Notes on the Bessel Process

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Abstract

Much of the analysis of the Schramm-Loewner evolution (SLE) boils down to estimates about the Bessel process. This is a self-contained summary of the one-dimensional Bessel process. Although the motivation and choice of topics come from applications to SLE, these notes do not use any facts about SLE. The notes do assume familiarity with stochastic calculus including Itô's formula, the product rule, Girsanov's theorem, and time changes of diffusions.

1 Introduction

This is an adaptation of one of the chapters of a long, far from finished, project about conformally invariant processes and the Schramm-Loewner evolution (SLE). Although the motivation and the choice of topics for these notes come from applications to SLE, the topic is the one-dimensional Bessel process. No SLE is assumed or discussed. It is assumed that the reader has a background in stochastic calculus including Itô's formula, the product rule, Girsanov's theorem, and time changes of diffusions. The Girsanov perspective is taken from the beginning.

The Bessel process is one of the most important one-dimensional diffusion processes. There are many ways that if arises. We will start by viewing the Bessel process as a Brownian motion "tilted" by a function of the current value.

We will give a summary of what is contained here as well as some discussion of the relevance to the Schramm-Loewner evolution (SLE). For an introduction to SLE, see [1]. The discussion here is for people who know about SLE; we emphasize that knowledge of SLE is not required for this paper.

The chordal Schramm-Loewner evolution with parameter $\kappa = 2/a > 0$ (SLE_κ) is the random path $\gamma(t)$ from 0 to infinity in the upper half-plane $\mathbb{H}$ defined as follows. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$ and let $g_t : H_t \to \mathbb{H}$ be the unique conformal transformation with $g(x) = z = o(1)$ as $z \to \infty$. Then

$$\partial_t g_t(z) = \frac{a}{g_t(z) + B_t}, \quad g_0(z) = z,$$

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where B_t is a standard Brownian motion. In fact, g_t can be extended by Schwarz reflection to a conformal transformation on H_t^*, defined to be the unbounded component of the complement of $\{z : z \in \gamma[0,t] \text{ or } \overline{z} \in \gamma[0,t]\}$. Note that $H_0^* = \mathbb{C} \setminus \{0\}$ and for fixed $z \in H_0^*$, the solution of the equation above exists for $t < T_z \in (0, \infty]$. In our parametrization,

$$g_t(z) = z + \frac{at}{|z|} + O(|z|^{-2}), \quad z \to \infty.$$

If $x > 0$ and $X_t = X_t^x = g_t(z) + B_t$, then X_t satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad X_0 = x.$$

This is the first place that the Bessel process arises; although only $a > 0$ is relevant here, other applications in SLE require understanding the equation for negative values as well, so we define the Bessel process for all $a \in \mathbb{R}$. We use a as our parameter throughout because it is easiest; however there two other more common choices: the index $\nu = a - \frac{1}{2}$ or the dimension $d = 2a + 1$. The latter terminology comes from the fact that the absolute value of a d-dimensional Brownian motion is a Bessel process with parameter $a = (d-1)/2$.

In the first section we consider the process only for $t < T_0$, that is, killed when it reaches the origin. While the process can be defined by the equation, we derive the equation by starting with X_t as a Brownian motion and then "weighting locally" by X_t^a. The equation comes from the Girsanov theorem. We start with some basic properties: scaling, the Radon-Nikodym derivative with respect to Brownian motion, and the phase transition at $a = 1/2$ for recurrence and transience (which corresponds to the phase transition at $\kappa = 4$ between simple paths and self-intersecting paths for SLE). Here we also consider the logarithm of the Bessel process and demonstrate the technique of using random time changes to help understand the process.

The transition density for the killed process is given in Section 2.2. It involves a special function; rather than writing it in terms of the modified Bessel function, we choose to write it in terms of the "entire" part of the special function that we label as h_a. The Radon-Nikodym derivative gives the duality between the process with parameters a and $1-a$ and allow us to compute the density only for $a \geq 1/2$. From this the density for the hitting time T_0 for $a < 1/2$ follows easily. We then consider the Green's function and also the process on geometric time scales which is natural when looking at the large time behavior of the process.

In Section 2.5, the Bessel process X_t^x is viewed as a function of its starting position x; indeed, this is how it appears in the Schramm-Loewner evolution. Here we prove when it is possible for the process starting at different points to reach the origin at the same time. For SLE, this corresponds to the phase transition at $\kappa = 2/a = 8$ between plane filling curves and non-plane filling curves. in the following subsection, we show how to give expectations of a certain functional for Brownian motion and Bessel process; this functional appears as a power of the spatial derivative of X_t^x. There is a basic technique to finding the asymptotic value of these functionals: use a (local) martingale of the form of the functional times a function of the current value of the process; find the limiting distribution under this martingale; then compute the expected value of the reciprocal of the function in the invariant distribution.

The next section discusses the reflecting Bessel process for $-1/2 < a < 1/2$. If $a \geq 1/2$, the process does not reach the origin and if $a \leq -1/2$ the pull towards the origin is too strong to allowed a reflected process to be defined. There are various ways to define the process that is characterized by:

- Away from the origin it acts like a Bessel process.
- The Lebesgue measure of the amount of time spent at the origin is zero.

We start by just stating what the transition density is. Proving that it is a valid transition density requires only computing some integrals (we make use of integral tables [2, 3] here) and this can be used to construct the process by defining on dyadics and establishing continuity of the paths. The form of the density also shows that for each t, the probability of being at the origin at time t is zero. We then show how one could derive this density from the knowledge of the density starting at the origin — if we start away from the origin we proceed as in the Bessel process stopped at time 0 and then we continue.

In the following two subsections we consider two other constructions of the reflecting process both of which are useful for applications:

- First construct the times at which the process is at the origin (a random Cantor set) and then define the process at other times using Bessel excursions. For $a = 0$, this is the Itô excursion construction for reflecting Brownian motion. Constructing Bessel excursions is an application of the Girsanov theorem.
- Consider the flow of particles $\{X_t^x : x > 0\}$ doing coupled Bessel processes stopped at the origin and define the reflected process by
$$\inf\{X_t^x : T_x > t\}.$$

The Bessel process relates to the chordal, or half-plane, Loewner equation. For the radial equation for which curves approach an interior point, a similar equation, the radial Bessel equation arises. It is an equation restricted to $(0, \pi)$ and the value often represents one-half times the argument of a process on the unit circle. We generalize some to consider processes that locally look like Bessel processes near 0 and π (the more general processes also arise in SLE.) The basic assumption is that the drift of the process looks like the Bessel process up to the first *two* terms of the expansion. In this case, the process can be defined in terms of the Radon-Nikodym derivative (locally) to a Bessel process. This approach also allows us to define a reflected process for $-1/2 < a < 1/2$. A key fact about the radial Bessel process is the exponentially fast convergence of the distribution to the invariant distribution. This is used to estimate functionals of the process and these are important in the study of radial SLE and related processes.

In the final section, we discuss the necessary facts about special functions that we use, again relying on some integral tables in [2, 3].

2 The Bessel process (up to first visit to zero)

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which is defined a standard one-dimensional Brownian motion X_t with filtration $\{\mathcal{F}_t\}$ with $X_0 > 0$. Let $T_x = \inf\{t : X_t = x\}$. Let $a \in \mathbb{R}$, and let $Z_t = X_t^a$. The Bessel process with parameter a will be the Brownian motion "weighted locally by X_t^a". If $a > 0$, then the Bessel process will favor larger values while for $a < 0$ it will favor smaller values. The value $a = 0$ will correspond to the usual Brownian motion. In this section we will stop the process at the time T_0 when it reaches the origin.

Itô's formula shows that if $f(x) = x^a$, then

$$\begin{aligned} dZ_t &= f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, dt \\ &= Z_t \left[\frac{a}{X_t}\, dX_t + \frac{a(a-1)}{2\, X_t^2}\, dt \right], \qquad t < T_0. \end{aligned}$$

For the moment, we will not consider $t \geq T_0$. Let

$$N_t = N_{t,a} = \left[\frac{X_t}{X_0} \right]^a \exp\left\{ -\frac{(a-1)a}{2} \int_0^t \frac{ds}{X_s^2} \right\}, \qquad t < T_0. \tag{1}$$

Then the product rule combined with the Itô calculation above shows that N_t is a local martingale for $t < T_0$ satisfying

$$dN_t = \frac{a}{X_t}\, N_t\, dX_t, \qquad N_0 = 1.$$

Suppose $0 < \epsilon < X_0 < R$ and let $\tau = \tau_{\epsilon,R} = T_\epsilon \wedge T_R$. For fixed ϵ, R and $t < \infty$, we can see that $N_{s\wedge\tau}$ is uniformly bounded for $s \leq t$ satisfying

$$dN_{t\wedge\tau} = \frac{a}{X_{t\wedge\tau}}\, 1\{t < \tau\}\, N_{t\wedge\tau}\, dX_t.$$

In particular, $N_{t\wedge\tau}$ is a positive continuous martingale.

We will use Girsanov's theorem and we assume the reader is familiar with this. This is a theorem about positive martingales that can also be used to study positive local martingales. We discuss now in some detail how to do this, but later on we will not say as much. For each t, ϵ, R, we define the probability measure $\hat{\mathbb{P}}_{a,\epsilon,R,t}$ on $\mathcal{F}_{t\wedge\tau}$ by

$$\frac{d\hat{\mathbb{P}}_{a,\epsilon,R,t}}{d\mathbb{P}} = N_{t\wedge\tau}, \qquad \tau = \tau_{\epsilon,R}.$$

Since $N_{t\wedge\tau}$ is a martingale, we can see that if $s < t$, then $\hat{\mathbb{P}}_{a,\epsilon,R,t}$ restricted to $\mathcal{F}_{s\wedge\tau}$ is $\hat{\mathbb{P}}_{a,\epsilon,R,s}$. For this reason, we write just $\hat{\mathbb{P}}_{a,\epsilon,R}$. Girsanov's theorem states that if

$$B_t = B_{t,a} = X_t - \int_0^t \frac{a\, ds}{X_s}, \qquad t < \tau,$$

then B_t is a standard Brownian motion with respect to $\hat{\mathbb{P}}_{a,\epsilon,R}$, stopped at time τ. In other words,

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad t \leq \tau.$$

Note that as X_t gets large, the drift term a/X_t gets smaller. By comparison with a Brownian motion with drift, we can therefore see that as $R \to \infty$,

$$\lim_{R\to\infty} \hat{\mathbb{P}}_{a,\epsilon,R}\{t \wedge \tau = T_R\} \to 0,$$

uniformly in ϵ. Hence we can write $\hat{\mathbb{P}}_{a,\epsilon}$, and see that

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad t \leq T_\epsilon.$$

Note that this equation does not depend on ϵ except in the specification of the allowed values of t. For this reason, we can write $\hat{\mathbb{P}}_a$, and let $\epsilon \downarrow 0$ and state that

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad t < T_0,$$

where in this case we define $T_0 = \lim_{\epsilon\downarrow 0} T_\epsilon$. This leads to the definition.

Definition X_t is a Bessel process starting at $x_0 > 0$ with parameter a stopped when it reaches the origin, if it satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad t < T_0, \quad X_0 = x_0, \tag{2}$$

where B_t is a standard Brownian motion.

We have written the parameter a to make the equation as simple as possible. However, there are two more common ways to parametrize the family of Bessel process.

- The *dimension* d,

  $$a = \frac{d-1}{2}, \quad d_a = 2a+1.$$

  This terminology comes from the fact that if W_t is a standard d-dimensional Brownian motion, then the process $X_t = |W_t|$ satisfies the Bessel process with $a = (d-1)/2$. We sketch the argument here. First note that

  $$dX_t^2 = \sum_{j=1}^{d} d[(W_t^j)^2] = 2\sum_{j-1}^{d} W_t^j\, dW_t^j + d\, dt,$$

  which we can write as

  $$dX_t^2 = d\, dt + 2\, X_t\, dZ_t,$$

for the standard Brownian motion

$$Z_t = \sum_{j=1}^{d} \int_0^t \frac{|W_s^j|}{X_s} \, dW_s^j.$$

(To see that Z_t is a standard Brownian motion, check that Z_t is a continuous martingale with $\langle Z \rangle_t = t$.) We can also use Itô's formula to write

$$dX_t^2 = 2\, X_t \, dX_t + d\langle X \rangle_t.$$

Equating the two expressions for dX_t^2, we see that

$$dX_t = \frac{\frac{d-1}{2}}{X_t} \, dt + dZ_t.$$

The process X_t^2 is called the *d-dimensional squared-Bessel process.*

- The *index* ν,
$$a = \frac{2\nu + 1}{2}, \quad \nu_a = \frac{2a-1}{2}.$$
Note that $\nu_{1-a} = -\nu_a$. We will see below that there is a strong relationship between Bessel processes of parameter a and parameter $1-a$.

As we have seen, to construct a Bessel process, we can start with a standard Brownian motion X_t on $(\Omega, \mathbb{P}, \mathcal{F})$, and then consider the probability measure $\hat{\mathbb{P}}_a$. Equivalently, we can start with a Brownian motion B_t on $(\Omega, \mathbb{P}, \mathcal{F})$ and define X_t to be the solution of the equation (2). There is a technical issue that the next proposition handles. The measure $\hat{\mathbb{P}}_a$ is defined only up to time T_{0+}. The next proposition shows that we can replace this with T_0 and get continuity at time T_0.

Proposition 2.1. *Let* $T_{0+} = \lim_{r \downarrow 0} T_r$.

1. *If* $a \geq 1/2$, *then for each* $x > 0, t > 0$.
$$\hat{\mathbb{P}}_a^x\{T_{0+} \leq t\} = 0.$$

2. *If* $a < 1/2$, *then for each* $t > 0$,
$$\hat{\mathbb{P}}_a^x\{T_{0+} \leq t\} > 0.$$
Moreover, on the event $\{T_{0+} \leq t\}$, *except on an event of* $\hat{\mathbb{P}}_a$*-probability zero,*
$$\lim_{t \uparrow T_0} X_t = 0.$$

Proof. If $0 < r < x < R < \infty$, let $\tau = T_r \wedge T_R$. Define $\phi(x) = \phi(x; r, R, a)$ by

$$\phi(x) = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}, \quad a \neq 1/2,$$

$$\phi(x) = \frac{\log x - \log r}{\log R - \log r}, \quad a = 1/2.$$

This is the unique function on $[r, R]$ satisfying the boundary value problem

$$x\,\phi''(x) = -2a\,\phi'(x), \quad \phi(r) = 0, \;\; \phi(R) = 1. \tag{3}$$

Using Itô's formula and (3), we can see that $\phi(X_{t\wedge\tau})$ is a bounded continuous $\hat{\mathbb{P}}_a$-martingale, and hence by the optional sampling theorem

$$\phi(x) = \hat{\mathbb{E}}_a^x\left[\phi(X_\tau)\right] = \hat{\mathbb{P}}_a^x\{T_R < T_r\}.$$

Therefore,

$$\hat{\mathbb{P}}_a^x\{T_R < T_r\} = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}, \quad a \neq 1/2, \tag{4}$$

$$\hat{\mathbb{P}}_a^x\{T_R < T_r\} = \frac{\log x - \log r}{\log R - \log r}, \quad a = 1/2.$$

In particular, if $x > r > 0$,

$$\hat{\mathbb{P}}_a^x\{T_r < \infty\} = \lim_{R\to\infty} \hat{\mathbb{P}}_a^x\{T_r < T_R\} = \begin{cases} (r/x)^{2a-1}, & a > 1/2 \\ 1, & a \leq 1/2, \end{cases} \tag{5}$$

and if $x < R$,

$$\hat{\mathbb{P}}_a^x\{T_R < T_{0+}\} = \lim_{r\downarrow 0} \hat{\mathbb{P}}_a^x\{T_R < T_r\} = \begin{cases} (x/R)^{1-2a}, & a < 1/2 \\ 1, & a \geq 1/2. \end{cases} \tag{6}$$

If $a \geq 1/2$, then (5) implies that for each $t, R < \infty$,

$$\hat{\mathbb{P}}_a^x\{T_{0+} \leq t\} \leq \hat{\mathbb{P}}_a^x\{T_{0+} < T_R\} + \hat{\mathbb{P}}_a^x\{T_R < T_{0+}; T_R < t\}.$$

Letting $R \to \infty$ (and doing an easy comparison with Brownian motion for the second term on the right), shows that for all t,

$$\hat{\mathbb{P}}_a^x\{T_{0+} \leq t\} = 0,$$

and hence, $\hat{\mathbb{P}}_a^x\{T_{0+} < \infty\} = 0$.

If $a < 1/2$, let $\tau_n = T_{2^{-2n}}$ and $\sigma_n = \inf\{t > \tau_n : X_t = 2^{-n}\}$. Then if $x > 2^{-2n}$, (6) implies that

$$\hat{\mathbb{P}}_a^x\{\sigma_n < T_{0+}\} = 2^{n(2a-1)}.$$

In particular,

$$\sum_{n=0}^{\infty} \hat{\mathbb{P}}_a^x\{\sigma_n < T_{0+}\} < \infty,$$

and by the Borel-Cantelli lemma, with $\mathbb{P}_a^x$-probability one, $T_{0+} < \sigma_n$ for all n sufficiently large. On the event that this happens, we see that

$$\lim_{t \uparrow T_{0+}} X_t = 0,$$

and hence $T_0 = T_{0+}$. On this event, we have $\max_{0 \leq t \leq T_0} X_t < \infty$, and hence

$$\hat{\mathbb{P}}_a^x\{T_{0+} = \infty\} \leq \hat{\mathbb{P}}_a^x \left\{ \sup_{0 \leq t < T_{0+}} X_t = \infty \right\} \leq \lim_{R \to \infty} \hat{\mathbb{P}}_a^x\{T_{0+} < T_R\} = 0.$$

$\square$

With this proposition, we can view $\hat{\mathbb{P}}_a^x$ for each t as a probability measure on continuous paths $X_s, 0 \leq s \leq t \wedge T_0$.

Proposition 2.2. *For each $x, t > 0$ and $a \in \mathbb{R}$, the measures $\mathbb{P}^x$ and $\hat{\mathbb{P}}_a^x$, considered as measures on paths $X_s, 0 \leq s \leq t$, restricted to the event $\{T_0 > t\}$ are mutually absolutely continuous with Radon-Nikodym derivative*

$$\frac{d\hat{\mathbb{P}}_a^x}{d\mathbb{P}^x} = \left[\frac{X_t}{x}\right]^a \exp\left\{-\frac{(a-1)a}{2} \int_0^t \frac{ds}{X_s^2}\right\}. \tag{7}$$

In particular,

$$\frac{d\hat{\mathbb{P}}_a^x}{d\mathbb{P}_{1-a}^x} = \left[\frac{X_t}{x}\right]^{2a-1}. \tag{8}$$

Proof. The first equality is a restatement of what we have already done. and the second follows by noting that the exponential term in (7) is not changed if we replace a with $1-a$. $\square$

Corollary 2.3. *Suppose $x < y$ and $a \geq 1/2$. Consider the measure $\tilde{\mathbb{P}}_a^y$ conditioned on the event $\{T_x < T_y\}$, considered as a measure on paths $X_t, 0 \leq t \leq T_x$. Then $\tilde{\mathbb{P}}_a^y$ is the same as $\tilde{\mathbb{P}}_{1-a}^y$, again considered as a measure on paths $X_t, 0 \leq t \leq T_x$.*

Proof. Using (7), we can see that the Radon-Nikodym derivative of the conditioned measure is proportional to the exponential term (the other term is the same for all paths). We also see from (8), a rederivation of the fact that $\tilde{\mathbb{P}}_a^y\{T_x < \infty\} = (x/y)^{2a-1}$. $\square$

For fixed t, on the event $\{T_0 > t\}$, the measures $\mathbb{P}^x$ and $\hat{\mathbb{P}}_a^x$ are mutually absolutely continuous. Indeed, if $0 < r < x < R$, and $\tau = T_r \wedge T_R$, then $\mathbb{P}^x$ and $\hat{\mathbb{P}}^x$ are mutually absolutely continuous on $\mathcal{F}_\tau$ with

$$\frac{d\hat{\mathbb{P}}^x}{d\mathbb{P}^x} = N_{\tau,a} \in (0, \infty).$$

However, if $a < b < 1/2$, the measures $\hat{\mathbb{P}}_a$ and $\hat{\mathbb{P}}_b$ viewed as measure on on curves $X_t, 0 \leq t \leq T_0$, can be shown to be singular with respect to each other.

Proposition 2.4 (Brownian scaling)**.** *Suppose X_t is a Bessel process satisfying* (2)*, $r > 0$, and*

$$Y_t = r^{-1} X_{r^2 t}, \quad t \le r^{-2} T_0.$$

Then Y_t is a Bessel process with parameter a stopped at the origin.

Proof. This follows from the fact that Y_t satisfies

$$dY_t = \frac{a}{Y_t}\, dt + dW_t,$$

where $W_t = r^{-1} B_{r^2 t}$ is a standard Brownian motion. □

2.1 Logarithm

When one takes the logarithm of the Bessel process, the parameter $\nu = a - \frac{1}{2}$ becomes the natural one to use. Suppose X_t satisfies

$$dX_t = \frac{\nu + \frac{1}{2}}{X_t}\, dt + dB_t, \quad X_0 = x_0 > 0.$$

If $L_t = \log X_t$, then Itô's formula shows that

$$dL_t = \frac{1}{X_t}\, dX_t - \frac{1}{2X_t^2}\, dt = \frac{\nu}{X_t^2}\, dt + \frac{1}{X_t}\, dB_t.$$

Let

$$\sigma(s) = \inf\left\{ t : \int_0^t \frac{dr}{X_r^2} = s \right\}, \qquad \hat{L}_s := L_{\sigma(s)}.$$

Then $\hat{L}_s$ satisfies

$$d\hat{L}_s = \nu\, ds + dW_s, \quad W_s := \int_0^{\sigma(s)} \frac{dB_r}{X_r},$$

Here W_s is a standard Brownian motion. In other words the logarithm of the Bessel process is a time change of a Brownian motion with constant drift. Note that $\sigma(\infty) = T_0$. If $\nu < 0$, it takes an infinite amount of time for the logarithm to reach $-\infty$ in the new parametrization, but it only takes a finite amount of time in the original parametrization.

2.2 Density

For most of the remainder of this paper, we will now drop the bulky notation $\hat{\mathbb{P}}_a^x$ and use $\mathbb{P}$ or $\mathbb{P}^x$. We assume that B_t is a standard Brownian motion and X_t satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad X_0 = x > 0$$

This is valid until time $T = T_0 = \inf\{t : X_t = 0\}$. When a is fixed, we will write L, L^* for the generator and its adjoint, that is, the operators

$$Lf(x) = \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x),$$

$$L^*f(x) = -\left[\frac{af(x)}{x}\right]' + \frac{1}{2}\, f''(x) = \frac{a}{x^2}\, f(x) - \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x).$$

For $x, y > 0$, let $q_t(x, y; a)$ denote the transition density for the Bessel process stopped when it reaches the origin. In other words, if $I \subset (0, \infty)$ is an interval,

$$\mathbb{P}^x\{T > t; X_t \in I\} = \int_I q_t(x, y; a)\, dy.$$

In particular,

$$\int_0^\infty q_t(x, y; a)\, dy = \mathbb{P}^x\{T > t\} \left\{ \begin{array}{ll} = 1, & a \geq 1/2, \\ < 1, & a < 1/2 \end{array} \right. .$$

If $a = 0$, this is the density of Brownian motion killed at the origin for which we know that $q_t(x, y; 0) = q_t(y, x; 0)$. We can give an explicit form of the density by solving either of the "heat equations"

$$\partial_t q_t(x, y; a) = L_x q_t(x, y; a), \qquad \partial_t q_t(x, y; a) = L_y^* q_t(x, y; a),$$

with appropriate initial and boundary conditions. The subscripts on L, L^* denote which variable is being differentiated with the other variables remaining fixed. The solution is given in terms of a special function which arises as a solution to a second order linear ODE. We will leave some of the computations to an appendix, but we include the important facts in the next proposition. We will use the following elementary fact: if a nonnegative random variable T has density $f(t)$ and $r > 0$, then the density of rT is

$$r^{-1}\, f(t/r). \tag{9}$$

Proposition 2.5. *Let $q_t(x, y; a)$ denote the density of the Bessel process with parameter a stopped when it reaches the origin. Then for all $x, y, t, r > 0$,*

$$q_t(x, y; 1 - a) = (y/x)^{1-2a}\, q_t(x, y; a), \tag{10}$$

$$q_t(x, y; a) = q_t(y, x; a)\, (y/x)^{2a}, \tag{11}$$

$$q_{r^2 t}(rx, ry; a) = r^{-1}\, q_t(x, y; a). \tag{12}$$

Moreover, if $a \geq 1/2$,

$$q_1(x, y; a) = y^{2a}\, \exp\left\{-\frac{x^2 + y^2}{2}\right\}\, h_a(xy), \tag{13}$$

where h_a is the entire function

$$h_a(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{a+2k-\frac{1}{2}}\, k!\, \Gamma(k+a+\frac{1}{2})}. \tag{14}$$

In particular,

$$q_1(0,y;a) := q_1(0+,y,a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, y^{2a}\, e^{-y^2/2}, \quad a \geq 1/2. \tag{15}$$

In the proposition we defined h_a by its series expansion (14), but it can also be defined as the solution of a particular boundary value problem.

Lemma 2.6. *h_a is the unique solution of the second order linear differential equation*

$$z\, h''(z) + 2a\, h'(z) - z\, h(z) = 0. \tag{16}$$

with $h(0) = 2^{\frac{1}{2}-a}/\Gamma(a+\frac{1}{2}), h'(0) = 0$.

Proof. See Proposition 5.1. □

Remarks.

- By combining (12) and (13) we get for $a \geq 1/2$,

$$q_t(x,y;a) = \frac{1}{\sqrt{t}}\, q_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a\right) = \frac{y^{2a}}{t^{a+\frac{1}{2}}}\, \exp\left\{-\frac{x^2+y^2}{2t}\right\}\, h_a\left(\frac{xy}{t}\right). \tag{17}$$

- The density is often written in terms of the modified Bessel function. If $\nu = a - \frac{1}{2}$, then $I_\nu(x) := x^\nu\, h_{\nu+\frac{1}{2}}(x)$ is the *modified Bessel function of the first kind of index ν*. This function satisfies the *modified Bessel equation*

$$x^2\, I''(x) + x\, I'(x) - [\nu^2 + x^2]\, I(x) = 0.$$

- The expressions (13) and (17) hold only for $a \geq 1/2$. For $a < 1/2$, we can use (10) to get

$$\begin{aligned} q_t(x,y;a) &= \frac{1}{\sqrt{t}}\, q_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a\right) \\ &= (y/x)^{2a-1}\, \frac{1}{\sqrt{t}}\, q_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; 1-a\right) \\ &= \frac{y}{x^{2a-1}\, t^{\frac{3}{2}-a}}\, \exp\left\{-\frac{x^2+y^2}{2t}\right\}\, h_{1-a}\left(\frac{xy}{t}\right). \end{aligned}$$

- In the case $a = 1/2$, we can use the fact that the radial part of a two-dimensional Brownian motion is a Bessel process and write

$$q_1(x, y; 1/2) = \frac{y}{2\pi} \int_0^{2\pi} \exp\left\{ \frac{(x - y\cos\theta)^2 + (y\sin\theta)^2}{2} \right\} d\theta = y\, e^{-(x^2+y^2)/2}\, h_{1/2}(xy),$$

  where

$$h_{1/2}(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{r\cos\theta}\, d\theta = I_0(r).$$

  The last equality is found by consulting an integral table. Note that $h_{1/2}(0) = 1, h'_{1/2}(0) = 0$.

- We can write (15) as

$$c_d\, y^{d-1}\, e^{-y^2/2} \qquad d = 2a + 1,$$

  which for positive integer d can readily be seen to be the density of the radial part of a random vector in $\mathbb{R}^d$ with density proportional to $\exp\{-|x|^2/2\}$. This makes this the natural guess for all d for which c_d can be chosen to make this a probability density, that is, $a > -1/2$. For $-1/2 < a < 1/2$, we will see that this is the density of the Bessel process reflected (rather than killed) at the origin.

- Given the theorem, we can determine the asymptotics of $h_a(x)$ as $x \to \infty$. Note that

$$\lim_{x\to\infty} q_1(x, x; a) = \frac{1}{\sqrt{2\pi}},$$

  since for large x and small time, the Bessel process looks like a standard Brownian motion. Therefore,

$$\lim_{x\to\infty} x^{2a}\, e^{-x^2}\, h_a(x^2) = \frac{1}{\sqrt{2\pi}},$$

  and hence

$$h_a(x) \sim \frac{1}{\sqrt{2\pi}}\, x^{-a}\, e^x, \qquad x \to \infty.$$

  In fact, there is an entire function u_a with $u_a(0) = 1/\sqrt{2\pi}$ such that for all $x > 0$,

$$h_a(x) = u(1/x)\, x^{-a}\, e^x.$$

  See Proposition 5.3 for details. This is equivalent to a well known asymptotic behavior for I_ν,

$$I_\nu(x) = x^\nu\, h_{\nu+\frac{1}{2}}(x) \sim \frac{e^x}{\sqrt{2\pi x}}.$$

- The asymptotics implies that for each $a \geq 1/2$, there exist $0 < c_1 < c_2 < \infty$ such that for all $0 < x, y, t < \infty$,

$$c_1 \left[\frac{y}{x}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x-y)^2/2t} \leq q_t(x, y; a) \leq c_2 \left[\frac{y}{x}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x-y)^2/2t}, \quad t \leq xy, \tag{18}$$

$$c_1 \left[\frac{y}{\sqrt{t}}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x^2+y^2)/2t} \leq q_t(x,y;a) \leq c_2 \left[\frac{y}{\sqrt{t}}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x^2+y^2)/2t}, \quad t \geq xy, \quad (19)$$

Proof. The relation (10) follows from (8). The relation (11) follows from the fact that the exponential term in the compensator $N_{t,a}$ for a reversed path is the same as for the path. The scaling rule (12) follows from Proposition 2.4 and (9).

For the remainder, we fix $a \geq 1/2$ and let $q_t(x,y) = q_t(x,y;a)$. The heat equations give the equations

$$\partial_t q_t(x,y;a) = \frac{a}{x}\,\partial_x\, q_t(x,y;a) + \frac{1}{2}\,\partial_{xx}\, q_t(x,y;a), \quad (20)$$

$$\partial_t q_t(x,y;a) = \frac{a}{y^2}\, q_t(x,y;a) - \frac{a}{y}\,\partial_y\, q_t(x,y;a) + \frac{1}{2}\,\partial_{yy}\, q_t(x,y;a), \quad (21)$$

In our case, direct computation (See Proposition 5.2) shows that if h_a satisfies (16) and q_t is defined as in (15), then q_t satisfies (20) and (21). We need to establish the boundary conditions on h_a. Let

$$q_t(0,y;a) = \lim_{x \downarrow 0} q_t(x,y;a)$$

Since

$$1 = \int_0^\infty q_1(0+,y;a)\,dy = \int_0^\infty h_a(0)\, y^{2a}\, e^{-y^2/2}\,dy,$$

we see that

$$\frac{1}{h_a(0)} = \int_0^\infty y^{2a}\, e^{-y^2/2}\,dy = \int_0^\infty (2u)^a\, e^{-u}\, \frac{du}{\sqrt{2u}} = 2^{a-\frac{1}{2}}\, \Gamma\left(a+\frac{1}{2}\right).$$

Note that

$$q_1(x,1;a) = q_1(0,1;a) + x\, h_a'(0)\, e^{-1/2} + O(x^2), \quad x \downarrow 0.$$

Hence to show that $h_a'(0) = 0$, it suffices to show that $q_1(x,1;0) = q_1(0,1;a) + o(x)$. Suppose $0 < r < x$, and we start the Bessel process at r. Let τ_x be the the first time that the process reaches x. Then by the strong Markov property we have

$$q_1(r,1;a) = \int_0^1 q_{1-s}(x;1;0)\,dF(s),$$

where F is the distribution function for τ_x. Using (20), we see that $q_{1-s}(x;1;0) = q_1(x,1;0) + O(s)$. Therefore,

$$|q_1(x,1;a) - q_1(r,1;a)| \leq c \int_0^1 s\,dF(s) \leq c\,\mathbb{E}^r[\tau_x].$$

Using the scaling rule, we can see that $\mathbb{E}^r[\tau_x] = O(x^2)$.

$\square$

We note that one standard way to solve a heat equation

$$\partial_t f(t,x) = L_x f(t,x),$$

with zero boundary conditions and given initial conditions, is to find a complete set of eigenfunctions for L

$$L\phi_j(x) = -\lambda_j \, \phi_j(x),$$

and to write the general solution as

$$\sum_{j=1}^{\infty} c_j \, e^{-\lambda_j t} \, \phi_j(x).$$

The coefficients c_j are found using the initial condition. This gives a series solution. This could be done for the Bessel process, but we had the advantage of the scaling rule (12), which allows the solution to be given in terms of a single special function h_a.

There are other interesting examples that will not have this exact scaling. Some of these, such as the radial Bessel process below, look like the Bessel process near the endpoints. We will use facts about the density of the Bessel process to conclude facts about the density of these other processes. The next proposition gives a useful estimate — it shows that the density of a Bessel process at $y \in (0, \pi/2]$ is comparable to that one would get by killing the process when it reaches level $7\pi/8$. (The numbers $3\pi/4 < 7\pi/8$ can be replaced with other values, of course, but the constants depend on the values chosen. These values will be used in discussion of the radial Bessel process.)

Proposition 2.7. *Let $\hat{q}_t(x,y;a)$ be the density of the Bessel process stopped at time $T = T_0 \wedge T_{7\pi/8}$. If $a \geq 1/2$, then for every $0 < t_1 < t_2 < \infty$, there exist $0 < c_1 < c_2 < \infty$ such that if $t_1 \leq t \leq t_2$ and $0 < x, y \leq 3\pi/4$, then*

$$c_1 \, y^{2a} \leq \hat{q}_t(x,y;a) \leq q_t(x,y;a) \leq c_2 \, y^{2a}.$$

This is an immediate corollary of the following.

Proposition 2.8. *Let $\hat{q}_t(x,y;a)$ be the density of the Bessel process stopped at time $T = T_0 \wedge T_{7\pi/8}$. For every $a \geq 1/2$ and $t_0 > 0$, there exists $0 < c_1 < c_2 < \infty$ such that for all $0 \leq x, y \leq 3\pi/4$ and $t \geq t_0$,*

$$\hat{q}_t(x,y;a) \geq c \, e^{-\beta t} \, q_t(x,y;a).$$

$$c_1 \mathbb{P}^x\{T > t - t_0; X_t \leq 3\pi/4\} \leq y^{-2a} \, \hat{q}_t(x,y;a) \leq c_2 \, \mathbb{P}^x\{T > t - t_0\}.$$

Proof. It suffices to prove this for t_0 sufficiently small. Note that the difference $q_t(x,y;a) - \hat{q}_t(x,y;a)$ represents the contribution to $q_t(x,y;a)$ by paths that visit $7\pi/8$ some time before t. Therefore, using the strong Markov property, we can see that

$$q_t(x,y;a) - \hat{q}_t(x,y;a) \leq \sup_{0 \leq s \leq t} [q_s(x,y;a) - \hat{q}_s(x,y;a)] \leq \sup_{0 \leq s \leq t} q_s(3\pi/4, y; a).$$

Using the explicit form of $q_t(x,y;a)$ (actually it suffices to use the up-to-constants bounds (18) and (19)), we can find $t'>0$ such that

$$c_1\, y^{2a} \leq \hat q_t(x,y;a) \leq c_2\, y^{2a}, \quad t' \leq t \leq 2t', \quad 0 < x,y \leq \pi/2.$$

If $s \geq 0$, and $t' \leq t \leq 2t'$,

$$\begin{aligned}
\hat q_{s+t}(x,y;a) &= \int_0^{7\pi/8} \hat q_s(x,z;a)\, \hat q_t(z,y;a)\, dy \\
&\leq c\, \mathbb{P}^x\{T > s\} \sup_{0 \leq z \leq 7\pi/8} q_t(z,y;a) \\
&\leq c\, \mathbb{P}^x\{T > s\}\, y^{2a},
\end{aligned}$$

$$\begin{aligned}
\hat q_{s+t}(x,y;a) &\geq \int_0^{3\pi/4} \hat q_s(x,z;a)\, \hat q_t(z,y;a)\, dy \\
&\geq c\, \mathbb{P}^x\{T > s, X_s \leq 3\pi/4\} \inf_{0 \leq z \leq 1} \hat q_t(z,y;a) \\
&\geq c\, \mathbb{P}^x\{T > s, X_s \leq 3\pi/4\}\, y^{2a}.
\end{aligned}$$

□

Proposition 2.9. *Suppose X_t is a Bessel process with parameter $a < 1/2$ with $X_0 = x$, then the density of T_0 is*

$$\frac{2^{a-\frac{1}{2}}}{\Gamma(\frac{1}{2}-a)}\, x^{1-2a}\, t^{a-\frac{3}{2}}\, \exp\{-x^2/2t\}, \tag{22}$$

Proof. The distribution of T_0 given that $X_0 = x$ is the same as the distribution of $x^2\, T_0$ given that $X_0 = 1$. Hence by (9), we may assume $x = 1$. Let

$$F(t) = \mathbb{P}\{T_0 \leq t \mid X_0 = 1\} = \mathbb{P}\{T_0 \leq tx^2 \mid X_0 = x\}.$$

Then the strong Markov property implies that

$$\begin{aligned}
s^{-1}\, \mathbb{P}\{t < T_0 \leq t+s\} &= s^{-1} \int_0^\infty q_t(1,y;a)\, F(s/y^2)\, dy \\
&= \int_0^\infty [x\sqrt{s}]^{-1} q_t(1, \sqrt{s}\, x; a)\, x\, F(1/x)\, dx.
\end{aligned}$$

Hence the density is

$$\lim_{s \downarrow 0} \int_0^\infty [x\sqrt{s}]^{-1} q_t(1, \sqrt{s}\, x; a)\, x\, F(1/x)\, dx.$$

We write $y_t = y/\sqrt{t}$ Using Proposition 2.5, we see that

$$\begin{aligned}
y^{-1}\, q_t(1,y;a) &= t^{-1/2}\, y^{-1}\, q_1(t^{-1/2}, y_t; a) \\
&= t^{-1/2}\, y^{2a-1}\, q_1(y_t, t^{-1/2}; a) \\
&= t^{-1/2}\, q_1(y_t, t^{-1/2}; 1-a).
\end{aligned}$$

Therefore,

$$\lim_{y\downarrow 0} y^{-1}\, q_t(1,y;a) = t^{-1/2} \lim_{z\downarrow 0} q_1(z,t^{-1/2};1-a) = h_{1-a}(0)\, t^{a-\frac{3}{2}}\, e^{-1/2t}.$$

This establishes the density up to a constant which is determined by

$$\int_0^\infty t^{a-\frac{3}{2}}\, e^{-1/2t}\, dt = 2^{\frac{1}{2}-a} \int_0^\infty u^{-\frac{1}{2}-a}\, e^{-u}\, du = 2^{\frac{1}{2}-a}\, \Gamma\left(\frac{1}{2}-a\right).$$

□

2.3 Geometric time scale

It is often instructive to consider the scaled Bessel process at geometric times (this is sometimes called the Ornstein-Uhlenbeck scaling). For this section we will assume $a \geq 1/2$ although much of what we say is valid for $a < 1/2$ up to the time that the process reaches the origin and for $-1/2 < a < 1/2$ for the reflected process.

Suppose X_t satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t,$$

and let

$$Y_t = e^{-t/2}\, X_{e^t} \qquad W_t = \int_0^{e^t} e^{-s/2}\, dB_s.$$

Note that

$$dX_{e^t} = \frac{a\, e^t}{X_{e^t}}\, dt + e^{t/2}\, dB_{e^t} = e^{t/2}\, \left[\frac{a}{Y_t}\, dt + dB_{e^t}\right],$$

or

$$dY_t = \left[\frac{a}{Y_t} - \frac{Y_t}{2}\right]\, dt + dW_t. \tag{23}$$

This process looks like the usual Bessel process near the origin, and it is not hard to see that processes satisfying (23) with $a \geq 1/2$, never reaches the origin. Of course, we knew this fact from the definition of Y_t and the fact that X_t does not reach the origin.

Not as obvious is the fact that Y_t is a recurrent process, in fact, a positive recurrent process with an invariant density. Let us show this in two ways. First, note that the process Y_t is the same as a process obtained by starting with a Brownian motion Y_t and weighting locally by the function

$$\phi(x) = x^a\, e^{-x^2/4}.$$

Indeed, using Itô's formula and the calculations,

$$\phi'(x) = \left[\frac{a}{x} - \frac{x}{2}\right]\, \phi(x),$$

$$\phi''(x) = \left(\left[\frac{a}{x} - \frac{x}{2}\right]^2 - \frac{a}{x^2} - \frac{1}{2}\right)\, \phi(x) = \left[\frac{a^2-a}{x^2} - (a+\frac{1}{2}) + \frac{x^2}{4}\right]\, \phi(x),$$

we see that if

$$M_t = \phi(X_t)\, \exp\left\{ \int_0^t \left(\frac{a-a^2}{Y_s^2} + (a+\frac{1}{2}) - \frac{Y_s^2}{4} \right) ds \right\},$$

then M_t is a local martingale satisfying

$$dM_t = \left[\frac{a}{Y_t} - \frac{Y_t}{2} \right] M_t \, dY_t.$$

The invariant probability density for this process is given by

$$f(x) = c\, \phi(x)^2 = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, x^{2a} e^{-x^2/2}. \tag{24}$$

where the constant is chosen to make this a probability density. The equation for an invariant density is $L^* f(x) = 0$, where L^* is the adjoint of the generator

$$\begin{aligned} L^* f(x) &= -\left(\left[\frac{a}{x} - \frac{x}{2} \right] f(x) \right)' + \frac{1}{2} f''(x) \\ &= \left(\frac{a}{x^2} + \frac{1}{2} \right) f(x) + \left[\frac{x}{2} - \frac{a}{x} \right] f'(x) + \frac{1}{2} f''(x). \end{aligned}$$

Direct differentiation of (24) gives

$$f'(x) = \left[\frac{2a}{x} - x \right] f(x),$$

$$f''(x) = \left(\left[\frac{2a}{x} - x \right]^2 - \frac{2a}{x^2} - 1 \right) f(x) = \left[\frac{4a^2 - 2a}{x^2} - 4a - 1 + x^2 \right] f(x),$$

so the equation $L^* f(x) = 0$ comes down to

$$\frac{a}{x^2} + \frac{1}{2} + \left[\frac{x}{2} - \frac{a}{x} \right] \left[\frac{2a}{x} - x \right] + \frac{1}{2} \left[\frac{4a^2 - 2a}{x^2} - 4a - 1 + x^2 \right] = 0,$$

which is readily checked.

Let $\phi_t(x,y)$ denote the density of a process satisfying (23). Then

$$\phi_t(x,y) = e^{t/2}\, q_{e^t}\left(x, e^{t/2}\, y; a\right) = y^{2a} \exp\left\{ -\frac{e^{-t}\, x^2 + y^2}{2} \right\} h_a\left(e^{-t/2}\, xy\right).$$

In particular,

$$\lim_{t\to\infty} \phi_t(x,y) = q_1(0,y;a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, y^{2a}\, e^{-y^2/2} = f(y).$$

2.4 Green's function

We define the Green's function (with Dirichlet boundary conditions) for the Bessel process by

$$G(x,y;a) = \int_0^\infty q_t(x,y;a)\,dt.$$

If $a \geq 1/2$, then

$$G(x,y;a) = \int_0^\infty q_t(x,y;a)\,dt = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\int_0^\infty \frac{y^{2a}}{t^{a+\frac{1}{2}}}\,\exp\left\{-\frac{x^2+y^2}{2t}\right\}\,h_a\left(\frac{xy}{t}\right),$$

and

$$G_{1-a}(x,y;a) = \int_0^\infty q_t(x,y;1-a)\,dt = (y/x)^{1-2a}\int_0^\infty q_t(x,y;a)\,dt = (y/x)^{1-2a}\,G(x,y;a).$$

In particular,

$$\begin{aligned} G(1,1;a) = G(1,1;1-a) &= \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\int_0^\infty \frac{1}{t^{a+\frac{1}{2}}}\,e^{-1/t}\,h_a(1/t)\,dt \\ &= \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\int_0^\infty \frac{1}{u^{\frac{3}{2}-a}}\,e^{-u}\,h_a(u)\,du. \end{aligned}$$

Proposition 2.10. *For all a, x, y,*

$$G(x,y;a) = (x/y)^{1-2a}\,G(x,y;1-a).$$

$$G(x,y;a) = (y/x)^{2a}\,G(y,x;a),$$

If $a = 1/2$, $G(x,y;a) = \infty$ for all x, y. If $a > 1/2$, then

$$G(r,ry;a) = C_a\,r\,\left[1 \wedge y^{1-2a}\right],$$

where

$$C_a = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\int_0^\infty \frac{1}{u^{\frac{3}{2}-a}}\,e^{-u}\,h_a(u)\,du < \infty.$$

Proof.

$$\begin{aligned} G(x,y;1-a) &= \int_0^\infty q_t(x,y;1-a)\,dt \\ &= (y/x)^{1-2a}\int_0^\infty q_t(x,y;a)\,dt = (y/x)^{1-2a}\,G(x,y;a). \end{aligned}$$

$$G(x,y;a) = \int_0^\infty q_t(x,y;a)\,dt = (y/x)^{2a}\int_0^\infty q_t(y,x;a)\,dt = (y/x)^{2a}\,G(y,x;a).$$

$$\begin{aligned} G(rx,ry;a) &= \int_0^\infty q_t(rx,ry;a)\,dt \\ &= \int_0^\infty \frac{1}{r}\,q_{t/r^2}(x,y;a)\,dt \\ &= \int_0^\infty r\,q_s(x,y;a)\,ds = r\,G(x,y;a). \end{aligned}$$

We assume that $a \geq 1/2$ and note that the strong Markov property implies that

$$G(x,1;a) = \mathbb{P}^x\{T_1 < \infty\}\ G(1,1;a) = \left[1 \wedge x^{1-2a}\right]\ G(1,1;a).$$

$$G(1,1;a) = \int_0^\infty \frac{1}{t^{(1-a)+\frac{1}{2}}}\,e^{-1/t}\,h_{1-a}(1/t)\,dt.$$

$\square$

2.5 Another viewpoint

The Bessel equation is

$$dX_t = \frac{a}{X_t}\,dt + dB_t,$$

where B_t is a Brownian motion and $a \in \mathbb{R}$. In this section we fix a and the Brownian motion B_t but vary the initial condition x. In other words, let X_t^x be the solution to

$$dX_t^x = \frac{a}{X_t^x}\,dt + dB_t, \qquad X_0 = x, \tag{25}$$

which is valid until time $T_0^x = \inf\{t : X_t^x = 0\}$. The collection $\{X_t^x\}$ is an example of a stochastic flow. If $t < T_0^x$, we can write

$$X_t^x = x + B_t + \int_0^t \frac{a}{X_s^x}\,ds.$$

If $x < y$, then

$$X_t^y - X_t^x = y - x + \int_0^t \left[\frac{a}{X_s^y} - \frac{a}{X_s^y}\right] ds = y - x - \int_0^t \left[\frac{a(X_s^y - X_s^x)}{X_s^x\,X_s^x}\right] ds. \tag{26}$$

In other words, if $t < T_0^x \wedge T_0^y$, then $X_t^y - X_t^x$ is differentiable in t with

$$\partial_t\left[X_t^y - X_t^x\right] = -[X_t^y - X_t^x]\,\frac{a}{X_t^x\,X_t^y},$$

which implies that

$$X_t^y - X_t^x = (y-x)\,\exp\left\{-a\int_0^t \frac{ds}{X_s^x\,X_s^y}\right\}.$$

From this we see that $X_t^y > X_t^x$ for all $t < T_0^x$ and hence $T_0^x \leq T_0^y$. By letting $y \to x$ we see that

$$\partial_x X_t^x = \exp\left\{ -a \int_0^t \frac{ds}{(X_s^x)^2} \right\}. \tag{27}$$

Although $X_t^x < X_t^y$ for all $t > T_0^x$, as we will see, it is possible for $T_0^x = T_0^y$.

Proposition 2.11. *Suppose $0 < x < y < \infty$ and X_t^x, X_t^y satisfy (25) with $X_0^x = x, X_0^y = y$.*

1. *If $a \geq 1/2$, then $\mathbb{P}\{T_0^x = \infty$ for all $x\} = 1$.*
2. *If $1/4 < a < 1/2$ and $x < y$, then*

$$\mathbb{P}\{T_0^x = T_0^y\} > 0.$$

3. *If $a \leq 1/4$, then with probability one for all $x < y$, $T_0^x < T_0^y$.*

Proof. If $a \geq 1/2$, then Proposition 2.1 implies that for each x, $\mathbb{P}\{T_0^x = \infty\} = 1$ and hence $\mathbb{P}\{T_0^x = \infty$ for all rational $x\} = 1$. Since $T_0^x \leq T_0^y$ for $x \leq y$, we get the first assertion.

For the remainder we assume that $a < 1/2$. Let us write $X_t = X_t^x, Y_t = X_t^y, T^x = T_0^x, T^y = T_0^y$. Let $h(x,y) = h(x,y;a) = \mathbb{P}\{T^x = T^y\}$. By scaling we see that $h(x,y) = h(x/y) := h(x/y,1)$. Hence, we may assume $y = 1$. We claim that $h(0+,1) = 0$. Indeed, T^r has the same distribution as $r^2\, T^1$ and hence for every $\epsilon > 0$ we can find r, δ such that $\mathbb{P}\{T^r \geq \delta\} \leq \epsilon/2, \mathbb{P}\{T^1 \leq \delta\} \leq \epsilon/2$, and hence $\mathbb{P}\{T^1 = T^r\} \leq \epsilon$.

Let $u = \sup_{t < T^x} Y_t/X_t$. We claim that

$$\mathbb{P}\{T^x < T^1; u < \infty\} = 0.$$

$$\mathbb{P}\{T^x = T^1; u = \infty\} = 0.$$

The first equality is immediate; indeed, if $Y_t \leq c X_t$ for all t, then $T^1 = T^x$. For the second equality, let $\sigma_N = \inf\{t : Y_t/X_t = N\}$. Then,

$$\mathbb{P}\{u \geq N; T^1 = T^x\} \leq \mathbb{P}\{T^1 = T^x \mid \sigma_N < \infty\} = h(1/N) \longrightarrow 0, \quad N \to \infty.$$

Let

$$L_t = \log\left(\frac{Y_t}{X_t} - 1\right) = \log(Y_t - X_t) - \log X_t.$$

Note that

$$d\log(Y_t - X_t) = -\frac{a}{X_t\, Y_t}\, dt,$$

$$d\log X_t = \frac{1}{X_t}\, dX_t - \frac{1}{2X_t^2}\, dt = \frac{a - \frac{1}{2}}{X_t^2}\, dt + \frac{1}{X_t}\, dB_t,$$

and hence

$$dL_t = \left[\frac{\frac{1}{2}-a}{X_t^2} - \frac{a}{X_t Y_t}\right] dt - \frac{1}{X_t}\, dB_t$$
$$= \frac{1}{X_t^2}\left[\frac{1}{2} - a - \frac{a}{e^{L_t}+1}\right] dt - \frac{1}{X_t}\, dB_t. \qquad (28)$$

In order to understand this equation, let us change the time parametrization so that the Brownian term has variance one. More precisely, define $\sigma(t), W_t$ by

$$\int_0^{\sigma(t)} \frac{ds}{X_s^2} = t, \qquad W_t = -\int_0^{\sigma(t)} \frac{1}{X_s}\, dB_s.$$

Then W_t is a standard Brownian motion and $\tilde{L}_t := L_{\sigma(t)}$ satisfies

$$d\tilde{L}_t = \left[\frac{1}{2} - a - \frac{aX_{\sigma(t)}}{\tilde{Y}_{\sigma(t)}}\right] dt + dW_t = \left[\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1}\right] dt + dW_t.$$

For every $a < 1/2$, there exists $u > 0$ and $K < \infty$ such that if $\tilde{L}_t \geq K$, then

$$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1} > u.$$

Hence, by comparison with a Brownian motion with drift u, we can see that if $\tilde{L}_t \geq K+1$, then with positive probability, $\tilde{L}_t \to \infty$ and hence $Y_t/X_t \to \infty$. Hence starting at any initial value $\tilde{L}_t = l$ there is a positive probability (depending on l) that $\tilde{L}_t \to \infty$.

If $a > 1/4$, then there exists $u > 0$ and $K < \infty$ such that if $\tilde{L}_t \leq -K$, then

$$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1} < -u.$$

Hence by comparison with a Brownian motion with drift $-u$, we can see that if $\tilde{L}_t \leq -(K+1)$, then with positive probability, $\tilde{L}_t \to -\infty$. Hence starting at any initial value $\tilde{L}_t = l$ there is a positive probability (depending on l) that $\tilde{L}_t \to -\infty$.

If $a \leq 1/4$, then

$$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1} > 0,$$

and hence by comparison with driftless Brownian motion, we see that

$$\limsup \tilde{L}_t \to \infty. \qquad (29)$$

But as mentioned before, if $\tilde{L}_t \geq K+1$ for some K there is a positive probability that $\tilde{L}_t \to \infty$. Since (29) shows that we get an "infinite number of tries" we see that $\tilde{L}_t \to \infty$ with probability one.

Using this argument, we also see that for $1/4 < a < 1/2$, then with probability one either $\tilde{L}_t \to -\infty$ or $\tilde{L}_t \to \infty$. Therefore,

$$\lim_{t \uparrow T^x} \frac{X_t}{Y_t} \in \{0, 1\}. \tag{30}$$

We note that by closer examination, we can see that if $1/4 < a < 1/2$, then

$$\lim_{y \downarrow x} \mathbb{P}\{T_0^x = T_0^y\} = 1.$$

□

Proposition 2.12. *In the notation of the previous proposition, if $1/4 < a < 1/2$ and $0 < x < y = 1$. Then,*

$$\psi(x) := \mathbb{P}\{T_0^x \neq T_0^1\} = \frac{\Gamma(2a)}{\Gamma(4a-1)\,\Gamma(1-2a)} \int_0^x \frac{ds}{(1-s)^{2-4a}\, s^{2a}}.$$

Proof. We note that $\psi(x)$ is the solution of the boundary value problem

$$\frac{1}{2}\,\psi''(x) + \left[\frac{1-2a}{1-x} - \frac{a}{x}\right] \psi'(x) = 0, \qquad \psi(0) = 0, \psi(1) = 1. \tag{31}$$

In the notation of the previous proof, let $R_t = X_t/Y_t,$. Itô's formula and the product rule give

$$dX_t = X_t \left[\frac{a}{X_t^2}\, dt + \frac{1}{X_t}\, dB_t\right],$$

$$\begin{aligned} d\left[\frac{1}{Y_t}\right] &= -\frac{1}{Y_t^2}\, dY_t + \frac{1}{Y_t^3}\, d\langle Y\rangle_t \\ &= \frac{1}{Y_t}\left[\frac{1-a}{Y_t^2}\, dt - \frac{1}{Y_t}\, dB_t\right]. \end{aligned}$$

$$\begin{aligned} dR_t &= R_t \left[\frac{1}{X_t^2}\left(a + (1-a)\; R_t^2 - R_t\right)\; dt + \frac{1}{X_t}\;(1-R_t)\; dB_t\right] \\ &= R_t \left[\frac{1}{X_t^2}\,((1-a)R_t - a)(R_t - 1)\, dt + \frac{1}{X_t}\;(1-R_t)\; dB_t\right] \end{aligned}$$

After a suitable time change, we see that $\hat{R}_t := R_{\sigma(t)}$ satisfies

$$d\hat{R}_t = \frac{(1-a)\hat{R}_t - a}{\hat{R}_t\,(1-\hat{R}_t)}\, dt + dW_t = \left[\frac{1-2a}{1-\hat{R}_t} - \frac{a}{\hat{R}_t}\right] dt + dW_t,$$

where W_t is a standard Brownian motion. Using (31), we see that $\psi(\hat{R}_t)$ is a bounded martingale, and using the optional sampling theorem and (30) we get the result.

□

2.6 Functionals of Brownian motion and Bessel process

In the analysis of the Schramm-Loewner evolution, one often has to evaluate or estimate expectations of certain functionals of Brownian motion or the Bessel process. One of the most important functionals is the one that arises as the compensator in the change-of-measure formulas for the Bessel process.

Suppose X_t is a Brownian motion with $X_t = x > 0$ and let

$$J_t = \int_0^t \frac{ds}{X_s^2}, \qquad K_t = e^{-J_t} = \exp\left\{-\int_0^t \frac{ds}{X_s^2}\right\},$$

which are positive and finite for $0 < t < T_0$. We have seen K_t in (27). Let I_t denote the indicator function of the event $\{T_0 > t\}$. The local martingale from (1) is

$$N_{t,a} = (X_t/X_0)^a \, K_t^{\lambda_a}, \qquad \text{where} \quad \lambda_a = \frac{a(a-1)}{2}.$$

Note that

$$a = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+8\lambda_a} \geq \frac{1}{2}. \tag{32}$$

In this section, we write $\mathbb{E}$ for Brownian expectations and $\hat{\mathbb{E}}_a$ for the corresponding expectation with respect to the Bessel process with parameter a. In particular, if Y is an $\mathcal{F}_t$-measurable random variable,

$$\hat{\mathbb{E}}_a^x \left[V \, I_t\right] = \mathbb{E}^x \left[V \, I_t \, N_{t,a}\right].$$

Proposition 2.13. *Suppose $\lambda \geq -1/8$ and*

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1+8\lambda} \geq \frac{1}{2},$$

is the larger root of the polynomial $a^2 - a - 2\lambda$. If X_t is a Brownian motion with $X_0 = x > 0$, then

$$\mathbb{E}^x[K_t^\lambda \, I_t] = x^a \, \hat{\mathbb{E}}_a^x \left[X_t^{-a} \, I_t\right] = x^a \int_0^\infty q_t(x,y;a) \, y^{-a} \, dy.$$

In particular, if $x = 1$, as $t \to \infty$,

$$\mathbb{E}^1[K_t^\lambda \, I_t] = t^{-\frac{a}{2}} \, \frac{\Gamma(\frac{a}{2}+\frac{1}{2})}{2^{a/2}\,\Gamma(a+\frac{1}{2})} \, [1 + O(t^{-1})].$$

Proof.

$$\begin{aligned}
\mathbb{E}^x\left[K_t^\lambda \, I_t\right] &= \mathbb{E}^x\left[N_{t,a} \, (X_t/X_0)^{-a} \, I_t\right] \\
&= x^a \, \hat{\mathbb{E}}_a^x \left[X_t^{-a} \, I_t\right] \\
&= x^a \int_0^\infty q_t(x,y;a) \, y^{-a} \, dy.
\end{aligned}$$

Since $q_t(x,y;a) \asymp y^{2a}$ as $y \downarrow 0$, the integral is finite.

We now set $x = 1$ and use Proposition 2.5 to get the asymptotics as $t \to \infty$. Note that for $a \geq 1/2$,

$$\begin{aligned}
\int_0^\infty q_t(1,y;a)\, y^{-a}\, dy &= t^{-1/2} \int_0^\infty q_1(1/\sqrt{t}, y/\sqrt{t}; a)\, y^{-a}\, dy \\
&= t^{-(a+\frac{1}{2})}\, e^{-1/2t} \int_0^\infty y^a\, e^{-y^2/2t}\, h\left(\frac{y}{t}\right)\, dy \\
&= t^{-(a+\frac{1}{2})}\, e^{-1/2t} \int_0^\infty (\sqrt{t}u)^a\, e^{-u^2/2} h\left(\frac{u}{\sqrt{t}}\right)\, \sqrt{t}\, du \\
&= t^{-\frac{a}{2}}\, [1 + O(t^{-1})] \int_0^\infty \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, u^a\, e^{-u^2/2}\, [1 + O(u^2/t)]\, du \\
&= t^{-\frac{a}{2}}\, \frac{\Gamma(\frac{a}{2}+\frac{1}{2})}{2^{a/2}\, \Gamma(a+\frac{1}{2})}\, [1 + O(t^{-1})].
\end{aligned}$$

$\square$

The next proposition is similar computing the same expectation for a Bessel process.

Proposition 2.14. *Suppose $b \in \mathbb{R}$ and*

$$\lambda + \lambda_b \geq -\frac{1}{8}. \tag{33}$$

Let

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8(\lambda + \lambda_b)} \geq \frac{1}{2}. \tag{34}$$

and assume that $a + b > -1$. Then, if X_t is a Bessel process with parameter b starting at $x > 0$,

$$\hat{\mathbb{E}}^x_b[K^\lambda_t\, I_t] = x^{a-b}\, \hat{\mathbb{E}}^x_a\left[X_t^{b-a}\right] = x^{a-b} \int_0^\infty y^{b-a}\, q_t(x,y;a)\, dy.$$

Note that if $b > -3/2$, then the condition $a + b > -1$ is automatically satisfied. If $b \leq -3/2$, then the condition $a + b > -1$ can be considered a stronger condition on λ than (33). If $b \leq -3/2$, then the condition on λ is

$$\lambda > 1 + 2b.$$

Proof. By comparing (32) and (34), we can see that

$$\begin{aligned}
\hat{\mathbb{E}}^x_b[K^\lambda_t\, I_t] &= x^{-b}\, \mathbb{E}^x\left[K_t^{\lambda_b+\lambda}\, X_t^b\, I_t\right] \\
&= x^{a-b}\, \mathbb{E}^x\left[N_{t,a}\, X_t^{b-a}\, I_t\right] \\
&= x^{a-b}\, \hat{\mathbb{E}}^x_a\left[X_t^{b-a}\right] \\
&= x^{a-b} \int_0^\infty y^{b-a}\, q_t(x,y;a)\, dy.
\end{aligned}$$

In the third equation, we drop the I_t since $I_t = 1$ with $\hat{\mathbb{P}}_a$-probability one. The condition $a+b > -1$ is needed to make the integral finite. $\square$

Proposition 2.15. *Let $\lambda > -1/8$ and let*

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1+8\lambda} \le \frac{1}{2},$$

be the smaller root of the polynomial $a^2 - a - 2\lambda$. Then if X_t is a Brownian motion starting at $x > 0$ and $0 < y < x$,

$$\mathbb{E}^x \left[K^\lambda_{T_y} \right] = (x/y)^a.$$

Proof. Let $n > x$ and let $\tau_n = T_y \wedge T_n$. Note that

$$\mathbb{E}^x \left[K^\lambda_{\tau_n} \right] = x^a \, \mathbb{E}^x \left[N_{\tau_n, a} \, X^{-a}_{\tau_n} \right] = x^a \, \hat{\mathbb{E}}^x_a \left[X^{-a}_{\tau_n} \right]$$

and similarly,

$$\mathbb{E}^x \left[K^\lambda_{\tau_n}; T_y < T_n \right] = x^a \, \hat{\mathbb{E}}^x_a \left[X^{-a}_{T_y}; T_y < T_n \right] = (x/y)^a \, \hat{\mathbb{P}}^x_a \left\{ T_y < T_n \right\},$$

$$\mathbb{E}^x \left[K^\lambda_{\tau_n}; T_y > T_n \right] = x^a \, \hat{\mathbb{E}}^x_a \left[X^{-a}_{T_n}; T_y > T_n \right] = (x/n)^a \, \hat{\mathbb{P}}^x_a \left\{ T_y > T_n \right\}.$$

Using (4), we see that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}^x \left[K^\lambda_{t\wedge\tau_n}; T_y > T_n \right] &= \lim_{n\to\infty} (x/n)^a \, \hat{\mathbb{P}}^x_a \left\{ T_y > T_n \right\} \\ &= \lim_{n\to\infty} (x/n)^a \, \frac{x^{1-2a} - y^{1-2a}}{n^{1-2a} - y^{1-2a}} = 0. \end{aligned}$$

Therefore,

$$\mathbb{E}^x \left[K^\lambda_{T_y} \right] = \lim_{n\to\infty} \mathbb{E}^x \left[K^\lambda_{T_y}; T_y < T_n \right] = (x/y)^a \lim_{n\to\infty} \hat{\mathbb{P}}^x_a \left\{ T_y < T_n \right\} = (x/y)^a.$$

The first equality uses the monotone convergence theorem and the last equality uses $a \le 1/2$. $\square$

Proposition 2.16. *Suppose $b \in \mathbb{R}$ and $\lambda + \lambda_b \ge -1/8$. Let*

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 8(\lambda+\lambda_b)} \le \frac{1}{2},$$

the smaller root of the polynomial $a^2 - a - 2(\lambda+\lambda_b)$. Then if X_t is a Bessel process with parameter b starting at $x > 0$ and $0 < y < x$,

$$\hat{\mathbb{E}}^x_b \left[K^\lambda_{T_y}; T_y < \infty \right] = (x/y)^{a-b}.$$

A special case of this proposition occurs when $b \geq 1/2$, $\lambda = 0$. Then $a = 1 - b$ and

$$\hat{\mathbb{E}}_b^x \left[K_{T_y}^{\lambda}; T_y < \infty \right] = \mathbb{P}_b^x\{T_y < \infty\} = (x/y)^{a-b} = (y/x)^{2b-1},$$

which is (5).

Proof.

$$\hat{\mathbb{E}}_b^x \left[K_{T_y}^{\lambda}; T_y < \infty \right] = x^{-b}\, \mathbb{E}^x \left[K_{T_y}^{\lambda}\, K_{T_y}^{\lambda_b}\, X_{T_y}^{b}; T_y < \infty \right] = (y/x)^b\, \mathbb{E}^x \left[K_{T_y}^{\lambda_a} \right] = (x/y)^{a-b}.$$

□

It is convenient to view the random variable J_t on geometric scales. Let us assume that $X_0 = 1$ and let

$$\hat{J}_t = J_{e^{-t}}.$$

Then if n is a positive integer, we can write

$$\hat{J}_n = \sum_{j=1}^{n} [\hat{J}_j - \hat{J}_{j-1}].$$

Scaling shows that the random variables $\hat{J}_j - \hat{J}_{j-1}$ are independent and identically distributed. More generally, we see that $\hat{J}_t$ is an increasing Lévy process, that is, it has independent, stationary increments. We will assume that $a \leq 1/2$ and write $a = \frac{1}{2} - b$ with $b = -\nu \geq 0$. Let Ψ_a denote the *characteristic exponent* for this Lévy process, which is defined by

$$\mathbb{E}[e^{i\lambda \hat{J}_t}] = \exp\{t\Psi_a(\lambda)\}.$$

It turns out that $\nu = a - \frac{1}{2}$ is a nicer parametrization for the next proposition so we will use it.

Proposition 2.17. *Suppose $b \geq 0$ and X_t satisfies*

$$dX_t = \frac{\frac{1}{2} - b}{X_t}\, dt + dB_t, \qquad X_0 = 1.$$

Then if $\lambda \in \mathbb{R}$,

$$\mathbb{E}^x \left[\exp\left\{ i\lambda \int_0^{T_y} \frac{ds}{X_s^2} \right\} \right] = y^{-r},$$

where

$$r = b - \sqrt{b^2 - 2i\lambda}.$$

is the root of the polynomial $r^2 - 2br + 2i\lambda$ with smaller real part. In other words, if $a \leq 1/2$

$$\Psi_{\frac{1}{2}-b}(\lambda) = b - \sqrt{b^2 - 2i\lambda},$$

where the square root denotes the root with positive real part. In particular,

$$\mathbb{E}\left[\hat{J}_t \right] = \frac{t}{b}. \tag{35}$$

Proof. We will assume $\lambda \neq 0$ since the $\lambda = 0$ case is trivial. If r_-, r_+ denote the two roots of the polynomial ordered by their real part, then $\mathrm{Re}(r_-) < b, \mathrm{Re}(r_+) > 2b$; we have chosen $r = r_-$.

Let $\tau_k = T_y \wedge T_k$. Using Itô's formula, we see that $M_{t \wedge \tau_k}$ is a bounded martingale where

$$M_t = \exp\left\{ i\lambda \int_0^t \frac{ds}{X_s^2} \right\} X_t^r.$$

Therefore,

$$\mathbb{E}[M_{\tau_k}] = 1.$$

If $b > 0$.

$$\mathbb{E}\left[|M_{\tau_k}|; T_k < T_y\right] \leq k^{\mathrm{Re}(r)}\, \mathbb{P}\{T_k < T_y\} \leq c(y)\, k^{\mathrm{Re}(r)}\, k^{2a-1} = c(r)\, k^{\mathrm{Re}(r)}\, k^{-2b},$$

and hence,

$$\lim_{k \to \infty} \mathbb{E}\left[|M_{\tau_k}|; T_k < T_y\right] = 0.$$

(One may note that if $\lambda \neq 0$ and we had used r_+, then $\mathrm{Re}(r_+) > 2b$ and this term does not go to zero.) Similarly, if $b = 0$,

$$\lim_{k \to \infty} \mathbb{E}\left[|M_{\tau_k}|; T_k < T_y\right] = 0.$$

Therefore,

$$1 = \lim_{k \to \infty} \mathbb{E}\left[M_{\tau_k}; T_k > T_y\right] = \mathbb{E}[M_{T_y}] = y^r\, \mathbb{E}\left[\exp\left\{ i\lambda \int_0^{T_y} \frac{ds}{X_s^2} \right\}\right].$$

The last assertion (35) follows by differentiating the characteristic function of $\hat{J}_t$ at the origin. $\square$

The moment generating function case is similar but we have to be a little more careful because the martingale is not bounded for $\lambda > 0$.

Proposition 2.18. *Suppose $b > 0$, X_t satisfies*

$$dX_t = \frac{\frac{1}{2} - b}{X_t}\, dt + dB_t, \qquad X_0 = 1,$$

and $2\lambda < b^2$. Then, if $0 < y < 1$,

$$\mathbb{E}^x\left[\exp\left\{ \lambda \int_0^{T_y} \frac{ds}{X_s^2} \right\}\right] = y^{-r},$$

where

$$r = b - \sqrt{b^2 - 2\lambda}.$$

is the smaller root of the polynomial $r^2 - 2br + 2\lambda$.

Proof. By scaling, it suffices to prove this result when $y = 1$. Let $\tau = T_1$ and let

$$K_t = \exp\left\{\lambda \int_0^t \frac{ds}{X_s^2}\right\}, \qquad M_t = K_t\, X_t^r.$$

By Itô's formula, we can see that M_t is a local martingale for $t < \tau$ satisfying

$$dM_t = \frac{r}{X_t}\, M_t\, dt, \quad M_0 = 1.$$

If we use Girsanov and weight by the local martingale M_t, we see that

$$dX_t = \frac{r + \nu + \frac{1}{2}}{X_t}\, dt + dW_t, \quad t < \tau$$

where W_t is a standard Brownian motion in the new measure which we denote by $\hat{\mathbb{P}}$ with expectations $\hat{\mathbb{E}}$. Since $r + \nu < 0$, then with probability one in the new measure $\hat{\mathbb{P}}^x\{\tau < \infty\} = 1$, and hence

$$\mathbb{E}^x\left[K_\tau; \tau < \infty\right] = x^r\, \mathbb{E}^x\left[M_\tau; \tau < \infty\right] = x^r\, \hat{\mathbb{E}}^x[1\{\tau < \infty\}] = x^r.$$

□

We can do some "multifractal" or "large deviation" analysis. We start with the moment generating function calculation

$$\mathbb{E}\left[e^{\lambda \hat{J}_t}\right] = e^{k\xi(\lambda)},$$

where

$$\xi(\lambda) = \xi_b(\lambda) = b - \sqrt{b^2 - 2\lambda}, \quad \xi'(\lambda) = \frac{1}{\sqrt{b^2 - 2\lambda}}, \quad \xi''(\lambda) = \frac{1}{(b^2 - 2\lambda)^{3/2}}.$$

This is valid provided that $\lambda < b^2/2$. Recall that $\mathbb{E}[\hat{J}_t] = t/b$. If $\theta > 1/b$, then

$$\mathbb{P}\{\hat{J}_t \geq \theta t\} \leq e^{-\lambda\theta t}\, \mathbb{E}[e^{\lambda \hat{J}_t}] = \exp\left\{t[\xi(\lambda) - \lambda\theta]\right\}, \quad .$$

This estimate is most useful for the value λ that minimizes the right-hand side, that is, at the value λ_θ satisfying $\xi'(\lambda_\theta) = \theta$, that is,

$$\lambda_\theta = \frac{1}{2}\left[b^2 - \theta^{-2}\right], \quad \xi(\lambda_\theta) = b - \frac{1}{\theta}$$

Therefore,

$$\mathbb{P}\{\hat{J}_t \geq \theta t\} \leq \exp\left\{t\rho(\theta)\right\}, \qquad \text{where} \qquad \rho(\theta) = b - \frac{1}{2\theta} - \frac{\theta b^2}{2}$$

While this is only an inequality, one can show (using the fact that ξ is C^2 and strictly concave in a neighborhood of λ_θ),

$$\mathbb{P}\{\hat{J}_t \geq \theta t\} \asymp \mathbb{P}\{\theta t \leq \hat{J}_t \leq \theta t + 1\} \asymp t^{-1/2}\, \exp\left\{t\rho(\theta)\right\}.$$

Similarly, if $\theta < 1/b$,

$$\mathbb{P}\{\hat{J}_t \leq \theta t\} \leq e^{\lambda\theta t}\,\mathbb{E}[e^{-\lambda\,\hat{J}_t}] = \exp\left\{k[\xi(-\lambda) + \lambda\theta]\right\}, \quad .$$

The right-hand side is minimized when $\xi'(-\lambda) = \theta$, that is, when

$$\lambda_\theta = \frac{1}{2}\,\left[\theta^{-2} - b^2\right], \quad \xi(-\lambda_\theta) = b - \sqrt{2b^2 - \theta^{-2}}$$

$$\mathbb{P}\{\hat{J}_t \leq \theta t\} \leq \exp\left\{t\rho(\theta)\right\}, \qquad \text{where} \qquad \rho(\theta) = \frac{1}{2\theta} - \frac{\theta b^2}{2} + b - \sqrt{2b^2 - \theta^{-2}}$$

3 The reflected Bessel process for $-1/2 < a < 1/2$

The Bessel process can be defined with reflection at the origin in this range. Before defining the process formally, let us describe some of of the properties. In this section, we assume that $-1/2 < a < 1/2$.

- The reflected Bessel process X_t is a strong Markov process with continuous paths taking values in $[0, \infty)$. It has transition density

$$\psi_t(x, y; a) = \frac{1}{\sqrt{t}}\,\psi_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a\right) = \frac{y^{2a}}{t^{a+\frac{1}{2}}}\,\exp\left\{-\frac{x^2+y^2}{2t}\right\}\,h_a\left(\frac{xy}{t}\right). \tag{36}$$

  Note that this is exactly the same formula as for $q_t(x, y; a)$ when $a \geq 1/2$. We use a new notation in order not to conflict with our use of $q_t(x, y; a)$ for the density of the Bessel process killed when it reaches the origin. We have already done the calculations that show that

$$\partial_t \psi_t(x, y; a) = L_x \psi_t(x, y; a),$$

  and

$$\partial_x \psi_t(x, y; a)\mid_{x=0} = 0.$$

  However, if $a \leq 1/2$, it is not the case that

$$\partial_y \psi_t(x, y; a)\mid_{y=0} = 0.$$

  In fact, for $a < 1/2$, the derivative does not exists at $y = 0$.

- Note that

$$\psi_1(0, y; a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\,y^{2a}\,e^{-y^2/2},$$

  and hence

$$\psi_t(0, y; a) = t^{-1/2}\,\psi_1(0, y/\sqrt{t}; a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\,t^{-\frac{1}{2}-a}\,y^{2a}\,e^{-y^2/2t}. \tag{37}$$

- We have the time reversal formula for $x, y > 0$.

$$\psi_t(x, y; a) = (y/x)^{2a}\, \psi_t(y, x; a). \tag{38}$$

Because of the singularity of the density at the origin we do not write $\psi_t(x, 0; a)$.

- A calculation (see Proposition 5.4) shows that for $x > 0$,

$$\int_0^\infty \psi_t(x, y; a)\, dy = 1. \tag{39}$$

We can view the process as being "reflected at 0" in a way so that the the total mass on $(0, \infty)$ is always 1.

- Another calculation (see Proposition 5.6) shows that the ψ_t give transition probabilities for a Markov chain on $[0, \infty)$.

$$\psi_{t+s}(x, y; a) = \int_0^\infty \psi_t(x, z; a)\, \psi_s(z, y; a)\, dz.$$

Note that this calculation only needs to consider values of $\psi_t(x, y; z)$ with $y > 0$.

- With probability one, the amount of time spent at the origin is zero, that is,

$$\int_0^\infty 1\{X_t = 0\}\, dt = 0.$$

This follows from (39) which implies that

$$\int_0^k 1\{X_t > 0\}\, dt = \int_0^k \int_0^\infty \psi_t(x, y; a)\, dy\, dt = k.$$

- For each $t, x > 0$, if $\sigma = \inf\{s \geq t : X_s = 0\}$, the distribution of $X_s, t \leq s \leq \sigma$, given X_t, is that of a Bessel process with parameter a starting at X_t stopped when it reaches the origin.

- The process satisfies the Brownian scaling rule: if X_t is the reflected Bessel process started at x and $r > 0$, then $Y_t = r^{-1}\, X_{r^2 t}$ is a reflected Bessel process started at x/r.

- To construct the process, we can first restrict to dyadic rational times and use standard methods to show the existence of such a process. With probability one, this process is not at the origin for any dyadic rational t (except maybe the starting point). Then, as for Brownian motion, one can show that with probability one, the paths are uniformly continuous on every compact interval and hence can be extended to $t \in [0, \infty)$ by continuity. (If one is away from the origin, one can argue continuity as for Brownian motion. If one is "stuck" near the origin, then the path is continuous since it is near zero.) The continuous extensions do hit the origin although at a measure zero set of times.

Here we explain why we need the condition $a > -1/2$. Assume that we have such a process for $a < 1/2$. Let $e(x) = e(x;a) = \mathbb{E}^0[T_x]$ and $j(x) = j(x;a) = \mathbb{E}^x[T_0 \wedge T_{2x}]$. We first note that $e(1) < \infty$; indeed, it is obvious that there exists $\delta > 0, s < \infty$ such that $\mathbb{P}^0\{T_1 < s\} \geq \delta$ and hence $\mathbb{P}^x\{T_x < s\} \geq \delta$ for every $0 \leq x < 1$. By iterating this, we see that $\mathbb{P}^0\{T_1 \geq ns\} \leq (1-\delta)^n$, and hence $\mathbb{E}^0[T_1] < \infty$. The scaling rule implies that $e(2x) = 4e(x), j(2x) = 4j(x)$. Also, the Markov property implies that

$$e(2x) = e(x) + j(x) + \mathbb{P}^x\{T_0 < T_{2x}\} e(2x),$$

which gives

$$4e(x) = e(2x) = \frac{e(x) + j(x)}{\mathbb{P}^x\{T_0 \geq T_{2x}\}}.$$

By (6), we know that

$$\mathbb{P}^x\{T_0 \geq T_{2x}\} = \min\{2^{2a-1}, 1\}.$$

If $a \leq -1/2$, then $\mathbb{P}^x\{T_0 \geq T_{2x}\} \leq 1/4$, which is a contradiction since $j(x) > 0$.

There are several ways to construct this process. In the bullets above we outline one which starts with the transition probabilities and then constructs a process with these transitions. In the next subsection, we will do another one which constructs the process in terms of excursions. In this section, we will not worry about the construction, but rather we will give the properties. We will write the measure as $\hat{\mathbb{P}}^x_a$ (this is the same notation as for the Bessel process killed at the origin — indeed, it is the same process just continued onward in time).

If $x > 0$, the scaling rule will imply

$$\psi_t(x, y; a) = t^{-1/2}\, \psi_1(x/\sqrt{t}, y/\sqrt{t}; a),$$

so we need only give $\psi_1(x, y; a)$. What we will show now is that if we assume that (37) holds and gives $\psi_t(0; y; a)$, then the value $\psi_t(x, y; a)$ must hold for all x. We will use T_0, the first time that the process reaches the origin and write

$$\begin{aligned} \psi_1(x, y; a) &= \tilde{\psi}_1(x, y; a) + q_1(x, y; a) \\ &= \tilde{\psi}_1(x, y; a) + (y/x)^{2a-1}\, q_1(x, y; 1-a) \end{aligned}$$

where

$$\tilde{\psi}_1(x, y; a) = \int_0^1 \psi_{1-s}(0, y; a)\, d\mathbb{P}^x\{T_0 = s\}. \tag{40}$$

The term $q_1(x, y; a)$ gives the contribution from paths that do not visit the origin before time 1, and $\tilde{\psi}_1(x, y; a)$ gives the contribution of those that do visit. The next proposition is a calculation. The purpose is to show that our formula for $\psi_1(x, y; a)$ must be valid for $x > 0$ provided that it is true for $x = 0$.

Proposition 3.1. *If $-\frac{1}{2} < a < \frac{1}{2}$, then*

$$\tilde{\psi}_1(x,y;a) = y^{2a}\, e^{-(x^2+y^2)/2}\, \left[h_a(xy) - (xy)^{1-2a}\, h_{1-a}(xy)\right].$$

Proof. Using (22), we see that

$$d\mathbb{P}^x\{T_0 = s\} = \frac{2^{a-\frac{1}{2}}}{\Gamma(\frac{1}{2}-a)}\, x^{1-2a}\, s^{a-\frac{3}{2}}\, \exp\{-x^2/2s\}\, ds,$$

and hence if (37) holds. Using equation 2.3.16 #1 of [2] (see also the top of page 790), and a well known identity for the Gamma function, we see that

$$\begin{aligned}
\int_0^\infty r^{-\nu-1} e^{-rz/2}\, e^{-z/2r}\, dr &= \frac{\pi}{\sin(-\pi\nu)}\, [I_\nu(z) - I_{-\nu}(z)] \\
&= \Gamma(\nu)\, \Gamma(1+\nu)\, [I_\nu(z) - I_{-\nu}(z)] \\
&= \Gamma\left(a - \frac{1}{2}\right)\, \Gamma\left(\frac{1}{2} - a\right)\, \left[z^{a-\frac{1}{2}}\, h_a(z) - z^{\frac{1}{2}-a}\, h_{1-a}(z)\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\tilde{\psi}_1(x,y;a) \\
&= \frac{1}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^1 s^{a-\frac{3}{2}}\, (1-s)^{-a-\frac{1}{2}} x^{1-2a}\, y^{2a}\, e^{-x^2/2s}\, e^{-y^2/2(1-s)}\, ds \\
&= \frac{x^{1-2a}\, y^{2a}\, e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^1 \left(\frac{1-s}{s}\right)^{-a-\frac{1}{2}} \exp\left\{-\frac{x^2}{2}\,\frac{1-s}{s}\right\} \exp\left\{-\frac{y^2}{2}\,\frac{s}{1-s}\right\} s^{-2}\, ds \\
&= \frac{x^{1-2a}\, y^{2a}\, e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^\infty u^{-a-\frac{1}{2}} \exp\left\{-\frac{x^2 u}{2}\right\} \exp\left\{-\frac{y^2}{2u}\right\} du \\
&= \frac{x^{\frac{1}{2}-a}\, y^{a+\frac{1}{2}}\, e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^\infty r^{-a-\frac{1}{2}} \exp\left\{-\frac{xyr}{2}\right\} \exp\left\{-\frac{xy}{2r}\right\} dr \\
&= y^{2a}\, e^{-(x^2+y^2)/2}\, \left[h_a(xy) - (xy)^{1-2a}\, h_{1-a}(xy)\right].
\end{aligned}$$

□

From the expression, we see that $\tilde{\psi}_1(x,y;a)$ is a decreasing function of x. Indeed, we could give another argument for this fact using coupling. Let us start two independent copies of the process at $x_1 < x_2$. We let the processes run until they collide at which time they run together. By the time the process starting at x_2 has reached the origin, the processes must have collided.

Recall that we are assuming $-1/2 < a < 1/2$. We will describe the measure on paths that we will write as $\hat{\mathbb{P}}^x_a$. Let $\psi_t(x,y;a), x \geq 0, y > 0, t > 0$ denote the transition probability for the process. We will derive a formula for this using properties we expect the process to have. First, the reversibility rule (11) will hold: if $x, y > 0$, then

$$\psi_t(x,y;a) = (y/x)^{2a}\, \psi_t(y,x;a).$$

In particular, we expect that $\psi_t(1,x;a) \asymp x^{2a}$ as $x \downarrow 0$. Suppose that $X_t = 0$ for some $1-\epsilon \leq t \leq \epsilon$. By Brownian scaling, we would expect the maximum value of X_t on that interval to be of order $\sqrt{\epsilon}$ and hence

$$\int_{1-\epsilon}^{1} 1\{|X_t| \leq \sqrt{\epsilon}\}\, dt \asymp \epsilon.$$

But,

$$\mathbb{E}\left[\int_{1-\epsilon}^{1} 1\{|X_t| \leq \sqrt{\epsilon}\}\, dt\right] = \int_{1-\epsilon}^{1}\int_0^{\sqrt{\epsilon}} \psi_t(0,x;a)\, dx\, dt \sim c\, \epsilon^{a+\frac{3}{2}}.$$

Hence, we see that we should expect $\hat{\mathbb{P}}_a^x\{X_t = 0 \text{ for some } 1-\epsilon \leq t \leq 1 \sim c'\, \epsilon^{\frac{1}{2}+a}$. Brownian scaling implies that $\hat{\mathbb{P}}_a^0\{X_t = 0 \text{ for some } ru \leq t \leq r\}$ is independent of r and from this we see that there should be a constant $c = c(a)$ such that

$$\hat{\mathbb{P}}_a^x\{X_t = 0 \text{ for some } 1-\epsilon \leq t \leq 1\} \sim c\, \epsilon^{\frac{1}{2}+a}.$$

In fact, our construction will show that we can define a *local time* at the origin. In other words, there is a process L_t that is a normalized version of "amount of time spent at the origin by time t" with the following properties. Let $Z = \{s : X_s = 0\}$ be the zero set for the process.

- L_t is continuous, nondecreasing, and has derivative zero on $[0,\infty) \setminus Z$.
- As $\epsilon \downarrow 0$,
$$\mathbb{P}^0\{Z \cap [1-\epsilon, 1] \neq \emptyset\} = \mathbb{P}^0\{L_1 > L_{1-\epsilon}\} \asymp \epsilon^{\frac{1}{2}+a}.$$
- The Hausdorff dimension of Z is $\frac{1}{2} - a$.
- $$\mathbb{E}[L_t] = c\int_0^t s^{-\frac{1}{2}-a}\, ds = \frac{c}{\frac{1}{2}-a}\, t^{\frac{1}{2}-a}.$$

We will use a "last-exit decomposition" to derive the formula for $\psi_t(0,x;a)$.

Proposition 3.2. *If $y > 0$, then*

$$\psi_1(0,y;a) = \frac{y}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^1 s^{-\frac{1}{2}-a}(1-s)^{a-\frac{3}{2}}\, e^{-y^2/2(1-s)}\, ds. \tag{41}$$

The proof of this proposition is a simple calculation,

$$\begin{aligned}
\int_0^1 s^{-\frac{1}{2}-a}(1-s)^{a-\frac{3}{2}}\, e^{-y^2/2(1-s)}\, ds &= \int_0^1 (1-s)^{-\frac{1}{2}-a} s^{a-\frac{3}{2}}\, e^{-y^2/2s}\, ds \\
&= e^{-y^2/2} \int_0^1 \left[\frac{1-s}{s}\right]^{-\frac{1}{2}-a} \exp\left\{-\frac{y^2}{2}\frac{1-s}{s}\right\} s^{-2}\, ds \\
&= e^{-y^2/2} \int_0^\infty u^{-\frac{1}{2}-a}\, e^{-u\,y^2/2}\, du \\
&= 2^{a+\frac{1}{2}}\, y^{2a}\, e^{-y^2/2} \int_0^\infty (uy^2/2)^{-\frac{1}{2}-a}\, e^{-uy^2/2}\, d(uy^2/2) \\
&= 2^{\frac{1}{2}-a}\, y^{2a-1} \int_0^\infty v^{-\frac{1}{2}-a}\, e^{-v}\, dv \\
&= 2^{\frac{1}{2}-a}\, y^{2a-1}\, \Gamma\left(\frac{1}{2}-a\right).
\end{aligned}$$

We would like to interpret the formula (41) in terms of a "last-exit" decomposition. What we have done is to split paths from 0 to t at the largest time $s < t$ at which $X_s = 0$. We think of $s^{-\frac{1}{2}-a}$ as being a normalized version of $\psi_s(0,0)$ and then $t^{a-\frac{3}{2}}\, e^{-y^2/2t}$ represents the normalized probability of getting to y at time t with no later return to the origin. To be more precise, let

$$q_t^*(y;a) = \lim_{x\downarrow 0} x^{2a-1}\, q_t(x,y;a),$$

and note that

$$\begin{aligned}
q_1^*(y;a) &= \lim_{x\downarrow 0} x^{2a-1}\, q_1(x,y;a) \\
&= \lim_{x\downarrow 0} x^{2a-1}\, (y/x)^{2a-1}\, q_1(x,y;1-a) \\
&= y^{2a-1}\, q_1(0,y;1-a) = c\, y\, e^{-y^2/2}.
\end{aligned}$$

$$\begin{aligned}
q_t^*(y;a) &= \lim_{x\downarrow 0} x^{2a-1}\, q_t(x,y;a) \\
&= t^{-\frac{1}{2}} \lim_{x\downarrow 0} x^{2a-1} q_1(x/\sqrt{t}, y/\sqrt{t};a) \\
&= t^{a-1} \lim_{z\downarrow 0} z^{2a-1} q_1(z, y/\sqrt{t};a) \\
&= t^{a-1}\, q_1^*(y/\sqrt{t};a) \\
&= c\, t^{a-\frac{3}{2}}\, y\, e^{-y^2/(2t)}.
\end{aligned}$$

Proposition 3.3. *For every $0 < t_1 < t_2 < \infty$ and $y_0 < \infty$, there exists c such that if $t_1 \le t \le t_2$ and $0 \le x, y \le y_0$, then*

$$c_1\, y^{2a} \le \psi_t(x,y;a) \le c_2\, y^{2a}.$$

Proof. Fix t_1, t_2 and y_0 and allow constants to depend on these parameter. It follows immediately from (37) that there exist $0 < c_1 < c_2 < \infty$ such that if $t_1/2 \leq t \leq t_2$ and $y \leq y_0$,

$$c_1\, y^{2a} \leq \psi_t(0, y; a) \leq c_2\, y^{2a}.$$

We also know that

$$\psi_t(x, y; a) = \tilde{\psi}_t(x, y; a) + q_t(x, y; a) \leq \tilde{\psi}_t(0, y; a) + q_t(x, y; a).$$

Using (10) and Proposition 2.7, we see that

$$q_t(x, y; a) = (y/x)^{2a-1}\, q_t(x, y; 1-a) \leq c\, y^{2a-1}\, y^{2(1-a)} = cy \leq c\, y^{2a}.$$

Also,

$$\tilde{\psi}_t(x, y; a) \geq \mathbb{P}^x\{T_0 \leq t_1/2\} \inf_{t_1/2 \leq s \leq t_2} \psi_s(0, y; a) \geq c\, y^{2a} \geq c\, y^{2a}\, \mathbb{P}^{y_0}\{T_0 \leq t_1/2\} \geq c\, y^{2a}.$$

$\square$

For later reference, we prove the following.

Proposition 3.4. *There exists $c < \infty$ such that if $x \geq 3\pi/4$ and $y \leq \pi/2$, then for all $t \geq 0$,*

$$\psi_t(x, y; a) \leq c\, y^{2a}. \tag{42}$$

Proof. Let $z = 3\pi/4$. It suffices to prove the estimate for $x = z$. By (38),

$$(z/y)^{2a}\, \psi_t(z, y; a) = \psi_t(y, z; a) \leq q_t(y, z; a) + \inf_{0 \leq s < \infty} \psi_t(0, z; a) \leq c.$$

$\square$

3.1 Excursion construction of reflected Bessel process

In this section we show how we can construct the reflected Bessel process using excursions. In the case $a = 0$ this is the Itô construction of the reflected Brownian motion in terms of local time and Brownian excursions. Let $0 < r = a + \frac{1}{2} < 1$ and let $\mathcal{K}$ denote a Poisson point process from measure

$$(r\, t^{-r-1}\, dt) \times \text{ Lebesgue }.$$

Note that the expected number of pairs (t, x) with $0 \leq x \leq x_0$ and $2^{-n} \leq t \leq 2^{-n+1}$ is

$$x_0 \int_{2^{-n}}^{2^{-n+1}} r\, t^{-r-1}\, dr = x_0\, (1 - 2^{-r})\, 2^{rn},$$

which goes to infinity as $n \to \infty$. However,

$$\mathbb{E}\left[\sum_{(t,x) \in \mathcal{K}; x \leq x_0, t \leq 1} t\right] = x_0 \int_0^1 r\, t^{-r}\, dr = \frac{r\, x_0}{1 - r} < \infty.$$

In other words, the expected number of excursions in $\mathcal{K}$ by time one is infinite (and a simple argument shows, in fact, that the number is infinite with probability one), but the expected number by time one of time duration at least $\epsilon > 0$ is finite. Also, the expected amount of time spent in excursions by time 1 of time duration at most one is finite. Let

$$T_x = \sum_{(t,x') \in \mathcal{K}; x' \leq x} t.$$

Then with probability one, $T_x < \infty$. Note that T_x is increasing, right continuous, and has left limits. It is discontinuous at x such that $(t, x) \in \mathcal{K}$ for some t. In this case $T_x = T_{x-} + t$. Indeed, the expected number of pairs (t, x') with $x' \leq x, t \geq 1$ is finite and hence with probability one the number of loops of time duration at least 1 is finite. We define L_t to be the "inverse" of T_x in the sense that

$$L_t = x \text{ if } T_{x-} \leq t \leq T_x.$$

Then L_t is a continuous, increasing function whose derivative is zero almost everywhere.

The density rt^{-r-1} is not a probability density because the integral diverges near zero. However we can still consider the conditional distribution of a random variable conditioned that it is at least k. Indeed we write

$$\mathbb{P}\{T \leq t \mid T \geq k\} = \frac{\int_k^t rs^{-r-1}}{\int_k^\infty rs^{-r-1}\, ds} = 1 - \left(\frac{k}{t}\right)^r,$$

which means that the "hazard function" is r/k,

$$\mathbb{P}\{T \leq k + dt \mid T \geq k\} = (r/k)\, dt + o(dt).$$

3.2 Excursions and bridges

Here we study the Bessel process with parameter $a < 1/2$ started at $x > 0$ "conditioned so that $T_0 = t$". We write

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad t < T.$$

where B_t is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and $T = T_0$ is the first hitting time of the origin. This is conditioning on an event of measure zero, but we can make sense of it using the Girsanov formula. Let

$$F(x, t) = x^{1-2a}\, t^{a-\frac{3}{2}} \exp\{-x^2/2t\}.$$

Up to a multiplicative constant, $F(x, \cdot)$ is the density of T_0 given $X_0 = x$ (see (22)). Let $M_s = F(X_s, t - s)$; heuristically, we think of M_s as the probability that $T = t$ given $\mathcal{F}_s$. Given this interpretation, it is reasonable to expect that M_s is a local martingale for $s < t$. Indeed, if we let $E_t = E_{t,\epsilon}$ be the event $E_t = \{t \leq T_0 \leq t + \epsilon\}$, and we weight by

$$F_\epsilon(x, t) = c\, \hat{\mathbb{P}}_a^x(E_t),$$

then $F_\epsilon(X_s, t-s) = c\,\hat{\mathbb{P}}_a^x(E_t \mid \mathcal{F}_s)$ which is a martingale. We can also verify this using Itô's formula which gives

$$dM_s = M_s \left[\frac{1-2a}{X_s} - \frac{X_s}{t-s}\right] dB_s.$$

Hence, if we tilt by the local martingale M_s, we see that

$$dX_s = \left[\frac{1-a}{X_s} - \frac{X_s}{t-s}\right] ds + dW_t, \tag{43}$$

where W_t is a Brownian motion in the new measure $\mathbb{P}^*$.

One may note that the SDE (43) gives the same process that one obtains by starting with a Bessel process X_t with parameter $1-a > 1/2$ and weighting locally by $J_s := \exp\{-X_t^2/2(t-s)\}$. Itô's formula shows that if X_s satisfies

$$dX_s = \frac{1-a}{X_s}\, ds + dB_s,$$

then

$$dJ_s = J_s \left[-\frac{X_s}{t-s}\, dB_s + \frac{a-\frac{3}{2}}{t-s}\, ds\right],$$

which shows that

$$N_s = \left(\frac{t}{t-s}\right)^{\frac{3}{2}-a} J_s,$$

is a local martingale for $s < t$ satisfying

$$dN_s = -\frac{X_s}{t-s}\, N_s\, dB_s.$$

There is no problem defining this process with initial condition $X_0 = 0$, and hence we have the distribution of a Bessel excursion from 0 to 0.

> We can see from this that if $a < 1/2$, then the distribution of an excursion X_s with $X_0 = X_t = 0$ and $X_s > 0$ for $0 < s < t$ is the same as the distribution of a Bessel process with parameter $1-a$ "conditioned to be at the origin at time t". More precisely, if we consider the paths up to time $t-\delta$, then the Radon-Nikodym derivative of the excursion with respect to a Bessel with parameter $1-a$ is proportional to $\exp\{-X_{t-\delta}^2/2(t-\delta)\}$.

There are several equivalent ways of viewing the excursion measure. Above we have described the probability measure associated to excursions starting and ending at the origin of time duration t. Let us write $\mu^\#(t;a)$ for this measure. Then the excursion measure can be given by

$$c\int_0^\infty \mu^\#(t,a)\, t^{a-\frac{3}{2}}\, dt.$$

The constant c is arbitary. This is an infinite measure on paths but can be viewed as the limit of the measure on paths of time duration at least s,

$$c \int_s^\infty \mu^\#(t,a)\, t^{a-\frac{3}{2}}\, dt,$$

which has total mass

$$c \int_s^\infty t^{a-\frac{3}{2}}\, dt = \frac{c}{\frac{1}{2}-a}\, s^{a-\frac{1}{2}}.$$

Another way to get this measure is to consider the usual Bessel process started at $\epsilon > 0$ stopped when it reaches the origin. This is a probability measure on paths that we will denote by $\tilde{\mu}^\#(\epsilon;a)$. The density of the hitting time T is a constant times $\epsilon^{1-2a}\, t^{a-\frac{3}{2}}\, \exp\{-\epsilon^2/2t\}$. Then the excursion measure can be obtained as

$$\lim_{\epsilon\downarrow 0} \epsilon^{2a-1}\, \tilde{\mu}^\#(\epsilon;a).$$

From this perspective it is easier to see that in the excursion measure has the following property: the distribution of the remainder of an excursion given that the time duration is at least s and $X_s = y$ is that of a Bessel process with parameter a started at y stopped when it reaches the origin.

We can also consider m_t which is the excursion measure restricted to paths with $T > t$ viewed as a measure on the paths $0 \leq X_s \leq t, 0 < s \leq t$. For each t this is a finite measure on paths, The density of the endpoint at time t (up to an arbitrary multiplicative constant) is given by

$$\psi_t(x) = \lim_{\epsilon\downarrow 0} \epsilon^{2a-1} q_t(\epsilon,x;a) = \lim_{\epsilon\downarrow 0} x^{2a-1}\, q_t(\epsilon,x;1-a) = x^{2a-1}\, q_t(0,x;1-a) = x\, t^{\frac{1}{2}-a}\, e^{-x^2/2t}.$$

Note that ψ_t is not a probability density; indeed,

$$\int_0^\infty \psi_t(x)\, dx = \int_0^\infty x\, t^{\frac{1}{2}-a}\, e^{-x^2/2t}\, dx = t^{\frac{3}{2}-a}.$$

Note that ψ_t satisfies the Chapman-Kolomogorov equations

$$\psi_{t+s}(x) = \int_0^\infty \psi_t(y)\, q_s(y,x;a)\, dx.$$

For example if $s = 1-t$, then this identity is the same as

$$\begin{aligned} x\, e^{-x^2/2} &= \int_0^\infty y\, t^{\frac{1}{2}-a}\, (1-t)\, e^{-x^2/2t}\, (y/x)^{2a-1}\, q_{1-t}(x,y;1-a)\, dy \\ &= \int_0^\infty y\, t^{\frac{1}{2}-a}\, e^{-x^2/2t}\, (y/x)^{2a-1}\, \frac{y^{2-2a}}{(1-t)^{\frac{3}{2}-a}} \exp\left\{-\frac{x^2+y^2}{1-t}\right\}\, h_a(xy/1-t). \end{aligned}$$

3.3 Another construction

Let us give another description of the reflected Bessel process using a single Brownian motion B_t. Suppose $a \in \mathbb{R}$, B_t is a standard Brownian motion, and for $x > 0$ let X_t^x satisfy

$$dX_t^x = \frac{a}{X_t^x}\, dt + dB_t, \quad X_0^x = x. \tag{44}$$

For a given x, this is valid up to time $T^x = \inf\{t : X_t^x > 0\}$. We define

$$Y_t = \inf\{X_t^y : y > 0, t < T^y\}, \quad .$$

We state the main result of the section here.

Theorem 1.

- *If $a \geq 1/2$, then Y_t has the distribution of the Bessel process with parameter a starting at the origin.*
- *If $-1/2 < a < 1/2$, then Y_t has the distribution of the reflected Bessel process starting at the origin.*
- *If $a \leq -1/2$, then with probability one $Y_t = 0$ for all t.*

For $a \geq 1/2$, this result is easy. By (26), if $x > 0$, then

$$X_t^x - x \leq Y_t \leq X_t^x.$$

Hence, for every $t > 0$, the distribution of $Y_s, s \geq t$, is that of the Bessel process starting at Y_t. It is not hard to see that $\mathbb{P}\{Y_t > 0\} = 1$ for each $t > 0$; indeed the density of Y_t is given by (15).

For the remainder of this section, we assume that $a < 1/2$. Note that this process is coalescing in the sense that if

$$Y_t^x = \inf\{X_t^y : y > x, t < T^y\},$$

then

$$Y_t^x = \begin{cases} X_t^x & t \leq T^x \\ Y_t & t \geq T^x \end{cases}$$

As an example, let us consider the case $a = 0$ for which the reflected Bessel process is the same as reflected Brownian motion. In this case $X_t^x = x + B_t$, and

$$T^x = \inf\{t : B_t = -x\}.$$

The set of times $\{T^x : x > 0\}$ are exactly the same as the set of times t at which the Brownian motion obtains a minimum, that is, $B_t < B_s, 0 \leq s < t$. This is also the set of times t at which $B_t \leq B_s, 0 \leq s < t$ (this is not obvious). The distribution of this set is the same as the distribution of the zero set of Brownian motion and is a topological Cantor set of Hausdorff dimension 1/2.

We will set up some notation. If B_t is a standard Brownian motion, and $t \geq 0$, we let $B_{s,t} = B_{t+s} - B_t$. Let $\mathcal{G}_t$ be the "future" σ-algebra at time t, that is, the σ-algebra generated by $\{B_{s,t} : s \geq 0\}$. Note that $\mathcal{G}_t$ is independent of $\mathcal{F}_t$, the σ-algebra generated by $\{B_s : s \leq t\}$. If $x > 0$ we write $X^x_{s,t}$ for the ($\mathcal{G}_t$-measurable) solution of

$$dX^x_{s,t} = \frac{a}{X^x_s}\, ds + dB_{s,t}, \qquad X^x_{0,t} = x.$$

This is valid up to time $T^x_{0,t} = \inf\{s : X^x_{s,t} = 0\}$ and for $s < T^x_{0,t}$, we have

$$X^x_{s,t} = B_{s,t} + a \int_0^s \frac{dr}{X^x_{r,t}}.$$

The Markov property can be written as

$$X^x_{s+r,t} = X^{X^x_{s,t}}_{r,t+s}, \qquad s + r < T^x_{0,t}.$$

If $\tau > 0$, we will say that t is a τ-*escape time* if for all $x > 0$,

$$X^x_{s,t} > 0, \qquad 0 \leq s \leq \tau.$$

We say that t is an *escape time* if it is a τ-escape time for some $\tau > 0$. Note that if $Y_t > 0$ and

$$z = \inf\{x : X^x_t > 0\},$$

then $T_z < t < \inf_{z<z'} T_{z'}$. In particular, T_z is an escape time. The next proposition proves our main theorem in the $a \leq -1/2$ case.

Proposition 3.5. *If $a \leq -1/2$, then with probability one there are no τ-escape times for any $\tau > 0$.*

Proof. By scaling it suffices to prove that there are no 1-escape times t with $t \leq 1$. For each integer n, let $J_{k,n}$ be the indicator function that $T^{2^{-n}}_{0,k2^{-2n}} \geq 1/2$. Note that $J_{k,n}$is $\mathcal{G}_{k2^{-2n}}$-measurable. Using (22) we can see that that $\mathbb{E}[J_{k,n}] = \mathbb{E}[J_{0,n}] \asymp 2^{(2a-1)n}$, and if

$$J_n = \sum_{k=0}^{2^{2n}} J(k,n),$$

then $\mathbb{E}[J_n] \asymp 2^{(2a+1)n}$. If $a < -1/2$, this immediately implies that

$$\mathbb{P}\{J_n \geq 1\} \leq \mathbb{E}[J_n] \leq c\, 2^{(2a+1)n}.$$

We claim that

$$\mathbb{P}\{J_n \geq 1\} \leq \frac{c}{n}, \qquad a = -\frac{1}{2}. \tag{45}$$

To see this, on the event $J_n \geq 1$, let $q = q_n$ be the largest index $k \leq 2^{2n}$ such that $J(k,n) = 1$ Note that

$$\mathbb{P}\{J_n \geq 1; q \leq 2^n\} \leq \sum_{k=1}^{2^n} \mathbb{E}[J(k,n)] \leq c\, 2^{-n}.$$

Now consider the $\mathcal{G}_{k2^{-2n}}$-measurable event $E(k,n) = \{J_n \geq 1, q = k\}$. Using the fact that $E(k,n)$ is independent of $\mathcal{F}_{k2^{-2n}}$ and using (22), we get

$$\begin{aligned} \mathbb{P}\{J(k-j,n) = 1 \mid E(k,n)\} &\geq \mathbb{P}^{2^{-n}}\{T_0 \geq (k-j)\, 2^{-2n}; T_{(k-j)\, 2^{-2n}} \geq 2^{-n}\} \\ &= \mathbb{P}^1\{T_0 \geq (k-j); T_{k-j} \geq 1\} \\ &\geq c/j. \end{aligned}$$

By summing over j, we see that

$$\mathbb{E}[J_n \mid J_n \geq 1, q > 2^n\} \geq c\, n,$$

and this gives (45).

We now consider the event that there exists a $t \leq 1$ that is a 1-escape time. It is not hard to see that the set of such t is closed and hence we can define σ to be the largest t. Note that σ is a backwards stopping time, that is, the event $\{\sigma \geq t\}$ is $\mathcal{G}_t$-measurable. If we take the largest dyadics smaller than σ for a given n, then we can see that given σ there is a positive probability of $J_n > 0$ (uniform in n for n large). But this contradicts the previous paragraph.

□

Proposition 3.6. *If $-1/2 < a < 1/2$, then with probability one, the set of escape times is a dense set of Hausdorff dimension $\frac{1}{2} + a$. In particular, it is a non-empty set of Lebesgue measure zero.*

Proof. We will only consider $t \in [0,1]$ and let R_τ denote the set of τ-escape times in $[0,1]$. If R is the set of escape times in $[0,1]$, then

$$R = \bigcup_{n=1}^{\infty} R_{1/n}.$$

Let us first fix τ. Let $\mathcal{Q}_n$ denote the set of dyadic rationals in $(0,1]$ with denominator 2^n,

$$\mathcal{Q}_n = \left\{ \frac{k}{2^n} : k = 1, 2, \ldots, 2^n \right\}.$$

We write $I(k,n)$ for the interval $[(k-1)2^{-n}, k2^n]$. We say that the interval $I(k,n)$ is *good* if there exists a time $t \in I(k,n)$ such that $X_{s,t}^{2^{-n/2}} > 0$ for $0 \leq s \leq 1$. Let

$$I_n = \bigcup_{I(k,n) \text{ good}} I(k,n), \quad I = \bigcap_{n=1}^{\infty} I_n.$$

Note that $I_1 \supset I_2 \supset \cdots$, and for each n, $R_1 \subset I_n$. We also claim that $I \subset R_{1/2}$. Indeed, suppose that $t \notin R_{1/2}$. Then there exists $x > 0$ such that $T^x_{0,t} \le 1/2$, and hence $T^y_{0,t} \le 1/2$ for $0 < y \le x$. Using continuity of the Brownian motion, we see that there exists $y > 0$ and $\epsilon > 0$ (depending on the realization of the Brownian motion B_t), such that $T^y_{0,s} \le 3/4$ for $|t - s| \le \epsilon$. (The argument is slightly different for $s < t$ and $s > t$.) Therefore, $t \notin I_n$ if $2^{-n} < \epsilon$.

Let $J(k,n)$ denote the corresponding indicator function of the event $\{I(k,n) \text{ good}\}$. We will show that there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1\, 2^{n(a-\frac{1}{2})} \le \mathbb{E}\left[J(j,n)\right] \le c_2\, 2^{n(a-\frac{1}{2})}, \tag{46}$$

$$\mathbb{E}[[J(j,n)\, J(k,n)] \le c_3\, 2^{n(a-\frac{1}{2})}\, [|j-k]+1]^{a-\frac{1}{2}}. \tag{47}$$

Using standard techniques (see, e.g., [1, Section A.3]), (46) implies that $\mathbb{P}\{\dim_h(R_1) \le a + \frac{1}{2}\} = 1$ and (46) and (47) imply that there exists $\rho = \rho(c_1, c_2, c_3, a) > 0$ such that

$$\mathbb{P}\left\{\dim_h(R_{1/2}) \ge a + \frac{1}{2}\right\} \ge \mathbb{P}\left\{\dim_h(I) \ge a + \frac{1}{2}\right\} \ge \rho.$$

Using stationarity of Brownian increments, it suffices to prove (46) and (47) for $j = 1$ and $k \ge 3$. Let us fix n and write $x = x_n = 2^{-n/2}, t = t_n = 2^{-n}$. The lower bound in (46) follows from

$$\mathbb{E}[J(1,n)] \ge \mathbb{P}\{T^x_{0,0} \ge 1\} \asymp 2^{n(a-\frac{1}{2})}.$$

Using $a \le 1/2$, we can see that $0 \le s \le t$,

$$\bar{X} := \max_{0 \le s \le t} X^x_{t-s,s} \le x + \frac{x}{2} + \max_{0 \le s \le t}[B_t - B_s].$$

Note that (for $n \ge 1$)

$$\mathbb{E}[J(1,n) \mid \bar{X} = z] \le c\, \mathbb{P}\{T^z_{0,t} \ge 1 - t\} \le c\,(z \wedge 1)^{1-2a}.$$

We then get the upper bound in (46) using a standard estimate (say, using the reflection principle) for the distribution of the maximum of a Brownian path.

For the second moment, let us consider the event that $I(1,n)$ and $I(k,n)$ are both good. Let V_k denote the event that there exists $0 \le s \le 2^{-n}$ such that $T^x_{s,(k-1)t-s} > 0$. Then V_k is independent of the event $\{I(k,n) \text{ good}\}$ and

$$\{I(1,n) \text{ good}, I(k,n) \text{ good}\} \subset V_k \cap \{I(k,n) \text{ good}\}\}.$$

Using the argument for the upper bound in the previous paragraph and scaling, we see that

$$\mathbb{P}(V_k) \le c\, k^{a-\frac{1}{2}}.$$

Using Brownian scaling, we see that the upper bound implies that for all $\tau > 0$,

$$\mathbb{P}\{\dim_h(R_\tau) \le a + \frac{1}{2}\} = 1,$$

and hence with probability one $\dim_h(R) \leq a + \frac{1}{2}$.. We claim that

$$\mathbb{P}\left\{\dim_h(R) = a + \frac{1}{2}\right\} = 1.$$

Indeed, if we consider the events

$$E_{j,n} = \left\{\dim_h[R_{2^{-(n+1)/2}} \cap I(j,n)] \geq a + \frac{1}{2}\right\}, \quad j = 1, 2, 3, \ldots, 2^{n-1},$$

then these are independent events each with probability at least ρ. Therefore,

$$\mathbb{P}\left\{E_{1,n} \cup E_{3,n} \cup \cdots \cup E_{2^n-1,n}\right\} \geq 1 - (1-\rho)^{2^{n-1}}.$$

Using this and scaling we see that with probability one for all rationals $0 \leq p < q \leq 1$, $\dim(R \cap [p,q]) = a + \frac{1}{2}$.

$\square$

Proof of Proposition 1. We follow the same outline as the previous proof, except that we define $I(k,n)$ to be β-good if if there exists $t \in I(k,n)$ such that $X_{s,t}^{2^{-n/2}} > 0$ for $0 \leq s \leq 1$ and

$$X_{s,t}^{2^{-n/2}} \geq 2\beta, \quad \frac{1}{4} \leq s \leq 1.$$

Arguing as before, we get the estimates (46) and (47), although the constant c_1 now depends on β. Let $R_{1/2,\beta}$ be the set of $t \in R_{1/2}$ such that

$$\lim_{x \downarrow 0} X_{1/2,t}^x \geq \beta.$$

Then $R_{1/2,\beta} \subset I^\beta$ where

$$I_n^\beta = \bigcup_{I(k,n)\, \beta\text{-good}} I(K,n), \quad I^\beta = \bigcap_{n=1}^\infty I_n^\beta.$$

There exists $\rho_\beta > 0$ such that

$$\mathbb{P}\left\{\dim_h(R_{1/2,\beta}) = \frac{1}{2} + a\right\} \geq \rho_\beta.$$

For each time $t \in R$, we define

$$X_{s,t}^0 = \inf\{X_{s,t}^x, x > 0\}$$

where the right-hand side is defined to be zero if $T_{0,t}^x \leq s$ for some $x > 0$. Recall that

$$\tilde{X}_t = \inf\{X_t^x : T_0^x > t\}.$$

Note that for every $0 \leq t \leq 1$,

$$\tilde{X}_1 \geq X^0_{1-t,t}.$$

We claim: with probability one, there exists $t < 1$ such that $X^0_{1-t,t} > 0$. To see this, consider the events V_n defined by

$$V_n = \{\exists t \in I(2^n - 1, n) \text{ with } X_{1-t,0} > 2^{-n/2}\}.$$

The argument above combined with scaling shows that $\mathbb{P}(V_n)$ is the same and positive for each n. Also if we choose a sequence $n_1 < n_2 < n_3 < \cdots$ going to infinity sufficiently quickly, the events V_{n_j} are almost independent. To be more precise, Let

$$V^j = \{\exists t \in I(2^{n_j} - 1, n_j) \text{ with } X_{1-t-2^{-(n+1)},t} > 2 \cdot 2^{-n_j/2}\}.$$

Then the events $V^1, V^2, \ldots$ are independent and there exists $\rho > 0$ with $\mathbb{P}(V^j) > 0$. Hence $\mathbb{P}\{V^j \text{ i.o.}\} = 1$. If we choose the sequence n_j to grow fast enough we can see that

$$\sum_{j=1}^{\infty} \mathbb{P}(V^j \setminus V_{n_j}) < \infty,$$

and hence, $\mathbb{P}\{V_{n_j} \text{ i.o.}\} > 0$.

□

4 Radial Bessel and similar processes

We will now consider similar processes that live on the bounded interval $[0, \pi]$ and arise in the study of the radial Schramm-Loewner evolution. These processes look like the Bessel process near the boundaries. One main example is the radial Bessel process. We will first consider the process restricted to the open interval $(0, \pi)$ and then discuss possible reflections on the boundary. As in the case of the Bessel process, we will define our process by starting with a Brownian motion and then weighting by a particular function.

4.1 Weighted Brownian motion on $[0, \pi]$

We will consider Brownian motion on the interval $[0, \pi]$ "weighted locally" by a positive function Φ. Suppose $m : (0, \pi) \to \mathbb{R}$ is a C^1 function and let $\Phi : (0, \pi) \to (0, \infty)$ be the C^2 function

$$\Phi(x) = c \exp\left\{ - \int_x^{\pi/2} m(y)\, dy \right\}.$$

Here c is any positive constant. Everything we do will be independent of the choice of c so we can choose $c = 1$ for convenience. Also, $\pi/2$ is chosen for convenience; choosing a different limit for the integral will change Φ by a constant. Note that

$$\Phi'(x) = m(x)\, \Phi(x), \quad \Phi''(x) = [m(x)^2 + m'(x)]\, \Phi(x).$$

Let X_t be a standard Brownian motion with $0 < X_0 < \pi$, $T_y = \inf\{t : X_t = y\}$ and $T = T_0 \wedge T_\pi = \inf\{t : X_t = 0 \text{ or } X_t = \pi\}$. For $t < T$, let

$$M_t = M_{t,\Phi} = \frac{\Phi(X_t)}{\Phi(X_0)} K_t, \quad K_t = K_{t,\Phi} = \exp\left\{ -\frac{1}{2} \int_0^t \left[m(X_s)^2 + m'(X_s) \right] ds \right\}. \tag{48}$$

Then Itô's formula shows that M_t is a local martingale for $t < T$ satisfying

$$dM_t = m(X_t)\, M_t\, dX_t, \quad M_0 = 1.$$

Using the Girsanov theorem (being a little careful since this is only a local martingale), we get a probability measure on paths $X_t, 0 \leq t < T$ which we denote by $\mathbb{P}_\Phi$. To be precise, if $0 < \epsilon < \pi/2$, $\tau = \tau_\epsilon = \inf\{t : X_t \leq \epsilon \text{ or } X_t \geq \pi - \epsilon\}$, then $M_{t\wedge\tau}$ is a positive martingale with $M_0 = 1$. Moreover, if V is a random variable depending only on $X_s, 0 \leq s \leq t \wedge \tau$, then

$$\mathbb{E}^x_\Phi[V] = \mathbb{E}^x[M_{t\wedge\tau}\, V].$$

The Girsanov theorem implies that

$$dX_t = m(X_t)\, dt + dB_t, \quad t < T,$$

where B_t is a standard Brownian motion with respect to $\mathbb{P}_\Phi$.
Examples

- If
$$\Phi(x) = x^a, \qquad m(x) = \frac{a}{x},$$
then X_t is the Bessel process with parameter a.

- If
$$\Phi(x) = (\sin x)^a, \quad m(x) = a \cot x,$$
then X_t is called the *radial Bessel process with parameter a*.

Note that the Bessel process and the radial Bessel process with the same parameter are very similar near the origin. The next definition makes this idea precise.

Definition

- We say that Φ (or the process generated by Φ) is *asymptotically Bessel-a at the origin* if there exists $c < \infty$ such that for $0 < x \leq \pi/2$,
$$\left| m(x) - \frac{a}{x} \right| \leq c\, x, \quad \left| m'(x) + \frac{a}{x^2} \right| \leq c.$$
Similarly, we say that Φ is *asymptotically Bessel-a at π* if $\tilde{\Phi}(x) := \Phi(\pi - x)$ is asymptotically Bessel-a at the origin.

- We let $\mathcal{X}(a,b)$ be the set of Φ that are asymptotically Bessel-a at the origin and Asymptotically Bessel-b at π.

If $\Phi \in \mathcal{X}(a,b)$, then as $x \downarrow 0$

$$\int_x^{\pi/2} m(y)\,dy = -a\log x + C + O(x^2), \quad C = \int_0^{\pi/2}\left[m(y) - \frac{a}{y}\right]\,dy,$$

and hence

$$\Phi(x) = e^{-C}\,x^a\,\left[1 + O(x^2)\right].$$

In particular, if $0 < r < 1$,

$$\Phi(x) = r^a\,\Phi(rx)\,\left[1 + O(x^2)\right].$$

Examples

- The radial Bessel-a process is in $\mathcal{X}(a,a)$.

- If
$$\Phi(x) = [\sin x]^a\,[1-\cos x]^v, \quad m(x) = (a+v)\,\cot x + \frac{v}{\sin x},$$
then Φ is in $\mathcal{X}(a, a+2v)$.

Lemma 4.1. *Suppose $\Phi \in \mathcal{X}(a,b)$ with martingale $M_t = M_{t,\Phi}$ and let $\tilde{\Phi}(x) = a/x$ with corresponding martingale $\tilde{M}_t$. There exists $c < \infty$ such that the following holds. Suppose $0 < x < y \leq 7\pi/8$, $X_0 = x$, $\tau = t \wedge T_0 \wedge T_y$. Then,*

$$|\log M_\tau - \log \tilde{M}_\tau| \leq c\,[t + y^2].$$

Proof. This follows from

$$\left|\int_0^\tau \left[m(X_s)^2 - \frac{a^2}{X_s^2}\right]\,ds\right| \leq c\,t,$$

$$\left|\int_0^\tau \left[m'(X_s) + \frac{a}{X_s^2}\right]\,ds\right| \leq ct$$

$$\frac{\Phi(X_\tau)}{\Phi(X_0)} = \frac{X_\tau^a}{X_0^a}\,\left[1 + O(y^2)\right].$$

□

Lemma 4.2. *For every $\Phi \in \mathcal{X}(a,b)$ with $a \geq 1/2$, there exists $c < \infty$ such that the following holds. Suppose $0 < x < y \leq 7\pi/8$ and let μ denote the measure on paths $X_t, 0 \leq t \leq T_y$. Let $\tilde{\mu}$ be the measure obtained by replacing X_t with a Bessel process $\tilde{X}_t$ with parameter a. Then the variation distance between μ and $\tilde{\mu}$ is less than $c\,y^2$.*

Proof. For the Bessel process there exists $\rho > 0$ such that for $z < y$, $\mathbb{P}^z\{T_y \leq y^2\} \geq \rho$. By Iterating this, we see that for every positive integer k

$$\mathbb{P}^z\{T_y \geq ky^2\} \leq \rho^k.$$

On the event $\{(k-1)\, y^2 < T_y \leq y^2\}$, we have

$$M_\tau = \tilde{M}_\tau\, [1 + O(ky^2)].$$

and hence the variation distance between μ and $\tilde{\mu}$ on the event $\{(k-1)y^2 < T_y \leq ky\}$ is less than $c\, \rho^k\, ky^2$. Summing over k gives the result.

□

It is sometimes more convenient to compare the asymptotically Bessel process to a Bessel process rather than to a Brownian motion. Suppose $a, b \geq 1/2$ and let us define the Bessel-(a, b) process on $(0, \pi)$ as follows. Let

$$\sigma_0 = 0, \quad \tau_0 = \inf\{t \geq 0 : X_t \geq 7\pi/8\},$$

and recursively,

$$\sigma_k = \inf\{t > \tau_{k-1} : X_t = \pi/8\}, \quad \tau_k = \inf\{t > \sigma_k : X_t = 7\pi/8\}.$$

Then the Bessel-(a, b) process on $(0, \pi)$ is defined to be the process X_t such that

- if $\sigma_j \leq t < \tau_j$, then X_t evolves as a Bessel process with parameter a;
- if $\tau_j \leq t < \sigma_j$, then $\pi - X_t$ evolves as a Bessel with parameter b.

If $\Phi \in \mathcal{X}(a, b)$ with corresponding m, then we define the martingale M_t by $M_0 = 1$ and

$$M_t = M_{\sigma_j}\, \frac{\Phi(X_t)/\Phi(X_{\sigma_j})}{(X_t/X_{\sigma_j})^a}\, \exp\left\{-\frac{1}{2}\int_{\sigma_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{a(a-1)}{X_s^2}\right] ds\right\},$$

if $\sigma_j \leq t \leq \tau_j$, and if $\tau_j \leq t \leq \sigma_{j+1}$,

$$M_t = M_{\tau_j}\, \frac{\Phi(X_t)/\Phi(X_{\tau_j})}{(X_t/X_{\tau_j})^b}\, \exp\left\{-\frac{1}{2}\int_{\tau_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{b(b-1)}{(\pi - X_s)^2}\right] ds\right\},$$

Note that there exists β with $e^{-\beta t} \leq M_t \leq e^{\beta t}$, so this is a martingale. We can say that the process titled by Φ is mutually absolutely continuous with the Bessel-(a, b) process with Radon-Nikodym derivative M_t. If $a < 1/2$ or $b < 1/2$ we can similarly define the Φ-process up to the first time that it leaves $[0, \pi]$.

Let

$$F(x) = F_\Phi(x) = \int_{\pi/2}^x \frac{dy}{\Phi(y)^2},$$

and note that

$$F'(x) = \frac{1}{\Phi(x)^2}, \quad F''(x) = -\frac{2\,\Phi'(x)}{\Phi(x)^3} = -\frac{2\, m(x)}{\Phi(x)^2}.$$

Using this and Itô's formula we see that $F(X_t)$ is a $\mathbb{P}_\Phi^x$ local martingale for $t < T$.

Proposition 4.3. *If $\Phi \in \mathcal{X}(a,b)$, $0 < x < z < \pi$, then*

$$\lim_{\epsilon \downarrow 0} \mathbb{P}^x_\Phi\{T_\epsilon < T_z\} = 0$$

if and only if $a \geq 1/2$. Similarly

$$\lim_{\epsilon \downarrow 0} \mathbb{P}^z_\Phi\{T_{\pi-\epsilon} < T_x\} = 0$$

if and only if $b \geq 1/2$.

Proof. We will prove the first; the second follows similarly. If $\Phi \in \mathcal{X}(a,b)$,

$$\Phi(x)^{-2} \sim c_1 \, x^{-2a} \, [1 + O(x^2)],$$

Note that F is strictly increasing on $(0,\pi)$ with $F(\pi/2) = 0$ and $F(0) = -\infty$ if and only if $a \geq 1/2$. Let $\tau = T_\epsilon \wedge T_z$. Since $F(X_{t\wedge\tau})$ is a bounded martingale, the optional sampling theorem implies that.

$$F(x) = F(z)\,\mathbb{P}\{T_z < T_\epsilon\} + F(\epsilon)\,\mathbb{P}\{T_\epsilon < T_z\} = F(\epsilon)\,\mathbb{P}\{T_\epsilon < T_z\},$$

$$\lim_{\epsilon \downarrow 0} \mathbb{P}\{T_\epsilon < T_z\} = \lim_{\epsilon \downarrow 0} \frac{F(z) - F(x)}{F(z) - F(\epsilon)} = \frac{F(z) - F(x)}{F(z) - F(0)}.$$

□

4.2 $a, b > 1/2$

In this section we consider $\Phi \in \mathcal{X}(a,b)$ with $a, b \geq 1/2$ so that the process does not hit the origin. Let

$$f(x) = c\,\Phi(x)^2, \quad 0 < x < \pi$$

where c is chosen so that f is a probability density. We will show that f is the invariant density for the process and the convergence to equilibrium is exponentially fast uniformly over the starting position.

The form of the invariant density follows almost immediately from the fact that the process is obtained from Brownian motion by tilting by Φ. Let $\tilde{p}_t(x,y)$ denote the density of a Brownian motion killed when it reaches $\{0,\pi\}$ and let $q_t(x,y)$ denote the transition density for X_t. Then

$$q_t(x,y) = \frac{\Phi(y)}{\Phi(x)}\,\tilde{p}_t(x,y)\,\mathbb{E}^*[K_t],$$

where K_t is as above, and $\mathbb{E}^*$ is the process corresponding to Brownian motion starting at x conditioned to be at y at time t and having not left the interval $(0,\pi)$ by that time. Using reversibility of Brownian motion, we see that

$$f(x)\,q_t(x,y) = f(y)\,q_t(y,x),$$

and hence

$$\int_0^\pi f(x)\, p_t(x,y)\, dx = \int_0^\pi f(y)\, p_t(y,x)\, dx = f(y).$$

The key to exponentially fast convergence to equilibrium is the following lemma.

Proposition 4.4. *If $\Phi \in \mathcal{X}(a,b)$ with $a, b \geq 1/2$ and $t_0 > 0$, then there exist $0 < c_1 < c_2 < \infty$ such that for all $x, y \in (0,\pi)$, and $t \geq t_0$,*

$$c_1\, f(y) \leq q_t(x,y) \leq c_2\, f(y). \tag{49}$$

Proof. Let

$$I_1 = \left[\frac{\pi}{4}, \frac{7\pi}{4}\right], \quad I_2 = \left[\frac{\pi}{8}, \frac{7\pi}{8}\right], \quad I_3 = \left[\frac{\pi}{16}, \frac{15\pi}{16}\right].$$

For $x, y \in I_3$, let $\tilde{q}_t(x,y)$ be the density for the process killed when it leaves I_3. We claim that there exist c_1, c_2 such that

$$c_1 \leq \tilde{q}_t(x,y) \leq c_2, \quad \frac{1}{4} \leq t \leq 1, \quad x, y \in I_2, \tag{50}$$

$$\tilde{q}_t(x,y) \leq c_2, \quad t > 0, \quad x \in \partial I_2, \;\; y \in I_1. \tag{51}$$

Indeed this is standard for Brownian motion killed when it leaves I_3 and the martingale is bounded uniformly away from 0 and ∞. To get an upper bound for $q_t(x,y)$ we split into excursions. Let

$$\sigma_1 = \inf\{t : X_t \in \partial I_3\}, \quad \tau_1 = \inf\{t > \sigma_1 : X_t \in I_2\},$$

and recursively,

$$\sigma_j = \inf\{t > \tau_{j-1} : X_t \in \partial I_3\}, \quad \tau_j = \inf\{t > \sigma_j : X_t \in I_2\}.$$

Then if $t \leq 1$, $x, y \in I_2$,

$$q_t(x,y) = \tilde{q}_t(x,y) + C \sum_{j=1}^{\infty} \mathbb{P}^x\{\tau_j < 1\},$$

where

$$C = \max\{\tilde{q}_t(z,w) : t \geq 1, z \in \partial I_2, w \in I_1\}.$$

Since the process starting on ∂I_3 has positive probability of not reaching I_2 by time 1, we see there exists $\rho < 1$ such that $\mathbb{P}^x\{\tau_j < 1\} \leq \rho^j$. Hence we get (50) and (51) with $\tilde{q}_t(x,y)$ replaced by $q_t(x,y)$ with a different value of c_2.

Since there exists uniform $\delta > 0$ such that for $x \in (0,\pi) \setminus I_3$, $\mathbb{P}\{\sigma_1 \leq 1/4\} > \delta$, we can use the strong Markov property to conclude that

$$q_{1/2}(x,y) \geq c_3, \quad x \in (0,\pi), y \in I_1$$

and hence also, $q_{1/2}(y,x) \geq c_4\, f(x)$ for these x,y. More generally, if $x,y \in (0,\pi)$,

$$q_1(x,y) \geq \int_{I_1} q_{1/2}(x,z)\, q_{1/2}(z,y)\, dz \geq c\, f(y).$$

We have upper bounds if one of x or y is in I_1 and if $x \geq \pi/4, y \geq 3\pi/4$, we can use the strong Markov property stopping the process at time $T_{\pi/4}$. The last case is the upper bound for $x,y \leq \pi/4$ (or $x,y \geq 7\pi/4$ that is done similarly). For this we compare to the Bessel process with parameter a using the estimates in Proposition 2.7.

□

Proposition 4.5. *If $\Phi \in \mathcal{X}(a,b)$ with $a,b \geq 1/2$, then there exist c,β such that for all $t \geq 1$ and $0 < x,y < \pi$,*

$$[1 - ce^{-\beta t}]\, f(y) \leq q_t(x,y) \leq [1 + ce^{-\beta t}]\, f(y).$$

In particular, if $g : (0,\pi) \to [0,\infty)$ with

$$\bar{g} := \int_0^\pi g(x)\, f(x)\, dx < \infty,$$

then

$$\bar{g}\, [1 - ce^{-\beta t}] \leq \mathbb{E}^x[g(X_t)] \leq \bar{g}\, [1 + ce^{-\beta t}].$$

Proof. It suffices to prove the result for positive integer t. Let us write the $1 - c_1$ in the last proposition with $t_0 = 1$ as $e^{-\beta}$. Let us fix x and write $f_t(y) = q_t(x,y)$. Then (49) implies that we can write

$$f_1(y) = [1 - e^{-\beta}]\, f(y) + e^{-\beta}\, g_1(y),$$

for some probability density g_1. By iterating this and using the fact that f is invariant, we see that we we can write for integer t

$$f_t(y) = [1 - e^{-\beta t}]\, f(y) + e^{-\beta t}\, g_t(y)$$

for a probability density g_t. Note at this point we have used only the lower bound in (49). From this equation we can conclude that the variation distance between the distribution at time t and the invariant measure decays exponentially. However, our claim is stronger. We appeal to the upper bound to get

$$[1 - e^{-\beta t}]\, f(y) \leq f_{t+1}(y) \leq [1 - e^{-\beta t} + c_2 e^{-\beta t}]\, f(y).$$

□

Proposition 4.6. *Suppose $\Phi \in \mathcal{X}(a,b)$ and $M_t = M_{t,\Phi}, K_t = K_{t,\Phi}$ is as defined in (48). There exists $\beta > 0$ such that if X_t is a standard Brownian motion, then*

$$\mathbb{E}^x[K_{t,\Phi}; T > t] = c_*\, \Phi(x)\, [1 + O(e^{-\beta t})], \quad c_* = \frac{\int_0^\pi \Phi(y)\, dy}{\int_0^\pi \Phi(y)^2\, dy}.$$

Proof.

$$\begin{aligned}
\mathbb{E}^x[K_{t,\Phi}; T > t] &= \Phi(x)\, \mathbb{E}^x \left[\Phi(X_t)^{-1}\, M_{t,\Phi}; T > t\right] \\
&= \Phi(x)\, \mathbb{E}^x_\Phi \left[\Phi(X_t)^{-1}; T > t\right] \\
&= \Phi(x)\, \mathbb{E}^x_\Phi \left[\Phi(X_t)^{-1}\right] \\
&= c_*\, \Phi(x)\, [1 + O(e^{-\beta t})]
\end{aligned}$$

The third equality uses $\mathbb{P}^x_\Phi\{T > t\} = 1$. □

Example. Suppose $\Phi(x) = (\sin x)^a$ with $a \geq 1/2$. Then,

$$m(x)^2 + m'(x) = a^2 \cot^2 x - \frac{a}{\sin^2 x} = \frac{a(a-1)}{\sin^2 x} - a^2,$$

$$K_t = e^{a^2 t/2} \exp\left\{\frac{a(1-a)}{2} \int_0^t \frac{ds}{\sin^2 X_s}\right\}.$$

We therefore get

$$\begin{aligned}
\mathbb{E}^x\left[\exp\left\{\frac{a(1-a)}{2}\int_0^t \frac{ds}{\sin^2 X_s}\right\}; T > t\right] &= e^{-a^2 t/2}\, \mathbb{E}^x[K_t; T > t] \\
&= e^{-a^2 t/2}\, c_*\, [\sin x]^a\, [1 + O(e^{-\beta t})],
\end{aligned}$$

where

$$c_* = \frac{\int_0^\pi [\sin y]^a\, dy}{\int_0^\pi [\sin y]^{2a}\, dy}.$$

4.3 Reflected process for $a, b > -1/2$

For future reference, we note that the unique cubic polynomial $g(x)$ satisfying $g(0) = 0, g'(0) = 0, g(\epsilon) = \epsilon\,\gamma, g'(\epsilon) = \theta$ is

$$g(x) = \epsilon\,[\theta - 2\gamma]\,(x/\epsilon)^3 + \epsilon[3\gamma - \theta]\,(x/\epsilon)^2.$$

Note that for $|x| \leq \epsilon$,

$$|g(x)| + \epsilon\,|g'(x)| \leq \epsilon\, [17|\gamma| + 7|\theta|]\,. \tag{52}$$

Here we will discuss how to define the reflecting Φ-process for $\Phi \in \mathcal{X}(a, b)$ with $a, b > -1/2$. As our definition we will give the Radon-Nikodym derivative with respect to the reflecting Bessel process. We start by defining the reflecting Bessel-(a, b) process X_t on $(0, \pi)$ in the same way that the Bessel-(a, b) was defined in Section 4.1 where the "Bessel process with parameter a (or b)" is defined to the the reflecting process. We then define the

Φ-process to be the process with Radon-Nikodym derivative given by the martingale defined by $M_0 = 1$ and

$$M_t = M_{\sigma_j} \, \frac{\Phi(X_t)/\Phi(X_{\sigma_j})}{(X_t/X_{\sigma_j})^a} \, \exp\left\{ -\frac{1}{2} \int_{\sigma_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{a(a-1)}{X_s^2} \right] ds \right\},$$

if $\sigma_j \leq t \leq \tau_j$, and if $\tau_j \leq t \leq \sigma_{j+1}$,

$$M_t = M_{\tau_j} \, \frac{\Phi(X_t)/\Phi(X_{\tau_j})}{(X_t/X_{\tau_j})^b} \, \exp\left\{ -\frac{1}{2} \int_{\tau_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{b(b-1)}{(\pi - X_s)^2} \right] ds \right\},$$

Note that there exists β with $e^{-\beta t} \leq M_t \leq e^{\beta t}$, so this is a martingale. We can see that the process tilted by Φ is mutually absolutely continuous with the Bessel-(a, b) process with Radon-Nikodym derivative M_t.

A technical issue here that might concern us is the fact that the proof that M_t is a martingale uses the Girsanov theorem. This is valid locally away from the boundary, but it may not be clear that it works at the boundary. If it were the case that for some $\epsilon > 0$

$$m(x) = \frac{a}{x}, \quad m(\pi - x) = -\frac{b}{\pi - x}, \quad 0 < x < \epsilon,$$

then this would not be a problem, since the process M_t would not change when $X_t < \epsilon$ or $X_t > \pi - \epsilon$. More generally, we can find a sequence m_n such that

$$m_n(x) = \frac{a}{x}, \quad m_n(\pi - x) = -\frac{b}{\pi - x}, \quad 0 < x < 2^{-n},$$

$$m_n(x) = m(x), \qquad 2^{-n+1} \leq x \leq \pi - 2^{-n+1},$$

and such that for all $x \leq 2^{-n+1}$,

$$|m_n(x) - m(x)| + |m_n(\pi - x) - m(\pi - x)| \leq c\, x,$$

$$|m_n'(x) - m'(x)| + |m_n'(\pi - x) - m'(\pi - x)| \leq c.$$

The comment above about cubic polynomials is useful in constructing a particular example.

One can check that the proof required more than the fact that $x \sim \sin x$ near the origin. We needed that

$$\sin x = x \, \left[1 - O(x^2)\right].$$

An error term of $O(x)$ would have not been good enough.

We can now describe how to construct the reflecting radial Bessel process.

- Run the paths until time $\tau = T_{3\pi/4}$. Then

$$\frac{d\mathbb{P}_a}{d\hat{\mathbb{P}}_a} = K_\tau.$$

- Let $\sigma = \inf\{t \geq \tau : X_t = \pi/4\}$. The measure on $X_t, \tau \leq t \leq \sigma$ is defined to be the measure obtained in the first step by reflecting the paths around $x = \pi/2$.

- Continue in the same way.

Let $\phi_t(x, y; a)$ denote the transition probability for the reflected process which is defined for $0 < x, y < \pi$ and $t > 0$. This is also defined for $x = 0$ and $x = \pi$ by taking the limit, but we restrict to $0 < y < \pi$. We will will use $p_t(x, y; a)$ for the transition probability for the process killed at 0.

If μ_0 is any probability distributition on $[0, \pi]$, let $\Phi_t\mu$ denote the distribution of X_t given X_0 has distribution μ_0.

Lemma 4.7. *If $-1/2 < a < 1/2$, there exists c, β and a probability distribution μ such that if μ_0 is any initial distribution and $\mu_t = \Phi_t\mu_0$, then*

$$\|\mu - \mu_t\| \leq c\, e^{-\beta t}. \tag{53}$$

Proof. This uses a standard coupling argument. The key fact is that there exists $\rho > 0$ such that for every $x \in [0, \pi]$, the probability that the process starting at x visits 0 by time 1 is at least ρ.

Suppose μ^1, μ^2 are two different initial distributions. We start processes X_1, X_2 independently with distributions μ^1, μ^2. When the particles meet we coalesce the particles and they run together. If $X_1 \leq X_2$, then the coalescence time will be smaller than the time for X_2 to reach the origin. If $X_1 \geq X_2$, the time will be smaller than the time for X_1 to reach the origin. Hence the coalescence time is stochastically bounded by the time to reach the origin. Using the strong Markov property and the previous paragraph, the probability that $T > n$ is bounded above by $(1-\rho)^n = e^{-\beta n}$ and $\|\mu_t^1 - \mu_t^2\|$ is bounded above by the probability that the paths have not coalesced by time t. If $s > t$, we can apply the same argument using initial probability distributions μ_{s-t}^1, μ_0^2 to see that

$$\|\mu_s^1 - \mu_t^2\| \leq c\, e^{-\beta t}, \quad s \geq t.$$

Using completeness, we see that the limit measure

$$\mu = \lim_{n\to\infty} \mu_n^1$$

exists and satisfies (53).

□

The construction of the reflected process shows that $\{t : \sin X_t = 0\}$ has zero measure which shows that the limiting measure must be carried on $(0, \pi)$.

We claim that the invariant density is given again by $f_a = C_{2a}\,(\sin x)^{2a}$. As mentioned before, it satisfies the adjoint equation

$$-[m(x)f_a(x)]' + \frac{1}{2}\,f_a''(x) = 0. \qquad \text{where } m(x) = a\,\cot x.$$

Another way to see that the invariant density is proportional to $(\sin x)^{2a}$ is to consider the process reflected at $\pi/2$. Let $p_t(z, x) = p_t(z, x) + p_t(z, \pi - x)$ be the probability density for this reflected process. Suppose that $0 < x < y < \pi/2$ and consider the relative amounts of time spent at x and y during an excursion from zero. If an excursion is to visit either x or y, it must start by visiting x. Given that it is x, the amount of time spent at x before the excursion ends is

$$\int_0^\infty \bar{p}_t(x, x; a)\,dt,$$

and the amount of time spent at y before the excursion ends is

$$\int_0^\infty \bar{p}_t(x, y; a)\,dt = \left[\frac{\sin y}{\sin x}\right]^{2a} \int_0^\infty \bar{p}_t(y, x; a)\,dt.$$

The integral on the right-hand side gives the expected amount of time spent at x before reaching zero for the process starting at y. However, if it starts at y it must hit x before reaching the origin. Hence by the strong Markov property,

$$\int_0^\infty \bar{p}_t(y, x; a)\,dt = \int_0^\infty \bar{p}_t(x, x; a)\,dt,$$

and hence,

$$\int_0^\infty \bar{p}_t(x, y; a)\,dt = \left[\frac{\sin y}{\sin x}\right]^{2a} \int_0^\infty \bar{p}_t(x, x; a)\,dt.$$

An important property of the radial Bessel process is the exponential rate of convergence to the equilibrium density. The next proposition gives a Harnack-type inequality that states within time one that one is within a multiplicative constant of the invariant density.

Proposition 4.8. *For every $-1/2 < a < 1/2$ and $t_0 > 0$, there exists $c = c < \infty$ such that for every $0 < x, y < 2\pi$ and every $t \geq t_0$,*

$$c^{-1}\,[\sin x]^{2a} \leq \phi_t(y, x; a) \leq c\,[\sin x]^{2a}.$$

Proof. By the Markov property it suffices to show that for each $s > 0$ there exists $c = c(s) < \infty$ such that

$$c^{-1}\,[\sin x]^{2a} \leq \phi_s(y, x; a) \leq c\,[\sin x]^{2a}.$$

We fix s and allow constants to depend on s. We write $z = \pi/2$. By symmetry, we may assume that $x \leq \pi/2$. for which $\sin x \asymp x$.

By comparison with Brownian motion, it is easy to see that

$$\inf\{\phi_t(z, y) : s/3 \leq t \leq s, \pi/4 \leq y \leq 3\pi/4\} > 0.$$

Therefore, for any $0 \leq x \leq \pi/2$, $2s/3 \leq t \leq s$, $\pi/4 \leq y \leq 3\pi/4$,

$$\phi_t(x, y; a) \geq \mathbb{P}^x_a\{T_z \leq s/3\}\, \inf\{\phi_r(z, y) : s/3 \leq r \leq s, \pi/4 \leq y \leq 3\pi/4\} \geq c,$$

and hence for such t, using ???,

$$\phi_t(z, x; a) = (x/z)^{2a}\, \phi_t(x, z; a) \geq c\, x^{2a}.$$

Hence, for every $0 \leq y \leq \pi$,

$$x^{-2a}\, \phi_s(y, x; a) \geq \mathbb{P}^y_a\{T_z \leq s/3\}\, \inf\{x^{-2a}\, \phi_r(z, x) : s/3 \leq r \leq s, 0 \leq y \leq \pi\} \geq c.$$

This gives the lower bound.

Our next claim is if $w = 3\pi/4$ and

$$\theta_1 := \sup\{x^{-2a}\, \phi_t(w, x) : 0 \leq t \leq s, 0 \leq x \leq \pi/2\},$$

then $\theta_1 < \infty$. To see this let $\phi^*_t(y, x)$ be the density of the process $X_{t\wedge T_{7\pi/8}}$. Using (42) and absolute continuity, we can see that

$$\phi^*_t(w, x) \leq c\, x^{2a}.$$

However, by the strong Markov property, we can see that

$$x^{-2a}\, \phi_t(y, x) \leq x^{-2a}\, \phi^*_t(y, x) + \mathbb{P}^{7\pi/8}\{T_w \leq s\}\, \theta_1 v, \leq x^{-2a}\, \phi^*_t(w, x) + (1 - \rho)\, \theta_1,$$

for some $\rho > 1$. Hence $\theta_1 \leq x^{-2a}\, \phi^*_t(w, x) \leq c/\rho < \infty$.

We now invoke Proposition 3.3 and absolute continuity,to see that for all $0 \leq y \leq 3\pi/4$ $\phi^*_s(y, x) \leq c\, x^{2a}$. Hence, by the Markov property,

$$\phi_s(y, x) \leq \phi^*_s(y, x) + \sup\{\phi_t(w, x) : 0 \leq t \leq s\} \leq c\, x^{2a}.$$

□

Proposition 4.9. *For every $-1/2 < a < 1/2$, there exists $\beta > 0$ such that for all $t \geq 1$ and all $0 < x, y < \pi$,*

$$\phi_t(x, y; a) = f_a(y)\, \left[1 + O(e^{-t\beta})\right].$$

More precisely, for every $t_0 > 0$, there exists $c < \infty$ such that for all x, y and all $t \geq t_0$,

$$f_a(y)\, [1 - ce^{-\beta t}] \leq \phi_t(x, y; a) \leq f_a(y)\, [1 + ce^{-\beta t}].$$

Proof. Exactly as in Proposition 4.5. □

4.4 Functionals of Brownian motion

Now suppose X_t is a Brownian motion with $X_0 = x \in (0,\pi)$. Let $T = \inf\{t : \Theta_t = 0 \text{ or } \pi\}$ and let I_t denote the indicator function of the event $\{T > t\}$. Suppose that $\lambda > -1/8$ and let

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1+8\lambda} \geq \frac{1}{2},$$

be the larger root of the polynomial $a^2 - a - 2\lambda$. Let

$$J_t = \exp\left\{-\int_0^t \frac{ds}{S_s^2}\right\}.$$

If $M_{t,a}$ denotes the martingale in (48), then we can write

$$M_{t,a} = \left[\frac{S_t}{S_0}\right]^a J^{\lambda_a}, \quad \text{where } \lambda_a = \frac{a(a-1)}{2}$$

Proposition 4.10. *Suppose $\lambda \geq -1/8$. Then there exists $\beta = \beta(\lambda) > 0$ such that*

$$\mathbb{E}^x[J_t^\lambda \, I_t] = [C_{2a}/C_a]\,(\sin x)^a\, e^{-at}\,[1 + O(e^{-\beta t})],$$

where

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1+8\lambda} \geq \frac{1}{2}. \tag{54}$$

Proof. Let a be defined as in (54). Then,

$$\begin{aligned}
\mathbb{E}^x[J_t^\lambda \, I_t] &= (\sin x)^a\, e^{-at}\, \mathbb{E}^x\left[M_{t,a}\, I_t\, S_t^{-a}\right] \\
&= (\sin x)^a\, e^{-at}\, \mathbb{E}_a^x\left[I_t\, S_t^{-a}\right] \\
&= (\sin x)^a\, e^{-at} \int_0^\pi p_t(x,y;a)\,[\sin y]^{-a}\, dy. \\
&= c'\,(\sin x)^a\, e^{-at}\,[1 + O(e^{-\beta t})].
\end{aligned}$$

Here $\beta = \beta_a$ is the exponent from Proposition 4.5 and

$$c' = \int_0^\pi f_a(y)\,[\sin y]^{-a}\, dy = C_{2a}/C_a.$$

Note that in the third line we could drop the I_t term since $\mathbb{P}_a^x\{I_t = 1\} = 1$. □

Proposition 4.11. *Suppose $b \in \mathbb{R}$ and*

$$\lambda + \lambda_b \geq -\frac{1}{8}. \tag{55}$$

Let

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8(\lambda+\lambda_b)} \geq \frac{1}{2}. \tag{56}$$

and assume that $a + b > -1$. Then, there exists $\beta = \beta(\lambda, b) > 0$ such that

$$\mathbb{E}_b^x[J_t^\lambda \, I_t] = [C_{2a}/C_{a+b}]\,(\sin x)^{a-b}\, e^{(b-a)t}\,[1 + O(e^{-\beta t})].$$

Proof. Let a be as in (56) and note that $\lambda_a = \lambda + \lambda_b$.

$$\begin{aligned}
\mathbb{E}_b^x[J_t^\lambda\, I_t] &= \mathbb{E}^x[M_{t,b}\, J_t^\lambda\, I_t] \\
&= (\sin x)^{-b}\, e^{bt}\, \mathbb{E}^x[S_t^b\, J_t^{\lambda_b+\lambda}\, I_t] \\
&= (\sin x)^{a-b}\, e^{(b-a)t}\, \mathbb{E}^x[S_t^{b-a}\, M_{t,a}\, I_t] \\
&= (\sin x)^{a-b}\, e^{(b-a)t}\, \mathbb{E}_a^x[S_t^{b-a}] \\
&= (\sin x)^{a-b}\, e^{(b-a)t} \int_0^\pi p_t(x,y;a)\, [\sin y]^{b-a}\, dy \\
&= c'\, (\sin x)^{a-b}\, e^{(b-a)t}\, [1 + O(e^{-\beta t})].
\end{aligned}$$

Here $\beta = \beta_a$ is the exponent from Proposition 4.5 and

$$c' = \int_0^\pi f_a(y)\, [\sin y]^{b-a}\, dy = C_{2a}/C_{a+b}.$$

The fourth equality uses the fact that $\mathbb{P}_a^x\{I_t = 1\} = 1$.

$\square$

4.5 Example

We consider Brownian motion on $(0, \pi)$ weighted locally by

$$\Phi(x) = [\sin x]^u\, [1 - \cos x]^v, \quad m(x) = \frac{v}{\sin x} + (u+v)\, \cot x.$$

When we tilt by the appropriate local martingale, we get

$$dX_t = \left[\frac{v}{\sin X_t} + (u+v)\, \cot X_t\right] dt + dB_t. \tag{57}$$

This process is asymptotically Bessel-$(u+2v)$ at the origin and asymptotically Bessel-u at π. we will assume that $u > -1/2$ and if $1/2 < u < 1/2$, we will consider the reflected process.

If $u, u+v > -1/2$, we have the invariant density

$$f_{u,v}(x) = c_{u,v}\, \Phi(x)^2 = c_{u,v}\, [\sin x]^{2u}\, [1 - \cos x]^{2v},$$

where

$$c_{u,v} = \left[\int_0^\pi [\sin x]^{2u}\, [1 - \cos x]^{2v}\, dx\right]^{-1} = \frac{\Gamma(2u + 2v + 1)}{2^{2u+2v}\, \Gamma(u + 2v + \frac{1}{2})\, \Gamma(u + \frac{1}{2})}.$$

We have used an integral identity. By first substituting $\theta = x/2$ and then $y = \sin^2\theta$ we see that

$$\begin{aligned}
\int_0^\pi [\sin x]^{2r}\,[1-\cos x]^s\,dx &= 2\int_0^{\pi/2}[\sin 2\theta]^{2r}\,[1-\cos 2\theta]^s\,d\theta \\
&= 2\int_0^{\pi/2}[2\,\sin\theta\,\cos\theta]^{2r}\,[2\,\sin^2\theta]^s\,d\theta \\
&= 2^{2r+s}\int_0^{\pi/2}[\sin^2\theta]^{r+s-\frac12}\,[\cos^2\theta]^{r-\frac12}\,[2\,\sin\theta\,\cos\theta\,d\theta] \\
&= 2^{2r+s}\int_0^1 y^{r+s-\frac12}\,(1-y)^{r-\frac12}\,dy \\
&= 2^{2r+s}\,\mathrm{Beta}\left(r+s+\frac12, r+\frac12\right) \\
&= 2^{2r+s}\,\frac{\Gamma(r+s+\frac12)\,\Gamma(r+\frac12)}{\Gamma(2r+s+1)}.
\end{aligned}$$

Proposition 4.12. *Suppose $u, u+v > -1/2$, and X_t satisfies (57). There exists $\alpha > 0$ such that if $p_t(x,y)$ denotes the density of X_t given $X_0 = x$, then for $t \geq 1$,*

$$p_t(x,y) = f_{u,v}(y)\,[1 + O(e^{-\alpha t})].$$

In particular, if F is a nonnegative function with

$$E_{u,v}(F) := \int_0^\infty F(y)\,f_{u,v}(y)\,dy < \infty$$

then

$$\mathbb{E}[X_t \mid X_0 = x] = E_{u,v}(F)\,\left[1 + O(e^{-\alpha t})\right].$$

For later reference, we note that if $k > -1-u-v$, and

$$F(x) = \left[\frac{1-\cos x}{2}\right]^k,$$

then

$$\begin{aligned}
\mathbb{E}_{u,v}(F) &= \int_0^\pi c_{u,v}\,[\sin x]^{2u}\,[1-\cos x]^{2v}\,\left[\frac{1-\cos x}{2}\right]^k\,dx \\
&= 2^{-k}\,\frac{c_{u,v}}{c_{u,v+\frac{k}{2}}} \\
&= \frac{\Gamma(2u+2v+1)\,\Gamma(u+2v+k+\frac12)}{\Gamma(u+2v+\frac12)\,\Gamma(2u+2v+k+1)}. \qquad (58)
\end{aligned}$$

5 Identities for special functions

5.1 Asymptotics of h_a

Suppose $a > -1/2$ and

$$h_a(z) = \sum_{k=0}^{\infty} c_k\, z^{2k} \quad \text{where } c_k = c_{k,a} = \frac{1}{2^{a+2k-\frac{1}{2}}\, k!\, \Gamma(k+a+\frac{1}{2})}.$$

We note that the modified Bessel function of the first kind of order ν is given by $I_\nu(z) = z^\nu\, h_{\nu+\frac{1}{2}}(z)$. What we are discussing in this appendix are well known facts about I_ν, but we will state and prove them for the analytic function h_a. Since c_k decays like $[2^k\, k!]^{-2}$, is easy to see that the series has an infinite radius of convergence, and hence h_a is an entire function. Note that the c_k are given recursively by

$$c_0 = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}, \quad c_{k+1} = \frac{c_k}{(2k+2)\,(2k+2a+1)}. \tag{59}$$

Proposition 5.1. *h_a is the unique solution to*

$$z\, h''(z) + 2a\, h'(z) - z\, h(z) = 0. \tag{60}$$

with

$$h(0) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}, \quad h'(0) = 0.$$

Proof. Using term-by-term differentiation and (59), we see that h_a satisfies (60). A second, linearly independent solution of (60) can be given by

$$\tilde{h}_a(z) = \sum_{k=1}^{\infty} \tilde{c}_{k-1} z^{2k-1},$$

where $\tilde{c}_k$ are defined recursively by

$$\tilde{c}_0 = 1, \quad \tilde{c}_k = \frac{\tilde{c}_{k-1}}{(2k+1)\,(2k+2a)}.$$

Note that $\tilde{h}_a(0) = 0, \tilde{h}'_a(0) = 1$. By the uniqueness of second-order linear differential equations, every solution to (60) can be written as $h(z) = \lambda\, h_a(z) + \tilde{\lambda}\, \tilde{h}_a(z)$, and only $\lambda = 1, \tilde{\lambda} = 0$ satisfies the initial condition. □

Proposition 5.2. *Suppose h satisfies* (60)*, and*

$$\phi(x,y) = y^{2a} \exp\left\{-\frac{x^2+y^2}{2}\right\} h(xy).$$

Let

$$q_t(x, y; a) = \frac{1}{\sqrt{t}}\, \phi(x/\sqrt{t}, y/\sqrt{t}).$$

Then for every t*,*

$$\partial_t q_t(x, y : a) = L_x q_t(x, y; a) = L_y^* q_t(x, y; a),$$

where

$$L f(x) = \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x),$$

$$L^* f(x) = \frac{a}{x^2}\, f(x) - \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x).$$

Proof. This is a straightforward computation. We first establish the equalities at $t = 1$. Note that

$$\partial_t q_t(x, y; a) \mid_{t=1} = -\frac{1}{2}\, \left[\phi(x, y) + x\, \phi_x(x, y) + y\, \phi_y(x, y)\right].$$

Hence we need to show that

$$\phi_{xx}(x, y) + \left[\frac{2a}{x} + x\right]\, \phi_x(x, y) + y\, \phi_y(x, y) + \phi(x, y) = 0,$$

$$\phi_{yy}(x, y) + x\, \phi_x(x, y) + \left[y - \frac{2a}{y}\right]\, \phi_y(x, y) + \left[\frac{2a}{y^2} + 1\right]\, \phi(x, y) = 0$$

Direct computation gives

$$\phi_x(x, y) = \left[-x + y\, \frac{h'(xy)}{h(xy)}\right] \phi(x, y),$$

$$\phi_{xx}(x, y) = \left[-1 + \frac{h''(xy)}{h(xy)}\, y^2 - 2xy\, \frac{h'(xy)}{h(xy)} + x^2\right]\, \phi(x, y),$$

$$\phi_y(x, y) = \left[\frac{2a}{y} - y + x\, \frac{h'(xy)}{h(xy)}\right]\, \phi(x, y),$$

$$\phi_{yy}(x, y) = \left[-1 - 4a + \frac{4a^2 - 2a}{y^2} + y^2 + x^2\, \frac{h''(xy)}{h(xy)} + \left(\frac{4ax}{y} - 2xy\right)\, \frac{h'(xy)}{h(xy)}\right]\, \phi(x, y).$$

If h satisfies (16), then

$$\frac{h''(xy)}{h(xy)} = 1 - \frac{2a}{xy}\, \frac{h'(xy)}{h(xy)},$$

so we can write

$$\phi_{xx}(x, y) = \left[-1 + x^2 + y^2 + \left(-2xy - \frac{2ay}{x}\right)\, \frac{h'(xy)}{h(xy)}\right]\, \phi(x, y),$$

$$\phi_{yy}(x, y) = \left[-1 - 4a + x^2 + y^2 + \frac{4a^2 - 2a}{y^2} + \left(\frac{4ax}{y} - 2xy\right)\, \frac{h'(xy)}{h(xy)}\right].$$

This gives the required relation.

For more general t, note that

$$\partial_t q_t(z,w;a) = \frac{1}{2t^{3/2}} \left[\phi(z/\sqrt{t}, w/\sqrt{t}; a) - \phi_x(z/\sqrt{t}, w/\sqrt{t}; a) - \phi_y(z/\sqrt{t}, w/\sqrt{t}; a)\right].$$

$$\partial_x q_t(z,w;a) = \frac{1}{t}\, \phi_x(z/\sqrt{t}, w/\sqrt{t}; a),$$
$$\partial_{xx} q_t(z,w;a) = \frac{1}{t^{3/2}}\, \phi_{xx}(z/\sqrt{t}, w/\sqrt{t}; a),$$
$$\partial_y q_t(z,w;a) = \frac{1}{t}\, \phi_y(z/\sqrt{t}, w/\sqrt{t}; a),$$
$$\partial_{yy} q_t(z,w;a) = \frac{1}{t^{3/2}}\, \phi_{yy}(z/\sqrt{t}, w/\sqrt{t}; a),$$
$$L_x q_t(z,w;a) = \frac{a}{(z/\sqrt{t})}\, \frac{1}{t^{3/2}}\, \phi_x(z/\sqrt{t}, w/\sqrt{t}; a) + \frac{1}{2}\, \frac{1}{t^{3/2}}\, \phi_{xx}(z/\sqrt{t}, w/\sqrt{t}; a),$$
$$L_y^* q_t(z,w;a) =$$
$$\frac{a}{(w/\sqrt{t})^2}\, \frac{1}{t^{3/2}}\, \phi(z/\sqrt{t}, w/\sqrt{t}; a) - \frac{a}{(w/\sqrt{t})}\, \frac{1}{t^{3/2}}\, \phi_y(z/\sqrt{t}, w/\sqrt{t}; a) + \frac{1}{2}\, \frac{1}{t^{3/2}}\, \phi_{yy}(z/\sqrt{t}, w/\sqrt{t}; a),$$

□

Proposition 5.3. *If h satisfies* (60)*, then exist an analytic function u with $u(0) \neq 0$ such that for all $x > 0$,*

$$h(x) = x^{-a}\, e^x\, u(1/x).$$

Proof. Let

$$v(x) = e^{-x}\, x^a\, h_a(x).$$

Then,

$$v'(x) = v(x) \left[-1 + \frac{a}{x} + \frac{h_a'(x)}{h_a(x)}\right],$$

$$\begin{aligned}
v''(x) &= v(x) \left(\left[-1 + \frac{a}{x} + \frac{h_a'(x)}{h_a(x)}\right]^2 - \frac{a}{x^2} + \frac{h_a''(x)}{h_a(x)} - \frac{h_a'(x)^2}{h_a(x)^2}\right) \\
&= v(x) \left[1 + \frac{a^2 - a}{x^2} - \frac{2a}{x} + \left(\frac{2a}{x} - 2\right) \frac{h_a'(x)}{h_a(x)} + \frac{h_a''(x)}{h_a(x)}\right] \\
&= v(x) \left[2 + \frac{a^2 - a}{x^2} - \frac{2a}{x} - 2\, \frac{h_a'(x)}{h_a(x)}\right] \\
&= -2v'(x) + \frac{a^2 - a}{x^2}\, v(x).
\end{aligned}$$

The third equality uses the fact that h_a satisfies (60). If $u_a(x) = v(1/x)$, then

$$u'(x) = -\frac{1}{x^2}\, v'(1/x),$$

$$\begin{aligned} u_a''((x) &= \frac{2}{x^3}\, v'(1/x) + \frac{1}{x^4}\, v''(1/x) \\ &= \left[\frac{2}{x^3} - \frac{2}{x^4}\right]\, v'(1/x) + \frac{a^2 - a}{x^2}\, v(1/x) \\ &= \left[\frac{2}{x^2} - \frac{2}{x}\right]\, u_a'((x) + \frac{a^2 - a}{x^2}\, u_a((x). \end{aligned}$$

In other words, u satisfies the equation

$$x^2\, u''(x) + (2 - 2x)\, u'(x) + (a^2 - a)\, u(x) = 0.$$

We can find two linearly independent entire solutions to this equation of the form

$$u(z) = \sum_{k=0}^{\infty} b_k\, z^k$$

by choosing $b_0 = 1, b_1 = 0$ or $b_0 = 0, b_1 = 1$, and the recursively,

$$b_{k+2} = \frac{(2k + a - a^2)\, b_k - 2\,(k+1)\, b_{k+1}}{(k+1)(k+2)}.$$

Then,

$$u(z) = \sum_{k=0}^{\infty} b_k\, z^k,$$

$$u'(x) = \sum_{k=0}^{\infty} (k+1) b_{k+1}\, z^k,$$

$$zu'(z) = \sum_{k=1}^{\infty} k\, b_k z^k,$$

$$u''(z) = \sum_{k=0}^{\infty} (k+1)\,(k+2)\, b_{k+2} z^k,$$

then the differential equation induces the relation

$$b_{k+2} = \frac{(2k + a - a^2)\, b_k - 2\,(k+1)\, b_{k+1}}{(k+1)(k+2)}.$$

Note that

$$|b_{k+2}| \leq \frac{2}{k+2}\, [|b_k| + |b_{k+1}|],$$

from which we can conclude that the power series converges absolutely for all z. By uniqueness $u_a(x)$ must be a linear combination of these solutions and hence must be the restriction of an entire function to the real line. $\square$

5.2 Some integral identities

In this subsection we establish two "obvious" facts about the density by direct computation. We first prove that $\psi_t(x,\cdot;a)$ is a probability density.

Proposition 5.4. *For every $a > -1/2$ and $x > 0$,*

$$\int_0^\infty \psi_t(x,y;a)\,dy = 1.$$

We use a known relation about special functions, that we state here.

Lemma 5.5. *If $a > -1/2$ and $x > 0$,*

$$\int_0^\infty z^{2a} \exp\left\{-\frac{z^2}{2x^2}\right\} h_a(z)\,dz = x^{2a-1}\, e^{x^2/2}.$$

Proof. If we let $\nu = a - \frac{1}{2}$ and $r = x^2$, we see this is equivalent to

$$\int_0^\infty z^{\nu+1} \exp\left\{-\frac{z^2}{2r}\right\} I_\nu(z)\,dz = r^{\nu+1}\, e^{r/2}.$$

Equation 1.15.5 #4 of [3] gives the formula

$$\int_0^\infty x^{b-1}\, e^{-px^2}\, I_\nu(x)\,dx = 2^{-\nu-1}\, p^{-\frac{b+\nu}{2}}\, \frac{\Gamma(\frac{b+\nu}{2})}{\Gamma(\nu+1)}\, {}_1F_1\left(\frac{b+\nu}{2}, \nu+1, \frac{1}{4p}\right),$$

where ${}_1F_1$ is the confluent hypergeometric function. In our case, $b = \nu + 2$, $p = 1/(2r)$, so the right-hand side equals

$$2^{-\nu-1}\,(1/2r)^{-\nu-1}\, {}_1F_1\left(\nu+1, \nu+1, r/2\right) = r^{\nu+1}\, e^{r/2}.$$

where the last equality comes from the well-known identity ${}_1F_1(b,b,z) = e^z$. □

Proof of 5.4. Since

$$\int_0^\infty \psi_t(x,y;a)\,dy = \int_0^\infty t^{-1/2}\, \psi_1\left(x/\sqrt{t}, y/\sqrt{t}; a\right)\,dy = \int_0^\infty \psi_1(x/\sqrt{t}, z; a)\,dz,$$

it suffices to show that for all x,

$$\int_0^\infty \psi(x,y)\,dy = 1$$

where

$$\psi(x,y) = \psi_1(x,y;a) = y^{2a} \exp\left\{-\frac{x^2+y^2}{2}\right\} h_a(xy).$$

Using the substitution $xy = z$, we see that

$$\begin{aligned}\int_0^\infty \psi(x,y)\,dy &= e^{-x^2/2}\int_0^\infty y^{2a}\,\exp\left\{-\frac{y^2}{2}\right\}\,h_a(xy)\,dy \\ &= x^{-2a-1}\,e^{-x^2/2}\int_0^\infty z^{2a}\,\exp\left\{-\frac{z^2}{2x^2}\right\}\,h_a(z)\,dz = 1\end{aligned}$$

□

We now show that $\psi_t(x,y;a)$ satisfies the Chapman-Kolomogrov equations.

Proposition 5.6. *if* $a > -1/2, 0 < t < 1$ *and* $x > 0$, *then*

$$\int_0^\infty \psi_t(x,z;a)\,\psi_{1-t}(z,y;a)\,dz = \psi_1(x,y;a).$$

Proof. Using (36), we see that the proposition is equivalent to the identity

$$[t(1-t)]^{-a-\frac{1}{2}}\int_0^\infty z^{2a}\,\exp\left\{-\frac{x^2+z^2}{2t}\right\}\,\exp\left\{-\frac{z^2+y^2}{2(1-t)}\right\}\,h_a\left(\frac{xz}{t}\right)\,h_a\left(\frac{zy}{1-t}\right)\,dz =$$

$$\exp\left\{-\frac{x^2+y^2}{2}\right\}\,h_a(xy).$$

We will use one integral identity which is equation 2.15.20 #8 in [3]: if $\nu > -1$, and $b, c > 0$,

$$\int_0^\infty x\,e^{-x^2/2}\,I_\nu(bx)\,I_\nu(cx)\,dx = \exp\left\{\frac{b^2+c^2}{2}\right\}\,I_\nu(bc). \tag{61}$$

If we set $\nu = a - \frac{1}{2}$, we have

$$\begin{aligned}&\int_0^\infty z^{2a}\,\exp\left\{-\frac{z^2}{2t(1-t)}\right\}\,h_a\left(\frac{xz}{t}\right)\,h_a\left(\frac{zy}{1-t}\right)\,dz \\ &= \left(\frac{t}{x}\right)^{a-\frac{1}{2}}\left(\frac{1-t}{y}\right)^{a-\frac{1}{2}}\int_0^\infty z\,\exp\left\{-\frac{z^2}{2t(1-t)}\right\}\,I_\nu\left(\frac{xz}{t}\right)\,I_\nu\left(\frac{zy}{1-t}\right)\,dz \\ &= \left(\frac{t}{x}\right)^{a-\frac{1}{2}}\left(\frac{1-t}{y}\right)^{a-\frac{1}{2}}t(1-t)\int_0^\infty u\,e^{-u^2/2}\,I_\nu\left(\frac{ux\sqrt{1-t}}{\sqrt{t}}\right)\,I_\nu\left(\frac{uy\sqrt{t}}{\sqrt{1-t}}\right)\,du \\ &= [t(1-t)]^{a+\frac{1}{2}}\,\exp\left\{\frac{x^2(1-t)}{2t}+\frac{y^2t}{2(1-t)}\right\}\,(xy)^{\frac{1}{2}-a}\,I_\nu(xy) \\ &= [t(1-t)]^{a+\frac{1}{2}}\,\exp\left\{\frac{x^2(1-t)}{2t}+\frac{y^2t}{2(1-t)}\right\}\,h_a(xy)\end{aligned}$$

□

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