Unit L-functions and a conjecture of Katz.

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Let p be a prime number, $\mathbb{F}_p$ the finite field of order p, and $\mathbb{Z}_p$ the ring of p-adic integers. Suppose that X is a separated finite type $\mathbb{F}_p$-scheme and that $\mathcal{L}$ is a lisse $\mathbb{Z}_p$-sheaf on the étale site of X. One defines in the usual way an L-function $L(X, \mathcal{L})$ attached to $\mathcal{L}$. This is a power series in a formal variable T, which by construction is an element of $1 + T\mathbb{Z}_p[[T]]$. If $f : X \to \operatorname{Spec} \mathbb{F}_p$ is the structural morphism of X then $f_!\mathcal{L}$ is a constructible complex of $\mathbb{Z}_p$-sheaves on the étale site of $\operatorname{Spec} \mathbb{F}_p$ of finite tor-dimension, and so we may form the L-function $L(\operatorname{Spec} \mathbb{F}_p, f_!\mathcal{L})$, which is also an element of $1 + T\mathbb{Z}_p[[T]]$ (and in fact a rational function). The ratio $L(X, \mathcal{L})/L(\operatorname{Spec} \mathbb{F}_p, f_!\mathcal{L})$ thus lies in $1 + T\mathbb{Z}_p[[T]]$, and so may be regarded as a nowhere-zero analytic function on the p-adic open unit disk $|T| < 1$. If $\mathcal{L}$ were a lisse $\mathbb{Z}_l$ sheaf with $l \neq p$ then in fact this ratio would be identically 1 (this is Grothendieck's approach to the rationality of the zeta function). One does not have this in the setting of $\mathbb{Z}_p$-sheaves. However, in this paper we prove the following result, which was conjectured by Katz ([11], 6.1):

THEOREM 0.1. *The ratio $L(X, \mathcal{L})/L(\operatorname{Spec} \mathbb{F}_p, f_!\mathcal{L})$ extends to a nowhere-zero function on the closed unit disc $|T| \leq 1$.*

In particular this implies that the L-function for $L(X, \mathcal{L})$ is p-adic meromorphic in the closed unit disc. According to Katz ([11], 6.1), when $X = \mathbb{A}^n$, this is an old result of Dwork's.

Katz also conjectured that $L(X, \mathcal{L})$ extends to a meromorphic function on the rigid-analytic affine T-line. The second part of Katz's conjecture was a generalization of a conjecture of Dwork predicting that $L(X, \mathcal{L})$ was p-adic meromorphic when the unit F-crystal $\mathcal{M}$ associated to $\mathcal{L}$ was the unit part of an ordinary, overconvergent F-crystal. Wan [13] has found counter-examples to the part of Katz's conjecture predicting that the L-function is meromorphic for general lisse sheaves. On the other hand, in a recent preprint [14] he has proven that Dwork's more cautious prediction is actually true. Thus, while the question of when the L-function is meromorphic is by now quite well understood, much less was known about the location of the zeroes and poles. In fact the only prior results in this direction that we are aware of are due to

Crew [3], who proved Katz's conjecture in the special case when X is an affine curve and the sheaf $\mathcal{L}$ has abelian monodromy, and Etesse and Le Stum [8], who proved Katz's conjecture under the strong assumption that $\mathcal{L}$ extends to some compactification of X.

In fact we prove a more general result showing that an analogue of Katz's conjecture is true even if one replaces $\mathbb{Z}_p$ by a complete noetherian local $\mathbb{Z}_p$-algebra Λ with finite residue field. We refer to Corollary 1.8 for the precise formulation.

Our methods also have some applications to results on lifting representations of arithmetic fundamental groups. In particular, we show the following

THEOREM 0.2.   *Let X be a smooth affine $\mathbb{F}_p$-scheme, Λ an artinian local $\mathbb{Z}_p$-algebra, having finite residue field, and*

$$\rho : \pi_1(X) \to GL_d(\Lambda)$$

a representation of the arithmetic étale fundamental group of X. Consider a finite flat local $\mathbb{Z}_p$-algebra $\tilde{\Lambda}$ and a surjection $\tilde{\Lambda} \to \Lambda$. There exists a continuous lifting of ρ

$$\tilde{\rho} : \pi_1(X) \to GL_d(\tilde{\Lambda}).$$

If X is an open affine subset of $\mathbb{P}^1$, then $\tilde{\rho}$ may be chosen so that the L-function of the corresponding lisse sheaf of $\tilde{\Lambda}$-modules is rational.

Let us now describe the contents of the paper in more detail.

In section 1 we define the necessary L-functions, and state our main results.

In section 2 we give the proof of some standard results on behaviour of L-functions in exact triangles, and under stratification of the underlying space. These are used later to reduce our calculations to a special case.

In section 3 we introduce a key ingredient in our work, which is the relationship between locally constant or lisse étale sheaves and unit F-crystals. We show that if Λ is an artinian (respectively a complete noetherian) finite local $\mathbb{Z}_p$-algebra with finite residue field and if X is a formally smooth p-adic formal scheme equipped with a lifting F of the absolute Frobenius of its special fibre, then there is a one-to-one correspondence between locally constant étale sheaves of finite free Λ-modules (respectively lisse étale sheaves of Λ-modules) on X and finite free $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X$-modules $\mathcal{E}$ on X equipped with a Λ-linear isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. This generalizes a result of Katz ([12], 4.1), which treats the cases when $\Lambda = \mathbb{Z}_p$ or $\mathbb{Z}/p^n$ for some positive integer n.

In section 4 we begin our proof of Katz's conjecture. For the usual reasons of technical convenience, we prove a more general result on complexes of p-adic constructible sheaves of Λ-modules. By passing to the inverse limit, it is enough to consider the case where Λ is artinian. Using standard techniques we reduce

ourselves to having to verify Katz's conjecture for a locally constant flat Λ-sheaf on an open subset of $\mathbb{G}_m^d$ with the additional property that the associated unit (Λ, F)-crystal $\mathcal{E}$ (as explained above) has an underlying locally free sheaf of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X$-modules which is actually free. Under this assumption we may choose a surjection $\tilde{\Lambda} \to \Lambda$, with $\tilde{\Lambda}$ a finite flat local $\mathbb{Z}_p$-algebra, and lift this Λ-sheaf to a lisse $\tilde{\Lambda}$-sheaf by lifting the associated unit (Λ, F)-crystal.

In this situation we prove Katz's conjecture directly, by using an explicit trace calculation in the style of Dwork [5]. To reduce to this trace calculation we utilize a technique of Deligne [4] in order to find an Artin-Schreier sequence for the extension by zero of a lisse $\tilde{\Lambda}$-sheaf under an open immersion, which allows us to relate p-adic étale cohomology with compact support to coherent cohomology of formal $\mathbb{Z}_p$-schemes. A key part of this calculation is the subject of section 5.

Finally, in section 6 we show how our methods can be used to lift representations of arithmetic fundamental groups.

Finally, let us mention that the methods used in this paper are a special instance of a more general theory developed by the authors in the papers [6], [7]. This theory describes the étale cohomology of p-power torsion or p-adic constructible sheaves on finite type $\mathbb{F}_p$-schemes in terms of quasi-coherent cohomology; more precisely, a certain Riemann-Hilbert correspondence is established between the appropriate derived categories of such étale sheaves, and a derived category of certain quasi-coherent analogues of unit F-crystals. However, we have written this paper so as to be independent of these more general techniques.

Acknowledgments. The authors would like to thank Pierre Berthelot for suggesting that they apply their techniques to the study of L-functions. The first author would also like to thank Mike Roth for useful discussions.

1. Notation and statement of results

(1.1) Let p be a prime number, fixed for the remainder of the paper, $\mathbb{F}_p$ denote a finite field of order p, and $\overline{\mathbb{F}}_p$ be a choice of an algebraic closure of $\mathbb{F}_p$ (fixed once and for all). For any positive integer n we denote by $\mathbb{F}_{p^n}$ the unique degree n extension of $\mathbb{F}_p$ contained in $\overline{\mathbb{F}}_p$. We denote by $\sigma : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ the (arithmetic) Frobenius automorphism $a \mapsto a^p$ of $\overline{\mathbb{F}}_p$, and denote by ϕ the geometric Frobenius automorphism of $\overline{\mathbb{F}}_p$, that is, the inverse of σ. Both σ and ϕ are topological generators of the Galois group of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$.

We denote by $\overline{W}$ the ring of Witt vectors of $\overline{\mathbb{F}}_p$, and by σ and ϕ respectively the canonical Frobenius automorphism of $\overline{W}$ and its inverse. If n is any positive integer, we let $W(\mathbb{F}_{p^n})$ denote the Witt ring of $\mathbb{F}_{p^n}$, regarded as a subring of $\overline{W}$. As usual we will denote by $\mathbb{Z}_p$ the ring $W(\mathbb{F}_p)$ of p-adic numbers.

(1.2) Let X be a finite type scheme over $\mathbb{F}_p$, and Λ a noetherian ring, with $m\Lambda = 0$ for some positive integer m. We denote by $D^-(X, \Lambda)$ the derived category of bounded above complexes of étale sheaves of Λ-modules on X, and we denote by $D^b_{ctf}(X, \Lambda)$ the full subcategory of $D^-(X, \Lambda)$ consisting of complexes of Λ-sheaves which have finite tor-dimension, and which are constructible (in the sense that their cohomology sheaves are constructible Λ-modules [1]). Given any object $\mathcal{L}^\bullet$ of $D^b_{ctf}(X, \Lambda)$, we may define its L-function in the usual way, which we briefly recall.

If $\mathcal{L}^\bullet$ is a constructible complex of Λ-sheaves of finite tor-dimension on X then by [4], proposition-définition 4.6 *(ii)*, p. 93, we may assume that $\mathcal{L}^\bullet$ is a finite length complex of flat constructible Λ-sheaves. If x is a closed point of X we let $\mathcal{L}^i_x$ denote the flat Λ-sheaf obtained by restricting $\mathcal{L}^i$ to the point x. If we choose an identification of the residue field of x with the subfield $\mathbb{F}_{p^{d(x)}}$ of $\overline{\mathbb{F}}_p$, and denote by $\bar{x}$ the $\overline{\mathbb{F}}_p$-valued point of X lying over x, then $\mathcal{L}^i_{\bar{x}}$ is a free Λ-module equipped with an action of the dth power ϕ^d of the geometric Frobenius automorphism. We let $|X|$ denotes the set of closed points of X, and define

$$L(X, \mathcal{L}^\bullet) = \prod_i \prod_{x \in |X|} \det{}_\Lambda(1 - \phi^{d(x)} T^{d(x)}, \mathcal{L}^i_{\bar{x}})^{(-1)^{i+1}}.$$

This is an element of $1 + T\Lambda[[T]]$.

From the definition we see that L-functions are compatible with change of ring: if Λ, Λ' are noetherian rings killed by some positive integer, $g : \Lambda \to \Lambda'$ a map of rings, and $\mathcal{L}^\bullet$ is as above, then we have

$$L(X, \mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes} \Lambda') = g(L(X, \mathcal{L}^\bullet)).$$

(1.3) Let X be a finite type $\mathbb{F}_p$-scheme and let $i : X_{red} \to X$ be the closed immersion of the underlying reduced subscheme of X into X. Then the functors $i_!$ and i^{-1} are quasi-inverse, and induce an equivalence of triangulated categories between $D^b_{ctf}(X_{red}, \Lambda)$ and $D^b_{ctf}(X, \Lambda)$ (preserving the canonical t-structure of both categories). If $\mathcal{L}^\bullet$ is any object of $D^b_{ctf}(X, \mathbb{Z}/p^n)$ then $L(X, \mathcal{L}^\bullet) = L(X_{red}, i^{-1}\mathcal{L}^\bullet)$.

(1.4) If $f : X \to Y$ is a separated morphism of finite type $\mathbb{F}_p$-schemes then push-forward with proper supports induces a functor $f_! : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(Y, \Lambda)$ ([4], théorème 4.9, p. 95).

THEOREM 1.5. *Let $f : X \to Y$ be a separated morphism of finite type $\mathbb{F}_p$-schemes, and Λ a finite, local, artinian $\mathbb{Z}_p$-algebra, with maximal ideal $\mathfrak{m}$. If $\mathcal{L}^\bullet$ is any object of $D^b_{ctf}(X, \Lambda)$ then the ratio of L-functions*

$$L(X, \mathcal{L}^\bullet)/L(Y, f_!\mathcal{L}^\bullet),$$

a priori an element of $1 + T\Lambda[[T]]$, in fact lies in $1 + \mathfrak{m}T\Lambda[T]$.

(1.6) Suppose that Λ is finite and reduced of characteristic p. Then from theorem 1.5 we conclude that the L-functions of objects of $D_{ctf}^b(X_{\acute{e}t}, \Lambda)$ are invariant under proper push-forward. This was originally proved by Deligne ([4], théorème 2.2, p. 116).

An interesting point is that Deligne gives a counterexample involving a locally constant sheaf of free rank one $\mathbb{F}_p[X]/X^2$-modules, to show that his formula does not hold without the hypothesis that Λ is reduced ([4], 4.5, p. 127). Thus if we take $\Lambda = \mathbb{F}_p[X]/X^2$, then Deligne's counterexample shows that, in general, the quotient of the L-functions in theorem 1.5 is not equal to 1. Nevertheless, our theorem asserts that it is equal to 1 modulo the principal ideal (X). This also follows from Deligne's theorem by "specialization of L-functions" (1.2), and in fact this same argument shows that the ratio of L-functions $L(X, \mathcal{L}^\bullet)/L(Y, f_! \mathcal{L}^\bullet)$ occurring in the statement of theorem 1.5 always lies in $1 + \mathfrak{m} T \Lambda[[T]]$. Thus the key result of theorem 1.5 is that this ratio is in fact a polynomial.

(1.7) We want to define L-functions for lisse sheaves, or more precisely, the lisse analogue of constructible sheaves. For this, suppose that we are given a noetherian ring Λ and an ideal $I \subset \Lambda$, such that Λ is I-adically complete and p is nilpotent in Λ/I. We define the category $D_{I-sm}^b(X, \Lambda)$ to be the 2-limit of the categories $D_{ctf}^b(X, \Lambda/I^n)$, $n = 1, 2, \ldots$. One has a formalism of $f_!$, f^{-1} and $\overset{\mathbb{L}}{\otimes}$ in $D_{I-sm}^b(X, \Lambda)$ (see [9] for details).

If $\mathcal{L}^\bullet$ is an object of $D_{I-sm}^b(X, \Lambda)$ then by construction $\mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^n$ belongs to $D_{ctf}^b(X, \Lambda/I^n)$ for each positive integer n. If $n \geq m$ then the compatibility of formation of L-functions with change of rings shows that

$$L(X, \mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^n) \equiv L(X, \mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^m) \pmod{I^m}.$$

Thus we may define $L(X, \mathcal{L}^\bullet) \in 1 + T\Lambda[[T]]$ to be the limit of the L-functions $L(X, \mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^n)$.

Now suppose that $f : X \to Y$ is a separable map of finite type $\mathbb{F}_p$-schemes. As before, we have a functor $f_! : D_{I-sm}^b(X, \Lambda) \to D_{I-sm}^b(Y, \Lambda)$. For each positive integer n there is a canonical isomorphism

$$(f_! \mathcal{L}^\bullet) \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^n \overset{\sim}{\longrightarrow} f_!(\mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^n),$$

so that

$$L(Y, f_! \mathcal{L}^\bullet) \equiv L(Y, f_!(\mathcal{L}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/I^n)) \pmod{I^n}.$$

If we now suppose that Λ is a complete local $\mathbb{Z}_p$-algebra with finite residue field (so that in the above discussion $I = \mathfrak{m}$ is the maximal ideal of Λ), by letting n tend to infinity in the above discussion we derive the following corollary of theorem 1.5, which includes Katz's conjecture as a special case (take Λ to be $\mathbb{Z}_p$ and $\mathcal{L}^\bullet$ to be a single lisse p-adic sheaf in degree zero):

COROLLARY 1.8. *Let Λ be a complete local $\mathbb{Z}_p$-algebra, with finite residue field, and maximal ideal $\mathfrak{m}$. Let $f : X \to Y$ be a separated morphism of finite type $\mathbb{F}_p$-schemes. If $\mathcal{L}^\bullet$ is any object of $D^b_{\mathfrak{m}-sm}(X, \Lambda)$ then the ratio of L-functions*

$$L(X, \mathcal{L}^\bullet)/L(Y, f_! \mathcal{L}^\bullet),$$

a priori an element of $1 + T\Lambda[[T]]$, in fact lies in $1 + \mathfrak{m}T\Lambda\langle T\rangle$, where $\Lambda\langle T\rangle$ denotes the $\mathfrak{m}$-adic completion of the polynomial ring $\Lambda[T]$

2. Preliminaries on L-functions

(2.1) Below Λ will be a noetherian $\mathbb{Z}/m\mathbb{Z}$-algebra for some integer m.

For any separated morphism $f : X \to Y$ of finite type $\mathbb{F}_p$-schemes and any object $\mathcal{L}^\bullet$ of $D^b_{ctf}(X, \Lambda)$, let us write

$$Q(f, \mathcal{L}^\bullet) = L(X, \mathcal{L}^\bullet)/L(Y, f_! \mathcal{L}^\bullet).$$

We begin by recalling some standard tools for analyzing such ratios of L-functions. Although the proofs are well-known and quite straightforward, for the sake of completeness we recall them.

LEMMA 2.2. *If $f : X \to Y$ is a separated morphism of finite type $\mathbb{F}_p$-schemes, then for any distinguished triangle*

$$\cdots \to \mathcal{L}^\bullet_1 \to \mathcal{L}^\bullet_2 \to \mathcal{L}^\bullet_3 \to \mathcal{L}^\bullet_1[1] \to \cdots$$

of objects of $D^b_{ctf}(X, \Lambda)$ we have the equality

$$Q(f, \mathcal{L}^\bullet_2) = Q(f, \mathcal{L}^\bullet_1)Q(f, \mathcal{L}^\bullet_3).$$

Proof. This follows immediately from the fact that $f_!$ takes distinguished triangles to distinguished triangles, together with the multiplicativity of L-functions of objects in a distinguished triangle. $\square$

If $f : X \to Y$ is a morphism of schemes and y is a point of Y we let $f_y : X_y \to y$ denote the fibre of f over the point y.

LEMMA 2.3. *For any separated morphism $f : X \to Y$ of finite type $\mathbb{F}_p$-schemes and any object $\mathcal{L}^\bullet$ of $D^b_{ctf}(X, \Lambda)$ we have the equality*

$$Q(f, \mathcal{L}^\bullet) = \prod_{y \in |Y|} Q(f_y, \mathcal{L}^\bullet_{|X_y})$$

Proof. By partitioning the elements of $|X|$ according to their image in $|Y|$, one sees that

$$L(X, \mathcal{L}^\bullet) = \prod_{y \in |Y|} L(X_y, \mathcal{L}^\bullet_{|X_y}).$$

By the proper base-change theorem, for any point y of $|Y|$,

$$(f_! \mathcal{L}^\bullet)_y = f_{y!} \mathcal{L}^\bullet_{|X_y},$$

and so

$$L(Y, f_! \mathcal{L}^\bullet) = \prod_{y \in |Y|} L(y, f_{y!} \mathcal{L}^\bullet_{|X_y}).$$

Thus

$$Q(f, \mathcal{L}^\bullet) = \prod_{y \in |Y|} L(X_y, \mathcal{L}^\bullet_{|X_y}) / L(y, f_{y!} \mathcal{L}^\bullet_{|X_y}) = \prod_{y \in |Y|} Q(f_y, \mathcal{L}^\bullet_{|X_y}),$$

proving the lemma. $\qquad\square$

LEMMA 2.4. *If $f : X \to Y$ and $g : Y \to Z$ are two separated morphisms of finite type $\mathbb{F}_p$-schemes and $\mathcal{L}^\bullet$ is any object of $D^b_{ctf}(X, \Lambda)$ then we have the equality*

$$Q(gf, \mathcal{L}^\bullet) = Q(g, f_! \mathcal{L}^\bullet) Q(f, \mathcal{L}^\bullet).$$

Proof. We compute (beginning with the right-hand side)

$$Q(g, f_! \mathcal{L}^\bullet) Q(f, \mathcal{L}^\bullet) = \frac{L(Y, f_! \mathcal{L}^\bullet)}{L(Z, g_! f_! \mathcal{L}^\bullet)} \frac{L(X, \mathcal{L}^\bullet)}{L(Y, f_! \mathcal{L}^\bullet)} = \frac{L(X, \mathcal{L}^\bullet)}{L(Z, (gf)_! \mathcal{L}^\bullet)} = Q(gf, \mathcal{L}^\bullet).$$

This proves the lemma. $\qquad\square$

LEMMA 2.5. *For any separated quasi-finite morphism $f : X \to Y$ of finite type $\mathbb{F}_p$-schemes and any object $\mathcal{L}^\bullet$ of $D^b_{ctf}(X, \Lambda)$, $Q(f, \mathcal{L}^\bullet) = 1$.*

Proof. Lemma 2.3 reduces us to the situation in which $Y = \operatorname{Spec} \mathbb{F}_{p^d}$ is a point, so that X is either empty, in which case there is nothing to prove, or else zero-dimensional. In this second case we may replace X by its underlying reduced subscheme (via (1.4)), and writing this as a disjoint union of points we reduce to the case that X is also a point, say $X = \operatorname{Spec} \mathbb{F}_{p^{d'}}$, with d dividing d'. Using lemma 2.2 one reduces to the case that $\mathcal{L}^\bullet = \mathcal{L}$ is a single flat Λ-module equipped with a Λ-linear $\phi^{d'}$ action. Then $f_! \mathcal{L}$ is simply the induced module

$$f_! \mathcal{L} = \mathbb{Z}/p^n[\phi^d] \otimes_{\mathbb{Z}/p^n[\phi^{d'}]} \mathcal{L},$$

and the equality of L-functions $L(X, \mathcal{L}) = L(Y, f_! \mathcal{L})$ is an elementary calculation. $\qquad\square$

LEMMA 2.6. *Suppose that $f : X \to Y$ is separated morphism of finite type $\mathbb{F}_p$-schemes, and that $X = S_0 \coprod S_1 \coprod \cdots \coprod S_n$ is a stratification of X by locally closed subsets S_i (more precisely, we take this to mean that each S_i is closed in the union $\coprod_{j=0}^{i} S_j$.) We give each of the S_i their reduced induced scheme structure. If $\mathcal{L}^\bullet$ is an object of $D_{ctf}^b(X, \Lambda)$ then let $\mathcal{L}_{|S_i}^\bullet$ denote the restriction of $\mathcal{L}^\bullet$ to each of the locally closed subsets S_i, and let $f_i : S_i \to Y$ denote the restriction of the morphism f to each of the locally closed subsets S_i. Then*

$$Q(f, \mathcal{L}^\bullet) = \prod_{i=0}^{n} Q(f_i, \mathcal{L}_{|S_i}^\bullet).$$

In particular, this applies if the stratification on X is obtained by pulling back a stratification on Y.

Proof. This follows from lemmas 2.2., 2.4 and 2.5, since immersions are quasi-finite. □

3. Etale sheaves and unit F-crystals

(3.1) We want to explain a generalization of a result of Katz relating étale sheaves and unit F-crystals. Although what we do can be done somewhat more generally, we restrict ourselves to smooth $\mathbb{Z}/p^n$-schemes.

Assume we have a smooth scheme X over $\mathbb{Z}/p^n$, and let Λ be a $\mathbb{Z}/p^n$-algebra. We assume that X is equipped with an endomorphism F lifting the absolute Frobenius on its reduced subscheme.

A unit (Λ, F)-crystal on X is a sheaf $\mathcal{E}$ of finite locally free $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$ modules, equipped with a $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$-linear isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Note that if the map $\mathbb{Z}/p^n \to \Lambda$ factors through $\mathbb{Z}/p^{n'}$ for some positive integer $n' < n$, then the notion of a unit (Λ, F)-crystal depends only on the reduction of X modulo $p^{n'}$.

PROPOSITION 3.2. *Suppose that Λ is noetherian, local, and finite over $\mathbb{Z}/p^n$. There is an equivalence of categories (explicitly described below) between the category of locally constant étale sheaves of finite free Λ-modules on X and the category of (Λ, F)-crystals on X.*

Proof. Let $\mathcal{L}$ be a locally constant étale sheaf of finite free Λ-modules on X (which is, of course equivalent to the data of such a sheaf on the reduction of X modulo p). We associate to $\mathcal{L}$ a (Λ, F)-crystal on X as follows. Consider the étale sheaf $\mathcal{E}_{\acute{e}t} =: \mathcal{L} \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}}$. This is a coherent sheaf of $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}}$-modules, which is in fact locally free of finite rank (since $\mathcal{L}$ is locally free of

finite rank over Λ). Regarding $\mathcal{E}_{\acute{e}t}$ as a coherent sheaf on $\mathcal{O}_{X_{\acute{e}t}}$, we see by [12], 4.1, that $\mathcal{E}_{\acute{e}t}$ comes by pull-back from a Zariski coherent $\mathcal{O}_X$-module $\mathcal{E}$. Moreover, $\mathcal{E}$ is equipped with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. If we denote by Φ the composite $\mathcal{E} \to F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$, then the induced map $\Phi_{\acute{e}t}$ on $\mathcal{E}_{\acute{e}t}$ has fixed subsheaf equal to $\mathcal{L}$. As the formation of $\mathcal{E}$ is functorial in $\mathcal{E}_{\acute{e}t}$, we see that $\mathcal{E}$ is equipped with the structure of a Λ-module, and that the isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is Λ-linear.

To see that this gives $\mathcal{E}$ the structure of a unit (Λ, F)-crystal we have to check that $\mathcal{E}$ is locally free as a $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$-module. If $x \in X$ denote by $\tilde{x}$ an étale point lying over x. Now $\mathcal{E}_{\acute{e}t,\tilde{x}} = \mathcal{E}_x \otimes_{(\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X,x})} (\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t},\tilde{x}})$ is certainly a free $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t},\tilde{x}}$ module. Since $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t},\tilde{x}}$ is faithfully flat over $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X,x}$ ($\mathcal{O}_{X_{\acute{e}t},\tilde{x}}$ being faithfully flat over $\mathcal{O}_{X,x}$), we conclude by descent that $\mathcal{E}_x$ is locally free of constant rank over $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X,x}$. Since Λ is finite over $\mathbb{Z}/p^n$ we see that $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$ is finite over the local ring $\mathcal{O}_{X,x}$, and so is semi-local. Thus the freeness of E_x over $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$ follows from the following (simple) commutative algebra fact: a finite module M over a semi-local ring A which is locally free of constant rank is in fact free. (To see this, observe that $M/rad(A)$ is locally free of constant rank over the direct sum of fields $A/rad(A)$, and so is certainly free. Lifting generators, Nakayama implies that M is itself free.)

Next we construct the quasi-inverse functor. Given a unit (Λ, F)-crystal $\mathcal{E}$ of rank m, we pull it and its Frobenius endomorphism Φ back to the étale site to get a coherent locally free $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}}$-module $\mathcal{E}_{\acute{e}t}$, equipped with a Λ-linear endomorphism $\Phi_{\acute{e}t}$. We have to show that if $\mathcal{L} = \ker(1 - \Phi_{\acute{e}t})$, then the natural map $\mathcal{L} \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}} \to \mathcal{E}_{\acute{e}t}$ is an isomorphism. Indeed, once we have this, then $\mathcal{L}$ is necessarily locally free over Λ, by flat descent, as $\mathcal{O}_{X_{\acute{e}t}}$ is flat over $\mathbb{Z}/p^n$, and $\mathcal{E}_{\acute{e}t}$ is locally free over $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}}$.

For this suppose first that Λ is flat (and necessarily finite) over $\mathbb{Z}/p^n$. In this case we may regard $\mathcal{E}$ as a unit $(\mathbb{Z}/p^n, F)$-crystal, so the map above is an isomorphism by Katz's theorem [12], 4.1.

In general, we can write Λ as a quotient of a finite flat local $\mathbb{Z}_p$-algebra $\tilde{\Lambda}$: as Λ is finite over $\mathbb{Z}_p$, there is a surjection $h : \mathbb{Z}_p[x_1, \ldots, x_r] \to \Lambda$, and for $i = 1, \ldots, r$ there exists a monic polynomial p_i with coefficients in $\mathbb{Z}_p$ such that $h(p_i(x_i)) = 0$. Thus Λ is a quotient of $\mathbb{Z}_p[x_1, \ldots, x_r]/(p_1(x_1), \ldots, p_r(x_r))$, which is a finite flat $\mathbb{Z}_p$-algebra, hence in particular semi-local. Thus we may take $\tilde{\Lambda}$ to be a localization of $\mathbb{Z}_p[x_1, \ldots, x_r]/(p_1(x_1), \ldots, p_r(x_r))$ at a suitable maximal ideal. Then $\tilde{\Lambda}$ is finite flat over $\mathbb{Z}_p$, as $\mathbb{Z}_p$ is a complete local ring (so localization does not destroy finiteness).

To show that we have $\mathcal{L} \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}} \xrightarrow{\sim} \mathcal{E}_{\acute{e}t}$ we may work locally. As $\mathcal{E}$ is a locally free $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$-module, after localizing on X, we may assume that $\mathcal{E}$

is free over $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$, say $\mathcal{E} = (\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X)^m$. Write $\tilde{\mathcal{E}} = (\tilde{\Lambda}/p^n \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X)^m$. In this case we can lift the isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ of free $\Lambda \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$-modules to a morphism $F^*\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of free $\tilde{\Lambda}/p^n \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$-modules, and any such lift is an isomorphism, by Nakayama's lemma. This gives $\tilde{\mathcal{E}}$ the structure of a unit $(\tilde{\Lambda}/p^n, F)$-crystal. We denote by $\tilde{\Phi} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ the induced $\tilde{\Lambda}$-linear, F^*-semi-linear endomorphism of $\tilde{\mathcal{E}}$, and by $\mathcal{E}_{\acute{e}t}, \tilde{\mathcal{E}}_{\acute{e}t}, \Phi_{\acute{e}t}, \tilde{\Phi}_{\acute{e}t}$ the pull-backs to the étale site of X of $\mathcal{E}, \tilde{\mathcal{E}}, \Phi, \tilde{\Phi}$. If x is any étale point of $X_{\acute{e}t}$, we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(1 - \tilde{\Phi}_{\acute{e}t})_x & \longrightarrow & \tilde{\mathcal{E}}_x & \xrightarrow{1 - \tilde{\Phi}_{\acute{e}t}} & \tilde{\mathcal{E}}_x & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(1 - \Phi_{\acute{e}t})_x & \longrightarrow & \mathcal{E}_x & \xrightarrow{1 - \Phi_{\acute{e}t}} & \mathcal{E}_x & \longrightarrow & 0.
\end{array}
$$

Here the bottom row is obtained by applying $\otimes_{\tilde{\Lambda}/p^n} \Lambda$ to the top row. By the previous discussion, applied to the finite flat $\mathbb{Z}/p^n$ algebra $\tilde{\Lambda}/p^n$, we know that the top row is exact, hence the bottom one is also, as $\tilde{\mathcal{E}}_x$ is flat over $\tilde{\Lambda}/p^n$, $\mathcal{O}_X$ being flat over $\mathbb{Z}/p^n$. Thus, letting $\tilde{\mathcal{L}}$ denote $\ker(1 - \tilde{\Phi}_{\acute{e}t})$, we see that the map $\mathcal{L} \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}} \to \mathcal{E}_{\acute{e}t}$ is obtained by applying $\otimes_{\tilde{\Lambda}/p^n} \Lambda$ to the isomorphism $\tilde{\mathcal{L}} \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{X_{\acute{e}t}} \xrightarrow{\sim} \mathcal{E}_{\acute{e}t}$, and so is an isomorphism. $\qquad\square$

(3.3) We may formulate a version of the above theorem for étale sheaves on a smooth $\mathbb{F}_p$-scheme X_0. We may define a unit (Λ, F)-crystal $\mathcal{E}$ to be a locally free sheaf of finite rank $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_0,crys}$-modules on the crystalline site of $X_0/\mathbb{Z}/p^n$ equipped with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ (here F denotes the Frobenius on the crystalline site, and $\mathcal{O}_{X_0,crys}$ is the structure sheaf on the crystalline site). If there exists a smooth $\mathbb{Z}/p^n$ scheme X whose special fibre is X_0 and a lift F of the absolute Frobenius on X, then the two notions are equivalent (exercise (!), but see also [7]), so our new definition is consistent with the previous one. Thus even if no global lift exists we get an equivalence of categories between locally constant étale sheaves of free Λ-modules and unit (Λ, F)-crystals. Indeed to construct this equivalence, we may work locally, and then we may assume ([10], III) that X_0 lifts to a smooth $\mathbb{Z}/p^n$ scheme, equipped with a lift of Frobenius. Now we can appeal to our previous results.

The referee has also remarked that there is a connection between the preceding theorem and the results of Berthelot and Messing which relate finite étale group schemes and Dieudonné crystals (see §2 of [2]).

(3.4) It will be convenient to have an analogue of proposition 3.2 for lisse sheaves. For this let X be a formally smooth p-adic formal scheme over $\mathbb{Z}_p$, equipped with a lift F of the absolute Frobenius on its reduced subscheme, and Λ a finite flat $\mathbb{Z}_p$-algebra. A unit (Λ, F)-crystal $\mathcal{E}$ is defined as in (3.1).

Namely, it is a locally free, coherent $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X$-module $\mathcal{E}$ equipped with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Then we have

COROLLARY 3.5. *Suppose that Λ is local, and finite flat over $\mathbb{Z}_p$. There is an equivalence of categories between p-adic lisse sheaves of Λ-modules on X and unit (Λ, F)-crystals on X.*

Proof. This follows immediately from proposition (3.3) once we note that a unit (Λ, F)-crystal $\mathcal{E}$ satisfies $\mathcal{E} \xrightarrow{\sim} \varprojlim \mathcal{E}/p^n\mathcal{E}$, because Λ is finite over $\mathbb{Z}_p$.

Explicitly, if $\mathcal{L}$ is a p-adic lisse sheaf of Λ-modules, write $\mathcal{L}_n = \mathcal{L}/p^n\mathcal{L}$. Then $\mathcal{L}_n \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\acute{e}t}}$ descends to a coherent, locally free Zariski sheaf of $\Lambda/p^n \otimes_{\mathbb{Z}_p} \mathcal{O}_X$-modules $\mathcal{E}_n$ on $\mathcal{O}_X$, equipped with an isomorphism $F^*\mathcal{E}_n \xrightarrow{\sim} \mathcal{E}_n$. Then we may attach $\mathcal{E} = \varprojlim \mathcal{E}_n$ to $\mathcal{L}$. It is locally free over $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X$ as each of the $\mathcal{E}_n$ is locally free over $\Lambda/p^n \otimes_{\mathbb{Z}_p} \mathcal{O}_X$. (In fact if $U \subset X$ is an open formal subscheme then $\mathcal{E}$ becomes free over U as soon as $\mathcal{E}_1$ is free over U).

Conversely, if $\mathcal{E}$ is a unit (Λ, F)-crystal, then we may attach a locally constant étale sheaf of free Λ/p^n-modules $\mathcal{L}_n$ to $\mathcal{E}/p^n\mathcal{E}$, and set $\mathcal{L} = \varprojlim \mathcal{L}_n$.

If $\mathcal{E}_{\acute{e}t}$ is the pull-back of $\mathcal{E}$ to $X_{\acute{e}t}$, we sometimes abuse notation, and write $\mathcal{E}_{\acute{e}t} = \mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\acute{e}t}}$. $\qquad\square$

4. Proof of theorem 1.5

(4.1) In this section we present the proof of theorem 1.5. We denote by Λ a finite, local, artinian $\mathbb{Z}_p$-algebra. We begin with some preliminaries on formal schemes, and liftings of Frobenius.

(4.2) Let d be a positive integer. We let $\mathbb{P}^d$ denote d-dimensional projective space over $\mathbb{F}_p$, equipped with homogeneous coordinates $Z_0, \ldots, Z_d$. Write $z_i := Z_i/Z_0$ for the affine coordinates corresponding to our choice of homogeneous coordinates; then we have

$$\mathbb{G}_m^d = \operatorname{Spec} \mathbb{F}_p[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}] \subset \mathbb{P}^n.$$

We denote by $\hat{\mathbb{P}}^d$ the formal, d-dimensional projective space over $\mathbb{Z}_p$. It is obtained by completing the d-dimensional projective space $\mathbb{P}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ along the special fibre $p = 0$. The underlying topological spaces of $\hat{\mathbb{P}}^d$ and $\mathbb{P}^d$ are equal. We denote by $\mathcal{O}_{\hat{\mathbb{P}}^d}$ the structure sheaf of $\hat{\mathbb{P}}^d$, thought of as a sheaf on the underlying topological space of $\mathbb{P}^d$, and if U is any open subset of $\mathbb{P}^d$, we write $\mathcal{O}_{\hat{U}} = \mathcal{O}_{\hat{\mathbb{P}}^d}|_U$. We will sometimes write $\hat{U}$ to denote U considered with the formal scheme structure given by $\mathcal{O}_{\hat{U}}$. We will employ these notations especially in the case that U is an open subset of $\mathbb{G}_m^d \subset \mathbb{P}^d$.

(4.3) Let X be of finite type over $\mathbb{F}_p$. If x is a closed point of such a scheme we let $\kappa(x)$ denote the residue field of x. This is a finite field of some degree

$d(x)$ over $\mathbb{F}_p$. We denote the ring of Witt vectors of $\kappa(x)$ by W_x. We again use σ (respectively ϕ) to denote the Frobenius automorphism of W_x (respectively its inverse automorphism). If we choose an isomorphism of $\kappa(x)$ with $\mathbb{F}_{p^{d(x)}}$, this induces an isomorphism of W_x with $W(\mathbb{F}_{p^{d(x)}})$, which is equivariant with respect to the automorphism σ of each of these rings.

Consider the endomorphism F of $\hat{\mathbb{P}}^d$ given by $Z_i \mapsto Z_i^p$. This is a lift of the absolute Frobenius to $\hat{\mathbb{P}}^d$, which induces an endomorphism of each formal open $\hat{U} \subset \hat{\mathbb{P}}^d$, and in particular of the formal d-dimensional multiplicative group

$$\hat{\mathbb{G}}_m^d = \mathrm{Spf}(\mathbb{Z}_p\langle z_1, z_1^{-1}, \dots, z_d, z_d^{-1}\rangle) \subset \hat{\mathbb{P}}^d,$$

where it is given by $F^*(z_i) = z_i^p$. (Here $\mathbb{Z}_p\langle z_1, z_1^{-1}, \dots, z_d, z_d^{-1}\rangle$ denotes the p-adic completion of the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[z_1, z_1^{-1}, \dots, z_d, z_d^{-1}]$).

If k is a finite field, and x is a k-valued point of $\hat{\mathbb{G}}_m^d$ corresponding to the morphism

$$\mathbb{F}_p[z_1, z_1^{-1}, \dots, z_d, z_d^{-1}] \to k$$

of $\mathbb{F}_p$-algebras, then x lifts to a morphism

$$\mathbb{Z}_p\langle z_1, z_1^{-1}, \dots, z_d, z_d^{-1}\rangle \to W(k)$$

(the Teichmüller lifting), characterized by the property that the diagram

$$
\begin{array}{ccc}
\mathbb{Z}_p\langle z_1, z_1^{-1}, \dots, z_d, z_d^{-1}\rangle & \longrightarrow & W(k) \\
\Big\downarrow{\scriptstyle F^*} & & \Big\downarrow{\scriptstyle \sigma} \\
\mathbb{Z}_p\langle z_1, z_1^{-1}, \dots, z_d, z_d^{-1}\rangle & \longrightarrow & W(k)
\end{array}
$$

commutes. We denote this $W(k)$-valued point of $\hat{\mathbb{G}}_m^d$ by $\tilde{x}$.

In particular this applies if x is a closed point of X and we take k to be $\kappa(x)$, the residue field at x, so that x is naturally a $\kappa(x)$-valued point of $\mathbb{G}_m^d$. If we choose an embedding of $\kappa(x)$ into a subfield $\mathbb{F}_{p^n}$ of $\overline{\mathbb{F}}_p$ (for some positive integer n) then we may regard x as an $\mathbb{F}_{p^n}$-valued point of $\mathbb{G}_m^d$, and the corresponding embedding of W_x as a subring of $W(\mathbb{F}_{p^n})$ realizes $\tilde{x}$ as a $W(\mathbb{F}_{p^n})$-valued point of $\hat{\mathbb{G}}_m^d$.

(4.4) We now begin the proof of theorem 1.5. Consider an arbitrary separated morphism $f : X \to Y$ of finite type $\mathbb{F}_p$-schemes. We may find a stratification of Y by locally closed affine schemes T_i. If $S_i = f^{-1}(T_i)$ then lemma 2.6 shows that it suffices to prove theorem 1.5 for each of the morphisms $f_i : S_i \to T_i$ obtained by restricting f to the subschemes S_i.

Since each T_i is affine, the structural morphisms $g_i : T_i \to \mathrm{Spec}\,\mathbb{F}_p$ and $h_i = f_i g_i : S_i \to \mathbb{F}_p$ are separated, and lemma 2.4 shows that for any object $\mathcal{L}^\bullet$ of $D_{ctf}^b(S_i, \Lambda)$,

$$Q(f_i, \mathcal{L}^\bullet) = Q(h_i, \mathcal{L}^\bullet)/Q(g_i, f_{i!}\mathcal{L}^\bullet).$$

Thus it suffices to prove theorem 1.5 for the morphisms g_i and h_i.

(4.5) The preceding section shows that we are reduced to considering theorem 1.5 in the case that $f : X \to \mathrm{Spec}\,\mathbb{F}_p$ is the structural morphism of a finite-type separated $\mathbb{F}_p$-scheme. We will prove theorem 1.5 by induction on the dimension of X. Thus we assume that the result holds for all separated $\mathbb{F}_p$-schemes of finite type and of dimension less than that of X.

Nothing is changed if we replace X by its reduced subscheme. We find a dense open subscheme U of X such that each connected component of U' admits a quasi-finite and dominant morphism to $\mathbb{G}_m^d$ for some natural number d (which is necessarily the dimension of this connected component, and so is less than or equal to the dimension of X). To find such a U we choose a dense smooth open subscheme of X. Each connected component of this smooth open subscheme is irreducible, and contains a non-empty (and so dense) open subset which admits an étale map to some $\mathbb{G}_m^d$. We take U be the union of these open subsets. Since $X \setminus U$ has lower dimension than X, lemma 2.6 together with our inductive hypothesis shows that it suffices to prove theorem 1.5 for each connected component of U. Fix one such connected component, V say, equipped with a quasi-finite dominant morphism $g : V \to \mathbb{G}_m^d$. Let $h : \mathbb{G}_m^d \to \mathrm{Spec}\,\mathbb{F}_p$ be the structural morphism of $\mathbb{G}_m^d$, and $k = gh : V \to \mathrm{Spec}\,\mathbb{F}_p$ the structural morphism of V. For any object $\mathcal{L}^\bullet$ of $D_{ctf}^b(V, \Lambda)$, lemmas 2.4 and 2.5 show that

$$Q(k, \mathcal{L}^\bullet) = Q(h, g_! \mathcal{L}^\bullet) Q(g, \mathcal{L}^\bullet) = Q(h, g_! \mathcal{L}^\bullet).$$

So it suffices to prove theorem 1.5 when $X = \mathbb{G}_m^d$, and $Y = \mathrm{Spec}(\mathbb{F}_p)$.

(4.6) Let $\mathcal{L}^\bullet$ be any object of $D_{ctf}^b(\mathbb{G}_m^d, \Lambda)$. As observed in 1.2, we may assume that each of the finitely many non-zero $\mathcal{L}^i$ is a constructible étale sheaf of flat Λ-modules. Lemma 2.2 allows us to verify theorem 1.5 for each of the $\mathcal{L}^i$ separately. Thus we assume that $\mathcal{L}^\bullet = \mathcal{L}$ is a single constructible étale sheaf of flat Λ-modules.

We may find a non-empty (and hence dense) open affine subscheme U of $\mathbb{G}_m^d$ such that $\mathcal{L}$ is locally constant when restricted to U. We form the tensor product $\mathcal{E} = \mathcal{L}_{|U} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{U}}$. Note that $\mathcal{E}$ is a coherent $\mathcal{O}_{\hat{U}}/p^n$-sheaf for sufficiently large integers n. Hence it descends to a coherent Zariski sheaf of locally free $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{U}}$-modules. In particular, U contains a non-empty (and hence dense) open subset V over which $\mathcal{E}$ becomes free as a $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{U}}$-module. We may find a global section a of $\mathcal{O}_{\mathbb{G}_m^d}$ such that the zero-set $Z(a)$ of a is a proper subset of $\mathbb{G}_m^d$ containing the complement of V. Writing $D(a) = \mathbb{G}_m^d \setminus Z(a) \subset V$, we see that $D(a)$ is non-empty and that $\mathcal{E}$ is free over $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{U}}$, when restricted to $D(a)$. Since $Z(a)$ has dimension less than d, our induction hypothesis, together with lemma 2.6, shows that it suffices to prove that $Q(f, \mathcal{L}_{|D(a)})$ lies in $1 + \mathfrak{m}T\Lambda[T]$.

(4.7) Since we will only be dealing with the locally constant sheaf $\mathcal{L}_{|D(a)}$ from now on, we write simply $\mathcal{L}$ and $\mathcal{E}$ rather than $\mathcal{L}_{|D(a)}$ and $\mathcal{E}_{|D(a)}$, so that $\mathcal{E} = \mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{D(\hat{a})}$.

Recall the endomorphism F of $D(\hat{a})$ defined in section (4.3) by restricting the endomorphism F of $\hat{\mathbb{P}}^d$. The tensor product of the identity morphism on $\mathcal{L}$ and the endomorphism $1 \otimes F^*$ of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{D(\hat{a})}$ induces an $1 \otimes F^*$-linear endomorphism of $\mathcal{E}$, which we denote by Φ, and endows $\mathcal{E}$ with the structure of a unit (Λ, F)-crystal, from which $\mathcal{L}$ may be recovered as the étale subsheaf of Φ-invariants (see section 3). For convenience we will often abbreviate $1 \otimes F^*$ by F^*.

The global section a is an element of $\mathbb{F}_p[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$. Let $\tilde{a}$ denote a lift of a to a global section of $\mathcal{O}_{\hat{\mathbb{G}}_m^d}$. Then *a priori* $\tilde{a}$ is an element of $\mathbb{Z}_p\langle z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}\rangle$, but we may and do choose $\tilde{a}$ to be an element of $\mathbb{Z}_p[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$ (since our only concern is that it reduce modulo p to a).

Choosing a basis of $\mathcal{E}$ induces an isomorphism $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{D(\hat{a})}^m \xrightarrow{\sim} \mathcal{E}$. With respect to this basis the F^*-linear endomorphism Φ may be written in the form $\Phi = (r_{ij}) \circ F^*$, where (r_{ij}) is an invertible $m \times m$ matrix of elements of $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}, \tilde{a}^{-1}]$.

As in the proof of proposition 3.2, there exists a finite flat local $\mathbb{Z}_p$-algebra $\tilde{\Lambda}$, and a surjection $\tilde{\Lambda} \to \Lambda$. We write $\tilde{\mathfrak{m}}$ for the maximal ideal of $\tilde{\Lambda}$. Now we may lift each of the rational functions r_{ij} to an element $\tilde{r}_{ij}$ of $\tilde{\Lambda}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}, \tilde{a}^{-1}]$. Using these lifts we may form the following unit $(\tilde{\Lambda}, F)$-crystal on the open subset $D(\hat{a})$ of $\hat{\mathbb{G}}_m^d$:

$$\tilde{\Phi} : \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{D(\hat{a})}^m \xrightarrow{(\tilde{r}_{ij}) \circ F^*} \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{D(\hat{a})}^m,$$

which reduces to $\mathcal{E}$ after $\otimes_{\tilde{\Lambda}} \Lambda$. Denote this unit $(\tilde{\Lambda}, F)$-crystal by $\tilde{\mathcal{E}}$, and let $\tilde{\mathcal{L}}$ denote the corresponding lisse étale $\mathbb{Z}_p$-sheaf on $D(\hat{a})$ obtained as the $\tilde{\Phi}$-fixed étale subsheaf of the étale sheaf induced by $\tilde{\mathcal{E}}$. We saw in section 3 that $\tilde{\mathcal{L}}$ is a lisse sheaf of Λ-modules of rank m. We will prove that the ratio $L(D(a), \tilde{\mathcal{L}})/L(\operatorname{Spec} \mathbb{F}_p, g_! \tilde{\mathcal{L}})$, belongs to $1 + \tilde{\mathfrak{m}} T \tilde{\Lambda}\langle T\rangle$. Specializing via the map $\tilde{\Lambda} \to \Lambda$, we conclude that

$$Q(f, \mathcal{L}) \in 1 + \mathfrak{m} T \Lambda/p^n[T],$$

completing the proof of theorem 1.5

(4.8) In this section we make a series of changes of basis of the unit F-crystal and its lift $\tilde{\mathcal{E}}$ in order to ensure that the matrices (r_{ij}) and $(\tilde{r}_{ij})$ have certain properties. Slightly abusing notation, we will again denote by $\tilde{a}$ the

image of $\tilde{a}$ in $\Lambda[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$. We begin by observing that

$$F^* \tilde{a} \equiv \tilde{a}^p \pmod{p}$$
$$\equiv 0 \pmod{\tilde{a}^p, p}.$$

Thus, if n is an integer such that $p^n = 0$ in Λ, then for some sufficiently large natural number N we have

$$(F^* \tilde{a})^N \equiv 0 \pmod{\tilde{a}^{N+1}, p^n},$$

and so

$$(F^* \tilde{a}/\tilde{a})^N \in \tilde{a}\Lambda[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}].$$

If we let M be the maximal power of $\tilde{a}$ occurring in the denominators of the elements r_{ij} of $\Lambda[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}, \tilde{a}^{-1}]$ then we see that

$$(F^* \tilde{a}/\tilde{a})^{(M+1)N} r_{ij} \in \tilde{a}\Lambda[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$$

for each pair i, j. Consider the commutative diagram

$$
\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{D(a)}} & \xrightarrow{(F^* \tilde{a}/\tilde{a})^{(M+1)N}(r_{ij}) \circ F^*} & \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{D(a)}} \\
\Big\downarrow {\tilde{a}^{(M+1)N}} & & \Big\downarrow {\tilde{a}^{(M+1)N}} \\
\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{D(a)}} & \xrightarrow{(r_{ij}) \circ F^*} & \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{D(a)}}
\end{array}
$$

in which the horizontal arrows are F^*-linear maps and the vertical maps are $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{D(a)}}$-linear isomorphisms (since they are simply multiplication by $\tilde{a}^{(M+1)N}$, a unit on $\widehat{D(a)}$). This shows that we may choose a basis for $\mathcal{E}$ with respect to which the matrix (r_{ij}) of Φ has entries in $\tilde{a}\Lambda[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$, and from now on we assume that we have made such a choice, so that $\mathcal{L}$ is described by the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{D(a)}} \xrightarrow{1-(r_{ij}(z)) \circ F^*} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{D(a)}} \longrightarrow 0,$$

in which $(r_{ij}(z))$ is a matrix of of polynomials vanishing precisely at the points of $Z(a)$.

Now as in (4.7) we may lift the polynomials r_{ij}, this time to elements $\tilde{r_{ij}}$ of $\tilde{a}\tilde{\Lambda}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$, to obtain a lift $\tilde{\mathcal{E}}$ of $\mathcal{E}$ and a lift $\tilde{\mathcal{L}}$ of $\mathcal{L}$. Let s be an integer chosen so that each $(z_1 \cdots z_d)^s \tilde{r_{ij}}$ belongs to $\tilde{\Lambda}[z_1, \ldots, z_d]$, and choose a second integer t so that $(p-1)t > s$. Then the diagram

$$
\begin{array}{ccc}
\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{\mathbb{G}_m^d}} & \xrightarrow{(z_1 \cdots z_d)^{(p-1)t}(\tilde{r}_{ij}) \circ F^*} & \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{\mathbb{G}_m^d}} \\
\Big\downarrow {(z_1 \cdots z_d)^t} & & \Big\downarrow {(z_1 \cdots z_d)^t} \\
\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}^m_{\widehat{\mathbb{G}_m^d}} & \xrightarrow{(\tilde{r}_{ij}) \circ F^*} & \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathbb{G}_m^d}},
\end{array}
$$

in which the horizontal arrows are F^*-linear morphisms and the vertical arrows are $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{G}}_m^d}$-linear isomorphisms (since $(z_1 \cdots z_d)^t$ is invertible on $\hat{\mathbb{G}}_m^d$), commutes. The matrix $(z_1 \cdots z_d)^{(p-1)t}(\tilde{r}_{ij})$ consists of elements of

$$(4.9) \qquad (z_1 \cdots z_d)\tilde{\Lambda}[z_1, \ldots, z_d] \cap \tilde{a}\tilde{\Lambda}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}],$$

by construction. Thus we see that we may choose a basis of $\tilde{\mathcal{E}}$ so that the matrix $\tilde{r}_{ij}$ describing the F^*-linear endomorphism $\tilde{\Phi}$ of $\tilde{\mathcal{E}}$ consists of elements of the intersection (4.9), and from now on we assume that we have done this, and that the matrix $\tilde{r}_{ij}$ is chosen with respect to this basis.

Let g denote the structural morphism $g : \mathbb{P}^d \to \operatorname{Spec} \mathbb{F}_p$, let h denote the open immersion of $\mathbb{F}_p$-schemes $h : D(a) \to \mathbb{P}^d$, underlying the open immersion of p-adic formal schemes $\hat{D(a)} \to \hat{\mathbb{P}}^d$, so that $f = gh$, and let $\mathcal{O}(-1)_{\hat{\mathbb{P}}^d}$ denote the ideal sheaf of the formal hyper-plane at infinity of $\hat{\mathbb{P}}^d$ (that is, the hyper-plane described by the equation $Z_0 = 0$). Let u be an integer chosen so that $(p-1)u$ is greater than the degree of each of the polynomials $\tilde{r}_{ij}$. By virtue of this choice of u, the matrix $(\tilde{r}_{ij})$ induces an F^*-linear endomorphism $(\tilde{r}_{ij}) \circ F^*$ of $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}$. Furthermore, since the $\tilde{r}_{ij}$ are divisible by each z_i as well as by $\tilde{a}$, and since $(p-1)u$ is in fact greater than (rather than just equal to) the degree of any of the $\tilde{r}_{ij}$, we see that the étale sheaf of $(\tilde{r}_{ij}) \circ F^*$-invariants of $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m$ is exactly equal to $h_!\tilde{\mathcal{L}}$. More precisely, write $\tilde{\mathcal{L}}_n = \tilde{\mathcal{L}}/p^n\tilde{\mathcal{L}}$, and $\tilde{\Lambda}_n = \tilde{\Lambda}/p^n\tilde{\Lambda}$. Then for each $n \geq 1$, we have have a short exact sequence of étale sheaves

$$0 \longrightarrow h_!\tilde{\mathcal{L}}_n \longrightarrow \tilde{\Lambda}_n \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m \xrightarrow{1 - (\tilde{r}_{ij}) \circ F^*} \tilde{\Lambda}_n \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m \longrightarrow 0.$$

This is the exact sequence which we will use to compute $f_!\tilde{\mathcal{L}} = g_! h_!\tilde{\mathcal{L}}$, and hence to compute its L-function, in order to compare it with the L-function of $\tilde{\mathcal{L}}$. (The construction of this exact sequence is an application of the technique of [4], lemma 4.5, p. 120. It is greatly generalized in [6], [7].)

(4.10) If we apply g_* to the short exact sequence of étale sheaves constructed above then keeping in mind that for coherent sheaves étale push-forward agrees with the coherent push-forward, and that for $i < d$,

$$R^i f_*(\tilde{\Lambda}_n \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}) = H^i(\hat{\mathbb{P}}^d, \tilde{\Lambda}_n \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}) = 0,$$

we obtain the exact sequence

$$0 \longrightarrow R^d f_!\tilde{\mathcal{L}}_n \longrightarrow H^d(\mathbb{P}^d, \tilde{\Lambda}_n \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m)$$

$$\xrightarrow{1 - H^d((\tilde{r}_{ij}) \circ F^*)} H^d(\mathbb{P}^d, \tilde{\Lambda}_n \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m) \longrightarrow R^{d+1} f_!\tilde{\mathcal{L}}_n \longrightarrow 0$$

of étale sheaves on $\operatorname{Spec} \mathbb{F}_p$. If we take the stalks of this exact sequence over the geometric point $\operatorname{Spec} \overline{\mathbb{F}}_p$ of $\operatorname{Spec} \mathbb{F}_p$, and pass to the inverse limit over n,

then we obtain the exact sequence

$$0 \longrightarrow (R^d f_! \tilde{\mathcal{L}})_{\overline{\mathbb{F}}_p} \longrightarrow H^d(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \overline{W}$$

$$\xrightarrow{1-H^d((\tilde{r}_{ij}) \circ F^*) \otimes \sigma} H^d(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \overline{W}$$

$$\longrightarrow (R^{d+1} f_! \tilde{\mathcal{L}})_{\overline{\mathbb{F}}_p} \longrightarrow 0.$$

The morphism $1 - H^d((\tilde{r}_{ij}) \circ F^*) \otimes \sigma$ is surjective, and so we see that $R^{d+1} f_! \tilde{\mathcal{L}} = 0$.

LEMMA 4.11. *The ratio*

$$L(\operatorname{Spec} \mathbb{F}_p, f_! \tilde{\mathcal{L}})/\det_{\tilde{\Lambda}}(1 - H^d((\tilde{r}_{ij}) \circ F^*)T, H^d(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m))^{(-1)^{d+1}}$$

is an element of $1 + \tilde{\mathfrak{m}} T \tilde{\Lambda} \langle T \rangle$, and is a rational function.

Proof. Let us denote the finite rank free $\tilde{\Lambda}$-module $H^d(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m)$ by M, for simplicity of notation, and let us denote the $\tilde{\Lambda}$-linear endomorphism $H^d((\tilde{r}_{ij}) \circ F^*)$ of M by Ψ. A little linear algebra shows that M has a canonical direct sum decomposition $M = M_{unit} \oplus M_{nil}$, determined by the property that M_{unit} is the maximal Ψ-invariant $\tilde{\Lambda}$-submodule of M on which Ψ acts surjectively (or equivalently, bijectively), while M_{nil} is the maximal Ψ-invariant $\tilde{\Lambda}$-submodule of M on which Ψ acts topologically nilpotently. Indeed, it is clear that such decomposition exists for M as a $\mathbb{Z}_p$-module. However, as it is canonical M_{unit} and M_{nil} are $\tilde{\Lambda}$ stable. Thus they are projective, and hence free $\tilde{\Lambda}$-modules.

We extend the endomorphism Ψ to the σ-linear endomorphism $\Psi \otimes \sigma$ of $M \otimes_{\mathbb{Z}_p} \overline{W}$. Since Ψ acts topologically nilpotently on M_{nil}, $1 - \Psi \otimes \sigma$ acts bijectively on $M_{nil} \otimes_{\mathbb{Z}_p} \overline{W}$. Thus we may replace the exact sequence

$$0 \to (R^d f_! \tilde{\mathcal{L}})_{\overline{\mathbb{F}}_p} \to M \otimes_{\mathbb{Z}_p} \bar{W} \xrightarrow{1-\Psi \otimes \sigma} M \otimes_{\mathbb{Z}_p} \bar{W} \to 0$$

of section (4.10) by the exact sequence

$$0 \to (R^d f_! \tilde{\mathcal{L}})_{\overline{\mathbb{F}}_p} \to M_{unit} \otimes_{\mathbb{Z}_p} \bar{W} \xrightarrow{1-\Psi \otimes \sigma} M_{unit} \otimes_{\mathbb{Z}_p} \bar{W} \to 0.$$

Since Ψ acts bijectively on M_{unit}, we see that Ψ makes M_{unit} into a unit $(\tilde{\Lambda}, F)$-crystal, and that $R^d f_! \tilde{\mathcal{L}}$ is the corresponding lisse p-adic étale sheaf on $\operatorname{Spec} \mathbb{F}_p$. Thus

$$L(\operatorname{Spec} \mathbb{F}_p, f_! \tilde{\mathcal{L}}) = \det_{\tilde{\Lambda}}(1 - \phi T, (R^d f_! \tilde{\mathcal{L}})_{\overline{\mathbb{F}}_p})^{(-1)^{d+1}}$$

$$= \det_{\tilde{\Lambda}}(1 - \Psi T, M_{unit})^{(-1)^{d+1}}$$

$$= \det_{\tilde{\Lambda}}(1 - \Psi T, M)^{(-1)^{d+1}} / \det_{\tilde{\Lambda}}(1 - \Psi T, M_{nil})^{(-1)^{d+1}}.$$

Thus the lemma will be proved once we show that $\det_{\tilde{\Lambda}}(1 - \Psi T, M_{nil})^{(-1)^{d+1}}$ belongs to $1 + \tilde{\mathfrak{m}}\tilde{\Lambda}\langle T \rangle$. But this follows from the fact that Ψ acts topologically nilpotently on M_{nil}, and so nilpotently on $M/\tilde{\mathfrak{m}}$, so that the characteristic polynomial of Ψ on M_{nil} satisfies the congruence

$$\det_{\tilde{\Lambda}}(1 - \Psi T, M_{nil}) \equiv 1 \pmod{\tilde{\mathfrak{m}}}. \qquad \square$$

(4.12) We now turn to describing the L-function of $\mathcal{L}$ in terms of the matrix $(\tilde{r}_{ij})$.

LEMMA 4.13. *The L-function of $\tilde{\mathcal{L}}$ on $D(a)$ is determined by the formula*

$$L(D(a), \tilde{\mathcal{L}}) = \prod_{x \in |D(a)|} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}.$$

(Recall from section 4.3 that $\tilde{x}$ denotes the Teichmüller lift of the closed point x.)

Proof. This follows from the fact that $\mathcal{O}_{\widehat{D(a)}}^m$ equipped with the F^*-linear endomorphism $(\tilde{r}_{ij}) \circ F^*$ is the unit F-crystal $\tilde{\mathcal{E}}$ corresponding to $\tilde{\mathcal{L}}$. More precisely, the Artin-Schreier short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{\widehat{D(a)}}^m \xrightarrow{1 - (\tilde{r}_{ij}) \circ F^*} \mathcal{O}_{\widehat{D(a)}}^m \longrightarrow 0$$

shows that for any point x of $D(a)$, if $\bar{x}$ denotes an étale point lying over x, then

$$\det(1 - \phi^{d(x)} T^{d(x)}, \mathcal{L}_{\bar{x}}) = \det(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m). \qquad \square$$

COROLLARY 4.14. *The L-function of $\tilde{\mathcal{L}}$ on $D(a)$ agrees with the product*

$$\prod_{x \in |\mathbb{G}_m^d|} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}$$

up to multiplication by an element of $1 + pT\tilde{\Lambda}\langle T \rangle$.

Proof. Lemma 4.13 shows that the ratio of $L(D(a), \tilde{\mathcal{L}})$ and the product in the statement of the corollary is equal to

$$\prod_{x \in |Z(a)|} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}.$$

Thus it suffices to show that this product belongs to $1 + p\tilde{\Lambda}\langle T \rangle$.

For any positive integer δ, let $|Z(a)|_\delta$ denote the (finite!) subset of $|Z(a)|$ consisting of those closed points x for which $d(x) = \delta$. By construction, each

$\tilde{r}_{ij}$ is an element of $\tilde{a}\tilde{\Lambda}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$. Thus for each x in $|Z(a)|_\delta$, $\tilde{r}_{ij}(\tilde{x})$ lies in $p\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x$, so that the entries of

$$((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} = ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^\delta$$

lie in $p^\delta\tilde{\Lambda}$, and the finite product

$$\prod_{x \in |Z(a)|_d} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((r_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}$$

lies in $1 + p^\delta T\Lambda[[p^\delta T]] \subset 1 + p^\delta T^\delta \tilde{\Lambda}\langle T^\delta \rangle$. From this we conclude that

$$\prod_{x \in |Z(a)|} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((r_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}$$
$$= \prod_{\delta > 0} \prod_{x \in |Z(a)|_\delta} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((r_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}$$

lies in $1 + pT\tilde{\Lambda}\langle T \rangle$, thus proving the corollary.   □

(4.15) Theorem 1.5 now follows from lemma 4.11 and corollary 4.14, together with the following result, whose proof is the subject of section 5.

PROPOSITION 4.16.   *Let $P(T)$ denote*

$$\det_{\tilde{\Lambda}}(1 - H^d((\tilde{r}_{ij}) \circ F^*)T, H^d(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m)) \in 1 + T\tilde{\Lambda}[T].$$

Then

$$\prod_{x \in |\mathbb{G}_m^d|} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}$$
$$= \prod_{i=0}^{d} P(p^i T)^{(-1)^{d+1-i}\binom{d}{i}},$$

and so in particular, the ratio

$$\left(\prod_{x \in |\mathbb{G}_m^d|} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1} \right) / P(T)^{(-1)^{d+1}}$$

lies in $1 + pT\tilde{\Lambda}\langle T \rangle$, and is a rational function.

5. Proof of proposition 4.16

(5.1) In order to prove proposition 4.16, we begin by describing the morphism which is Grothendieck-Serre dual to the map $1 - H^d((\tilde{r}_{ij}) \circ F^*)$. The free $\tilde{\Lambda}$-module $H^d(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m) \xrightarrow{\sim} H^d(\mathbb{P}^d, \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \tilde{\Lambda}$ has

Grothendieck-Serre dual equal to $H^0(\mathbb{P}^d, \Omega^d(u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \tilde{\Lambda}$. Let $\mathcal{C}$ denote the Cartier operator on $\Omega_{\mathbb{P}^d}^d$ corresponding to the lift of Frobenius F, defined by the formula

$$\mathcal{C}(z_1^{i_1} \cdots z_d^{i_d} \frac{dz_1 \wedge \cdots \wedge dz_d}{z_1 \cdots z_d})$$
$$= \begin{cases} z_1^{\frac{i_1}{p}} \cdots z_d^{\frac{i_d}{p}} \frac{dz_1 \wedge \cdots \wedge dz_d}{z_1 \cdots z_d} & \text{if } i_j \equiv 0 \pmod{p} \text{ for all } j \\ 0 & \text{if } i_j \not\equiv 0 \pmod{p} \text{ for some } j. \end{cases}$$

By virtue of our choice of u, the composite $\mathcal{C} \circ (\tilde{r}_{ij})$ defines an endomorphism of $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)$:

$$\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u) \xrightarrow{\mathcal{C} \circ (\tilde{r}_{ij})} \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u),$$

and the dual of the endomorphism $H^d((\tilde{r}_{ij})) \circ F^*$ of $H^d(\mathbb{P}^d, \mathcal{O}(-u)_{\hat{\mathbb{P}}^d}^m)$ is the endomorphism $H^0(\mathcal{C} \circ (\tilde{r}_{ij}))$ of $H^0(\mathbb{P}^d, \Omega^d(u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \tilde{\Lambda}$. Since a matrix and its adjoint have the same characteristic polynomial we see that

$$P(T) = \det_{\tilde{\Lambda}}(1 - H^0(\mathcal{C} \circ (\tilde{r}_{ij}))T, H^0(\mathbb{P}^d, \Omega^d(u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \tilde{\Lambda}).$$

(5.2) The usual formula for the logarithm of a determinant shows that

$$(5.3) \quad \log P(T) = -\sum_{n=1}^{\infty} \text{trace}_{\tilde{\Lambda}}(H^0(\mathcal{C} \circ (\tilde{r}_{ij}))^n, H^0(\mathbb{P}^d, \Omega^d(u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \tilde{\Lambda}) \frac{T^n}{n}.$$

For any sections a of $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^d}$ and ω of $\Omega_{\hat{\mathbb{P}}^d}^d$ we have the identity

$$a\mathcal{C}\omega = \mathcal{C}F^*(a)\omega,$$

and so for any positive integer n we have

$$(\mathcal{C} \circ (\tilde{r}_{ij}))^n = \mathcal{C}^n \circ \prod_{i=0}^{n-1}((F^*)^i \tilde{r}_{ij}).$$

This allows us to rewrite (5.2) in the form
(5.4)
$$\log P(T) = -\sum_{n=1}^{\infty} \text{trace}_{\tilde{\Lambda}}(H^0(\mathcal{C}^n \circ \prod_{k=0}^{n-1}(F^*)^k(\tilde{r}_{ij})), H^0(\mathbb{P}^d, \Omega^d(u)_{\hat{\mathbb{P}}^d}^m) \otimes_{\mathbb{Z}_p} \tilde{\Lambda}) \frac{T^n}{n}.$$

(5.5) Fix a positive integer n, and suppose that (f_{ij}) is any $m \times m$ matrix of elements of $(z_1 \cdots z_d)\tilde{\Lambda}[z_1, \dots, z_d]$, such that each of the f_{ij} is of degree less than $(p^n - 1)u$. (For example, $\prod_{k=0}^{n-1}((F^*)^k \tilde{r}_{ij})$ is such a matrix.) We have a morphism

$$\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d}^m \xrightarrow{\mathcal{C}^n \circ (f_{ij})} \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d}^m,$$

which induces an endomorphism $H^0(\mathcal{C}^n \circ (f_{ij}))$ of $H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d}^m)$. We may also form the sum of matrices

$$\sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} (f_{ij}(\tilde{x})),$$

which will be an $m \times m$ matrix of elements of $\mathbb{Z}_p$. In this situation we have the following lemma, which is a slight modification of lemma 2 of [5]:

LEMMA 5.6. *For any matrix (f_{ij}) as above we have the following formula:*

$$\text{trace}_{\tilde{\Lambda}}(H^0(\mathcal{C}^n \circ (f_{ij})), H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d}^m))$$
$$= \frac{1}{(p^n-1)^d} \text{trace}_{\tilde{\Lambda}}\left(\sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} (f_{ij}(\tilde{x})), \tilde{\Lambda}^m \right).$$

(Recall from section 4.3 that $\tilde{x}$ denotes the Teichmüller lift of the $\mathbb{F}_{p^n}$-valued point x of $\mathbb{G}_m^d$.)

Proof. We first reduce the proof of the formula to the case $m = 1$, as follows. It is clear that the trace on the left hand side is given by the formula

$$\text{trace}_{\tilde{\Lambda}}(H^0(\mathcal{C}^n \circ (f_{ij})), H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d}^m))$$
$$= \sum_{i=1}^{m} \text{trace}_{\tilde{\Lambda}}(H^0(\mathcal{C}^n \circ f_{ii}), H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d})),$$

while the trace on the right hand side is given by the formula

$$\frac{1}{(p^n-1)^d} \sum_{i=1}^{m} \sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} f_{ii}(\tilde{x}).$$

Thus if we knew that for any f in $z_1 \cdots z_d \tilde{\Lambda}[z_1, \ldots, z_d]$ of degree less than $(p^n - 1)u$ we had the equation

(5.7) $\text{trace}_{\tilde{\Lambda}}(H^0(\mathcal{C}^n \circ f), H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d})) = \dfrac{1}{(p^n-1)^d} \displaystyle\sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} f(\tilde{x}),$

the lemma would follow. This is what we now prove.

The set of all such f form a $\tilde{\Lambda}$-submodule of $\tilde{\Lambda}[z_1, \ldots, z_d]$, and both sides of equation (5.7) are $\tilde{\Lambda}$-linear in f. Thus it suffices to prove the formula in the case that f is a single monomial

$$f = z_1^{\alpha_1} \cdots z_d^{\alpha_d},$$

with

(5.8) $\qquad\qquad 0 < \alpha_1, \ldots, \alpha_d, \quad \alpha_1 + \cdots + \alpha_d < (p^n - 1)u.$

The $\tilde{\Lambda}$-module $H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d})$ has as a basis the differentials

$$z_1^{\beta_1} \cdots z_d^{\beta_d} \frac{dz_1 \wedge \cdots \wedge dz_d}{z_1 \cdots z_d},$$

with

(5.9) $$0 < \beta_1, \ldots, \beta_d, \quad \beta_1 + \cdots + \beta_d < u.$$

We see that

$$\mathcal{C}^n(z_1^{\alpha_1} \cdots z_d^{\alpha_d} z_1^{\beta_1} \cdots z_d^{\beta_d} \frac{dz_1 \wedge \cdots \wedge dz_d}{z_1 \cdots z_d})$$

$$= \mathcal{C}^n(z_1^{\alpha_1+\beta_1} \cdots z_d^{\alpha_d+\beta_d} \frac{dz_1 \wedge \cdots \wedge dz_d}{z_1 \cdots z_d})$$

$$= \begin{cases} z_1^{\frac{\alpha_1+\beta_1}{p^n}} \cdots z_d^{\frac{\alpha_d+\beta_d}{p^n}} \frac{dz_1 \wedge \cdots \wedge dz_d}{z_1 \cdots z_d} & \text{all } \alpha_i + \beta_i \equiv 0 \pmod{p^n} \\ 0 & \text{some } \alpha_i + \beta_i \not\equiv 0 \pmod{p^n}. \end{cases}$$

From this we see that the matrix of $H^0(\mathcal{C}^n \circ f)$ with respect to the given basis of $H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d})$ has entries which are either 0 or 1, and so the trace of $H^0(\mathcal{C}^n \circ f)$ is equal to the number of 1's on the diagonal, which is given by the number of d-tuples $\beta_1, \cdots, \beta_d$ which satisfy

(5.10) $$\alpha_i + \beta_i = p^n \beta_i, \text{ for } 1 \leq i \leq d,$$

together with (5.9). We see immediately that the system of inequalities and equations (5.9), (5.10) has no solutions unless

$$\alpha_1 \equiv \cdots \equiv \alpha_d \equiv 0 \pmod{p^n - 1},$$

in which case there is a unique solution (here we are using the fact that $\alpha_1, \ldots, \alpha_d$ satisfies (5.8)), so that
(5.11)

$$\text{trace}_{\tilde{\Lambda}}(\mathcal{C}^n \circ f, H^0(\mathbb{P}^d, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Omega^d(u)_{\hat{\mathbb{P}}^d})) = \begin{cases} 1 & \text{all } \alpha_i \equiv 0 \pmod{p^n - 1} \\ 0 & \text{some } \alpha_i \not\equiv 0 \pmod{p^n - 1}. \end{cases}$$

On the other hand, as x ranges over all the $\mathbb{F}_{p^n}$-valued points of $\mathbb{G}_m^d$, the set of Teichmüller lifts $\tilde{x}$ ranges over all d-tuples of $(p^n - 1)^{\text{st}}$ roots of unity in $W(\mathbb{F}_{p^n})$. Thus

(5.12)
$$\frac{1}{(p^n-1)^d} \sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} f(\tilde{x}) = \frac{1}{(p^n-1)^d} \sum_{\zeta_1, \cdots, \zeta_d \in \mu_{p^n-1}} \zeta_1^{\alpha_1} \cdots \zeta_d^{\alpha_d}$$

$$= \begin{cases} 1 & \text{all } \alpha_i \equiv 0 \pmod{p^n - 1} \\ 0 & \text{some } \alpha_i \not\equiv 0 \pmod{p^n - 1}. \end{cases}$$

Comparing (5.11) and (5.12), we see that (5.7) is proved, and with it the lemma. $\qquad\square$

(5.13) Applying lemma 5.6 to the matrices $\prod_{k=0}^{n-1}((F^*)^k \tilde{r}_{ij})$ which appear in the expression on the right side of equation (5.4) we obtain the equation

$$\log P(T) = -\sum_{n=1}^{\infty} \mathrm{trace}_{\tilde{\Lambda}}\Big(\sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} \prod_{k=0}^{n-1}(\tilde{r}_{ij}(\sigma^k(\tilde{x}))), \tilde{\Lambda}^m \Big)\frac{T^n}{n(p^n-1)^d}$$

$$= -\sum_{n=1}^{\infty} \mathrm{trace}_{\tilde{\Lambda}}\Big(\sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^n, \tilde{\Lambda}^m \Big)\frac{T^n}{n(p^n-1)^d}.$$

This in turn implies that

$$\log \prod_{i=0}^{d} P(p^i T)^{(-1)^{d+1-i}\binom{d}{i}} = \sum_{n=1}^{\infty} \mathrm{trace}_{\tilde{\Lambda}}\Big(\sum_{x \in \mathbb{G}_m^d(\mathbb{F}_{p^n})} ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^n, \tilde{\Lambda}^m \Big)\frac{T^n}{n}.$$

Exponentiating both sides yields proposition 4.16.

6. Lifting representations of arithmetic fundamental groups

(6.1) In this section we discuss some applications of our method of lifting locally constant étale sheaves of finite free Λ modules by lifting the associated (Λ, F)-crystals. For the duration of this section Λ will denote an artinian local $\mathbb{Z}_p$-algebra with finite residue field and X will denote a smooth affine $\mathbb{F}_p$-scheme. We have seen above that there exists a surjection $\tilde{\Lambda} \to \Lambda$ with $\tilde{\Lambda}$ a finite flat local $\mathbb{Z}_p$-algebra. With these notations, we have the following theorem:

THEOREM 6.2.   *Let*

$$\rho : \pi_1(X) \to GL_d(\Lambda)$$

be a continuous representation of the arithmetic étale fundamental group of X. There exists a continuous lifting of ρ

$$\tilde{\rho} : \pi_1(X) \to GL_d(\tilde{\Lambda}).$$

Proof. The representation ρ corresponds to a locally constant étale sheaf of finite free Λ-modules on X. We denote this sheaf by $\mathcal{L}$. Now we claim that X can be lifted to a formally smooth p-adic formal scheme $\hat{X}$ over $\mathbb{Z}_p$ equipped with a lift F of the absolute Frobenius. Indeed by [10], III the obstructions to the existence of such a lifting are contained in the cohomology of certain coherent sheaves on X, and so vanish, since X is affine.

Fix $\hat{X}$ and F, and denote by $\mathcal{E}$ the unit (Λ, F)-crystal on $\hat{X}$ corresponding to $\mathcal{L}$, so that $\mathcal{E}$ is a finite free $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{X}}$ sheaf equipped with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. We have already seen in the proof of proposition 3.2, and also in

(4.7), that locally on $\hat{X}$, $\mathcal{E}$ can be lifted to a finite free $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{X}}$-module $\tilde{\mathcal{E}}$ equipped with an isomorphism $F^*\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ lifting $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. The obstruction to the existence of such a global lift is contained in the cohomology of certain coherent $\mathcal{O}_{\hat{X}}$-modules (because $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{X}}$ is $\mathcal{O}_{\hat{X}}$ coherent), and so vanishes, since $\mathcal{O}_{\hat{X}}$ is formal affine.

This proves the proposition, because by proposition 3.2, $\tilde{\mathcal{E}}$ corresponds to a lisse sheaf of $\tilde{\Lambda}$-modules on $\hat{X}$, which gives the required representation $\tilde{\rho}$. $\square$

THEOREM 6.3.   *With the notation of theorem 6.2, suppose that $X \subset \mathbb{A}^1$ is an open subset of the affine $\mathbb{F}_p$-line. Then $\tilde{\rho}$ may be chosen so that the corresponding lisse sheaf of $\tilde{\Lambda}$-modules $\mathcal{L}$ has rational L-function.*

Proof. As in section 4, we let F denote the lift of Frobenius on $\hat{\mathbb{P}}^1$ given (in affine coordinates) by $z \mapsto z^p$, and let $\hat{X} \subset \hat{\mathbb{P}}^1$ denote the open formal subscheme corresponding to $X \subset \mathbb{P}^1$. We may extend the unit (Λ, F)-crystal $\mathcal{E}$ attached to ρ to a sheaf $\mathcal{E}^+$ of locally free $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}$-modules on $\mathbb{P}^1$. Indeed to extend across a point of $\mathbb{P}^1 - X$, we may work in a neighbourhood of this point, and hence assume that $\mathcal{E}$ is free over $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{X}}$ over this neighbourhood, when the existence of the extension is clear. Moreover, if D denotes the divisor on $\mathbb{P}^1$ corresponding to $\mathbb{P}^1 - X$, and $\hat{D}$ denotes the divisor on $\hat{\mathbb{P}}^1$ corresponding to the Teichmüller lift of D, then replacing $\mathcal{E}^+$ by $\mathcal{E}^+(-n\hat{D})$, for some large integer n, we may assume that isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ extends to a map $F^*\mathcal{E}^+ \to \mathcal{E}^+$ (which cannot be an isomorphism unless ρ is trivial). Now set

$$M = \underline{Hom}_{\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}}(F^*\mathcal{E}^+, \mathcal{E}^+).$$

Replacing $\mathcal{E}^+$ by $\mathcal{E}^+(-n\hat{D})$ for some large integer n replaces M by $M((p-1)n\hat{D})$, and so we may assume that we have $H^1(\hat{\mathbb{P}}^1, M) = 0$. (This will be used later).

Now we claim that $\mathcal{E}^+$ can be lifted to a locally free $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}$-module $\tilde{\mathcal{E}}^+$. Indeed, such a lifting exists locally, and the obstruction to a global lifting is contained in H^2 of certain coherent sheaves on $\mathbb{P}^1$, hence vanishes (cf. [10] III, 7.1).

Let $\mathfrak{n}$ be the kernel of $\tilde{\Lambda} \to \Lambda$. We claim that $F^*\mathcal{E}^+ \xrightarrow{\sim} \mathcal{E}^+$ lifts to a morphism $F^*\tilde{\mathcal{E}}^+/\mathfrak{n}^2 \to \tilde{\mathcal{E}}^+/\mathfrak{n}^2$ which is an isomorphism when restricted to X. To see this, we begin by observing that such a lift exist locally. More precisely, such a lift exists over any open $U \subset \mathbb{P}^1$ which has the property that $\tilde{\mathcal{E}}^+/\mathfrak{n}^2$ becomes a free $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}$-module when restricted to U. Now let $U_0, U_1, \ldots U_r$ be a collection of open subsets of $\mathbb{P}^1$ such that the U_i cover $\mathbb{P}^1$ and such that $\tilde{\mathcal{E}}^+/\mathfrak{n}^2$ becomes a free $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}$-module when restricted to each U_i. Let $\phi_i : F^*\tilde{\mathcal{E}}^+ \to \tilde{\mathcal{E}}^+$ be a lift of $F^*\mathcal{E}^+ \xrightarrow{\sim} \mathcal{E}^+$ over each U_i. The elements $\phi_i - \phi_j$ on the intersections $U_i \cap U_j$ give rise to an element of $H^1(\mathbb{P}^1, N)$, where $N = \underline{Hom}_{\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}}(F^*\mathcal{E}^+, \mathfrak{n}\tilde{\mathcal{E}}^+/\mathfrak{n}^2\tilde{\mathcal{E}}^+)$. We claim that $H^1(\mathbb{P}^1, N) = 0$. Indeed,

choosing a set of generators $n_1, \ldots, n_s \in \tilde{\Lambda}$ for the ideal $\mathfrak{n}$, we can define a surjection of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}$-modules

$$(\mathcal{E}^+)^s \xrightarrow{(e_1,\ldots,e_s) \mapsto e_1 n_1 + \cdots + e_s n_s} \mathfrak{n}\mathcal{E}^+/\mathfrak{n}^2\mathcal{E}^+.$$

Since $F^*\mathcal{E}^+$ is a locally free $\Lambda \otimes_{\mathbb{Z}_p} \hat{\mathbb{P}}^1$-module, the above surjection yields a surjection $M^s \to N$. Taking cohomology and recalling that H^2 of a coherent sheaf of $\mathcal{O}_{\hat{\mathbb{P}}^1}$-modules vanishes, we see that $H^1(\mathbb{P}^1, N)$ is a quotient of $H^1(\mathbb{P}^1, M)^s$, which vanishes by assumption. The vanishing of $H^1(\mathbb{P}^1, N)$ implies that we can modify the ϕ_i so that they agree on overlaps. This gives the required lift, since over each of U_i the ϕ_i are automatically isomorphisms by Nakayama's lemma, as this is true modulo $\mathfrak{n}$.

Let $M_2 = \underline{Hom}_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathbb{P}}^1}}(F^*\tilde{\mathcal{E}}^+/\mathfrak{n}^2, \tilde{\mathcal{E}}^+/\mathfrak{n}^2)$. We also have $H^1(\hat{\mathbb{P}}^1, M_2) = 0$, as we by taking the long exact cohomology sequence attached to the short exact sequence $0 \to N \to M_2 \to M \to 0$, and recalling that $H^1(\mathbb{P}^1, M) = H^1(\mathbb{P}^1, N) = 0$.

Thus we can replace Λ by $\tilde{\Lambda}/\mathfrak{n}^2$ and $\mathcal{E}^+$ by $\tilde{\mathcal{E}}^+/\mathfrak{n}^2$. Repeating the above arguments and taking an inverse limit we obtain a lifting $F^*\tilde{\mathcal{E}}^+ \to \tilde{\mathcal{E}}^+$ of $F^*\mathcal{E}^+ \to \mathcal{E}^+$ which is an isomorphism when restricted to X. Now this restriction $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}^+|_X$ gives rise to a lisse étale sheaf $\tilde{\mathcal{L}}$ of finite free $\tilde{\Lambda}$-modules on X which induces the required representation $\tilde{\rho}$. To see that the L-function of $\tilde{\rho}$ has the required property, first note we may replace $\hat{X}$ by a non-empty open subset, and $\mathcal{L}$ by its restriction to this open subset, as this changes the L-function only by a rational function. Thus we may assume that $X \subset \mathbb{G}_m$, and that $\mathcal{E}$ is free over $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{X}}$ (then $\tilde{\mathcal{E}}$ is free over $\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{X}}$, as one can lift a basis for $\mathcal{E}$.) Now we are in exactly the situation of (4.8) (note that Grothendieck's algebraization theorem guarantees that the matrix entries describing the morphism $F^*\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ are rational functions, since by construction this extends to a morphism of coherent sheaves $F^*\tilde{\mathcal{E}}^+ \to \tilde{\mathcal{E}}^+$) and the argument of section 4 shows that $L(X, \tilde{\mathcal{L}})$ is a rational function over $\tilde{\Lambda}$. Indeed in the calculation of this L-function in section 4 the only possible non-rational contribution comes from corollary 4.14, where (in the notation of the proof of that corollary) one encounters the product $\prod_{x \in Z(a)} \det_{\tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x}(1 - ((\tilde{r}_{ij}(\tilde{x})) \circ \sigma)^{d(x)} T^{d(x)}, \tilde{\Lambda} \otimes_{\mathbb{Z}_p} W_x^m)^{-1}$. In the general context of that corollary, this product could be infinite. However, in the current situation $Z(a)$ is finite (being a proper closed subset of a curve), and so this is a finite product, and hence is indeed a rational function! □

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REFERENCES

[1] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Math., vol. **305**, Springer-Verlag, 1973.

[2] P. Berthelot and W. Messing, Théorie de Dieudonné cristalline. III. Théorèmes d'équivalence et de pleine fidélité, *The Grothendieck Festschrift, Vol. I*, Progress in Math., vol. **86**, Birkhäuser, 1990, 173–247.

[3] R. Crew, *L*-functions of *p*-adic characters and geometric Iwasawa theory, Inv. Math. 88 (1987), 395–403.

[4] P. Deligne, *Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$*, Lecture Notes in Math., vol. **569**, Springer-Verlag, 1977.

[5] B. Dwork, On the rationality of the zeta function of an algebraic variety, Am. J. Math. 82 (1960), 631–648.

[6] M. Emerton and M. Kisin, A Riemann-Hilbert correspondence for unit *F*-crystals. I, in preparation.

[7] M. Emerton and M. Kisin, A Riemann-Hilbert correspondence for unit *F*-crystals. II, in preparation.

[8] J.-Y. Etesse and B. Le Stum, Fonctions *L* associées aux *F*-isocristaux surconvergents II: Zéros et pôles unités, Inv. Math. 127 (1997), 1–31.

[9] T. Ekedahl, On the adic formalism, *The Grothendieck Festschrift, Vol. II*, Progress in Math., vol. **87**, Birkhäuser, 1990, 197–218.

[10] A. Grothendieck, *Revêtments Étales et Groupe Fondamental*, Lecutre Notes in Math., vol. **224**, Springer-Verlag, 1970.

[11] N. Katz, Travaux de Dwork, *Seminaire Bourbaki, 24ème année (1971/1972)*, Lecture Notes in Math., vol. **317**, Springer-Verlag, 1973, 69–190.

[12] N. Katz, *p*-adic properties of modular schemes, and modular forms, *Modular functions of one variable III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp 1972)*, Lecture Notes in Math., vol. **350**, Springer-Verlag, 1973, 69–190.

[13] D. Wan, Meromorphic continuation of *L*-functions of *p*-adic representations, Ann. Math. 143 (1996), 469–498.

[14] D. Wan, An Embedding Approach to Dwork's Conjecture, preprint.