where the transition maps in this inductive limit are injective. Since $A_{m'}$ is Noetherian and $M_{m'}$ is finitely generated over $A_{m'}$, [6, Prop. 1.2.4] implies that the kernel of (A.7) is a closed subspace of $M_{m'}$, and thus that each $N_{m'}$ is a Banach space. Since each $N_{m'}$ is finitely generated over $A_{m'}$, and since the transition maps between the $A_{m'}$ are compact, we see that the transition maps in the inductive limit of (A.9) are furthermore compact. Thus (A.9) presents $A \otimes_{A_m} M$ as an inductive limit of a sequence of Banach spaces with compact and injective transition maps. This proves the first claim of the lemma. The claim of finite presentability follows directly from the fact that M_m is finitely generated over the Noetherian Banach algebra A_m, and hence in fact finitely presented. □

In the context of the preceding lemma, the ring A is coherent (being the inductive limit of a sequence of Noetherian rings via flat transition maps). Thus the category of finitely presented A-modules (i.e the full subcategory of the category of A-modules whose objects are the finitely presented A-modules) is abelian (and the formation of kernels, cokernels, and images in this category coincide with the corresponding constructions in the category of all A-modules). The following result provides some useful functional analytic information about the category of finitely presented A-modules.

Proposition A.10. *Let $\{A_m\}$ and A be as in the statement of Lemma A.6.*

(i) *Every finitely presented A-module has a unique structure of compact type space making it a topological A-module.*

(ii) *If $f : M \to N$ is an A-linear morphism between finitely presented A-modules, then f is continuous and strict with respect to the topologies on M and N given by* (i).

Proof. If M is a finitely generated A-module, then we may find a surjection of A-modules $A^r \to M$ for some $r \geq 0$. If M is furthermore equipped with a locally convex topology with respect to which it becomes a topological A-module, then this map is necessarily continuous. If the topology on M is of compact type, then the open mapping theorem shows that this surjection is a strict map of convex spaces, and the so the topology on M coincides with the topology it inherits as a quotient of A^r. This proves the uniqueness statement of (i).

Suppose now that M is finitely presented. It is a standard (and easily verified) fact that we may write $M \cong A \otimes_{A_m} M_m$ for some value of m, and some finitely generated A_m-module M_m. From [6, Prop. 1.2.4 (i)], we see that M_m may be equipped with a canonical A_m-Banach module structure, and Lemma A.6 then shows that the tensor product topology on M (which certainly makes M a topological A-module) endows M with a compact type space structure. This establishes the existence claim of (i).

Suppose now that $M \to N$ is an A-linear morphism of finitely presented A-modules. We may fit this morphism into a commutative diagram of A-linear morphisms

$$\begin{array}{ccc} A^r & \longrightarrow & A^s \\ \downarrow & & \downarrow \\ M & \longrightarrow & N, \end{array}$$

for some $r, s \geq 0$. If we endow each of M and N with the topological A-module structure given by part (i), then we have already observed that the vertical arrows must be strict, while the upper horizontal arrow is certainly continuous (since A is a topological algebra). Thus the lower horizontal arrow must be continuous. This proves the continuity claim of (ii).

Now consider a short exact sequence of finitely presented A-modules

$$0 \to M \to N \to P \to 0.$$

If we endow each of these modules with the topological A-module structure given by part (i), then the result of the preceding paragraph shows that the morphisms in this sequence are necessarily continuous. Since the topologies on M, N, and P are of compact type, the open mapping theorem shows that the morphisms are furthermore necessarily strict.

As was noted above, the kernel or cokernel of a morphism of finitely presented A-modules is again finitely presented (since A is coherent), and so any injection or surjection of finitely presented A-modules may be fitted into a short exact sequence of finitely presented A-modules. The result of the preceding paragraph thus implies the strictness claim of (ii) for injections and surjections of finitely presented A-modules.

Any morphism of finitely presented A-modules may be factored as the composite of a surjection of its source onto its image, and an injection of its image into its target. Furthermore, as was noted above, this image is again finitely presented (since A is coherent). The strictness claim of (ii) (in full generality) thus follows from the result for injections and surjections already established. □

The isomorphism (A.5) and [6, Prop. 5.3.11] allow us to apply the preceding results in the particular context when $A = \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ and $A_m = \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)}$ for $m \geq 0$.

Lemma A.11. *If H is a good analytic open subgroup of G, then $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ is flat over $\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b$, and the natural map*

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M \tag{A.12}$$

is an isomorphism for any coadmissible $\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b$-module M. Furthermore, the topological tensor product $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$ is of compact type, and is finitely presented as a $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$-module.

Proof. We employ the same notation as in the proof of Proposition A.1. From [6, Prop. 5.2.3] we obtain an isomorphism $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \xrightarrow{\sim} \varinjlim_m \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)}$, and [6, Prop. 5.3.18] implies that the map (A.4) (for $n = 1$) factors as

$$\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b \to D(\mathbb{H}_r^\circ, H^\circ) \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)} \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$$

for m sufficiently large. The second and third of these maps are flat, by [6, Prop. 5.3.18] and [6, Prop. 5.3.11 (ii)] respectively, and the first is flat, by [16, Rem. 3.2]. Thus $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$ is indeed a flat $\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b$-module.

It follows from [6, Prop. 5.3.11 (i)] that $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)}$ is a Noetherian Banach algebra for any value of m. As was observed in the proof of [6, Thm. 1.2.11], the tensor product $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)} \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$ is a finitely generated $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)}$-Banach module, and hence the natural map

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)} \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b^{(m)} \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$$

is an isomorphism for any m. Since the natural maps

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}} (\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M) \\ \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$$

and

$$\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \hat{\otimes}_{\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)}} (\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)^{(m)} \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M) \\ \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \hat{\otimes}_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$$

are also both isomorphisms, it thus follows from Lemma A.6 that (A.12) is indeed an isomorphism, and also that $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$ is finitely presented as a $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$-module. □

Corollary A.13. *If H is a good analytic open subgroup of G, then the functor $V \mapsto V_{\mathbb{H}^\circ-\mathrm{an}}$ on the category of admissible locally analytic G-representations is exact (in the strong sense that it takes an exact sequence of admissible locally analytic representations to a strict exact sequence of nuclear Fréchet spaces).*

Proof. By Proposition A.1 (and the fact that passing to duals is exact, by the Hahn-Banach theorem), to prove the corollary it suffices to show that the functor

$$M \mapsto \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \mathop{\hat{\otimes}}_{\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b} M$$

is an exact functor on the category of coadmissible $\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b$-modules.

If $0 \to M \to N \to P \to 0$ is a short exact sequence of coadmissible $\mathcal{D}^{\mathrm{la}}(H^\circ, K)_b$-modules, then it follows from Lemma A.11 that

$$0 \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ,K)_b} M \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ,K)_b} N \\ \to \mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b \otimes_{\mathcal{D}^{\mathrm{la}}(H^\circ,K)_b} P \to 0$$

is a sequence of continuous morphisms of finitely presented compact type topological $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)_b$-modules, which is short exact in the algebraic sense (i.e. ignoring topologies). Proposition A.10 then implies that the maps are strict. □

Corollary A.14. *If V is an admissible locally analytic representation of G, and if H is a good analytic open subgroup of G, then $(V_{\mathbb{H}^\circ-\mathrm{an}})'$ is finitely presented as a $\mathcal{D}^{\mathrm{an}}(\mathbb{H}^\circ, K)$-module.*

Proof. This was noted in the proof of the preceding corollary. □

References

1. Y. Amice, J. Vélu, *Distributions p-adiques associées aux séries de Hecke*, Astérisque **24/25** (1975), 119–131.
2. N. Bourbaki, *Elements of Mathematics. Topological Vector Spaces. Chapters 1-5*, Springer-Verlag, 1987.
3. C. Breuil, M. Emerton, *Représentations p-adiques ordinaires de* $\mathrm{GL}_2(Q_p)$ *et compatibilité local-global*, preprint, available at `http://www.ihes.fr/~breuil/publications.html`.
4. P. Cartier, *Representations of* $\mathfrak{p}$*-adic groups: A survey*, Automorphic forms, representations, and L-functions, Proc. Symp. Pure Math., vol. 33, part 1, Amer. Math. Soc., 1979, pp. 111–155.
5. W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, unpublished notes distributed by P. Sally, draft dated May 7, 1993.
6. M. Emerton, *Locally analytic vectors in representations of locally p-adic analytic groups*, to appear in Memoirs of the AMS.
7. M. Emerton, *Jacquet modules of locally analytic representations of p-adic reductive groups I. Construction and first properties*, Ann. Sci. Ec. Norm. Sup **39** (2006), 353–392.
8. M. Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. **164** (2006), 1–84.
9. M. Emerton, *Locally analytic representation theory of p-adic reductive groups: A summary of some recent developments*, to appear in the proceedings of the Durham conference on L-functions and Galois representations (Durham, 2004).
10. C. T. Féaux de Lacroix, *Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper*, Thesis, Köln 1997, Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, Heft 23 (1999), 1-111.
11. P. Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Math., Springer-Verlag, 2002.

12. P. Schneider, *p-adische analysis*, Vorlesung in Münster, 2000; available electronically at http://www.math.uni-muenster.de/math/u/schneider/publ/lectnotes.
13. P. Schneider, J. Teitelbaum, *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Amer. Math. Soc. (2001).
14. P. Schneider, J. Teitelbaum, $U(\mathfrak{g})$-*finite locally analytic representations*, Represent. Theory **5** (2001), 111–128.
15. P. Schneider, J. Teitelbaum, *Banach space representations and Iwasawa theory*, Israel J. Math. **127** (2002), 359–380.
16. P. Schneider, J. Teitelbaum, *Algebras of p-adic distributions and admissible representations*, Invent. Math. **153** (2003), 145–196.
17. J.-P. Serre, *Lie algebras and Lie groups*, W. A. Benjamin, 1965.
18. M. Vishik, *Nonarchimedean measures connected with Dirichlet series*, Math. USSR Sb. (1976), 216–228.

Northwestern University
Department of Mathematics
2033 Sheridan Rd.
Evanston, IL 60208-2730, USA
E-mail address: emerton@math.northwestern.edu