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A REPORT ON THE AIM WORKING GROUP ON MOD p LOCAL LANGLANDS

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1. Introduction

This is a report on one of the working groups at the AIM workshop "p-adic representations, modularity and beyond". The goal of the working group was to make some progress on understanding the (conjectural) mod p local Langlands correspondence for G_F when F is a (proper) finite extension of $\mathbb{Q}_p$. The group's discussion took place on the afternoon of Wednesday, 22nd of February.

The members of the group were motivated by the following ideas:

(a) The theory of the mod p correspondence for $G_{\mathbb{Q}_p}$ is now fairly well established, and gives one confidence that a similar theory should exist when $\mathbb{Q}_p$ is replaced by some finite extension F.

(b) Local-global compatibility considerations suggest that any such correspondence should be related to Serre-type conjectures for global Galois representations, say of G_K for some number field K having F as a completion at some prime above p. More precisely, the structure of the local correspondence should be related to the weight part of such a Serre-type conjecture. When K is a totally real field unramified above p, such a conjecture on the weights has been recently been made by Buzzard, Diamond, and Jarvis. This conjecture is furthermore phrased in representation-theoretic terms, making a comparison with the conjectural mod p local Langlands correspondence particularly natural.

The main participants in the working group were Breuil, Buzzard, Diamond, Emerton, Gee, Kisin, Schein, Schneider, and Taylor.

I would like to thank Fred Diamond for his helpful remarks on an earlier draft of this report.

2. The role of local-global considerations

2.1. **The Buzzard-Diamond-Jarvis Serre-type conjecture on the weights.** Let K be a totally real field in which p is inert, and fix a quaternion algebra D over K unramified at one real prime, and unramified at p. Fix some level structure away from p, and let X_r denote the Shimura curve with this given level structure away from p, and of level p^r at p. Write $H^1 := \varinjlim_r H^1_{\text{ét}}(X_{r/\overline{K}}, \overline{\mathbb{F}}_p)$. This is a $G_K \times \mathrm{GL}_2(F)$-representation. Fix a two-dimensional continuous absolutely irreducible representation $\overline{\rho}$ of G_K over $\overline{\mathbb{F}}_p$ which appears in the mod p cohomology of the Shimura curve at level pN. Define $\mathcal{M}(\overline{\rho}) := \mathrm{Hom}_{G_K}(\overline{\rho}, H^1)$; this is a $\mathrm{GL}_2(F)$-representation, and the evaluation map gives a natural $G_K \times \mathrm{GL}_2(F)$-equivariant embedding

$$\overline{\rho} \otimes_{\overline{\mathbb{F}}_p} \mathcal{M}(\overline{\rho}) \hookrightarrow H^1,$$

whose image is exactly the $\overline{\rho}$-isotypical component of H^1. Since $\overline{\rho}$ is absolutely irreducible, one finds that for any congruence subgroup $\Gamma(p^r)$, one has

$$M(\overline{\rho})^{\Gamma(p^r)} \xrightarrow{\sim} \mathrm{Hom}_{G_K}(\overline{\rho}, H^1_{\text{ét}}(X_r, \overline{\mathbb{F}}_p)),$$

or equivalently,

$$\overline{\rho} \otimes_{\overline{\mathbb{F}}_p} M(\overline{\rho})^{\Gamma(p^r)} \xrightarrow{\sim} \overline{\rho}\text{-isotypical part of } H^1_{\text{ét}}(X_r, \overline{\mathbb{F}}_p).$$

Taking $r = 1$, we find in particular that $\mathcal{M}(\overline{\rho})^{\Gamma(p)}$ is a representation of $\mathrm{GL}_2(k)$. The Serre-type conjecture of Buzzard, Diamond, and Jarvis (which is a theorem of Gee in many cases) predicts exactly which irreducible representations of $\mathrm{GL}_2(k)$ over $\overline{\mathbb{F}}_p$ appear as subrepresentations of $M(\overline{\rho})^{\Gamma(p)}$. The list of V depends only on $\overline{\rho}_{|G_F}$ (in fact, only on the restriction to the inertia subgroup I_F of G_F). If $\overline{\rho}_{|G_F}$ is semi-simple, then the number of V that can appear is roughly 2^f, where $\mid k \mid = p^f$. If $\overline{\rho}_{|G_F}$ is non semi-simple, then the V that can appear is a certain subset of the corresponding set of weights attached to $(\overline{\rho}_{|G_F})^{\text{ss}}$.

Recall that for any representation V of $\mathrm{GL}_2(k)$ over $\overline{\mathbb{F}}_p$ (which we can also think of a $\mathrm{GL}_2(\mathcal{O}_F)$ representation, via the reduction map), there is an adjointness

$$\mathrm{Hom}_{\mathrm{GL}_2(k)}(V, \mathcal{M}(\overline{\rho})^{\Gamma(p)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(F)}(c\mathrm{-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V, \mathcal{M}(\overline{\rho})).$$

(Here $c-$Ind means compact induction, and Z denotes the centre of $\mathrm{GL}_2(F)$, which is acting via some appropriate central character.) Thus the B-D-J conjecture equally well predicts which possible maps

$$c\mathrm{-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V \to \mathcal{M}(\overline{\rho}) \tag{1}$$

can exist (for V an irreducible $\mathrm{GL}_2(k)$-representation).

Actually, one can make a slightly stronger conjecture: A consideration of how Hecke operators act shows that a map of the form (1) should factor through a map

$$c\mathrm{-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V/(T-a) \to \mathcal{M}(\overline{\rho}),$$

where T is the usual Hecke operator on the compact induction, a is some element of $\overline{\mathbb{F}}_p$, equal to zero always in the case when $\overline{\rho}_{|G_F}$ is irreducible, and either zero or else related to the eigenvalue of Frobenius on an unramified line in some twist of $\overline{\rho}_{|G_F}$, in the case when $\overline{\rho}_{|G_F}$ is reducible.

2.2. **Local-global compatibility.** Suppose that the level structure away from p is chosen to precisely match with the prime-to-p Artin conductor of $\overline{\rho}$. Then one might optimistically hope that the $\mathrm{GL}_2(F)$-representation $\mathcal{M}(\overline{\rho})$ depends only on $\overline{\rho}_{|G_F}$, and *is* the representation attached to $\overline{\rho}_{|G_F}$ via the conjectural mod p local Langlands correspondence for G_F. In fact, since one doesn't (yet) have an *a priori* characterization of this correspondence, this can possibly be taken as a defining property of the correspondence (assuming that the hope is borne out).

Thus one can try to get information about the correspondence by getting information about the $\mathcal{M}(\overline{\rho})$. One in turn can try to get information about these via the B-D-J conjecture, which restricts the possible maps $c\mathrm{-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V \to \mathcal{M}(\overline{\rho})$.

3. Calculations

3.1. **Preliminaries.** One expects, based on experience from the case $K = \mathbb{Q}$, $F = \mathbb{Q}_p$, that if $\overline{\rho}_{|G_F}$ is not semisimple, then $\mathcal{M}(\overline{\rho})$ will be a non-trivial extension of $\mathrm{GL}_2(F)$-representations, whose semi-simplification will, however, match with the semi-simplification of $\overline{\rho}_{|G_F}$. (This is related to the fact that in this case the list of Serre weights is smaller than in the semi-simple case.)

Thus, so as to not have to think about extensions, and in order to get the maximum number of Serre weights (and hence to derive as much information about $\mathcal{M}(\overline{\rho})$ as we can from the B-D-J conjecture) we will suppose that $\overline{\rho}_{|G_F}$ is semi-simple.

Since from now on we will be focusing on $\overline{\rho}_{|G_F}$ we will write simply $\overline{\rho}$ to denote this local Galois representation. Since we expect $\mathcal{M}(\overline{\rho})$ (where $\overline{\rho}$ is momentarily still referring to the global $\overline{\rho}$) to be related to $\overline{\rho}$ (which now, and from now on, refers to the restriction of our global $\overline{\rho}$ to G_F) via a local correspondence, we will denote it by $\overline{\pi}(\overline{\rho})$.

To fix ideas, we will assume that $F = \mathbb{Q}_{p^2}$. Then $\overline{\rho}_{|I_F}$ (assuming it is semi-simple) can have one of the following two forms (up to twist):

$$\begin{pmatrix} \omega_4^{r_0+1+p(r_1+1)} & 0 \\ 0 & \omega_4^{p^2(r_0+1+p(r_1+1))} \end{pmatrix} \quad \text{(when } \overline{\rho}_{|G_F} \text{ is irreducible)}$$

$$\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(when } \overline{\rho}_{|G_F} \text{ is reducible)}$$

In each case, $0 \le r_0, r_1 \le p-1$.

We now prepare to give the list of Serre weights (according to B-D-J) in each case (provided we are not in certain "boundary cases"). We adopt the convention that Sym_0^r stands for the representation of $\mathrm{GL}_2(k)$ on $\mathrm{Sym}^r \overline{\mathbb{F}}_p^2$, and that Sym_1^r stands for the twist of this representation by the Frobenius automorphism (acting on $\mathrm{GL}_2(k)$, induced by Frobenius on k). We will similarly write $\det_0$ and $\det_1$ for the determinant, and for its twist by Frobenius.

The following tables record each Serre weight, together with the associated T-eigenvalue. (A caveat: in the B-D-J paper, they work with p-torsion in Jacobians rather than étale cohomology with $\overline{\mathbb{F}}_p$ coefficients, and take invariants after tensoring with V, rather than taking equivariant homs from V. Thus to convert from B-D-J to the cohomological conventions used above, one should twist by a cyclotomic character, and replace V by V^*. From now on we follow the B-D-J conventions, and so write down the weights as they appear in B-D-J.)

Serre weights in the irreducible case:

$$\mathrm{Sym}_0^{r_0} \otimes \mathrm{Sym}_1^{r_1} \qquad (T = 0)$$

$$\det\nolimits_0^{r_0+1} \det\nolimits_1^{-1} \mathrm{Sym}_0^{p-2-r_0} \otimes \mathrm{Sym}_1^{r_1+1} \qquad (T = 0)$$

$$\det\nolimits_1^{r_1+1} \mathrm{Sym}_0^{r_0-1} \otimes \mathrm{Sym}_1^{p-2-r_1} \qquad (T = 0)$$

$$\det\nolimits_0^{r_0} \det\nolimits_1^{r_1+1} \mathrm{Sym}_0^{p-1-r_0} \otimes \mathrm{Sym}_1^{p-3-r_1} \qquad (T = 0).$$

Serre weights in the reducible case:

$$\mathrm{Sym}_0^{r_0} \otimes \mathrm{Sym}_1^{r_1} \qquad (T = a_1 \neq 0)$$

$$\det{}_0^{r_0+1} \det{}_1^{r_1+1} \mathrm{Sym}_0^{p-3-r_0} \otimes \mathrm{Sym}_1^{p-3-r_1} \qquad (T = a_2 \neq 0)$$

$$\det{}_0^{p-1} \det{}_1^{r_1} \mathrm{Sym}_0^{r_0+1} \otimes \mathrm{Sym}_1^{p-2-r_1} \qquad (T = 0)$$

$$\det{}_0^{r_0} \det{}_1^{p-1} \mathrm{Sym}_0^{p-2-r_0} \otimes \mathrm{Sym}_1^{r_1+1}. \qquad (T = 0).$$

In summary, for each of these V and a (and only these V and a), we get a corresponding map $c\text{−}\mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V/(T-a) \to \pi(\overline{\rho})$.

3.2. **Some results of Paskunas.** To save typing, from now on we write simply $c\text{−}\mathrm{Ind}\ V$ to denote $c\text{−}\mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V$.

Paskunas has shown (maybe excluding some boundary cases for r_0, r_1) that there is an embedding

(2)
$$c\text{−}\mathrm{Ind}\ \det{}_1^{r_1+1} \mathrm{Sym}_0^{r_0-1} \otimes \mathrm{Sym}_1^{p-2-r_1} \bigoplus c\text{−}\mathrm{Ind}\ \det{}_0^{r_0+1} \mathrm{Sym}_0^{p-2-r_0} \otimes \mathrm{Sym}_1^{r_1-1} \hookrightarrow c\text{−}\mathrm{Ind}\ \mathrm{Sym}_0^{r_0} \otimes \mathrm{Sym}_1^{r_1}/T.$$

Replacing r_0, r_1 by $p-1-r_0$, $p-1-r_1$, and twisting appropriately, we similarly obtain an embedding

(3)
$$c\text{−}\mathrm{Ind}\ \det{}_1^{r_1+1} \mathrm{Sym}_0^{r_0-1} \otimes \mathrm{Sym}_1^{p-2-r_1} \bigoplus c\text{−}\mathrm{Ind}\ \det{}_0^{r_0+1} \mathrm{Sym}_0^{p-2-r_0} \otimes \mathrm{Sym}_1^{r_1-1} \hookrightarrow c\text{−}\mathrm{Ind}\ \det{}_0^{r_0} \det{}_1^{r_1} \mathrm{Sym}_0^{p-1-r_0} \otimes \mathrm{Sym}_1^{p-1-r_1}/T.$$

Paskunas has furthermore shown that the cokernels of (2) and (3) are isomorphic. Finally, he has constructed a certain irreducible quotient π_{r_0,r_1} ($\xrightarrow{\sim} \det{}_0^{r_0} \det{}_1^{r_1} \otimes \pi_{p-1-r_0,p-1-r_1}$) of these cokernels.

Currently, no other irreducible quotient of $c\text{−}\mathrm{Ind}\ \mathrm{Sym}_0^{r_0} \otimes \mathrm{Sym}_1^{r_1}/T$ seems to have been described in the literature. (But see the Postscript below.)

3.3. **Some deductions: the reducible case.**

Let's consider the reducible case first. Then considering the first two Serre weights, we find maps

$$c\text{−}\mathrm{Ind}\ \mathrm{Sym}_0^{r_0} \otimes \mathrm{Sym}_1^{r_1}/(T-a_1) \to \overline{\pi}(\overline{\rho})$$

and

$$c\text{−}\mathrm{Ind}\ \det{}_0^{r_0+1} \det{}_1^{r_1+1} \mathrm{Sym}_0^{p-3-r_0} \otimes \mathrm{Sym}_1^{p-3-r_1}/(T-a_2) \to \overline{\pi}(\overline{\rho}),$$

where $a_1, a_2 \neq 0$.

If we consider the third and fourth Serre weights, they require $T = 0$, and also match under the Paskunas intertwining. Thus it is natural to guess that they are simultaneously explained by a map

$$\det{}_0^{p-1} \det{}_1^{r_1} \otimes \pi_{r_0+1,p-2-r_1} \to \overline{\pi}(\overline{\rho}).$$

The upshot is that for reducible $\overline{\rho}$, we guess that

$$\overline{\pi}(\overline{\rho}) \overset{?}{=} c\text{−Ind Sym}_0^{r_0} \otimes \text{Sym}_1^{r_1}/(T - a_1)$$
$$\bigoplus c\text{−Ind } \det_0^{r_0+1} \det_1^{r_1+1} \text{Sym}_0^{p-3-r_0} \otimes \text{Sym}_1^{p-3-r_1}/(T - a_2)$$
$$\bigoplus \det_0^{p-1} \det_1^{r_1} \pi_{r_0+1,p-2-r_1}.$$

3.4. **Some deductions: the irreducible case.** In this case, we find that the weights that appear for $\overline{\rho}$ are not preserved under the Paskunas intertwiners. Thus for any weight V, say the first – namely $\text{Sym}_0^{r_0} \otimes \text{Sym}_1^{r_1}$ – whatever the corresponding map

$$c\text{−Ind Sym}_0^{r_0} \otimes \text{Sym}_1^{r_1} \to \overline{\pi}(\overline{\rho})$$

is, it must not factor through the cokernel of (2) (since otherwise, using the Paskunas intertwiner, it would also factor through the cokernel of (3), which would give an impossible weight).

Thus it must be non-zero on the source of (2). Now again a consideration of weights shows that it is trivial on the second summand in this source, and so must be non-trivial on the first summand – which fortunately does correspond to an allowed weight! It must furthermore factor through the $T = 0$ quotient of this summand. Again, it cannot factor through the Paskunas cokernel, and so this process iterates.

During this iteration, we find each weight of $\overline{\rho}$ appearing, and so the first weight actually “explains” all of the others (in the sense that it forces them to appear).

So it seems that $\overline{\pi}(\overline{\rho})$ is a certain quotient of $c\text{−Ind Sym}_0^{r_0} \otimes \text{Sym}_1^{r_1}$ whose precise structure we don't yet understand.

3.5. **Further computations.** There are some boundary cases of the B-D-J conjecture (e.g. $r_0 = r_1 = 0$) when we get additional intertwiners in the compact induction, for example

$$c\text{−Ind } 1 \otimes 1/T \xrightarrow{\sim} c\text{−Ind}\,\text{Sym}_0^{p-1} \otimes \text{Sym}_1^{p-1},$$

which one finds are compatible with the weight predictions of B-D-J. This gives further evidence for the consistency/compatibility of the B-D-J conjectures and mod p Langlands.

3.6. **Postscript.** Following the discussion on Feb 22, further discussions took place, involving Breuil, Buzzard, Diamond, Emerton, Gee, and Taylor (in various permutations). These discussions led to a much firmer understanding of the case $\mathbb{Q}_{p^2}$, as well as to significant progress towards understanding the case of $\mathbb{Q}_{p^f}$ for general f. It now seems very clear that the structure of the $c\text{−Ind Sym}^{\vec{r}}/T$ and their irreducible quotients is intimately related to – and to a surprisingly large extent controlled by – the B-D-J weight conjecture.

One natural question that was raised was whether the weights appear in $\mathcal{M}(\overline{\rho})^{\Gamma(p)}$ with multiplicity one (i.e. whether $\text{Hom}_{\text{GL}_2(k)}(V, \mathcal{M}(\overline{\rho})^{\Gamma(p)})$ is one-dimensional), at least generically. It seems to be accepted by the experts that this should be so, but it would be worthwhile to have precise statements and arguments.

If one grants this multiplicity one statement in the case when $F = \mathbb{Q}_{p^2}$ for some global $\overline{\rho}$ for which $\overline{\rho}_{|G_F}$ is irreducible (which is not of exceptional weight from the point of view of the B-D-J conjecture) then the discussion of Subsection 3.4 implies that for any Serre weight V of $\overline{\rho}$, the image of the corresponding map

$$c-\mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_F)Z}^{\mathrm{GL}_2(F)} V/T \to \mathcal{M}(\overline{\rho})$$

is independent of V, and is irreducible, but is not one of the irreducible representations constructed by Paskunas. It would naturally be of great interest to construct such a quotient directly.

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