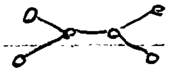
Intro to cat-l Kac-Moody actions; U. Chicago
unclear time

Rel-d text: 1209.1067

0) Intro: Goal: $\underbrace{\text{Kac-Moody alg-s}}_{\text{type A: } \mathfrak{sl}_n,\ \hat{\mathfrak{sl}}_n,\ \mathfrak{gl}_\infty \text{ (gen. case harder)}} \circlearrowright$ cat-s

Reminder: (simply laced) KM alg-s:

I-graph, e.g.  $\rightsquigarrow$ Cartan matrix $(a_{ij})_{i,j\in I}$: $a_{ii}=2$, $a_{ij}=-1$, $i - j$
$a_{ij}=0$, $i \not- j$

$\mathfrak{g}(I)$ = gen-rs e_i, f_i, h_i, $i\in I$,
rel-ns $[h_i,e_j]=a_{ij}e_j$, $[h_i,f_j]=-a_{ij}f_j$ (*)
$[e_i,f_j]=\delta_{ij}h_i$, $e_ie_j=e_je_i$, $f_if_j=f_jf_i$ $i \not- j$
$e_i^2e_j+e_je_i^2=2e_ie_je_i$, $f_i^2f_j+f_jf_i^2=2f_if_jf_i$ $i - j$ (2 rel-ns)

Rep-n of $\mathfrak{g}(I)$ on vect. sp. V = op-rs e_i, f_i, h_i subj. to rel-ns (enough e_i, f_i)

wt rep-n on V: $V=\bigoplus_{\mu\in\mathbb{C}^I} V_\mu$ w $h_i|_{V_\mu}=\mu_i\,\mathrm{id}_{V_\mu}$, need e_i, f_i

(*) $\Leftrightarrow$ $e_iV_\mu\subset V_{\mu+\alpha_i}$, $f_iV_\mu\subset V_{\mu-\alpha_i}$

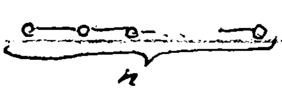
Q: What's $\mathfrak{g}(I)$-action on a cat-y (abelian, artinian - fin. length $\mathbb{F}$-lin, $\mathbb{F}$ is field) $\mathcal{C}$

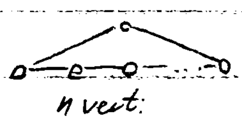
Groth. gr-p $[\mathcal{C}]:=\mathbb{C}\otimes_{\mathbb{Z}}K_0(\mathcal{C})$

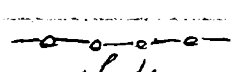
Rough idea: $\mathcal{C}=\bigoplus_\mu \mathcal{C}_\mu$ + exact funct-s E_i, F_i, $i\in I$ s.t. $[E_i],[F_i] \rightsquigarrow$ $\mathfrak{g}(I)\circlearrowright[\mathcal{C}]$

- appears in many examples, not powerf. for inter theory

Idea (Chuang & Rouquier) include transf-s of functors (also app in examples)

Graphs for today:  $\rightsquigarrow \mathfrak{sl}_{n+1}$

 $\rightsquigarrow \hat{\mathfrak{sl}}_n$

 $\rightsquigarrow \mathfrak{gl}_\infty$

Start w. main motiv. examples

1) Examples: symmetr. gr-ps, type A Hecke alg-s, cat-y $\mathcal{O}$ for $\mathfrak{gl}_n$

1.1) Symm. gr-s:
$\mathfrak{S}_n$ - symm. gr-p on $\{1,2,\dots,n\}$, field $\mathbb{F}$ alg. clos. $\rightsquigarrow \mathbb{F}\mathfrak{S}_n \rightsquigarrow Rep(\mathbb{F}\mathfrak{S}_n)$ - fin dim. reps; char $\mathbb{F}=0$ (s/s cat-y w. simples $\xleftrightarrow{\sim} \{\lambda \vdash n\}$)

char $\mathbb{F}>0$ - hard! (unkn dim-s of simples):

Basic idea: induct. based on: $\underset{\{1\}}{\mathfrak{S}_0} \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \dots \subset \underset{\text{fixes } n}{\mathfrak{S}_{n-1}} \subset \mathfrak{S}_n \subset \dots$

$\rightsquigarrow$ res funct: $Res_n^{n-1}: Rep(\mathbb{F}\mathfrak{S}_n) \to Rep(\mathbb{F}\mathfrak{S}_{n-1})$

1.1.1) Functors E_i

Goal: funct-l decomp. of $Res_n^{n-1}(M)$

Key obs-n: $JM_n := \sum_{i<n} (in) \in (\mathbb{F}\mathfrak{S}_n)^{\mathfrak{S}_{n-1}} \Rightarrow [JM_n, \mathbb{F}\mathfrak{S}_{n-1}]=0$

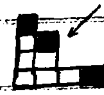
$\rightsquigarrow$ funct. (in M) end-m of $Res_n^{n-1}(M) \rightsquigarrow$ e-decomp $Res_n^{n-1}(M) = \bigoplus_{i\in\mathbb{F}} Res_n^{n-1}(M)_i$

$\rightsquigarrow$ functor $Res_n^{n-1}(\bullet)_i : Rep(\mathbb{F}\mathfrak{S}_n) \to Rep(\mathbb{F}\mathfrak{S}_{n-1})$

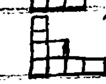
Example (char $\mathbb{F}=0$) $M=M_\lambda$ - simple

$Res_n^{n-1}(M) = \bigoplus_{\substack{\mu\vdash n-1 \\ \lambda\setminus\mu = \text{box}}} M_\mu$, e-val of JM_n on M_μ = cont$(\lambda\setminus\mu)$ = x-coord - y-coord

remov. boxes

e.g. $\lambda=(4\,2\,2\,1)\vdash 9 \rightsquigarrow$  $\rightsquigarrow Res_n^{n-1}(M) = \bigoplus$ 3 simples

$\mu_1 =$ , cont $= 4-1=3$, $\mu_3 =$ , cont $= 1-4=-3$

$\mu_2 =$ , cont $= 2-3=-1$

Cor (char p) $Res_n^{n-1}(\cdot)_i \neq 0 \Rightarrow i \in \mathbb{F}_p \subset \mathbb{F}$ (Exer: $\exists\, \alpha_i \in \mathbb{Z}$ s.t. $\prod(JM_n - \alpha_i)=0$ is $\mathbb{C}\mathfrak{S}_n \Rightarrow$ in $\mathbb{Z}\mathfrak{S}_n \Rightarrow$ in $\mathbb{F}\mathfrak{S}_n$)

Not-n. $\mathcal{C} = \bigoplus_{n\geq 0} Rep(\mathbb{F}\mathfrak{S}_n)$, $E := \bigoplus_{n\geq 0} Res_n^{n-1}: \mathcal{C}\to\mathcal{C}$ ($Res_0^{-1}=0$)

$E_i := \bigoplus_{n\geq 0} Res_n^{n-1}(\cdot)_i : \mathcal{C}\to\mathcal{C}$ ($JM_1=0$)

Have $X\in End(E)$, $X|_{Res_n^{n-1}} = JM_n \Rightarrow E_i$ is e-funct for X

1.1.2) Functors F_i;
left adj to E, $F := \bigoplus_{n\geq 0}^{+\infty} Ind_n^{n+1}$ - isom to right adj $\bigoplus_{n\geq 0} Coind_n^{n+1}$ b/c $\mathbb{F}\mathfrak{S}_n \underset{\text{bimod}}{\cong} (\mathbb{F}\mathfrak{S}_n)^*$ via $(a,b) := tr(ab)$, $tr: \mathbb{F}\mathfrak{S}_n \to \mathbb{F}$, $tr(g) = \delta_{g,1}$

F left adj to $E = \bigoplus E_i \rightsquigarrow F = \bigoplus_i F_i$ - left adj to E_i

Rem: $F_i \cong$ right adj to E_i

Reason: $J^n = \sum_{i=1}^{n} JM_i = \sum_{i<j} (ij) \in$ center of $\mathbb{F}\mathfrak{S}_n$; $\beta \in \mathbb{F} \rightsquigarrow Rep(\mathbb{F}\mathfrak{S}_n)^\beta = \{M \mid$ e-vals of $J^n = \beta\} \rightsquigarrow Rep(\mathbb{F}\mathfrak{S}_n) = \bigoplus_\beta Rep(\mathbb{F}\mathfrak{S}_n)^\beta$; $M \in Rep(\mathbb{F}\mathfrak{S}_n)^\beta$, $E_i M =$ proj-n of EM to $Rep(\mathbb{F}\mathfrak{S}_n)^{\beta - i}$ $\Rightarrow F_i$ is right adj to E_i (exer)

1.1.3) Behavior on K_0: E_i, F_i - biadj $\Rightarrow$ exact $\rightsquigarrow$ op-rs $[E_i], [F_i] \subset [\mathcal{C}]$, $i \in \mathbb{Z}$

char 0: $[E_i][M_\lambda] = \begin{cases} [M_\mu], & \lambda = \mu \sqcup \text{box of cont } i \\ 0, & \text{no such } \mu \end{cases}$

$\Downarrow$ adj-s

$[F_i][M_\lambda] = \begin{cases} [M_\nu], & \nu = \lambda \sqcup \text{box of cont } i \\ 0, & \text{no such } \nu \end{cases}$

- Fock space rep-n of $\mathfrak{gl}_\infty$, (irred-le) wt. spaces spanned by diag-s w. given cont = multiset of cont's of boxes, 1-dim-l

char p: $i \in \mathbb{F}_p$, still have $[M_\lambda] \in \mathcal{C}$: $M_\lambda \rightsquigarrow \mathbb{Z}$-form $\rightsquigarrow$ red mod p - well def in K_0

Comment: (1) $\left[\begin{array}{l} [E_i][M_\lambda] = \sum_\mu [M_\mu], \ \lambda = \mu \sqcup \text{box of cont. } i \text{ mod } p \\ [F_i][M_\lambda] = \sum [M_\nu], \ \nu = \lambda \sqcup - - - - - - \end{array}\right.$

Example: $\lambda =$, $p = 3$, $i = 0$: $E_0[M_\lambda] = [M] + [M]$

$F_0[M_\lambda] = 0$

$F_1[M_\lambda] = [\ ] + [\ ] + [\ ]$

Def-claim Let Fock space rep-n of $\widehat{\mathfrak{sl}}_p$ ($[M_\lambda]$-basis), $\mathcal{F}$

Fact: $[M_\lambda]$ span $[\mathcal{C}]$ (even classes of projectives lie in the span)

$\rightsquigarrow [\mathcal{C}]$ is $\widehat{\mathfrak{sl}}_p$-module w. $\mathcal{F} \twoheadrightarrow [\mathcal{C}]$

Thm (Lascoux, Leclerc, Thibon) $[\mathcal{C}]$ - irred h.w. module w. weight ω_0

wt. spaces = spanned by: $[M_\lambda]$, λ w. given cont. mod p

Simples w. spec'd action of symm polyn-s of JM's - central el-ts ~~$\Rightarrow$ class of simple is wt vector~~

1.1.4) $\exists$ Hom. E^N: $E^N|_{Rep(\mathbb{F}\mathfrak{S}_n)} = Res_n^{n-N}: Rep(\mathbb{F}\mathfrak{S}_n) \to Rep(\mathbb{F}\mathfrak{S}_{n-N})$

Have endom-s - from el-ts of $(\mathbb{F}\mathfrak{S}_n)^{\mathfrak{S}_{n-N}}$:

$JM_{n-N+i} = \sum_{j < n-N+i} (j, n-N+i)$, $(n-N+i \ \ n-N+i+1)$: $\mathcal{Q}$ - rel-ns

Def (deg. aff. Hecke alg-a: dAHA) $\mathcal{H}_N^{aff}(1) = \mathbb{C}\langle X_1, \dots X_N, T_1, \dots T_{N-1}\rangle$

$$(*) \quad \begin{cases} X_iX_j = X_jX_i, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \\ T_iT_j = T_jT_i \quad |i-j|>1 \\ T_i^2 = 1, \quad T_iX_i + 1 = X_{i+1}T_i \end{cases}$$

Exer: $\mathcal{H}_N^{aff}(1) \longrightarrow (\mathbb{F}\mathfrak{S}_n)^{\mathfrak{S}_{n-N}}$, $X_i \mapsto JM_{n-N+i}$, $T_i \mapsto (n-N+i, n-N+i+1)$

ext-s to alg. homom

Concl: $\mathcal{H}_N^{aff}(1) \longrightarrow End(E^N)$-alg. homom

Rem: $\cdot\, T \in End(E^2)$ - transp-n; $X_i \mapsto 1_E^{i-1} X 1_E^{n-i}$, $T_i \mapsto 1_E^{i-1} T 1_E^{n-i-1}$

center of $\mathcal{H}_n^{aff}(1) \supset \mathbb{F}[X_1 \dots X_n]^{\mathfrak{S}_n}$

$\cdot\, \mathbb{F}\mathfrak{S}_n = \mathcal{H}_n^{aff}/(X_1 = 0)$, $X_i \mapsto JM_i$

1.2) Type A Hecke alg-s: same picture but entirely in char. 0

1.2.1) AHA & HA: $\mathcal{H}_N^{aff}(q) = \mathbb{C}\langle X_1^{\pm 1}, \dots X_N^{\pm 1}, T_1, \dots T_{N-1}\rangle / (*) +$
$q \in \mathbb{F}^\times \setminus \{1\}$ $\quad (T_i - q)(T_i + 1) = 0, \ T_iX_iT_i = qX_{i+1}$

finite HA: $\mathcal{H}_N(q) = \mathcal{H}_N^{aff}(q)/(X_1 = 1)$

$JM_i = $ image of $X_i \in \mathcal{H}_N(q)$

1.2.2) Same story: $\mathbb{C} = \mathcal{H}_0(q) \hookrightarrow \mathcal{H}_1(q) \hookrightarrow \mathcal{H}_2(q) \hookrightarrow$

$\rightsquigarrow Res_n^{n-1}$, e-vals of $JM_i = q^i$, $i \in \mathbb{Z} \rightsquigarrow Res_n^{n-1}(\cdot)_i$

$q \neq \sqrt{1} \rightsquigarrow i \in \mathbb{Z}$, $q = \sqrt[r]{1}$ (primit.) $\rightsquigarrow i \in \mathbb{Z}/r\mathbb{Z}$

have $\mathcal{C}, E, E_i$; biadj $F, F_i \rightsquigarrow [F_i], [E_i] \circlearrowright [\mathcal{C}] =$

$\cdot$ Fock space for $\mathfrak{gl}_\infty$, $q \neq \sqrt{1}$

$\cdot$ irred. $\hat{\mathfrak{sl}}_r$-mod w. h.wt. ω_0, $q = \sqrt[r]{1}$ (prim.)

$+ \mathcal{H}_N^{aff}(q) \longrightarrow End(E^N)$

1.3) BGG cat. $\mathcal{O}$ for $\mathfrak{gl}_n$

1.3.0) Not-n. $G = GL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \left[\begin{array}{l}\text{triang. decomp} \\ \text{as vect. spaces}\end{array}\right] = \underbrace{\mathfrak{n}^-}_{\left\{\begin{pmatrix} 0 & & 0 \\ * & \ddots & \\ * & * & 0\end{pmatrix}\right\}} \oplus \underbrace{\mathfrak{t}}_{\begin{pmatrix} * & & 0 \\ & * & \\ 0 & & *\end{pmatrix}} \oplus \underbrace{\mathfrak{n}^+}_{\begin{pmatrix} 0 & * & * \\ & 0 & * \\ 0 & & 0\end{pmatrix}}$, $\mathfrak{t} \oplus \mathfrak{n}^+ = \mathfrak{b}$

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{t}) \otimes \mathcal{U}(\mathfrak{n}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{b})$$

$E_{11}, \dots, E_{nn} \in \mathfrak{t}$-basis $\rightsquigarrow$ dual basis $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^* \rightsquigarrow \mathfrak{t}^* = \mathbb{C}^n$

$M \in \mathfrak{g}$-mod, $\lambda \in \mathfrak{t}^* \rightsquigarrow M_\lambda$ = e-space $\{m \in M \mid xm = \langle \lambda, x \rangle m\}$

1.3.1) Cat-y $\mathcal{O} = \{M \in \mathfrak{g}$-mod s.t. i) $M = \bigoplus_{\lambda \in \mathbb{Z}^n} M_\lambda$

ii) $\mathfrak{n}$ acts loc. nilp on M

iii) M is fin gen / $U(\mathfrak{n}^-)$ $\}$

- abel. cat-y.

Example of obj-s: Verma $\Delta(\lambda)$; $\rho := -(0, 1, \dots, n-1) \in \mathfrak{t}^*$; $\lambda \in \mathbb{Z}^n$

$\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda - \rho}$ ← 1-dim $\mathfrak{b}$-module, $\mathfrak{n}$ acts by 0, $\mathfrak{t}$ by $\lambda - \rho$

$\Downarrow$

$L(\lambda)$ - unique simple quot-t. $\mathbb{Z}^n \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O})$, $\lambda \mapsto L(\lambda)$

Rem: $\forall M \in \mathcal{O}$, $\Rightarrow$ length$(M) < \infty$ ($\mathcal{O}$ is artin. cat-y)

1.3.2) Special proj-ve functors E_i, F_i

Obs-n: L - fin. dim. G-mod, $M \in \mathcal{O} \Rightarrow L \otimes M \in \mathcal{O}$.

$\rightsquigarrow$ functor $L \otimes \bullet : \mathcal{O} \to \mathcal{O}$ w. biadj-t $L^* \otimes \bullet$

$E = \mathbb{C}^n \otimes \bullet$, $F = \mathbb{C}^{n*} \otimes \bullet$

Decomp. of E - using $X \in \mathrm{End}(E)$ given by $C_{tens} = \sum_{i,j} E_{ij} \otimes E_{ji} \in [U(\mathfrak{g}) \otimes U(\mathfrak{g})]^G$

$X_M(v \otimes m) = \sum_{i,j} E_{ij} v \otimes E_{ji} m$ - endom b/c C_{tens} is $\mathfrak{g}$-invar.

usual Casimir $C = m(C_{tens})$, $m: U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ - product

$C_{tens} = \frac{1}{2}(\delta(C) - C \otimes 1 - 1 \otimes C)$, $\delta: U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ - coprod.

E_i, $i \in \mathbb{C}$, e-funct. for X

Claim: $i \in \mathbb{Z}$ (reason C acts w. int-l e-values)

$\rightsquigarrow E = \bigoplus_{i \in \mathbb{Z}} E_i \xrightarrow{F \text{ left adj to } E} F = \bigoplus_{i \in \mathbb{Z}} F_i$, F_i is left adj to E_i (but also right adj - exer)

1.3.3) Behavior on K_0: - via comput. of $E_i \Delta(\lambda)$, $F_i \Delta(\lambda)$

Obs-n: $L \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (L \otimes N)$

filt. w. 1-dim quot.

e.g. $L\otimes\Delta(\lambda) = U(\mathfrak{g})\otimes_{U(\mathfrak{b})}(L\otimes\mathbb{C}_{\lambda-\rho})$ $\rightsquigarrow$ filtr on $L\otimes\Delta(\lambda)$ w. Verma factors

$\rightsquigarrow L=\mathbb{C}^n$: filtr quot-s $\Delta(\lambda+\epsilon_i)$, $i=1,\dots,n$

$L=\mathbb{C}^{n*}$: $\Delta(\lambda-\epsilon_i)$, $i=1,\dots,n$

Exer: X preserves filtr-n & acts on $\Delta(\lambda+\epsilon_i)$ w. e-val. λ_i

$\Rightarrow E_j\Delta(\lambda)$ - - - - - $\Delta(\lambda+\epsilon_i)$, $\lambda_i=j$

$F_j\Delta(\lambda)$ - - - - $\Delta(\lambda-\epsilon_i)$, $\lambda_i=j+1$ $\leftarrow$ from exer

Back to $[\mathcal{O}]$: $[\Delta(\lambda)]$ - basis in $\mathcal{O}$ $\rightsquigarrow$ get descr-n of $[E_j]$, $[F_j]$

$[\mathcal{O}] \longrightarrow (\mathbb{C}^{\mathbb{Z}})^{\otimes n}$, $[\Delta(\lambda)] \mapsto v_{\lambda_1}\otimes v_{\lambda_2}\otimes\dots\otimes v_{\lambda_n}$

$\mathfrak{gl}_\infty \curvearrowright \mathbb{C}^{\mathbb{Z}}$: $e_j v_i = \delta_{ij} v_{i+1}$, $f_j v_i = \delta_{j+1,i} v_{i-1}$

isom. of $\mathfrak{gl}_\infty$-modules

Rem: weight spaces - spanned by $[\Delta(\lambda)]$ w. fixed multiset assoc. to λ

- param by unord n-tuples

corresp to infinit. blocks = subcat-s w. single e-value for action of center of $U(\mathfrak{g})$ $\Rightarrow$ ~~$[L(\lambda)]$ ... wt. vector ...~~

1.3.4) Endom-s of E^N: $X\in\mathrm{End}(E)$, $T\in\mathrm{End}(E^2)$ - permute tens factors

$T_M\in\mathrm{End}_{\mathfrak{g}}(\mathbb{C}^n\otimes\mathbb{C}^n\otimes M)$, $T_M(v\otimes v'\otimes m)=v'\otimes v\otimes m$

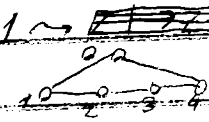
$\rightsquigarrow 1_E^{i-1}X1_E^{n-i}$, $1_E^{i-1}T1_E^{n-i-1}$

Exer: $X_i\mapsto 1_E^{i-1}X1_E^{n-i}$, $T_i\mapsto -1_E^{i-1}T1_E^{n-i-1}$ ext-s to homom $H_n^{aff}(1)\to\mathrm{End}(E^n)$

2) Axiomatic description of cat-l $\mathfrak{g}(I)$-action (I as before - type A)

on abelian, artinian, $\mathbb{F}$-lin cat-y $\mathcal{C}$ (e.g. $\mathcal{C}=\bigoplus_{n\geq 0}\mathrm{Rep}(\mathbb{F}S_n)$, $\bigoplus_{n\geq0}\mathrm{Rep}(H_n(q))$, $\mathcal{O}$) - data + axioms

Data: • $q\in\mathbb{F}^\times$ embed $I\hookrightarrow\mathbb{F}$: $q=1\rightsquigarrow$ $\mathbb{F}\cdot 1\subset\mathbb{F}$ (char $\mathbb{F}=0$ for I -∘-∘-∘-, char $\mathbb{F}=p$ for cycl. I w. p vert.)



$q\neq 1\rightsquigarrow z\mapsto q^z$

endof-s: $E, F: \mathcal{C} \to \mathcal{C}$

$X \in End(E)$, $T \in End(E^2)$

Comment (2): Axioms: 1) E, F - biadj

2) $E = \bigoplus_{i \in I} E_i$ ← e-funct. for X w. e-val $i (\in I \subset \mathbb{F}^\times)$

2) ⇝ [left adj-s] $F = \bigoplus_{i \in I} F_i$

3) $[E_i], [F_i] \rightsquigarrow \mathfrak{g}(I) \subset [\mathcal{C}]$, integr., s.t. $\mathcal{C} = \bigoplus_\mu \mathcal{C}_\mu$ w. $[\mathcal{C}_\mu] = [\mathcal{C}]_\mu$

3) ⇒ F_i, E_i - biadj

4) $X_i \mapsto 1_E^{i-1} X 1_E^{n-i}$, $T_i \mapsto 1_E^{i-1} T 1_E^{n-i-1}$ ext-s to alg. homom

$\mathcal{H}_N^{aff}(q) \longrightarrow End(E^N)$

Termin: $\mathcal{C}$ is categ-n of $[\mathcal{C}]$.

Rem: E & F are symmetr: $End(F^n) \simeq End(E^n)^{opp}$, $\mathcal{H}_q^{aff}(n) \xrightarrow{\sim} \mathcal{H}_{q^{-1}}^{aff}(n)^{opp}$

$T_i \mapsto$ [illegible] T_{n-i}^{-1}, $X_i \mapsto X_{n+1-i}$

2.3*) Cat-s w. type A cat-l actions: 2 types

everything in type A. Hecke type: $\bigoplus$ Rep-s / cyclot. HA's (next time); $\bigoplus$ cat-s $\mathcal{O}$ / cyclot. Rat-l Cherednik alg-s; E - restr-n, F - ind-n

Lie type: Cat. $\mathcal{O}$ & ramif-ns (parab. cat. $\mathcal{O}$, cat. $\mathcal{O}$ / quant gr-ps, affine cat-s $\mathcal{O}$, ~~cat~~ rep-s of $\mathcal{C}$ ~~...~~), E, F - tens products w. taut. rep-n / its dual

2.4*) Other types / cat-ns of quant gr-ps

I - arb. graph ⇝ symmetr. Kac-Moody alg

Key idea: $\mathcal{H}_N^{aff}(q)$ ⇝ KLR alg a.k.a. quiver affine HA - very diff. gen-rs & rel-ns

(Brundan's review for KLR, Rouquier's for cat-n)

Rel-n KLR vs usual (type A) - Brundan-Kleshchev

Q: cat-n of $U_v(\mathfrak{g}(I))$ - mods, v - indep. variable

A: graded $\mathcal{C}$ (e.g. Rep's of graded alg.) ⇝ grading shift functor [1]

⇝ $[\mathcal{C}]$ is $\mathbb{C}[v^{\pm 1}]$-module

KLR alg-s are graded ⇝ require all homom-s in 4) to be of grad. alg-s

3) Consequences of axioms: divided powers, cat-ns of simple reflections, crystals, Serre rel-ns, - categorically.

3.1) Discussion: axioms 1, 3 - natural, 2 - tool to def. E_i from E. Why need 4)?

A: rep. th. of $\mathcal{H}_n^{aff}(q)$ is nice $\rightsquigarrow$ stronger struct. results

E.g: how to cat-fy simple refl-n?

$s_i \rightsquigarrow E_i, F_i$; im. of $\begin{pmatrix}0&1\\-1&0\end{pmatrix}\in SL_2(\mathbb{C})$

V - int-le $\mathfrak{sl}_2$-rep-n, V_λ - wt space, $v\in V_\lambda$

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}.v = \sum_{j\geq \max(0,-\lambda)} (-1)^j \frac{e_i^j}{j!}\frac{f_i^{j+\lambda}}{(j+\lambda)!}$$

$\pm \rightsquigarrow$? complex; how about division? $\mathcal{E}$ - funct, $n\in\mathbb{Z}_{>0}$, $\mathcal{E}/n$ makes sense if $\mathcal{E}=\mathcal{E}_1^{\oplus n}$

Q: Why $E_i^n = n!\cdot$ other funct.

3.2) Divided powers $E_i^{(n)}, F_i^{(n)}$

$E^n M \in \mathcal{H}_n^{aff}(q)$-mod

$E_i^n M$ - submodule: $\mathbb{F}[X_1,\dots,X_n]^{\mathfrak{S}_n}\subset \mathcal{H}_n^{aff}(q)$ - central, $E_i^n M \subset E^n M$ - gen e-space for act. on center: $X_1,\dots,X_n$ all act w. e-val i.

Q: str-re of such modules?

Basic result: only one simple: $\mathrm{Ind}_{P_n := \mathbb{F}[X_1^{\pm1},\dots,X_n^{\pm1}]}^{\mathcal{H}_n^{aff}(q)} \mathbb{F} \underset{\mathcal{H}_n(q)}{\simeq} \mathcal{H}_n(q)$

Cat-s: $\mathcal{H}_n^{aff}(q)\text{-mod}_i \subset \mathcal{H}_n^{aff}(q)$-mod (fin dim), all X_j's act w. gen e-val i

$P_n^{\mathfrak{S}_n}\text{-mod}_i \subset P_n^{\mathfrak{S}_n}$-mod - similar

in fact, equiv

Constr-n of equiv's: $w\in\mathfrak{S}_n \rightsquigarrow T_w := T_{i_1}\cdots T_{i_k} \in \mathcal{H}_n(q)$

red. expr-n $s_{i_1}\cdots s_{i_k}$, $k:=\ell(w)$

$$e_+ := \sum_{w\in\mathfrak{S}_n} T_w, \quad e_- = \sum_{w\in\mathfrak{S}_n}(-q)^{-\ell(w)}T_w$$

$\mathcal{H}_n^{aff}(q)e_+$ - $\mathcal{H}_n^{aff}(q)$-$P_n^{\mathfrak{S}_n}$-bimod - free rk $= n!$ / $P_n^{\mathfrak{S}_n}$

$$P_n^{\mathfrak{S}_n}\text{-mod}_i \underset{e_\pm \bullet}{\overset{\mathcal{H}_n^{aff}(q)e_\pm \otimes_{P_n^{\mathfrak{S}_n}} \bullet}{\rightleftarrows}} \mathcal{H}_n^{aff}(q)\text{-mod}_i$$

Functors $E_i^{(n)}, F_i^{(n)}$: $E_i^{(n,\pm)} = e_\pm E_i^n \Rightarrow E^n \simeq \mathcal{H}_n^{aff}(q)e_\pm \otimes_{P_n^{\mathfrak{S}_n}} E^{(n,\pm)} \simeq \left(E^{(n,\pm)}\right)^{\oplus n!}$

$\Rightarrow E_i^{(n,+)} \simeq E_i^{(n,-)}$; $F_i^{(n)}$ - similar, using $H_n^{aff}(q) \longrightarrow End(E_i^n)^{opp}$

3.3) Rickard complexes - cat-ns of s_i

Θ_λ: $D^b(\mathcal{C}_\lambda) \longrightarrow D^b(\mathcal{C}_\lambda)$ complex $\to E_i^{(\lambda+r,-)}F_i^{(r,+)} \to E_i^{(\lambda+r-1,-)}F_i^{(r-1,+)} \to$

constr-n of differ-s:

$$\begin{tikzcd} E_i^{(\lambda+r,+)}F_i^{(r,-)} \arrow[r, hook, "\text{direct summ.}"'] \arrow[d, dashed, "(*)"'] \arrow[dr, phantom, "\square"] & E_i^{\lambda+r}F_i^{r} = E_i^{\lambda+r-1}EFF_i^{r-1} \arrow[d, "\text{adj-n}"] \\ E_i^{(\lambda+r-1,+)}F_i^{(r-1,-)} \arrow[r, hook, "\text{direct summand}"'] & E_i^{\lambda+r-1}F_i^{r-1} \end{tikzcd}$$

(*) $e_{+,\lambda+r} = e_{+,\lambda+r-1}\cdot$ smth, $e_{-,r} =$ smth $\cdot\, e_{-,r-1}$

- get complex $d^2=0$: $e_{+,\lambda+r} =$ smth $\cdot\, e_{+,2}$, $e_{-,r} = e_{-,2}\cdot$ smth

d^2 incl-s mult-n by $e_{+,2}\cdot e_{-,2} = 0$

Thm: $\Theta_\lambda: D^b(\mathcal{C}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{C}_\lambda)$

3.4) Crystals

Questions: 1) $L \in Irr\,\mathcal{C}$, what can be said about E_iL, F_iL, or gen. $E_i^{(k)}L, F_i^{(k)}L$

~~2) Can we split E_i, F_i further~~

2) int-le $\mathfrak{g}(I) \rightsquigarrow$ crystal (combin. object). Can we see it from categ-n?

3) Why is $[\bigoplus_{n\geq 0} Rep(\mathbb{F}\mathfrak{S}_n)]$ irred-le?

3.4.1) A to Q1.

Reminder: $M \in \mathcal{C} \rightsquigarrow head(M) =$ max s/s quot.

$soc(M) =$ max s/s sub

~~$i \rightsquigarrow m_i \subset P_n$ - max ideal;~~

Thm: $L \in Irr\,\mathcal{C}$, n - max w. $E_i^nL \neq 0$, $j \leq n$

(a) $head(E^{(j)}L) \simeq soc(E^{(j)}L)$ is simple, say L'.

b) $H_j^{aff}(q) \twoheadrightarrow \mathrm{End}(E_i^j L)$, ker is gen-d by $P_j^{\mathfrak{S}_j} \cap \mathfrak{m}_n P_n$, w. $P_j \hookrightarrow P_n$ as usual & $\mathfrak{m}_n \subset P_n^{\mathfrak{S}_n}$ - ideal corresp to $(i,i,\dots i)$.

c) mult. of L' in $E_i^{(j)}L$ is $\binom{n}{j}$, for $\forall\, L'' \neq L'$ ~~~~ ext. $E_i^{(j)}L$ have $E_i^{n-j}L'' = 0$ ($\Rightarrow E_i^{n-j}L' \neq 0$)

Similar for F_i

Rem. b) $\Rightarrow$ $\mathrm{End}(E_i^{(j)}L) = P_j^{\mathfrak{S}_j} / P_j^{\mathfrak{S}_j} \cap \mathfrak{m}_n P_n$

Example: $j=1$ $\mathrm{End}(EL) = \mathbb{C}[X]/(X^n)$, X acts a Jordan block on n copies of L'

3.4.2) A to Q2: crystals & perfect bases

Maps $\tilde{e}_i, \tilde{f}_i : \mathrm{Irr}\, \mathcal{C} \longrightarrow \mathrm{Irr}\, \mathcal{C} \sqcup \{0\}$

$\tilde{e}_i L = \begin{cases} \mathrm{soc}(E_i L) = \mathrm{head}(E_i L), & \cancel{E_i L \neq 0} \\ \nexists & \cancel{E_i L = 0} \end{cases}$, $\tilde{f}_i L$ - similar

Exercise: $\tilde{e}_i L \neq 0 \Rightarrow \tilde{f}_i \tilde{e}_i L = L$ - exercise (hint: biadj-s + Chuang-Rouquier thm)

Cor of c: $(\mathrm{Irr}\,\mathcal{C}, \tilde{e}_i, \tilde{f}_i, i \in I) \simeq$ Kashiwara's crystal of $[\mathcal{C}]$ - defined using quant. gr-ps but can be seen w/o them - via perfect bases (Berenstein-Kazhdan).

Idea: V - irred $\mathfrak{sl}_2$-mod, $\dim V = n+1$, have basis $v_0, v_1, \dots v_n$ w. $fv_{j-1} = \binom{n}{j} v_j$, $j = 1 \dots n$, $ev_{j+1} = \binom{n}{n-j} v_j$, $j = 0 \dots n-1$.

$\forall$ $\mathfrak{sl}_2$-module $\rightsquigarrow$ operators $\tilde{e}, \tilde{f}$ ~~~~ on set of basis el-ts in $\rightsquigarrow$ (non-unique) basis, call good

$\mathfrak{g}(I)$-action $\rightsquigarrow$ $|I|$ $\mathfrak{sl}_2$-actions but cannot choose basis ~~compat~~ good for all of them

Have weaker result: V - $\mathfrak{g}(I)$-module, $i \in I \rightsquigarrow V_i^{\leq n}$ = sum of all (e_i, h_i, f_i)-submods of dim $\leq n+1$

Def (perfect basis): weight basis $\mathcal{B}$ in V compat w. all filtr-ns $V_i^{\leq n}$ (el-ts lying in $V_i^{\leq n}$ form basis there), & part of $\mathcal{B}$ belong to $V_i^{\leq n}/V_i^{\leq}$

| Rem: Basis of simples in $[\mathcal{C}]$ is perfect

forms good basis there

$\rightsquigarrow \tilde{e}_i, \tilde{f}_i, i\in I, \mathcal{B}\to\mathcal{B}\sqcup\{0\}$

Thm (Berenstein-Kazhdan) • perfect basis exists $\forall V$.

• $\mathcal{B}_1, \mathcal{B}_2$ - 2 perf bases $(\mathcal{B}_1, \tilde{e}_i, \tilde{f}_i) \xrightarrow{\sim} (\mathcal{B}_2, \tilde{e}_i, \tilde{f}_i)$ (and iso to Kashiwara's crystal).

Cor: V is reduc. $\Rightarrow b_1\neq b_2\in\mathcal{B}$ s.t. $\tilde{e}_i b_1 = \tilde{e}_i b_2$ $\forall i\in I$

3.4.3) $[\bigoplus_{n\geq 0} Rep(\mathbb{F}\mathfrak{S}_n)]$ is irred

Proof: $L\in Irr(\bigoplus_{n\geq 0}\cdot)$, $\tilde{e}_i L=0 \Rightarrow E_i L=0 \Rightarrow Res_n^{n-1} L=0 \Rightarrow L\in Rep(\mathbb{F}\mathfrak{S}_0)$

3.5) Cat-l Serre rel-ns

3.5.1) $[e_i,f_i]=h_i$ - Chuang-Rouquier

$\alpha\in\mathbb{Z}\rightsquigarrow$ wt. space $[\mathcal{C}]_\alpha$: $e_if_i - f_ie_i|_{[\mathcal{C}]_\alpha}=\alpha$

So want: $E_iF_i = F_iE_i\oplus id^{\oplus\alpha}, \alpha\geq 0$ (1)

$E_iF_i\oplus id^{\oplus -\alpha} = F_iE_i, \alpha\leq 0$

Have $id \xrightarrow{\eta} F_iE_i$, $E_iF_i \xrightarrow{\varepsilon} id \rightsquigarrow$

$\sigma: E_iF_i \xrightarrow{\eta 1_{E_iF_i}} F_iE_iE_iF_i \xrightarrow{T} F_iE_iE_iF_i \xrightarrow{1_{F_iE_i}\varepsilon} F_iE_i$

$(1) = \sigma\oplus\bigoplus_{j=0}^{\alpha-1}\varepsilon\circ(X^j 1_F)$ - isom from str-re of rep-ns in $H_n^{aff}(q)$-mod$_i$

3.5.2) Rel-ns for E_i's: $i\neq j$: $E_iE_j=E_jE_i$

$i-j$: $E_iE_jE_i \simeq E_i^{(2)}E_j\oplus E_jE_i^{(2)}$

- from rep-n th. of $H_3^{aff}(q)$, where X_1,X_2,X_3 act w. 2 e-vals i & 1-eval j

3.5.3) $E_iF_j\simeq F_jE_i$ $i\neq j$

4) Str-re of cat-l actions

Q: Fin. dim rep-s of s/s Lie alg-s are compl. reduc & simply are classif. Same for highest wt integrable rep-s of gen. $\mathfrak{g}(I)$

To what ext. does this hold for cat-l actions

A: 1) Can construct cat-ns of irreps - from cyclot HA

2) Unique under minimality assumption

3) In gen. cat-l action is not compl. reduc, but have filtr-n

4.1) Cyclot. HA & cat-ns

I-graph as before, $\omega = \sum_{i \in I} m_i \omega_i$ - highest wt $\rightsquigarrow$ irred. L^ω w. h.wt ω

Goal: cat-n $\mathcal{C}$ of L^ω = $\mathcal{C}$ + cat-l $\mathfrak{g}(I)$-action w. $[\mathcal{C}] = L^\omega$

First consider $\mathfrak{g}(I) = \hat{\mathfrak{sl}}_p$ or $\mathfrak{gl}_\infty$

Def-n: cyclot HA (a.k.a. Ariki-Koike algebra) $\mathcal{H}_q^\omega(n) = \dfrac{\mathcal{H}_q^{aff}(n)}{\prod_{i \in I} (X_1 - i)^{m_i}}$

Facts: • $\mathbb{C} = \mathcal{H}_q^\omega(0) \hookrightarrow \mathcal{H}_q^\omega(1) \hookrightarrow \mathcal{H}_q^\omega(2) \hookrightarrow \ldots \hookrightarrow \mathcal{H}_q^\omega(n-1) \hookrightarrow \mathcal{H}_q^\omega(n) \hookrightarrow$

(naive homom-s)

• $X_1^{j_1} \ldots X_n^{j_n} T_w$ - basis, $0 \le j_i \le m := \sum_{i \in I} m_i$, $w \in S_n$

• symmetric alg: $\mathcal{H}_q^\omega(n) \underset{bimod}{\cong} \mathcal{H}_q^\omega(n)^*$

via $tr: \mathcal{H}_q^\omega(n) \longrightarrow \mathbb{C}$

$tr(X_1^{j_1} \ldots X_n^{j_n} T_w) = \delta_{j_1,0} \delta_{j_2,0} \ldots \delta_{j_n,0} \delta_{w,1}$

Ind = Coind

$\mathcal{C}^\omega := \bigoplus_{n \ge 0} \mathcal{H}_q^\omega(n)\text{-mod}$, $E = \bigoplus_{n \ge 0} Res_n^{n-1}$, $F = \bigoplus_{n \ge 0} Ind_n^{n+1}$

$X \in End(E)$, $T \in End(E^2)$ using $\mathcal{H}_q^{aff}(n) \twoheadrightarrow \mathcal{H}_q^\omega(n)$, $X \rightsquigarrow E = \bigoplus_{i \in I} E_i$

Wt. decomp $\mathcal{C}^\omega = \bigoplus_\mu \mathcal{C}_\mu^\omega$ - from action of center

Q: How to see $[E_i], [F_i] \curvearrowright [\mathcal{C}^\omega] \rightsquigarrow \mathfrak{g}(I) \curvearrowright [\mathcal{C}^\omega]$

A: via deform-n (of q & factors of cyclot polyn - as in symm gr, case)

Q: What's rep-n $\mathfrak{g}(I) \curvearrowright [\mathcal{C}^\omega]$

A: irred - same reasons as for $[\bigoplus_{n\geq 0} Rep(\mathbb{F}\mathfrak{S}_n)]$

. highest wt: $\mathbb{1}\in Rep(\mathcal{H}_q^\omega(0)=\mathbb{C})$, [redacted] $F\mathbb{1} = \mathcal{H}_q^\omega(1) = \mathbb{C}[X_1]/\prod_{i\in I}(X_1-i)^{m_i}$

$F_i\mathbb{1} = \mathbb{C}[X_1]/(X_1-i)^{m_i} \Rightarrow EF_i\mathbb{1} = E_iF_i\mathbb{1} = \mathbb{1}^{\oplus m_i} \Rightarrow h_i[\mathbb{1}] = m_i[\mathbb{1}]$

$\Rightarrow$ h. wt $=\omega$.

Rem: 1) What about $\hat{\mathfrak{sl}}_p$?

A: take blocks - wt. subcat-s $\underbrace{\overset{0}{\circ} - \overset{1}{\circ} \ldots - \overset{p-1}{\circ}}_{\omega \text{ supp. here}}$

$\rightsquigarrow \mathcal{H}_q^\omega(n) \rightsquigarrow \underline{\mathcal{C}}^\omega(n) = \{M\in\mathcal{C}^\omega(n) \mid X_1\ldots X_n$ act w. e-values $\in\{0,1,\ldots,p-1\} \subset I\subset\mathbb{C}\} \rightsquigarrow \underline{\mathcal{C}}^\omega = \bigoplus_{n\geq 0}\underline{\mathcal{C}}^\omega(n)$ (-fin. sum) - categorif. L^ω for $\hat{\mathfrak{sl}}_p$.

2) Dependence on q? - no - algebras $\mathcal{H}_q^\omega(n)$ are isom (depend on ω, not on q) - via isom w. KLR-alg-s (cyclot. quotients of)

4.2) Uniqueness of min cat-ns

Q: Is cat-n of L^ω unique?

A: no b/c cat-n of $\mathbb{C}$ (w/o actions) in not unique (but is unique if s/s - Vect)

Def: Cat-n $\mathcal{C}$ of L^ω is minimal if $\mathcal{C}_\omega$ = Vect.

Thm (Rouquier): Min-l cat-n is unique

Rem: Have univ cat-n $\mathcal{L}^\omega$ of L^ω, other are obt. by base change.

Prf: Have functor $\mathcal{C}\longrightarrow\mathcal{C}^\omega$

$P = \bigoplus_{k\geq 0} F^k\underset{\in\mathcal{C}_\omega}{\mathbb{1}} \rightsquigarrow Hom(P,\bullet): \mathcal{C}\longrightarrow End(P)^{op}\text{-mod}$

$$\begin{tikzcd} End(F^k\mathbb{1})^{op} & & \mathcal{H}_q^{aff}(n) \arrow[ll] \arrow[dl] \\ & \mathcal{H}_q^\omega(n) \arrow[ul] & \end{tikzcd} \rightsquigarrow End(P)^{op}\text{-mod}\longrightarrow\mathcal{C}^\omega$$

composition is equival

4.3) Filtr-n $\mathfrak{sl}_2$ case: Thm, part c: $d \in \mathbb{Z}_{\geq 0} \rightsquigarrow \mathcal{C}_{\leq d}$ = ~~Serre span of~~ all ~~of~~ obj M s.t. $[M] \in \mathfrak{sl}_2$-submod of dim $\leq d$

$\mathcal{C}_{\leq d}$-stable wrt cat-n functors

General: Λ - wts, Λ^+ - domin. wts for $\mathfrak{g}(I)$

- poset wrt $\leq$: $\lambda \leq \mu$ if $\mu - \lambda$ = sum of posit. roots

- filtr-n wrt poset Λ^+ on $\mathcal{C}$ (Rouquier - warning - has filtr-n in opposit. direction but works w. category $\mathcal{C}$-proj - "dual" to $\mathcal{C}$)

5) Tensor products

V_1, V_2 - $\mathfrak{g}(I)$-modules $\rightsquigarrow V_1 \otimes V_2$ - $\mathfrak{g}(I)$-module

$\mathcal{C}_1, \mathcal{C}_2$ - $\mathfrak{g}(I)$-cat-ns $\rightsquigarrow \mathcal{C}_1 \otimes \mathcal{C}_2$? (shouldn't be symmetric)

Motivation: (cat-n of) quant. knot invar-s (come from tensor products of quant. gr-p rep-ns)

Have seen example: BGG cat. $\mathcal{O}$ cat-s $(\mathbb{C}^{\mathbb{Z}})^{\otimes m}$

$\mathfrak{sl}_n \curvearrowright$ sum of blocks $\rightsquigarrow (\mathbb{C}^n)^{\otimes m}$

Idea (I. L-B. Webster) axiomatic descr-n using add-l struct.

- motiv. by example

5.1) Highest wt structure: $\mathcal{C}$ - abel. artin. $\mathbb{F}$ - lin. cat-y

$\lambda \mapsto L(\lambda)$ $\rightarrow \Lambda \xrightarrow{\sim} \mathrm{Irr}\,\mathcal{C}$, assume $\mathcal{C}$ = sum of cat-s of reps of fin. dim. alg-s

H. wt. str: data + axioms | e.g. enough proj-s

Data: poset str-re on Λ

[$\lambda \in \Lambda \rightsquigarrow \mathcal{C}_{\leq \lambda}$ = Serre span of $L(\mu), \mu \leq \lambda$

$\mathcal{C}_{<\lambda}$

$\rightsquigarrow \mathcal{C}_\lambda := \mathcal{C}_{\leq\lambda} / \mathcal{C}_{<\lambda} \xleftarrow{\pi} \mathcal{C}_{\leq\lambda} \rightsquigarrow$ left adj $\mathcal{C}_\lambda \xrightarrow{\pi_\lambda^!} \mathcal{C}_{\leq\lambda}$

$P(\lambda)$ = proj cover of $L(\lambda)$]

Axioms: 1) $\mathcal{C}_\lambda \simeq \mathrm{Vect}$

$\Delta(\lambda) := \pi_\lambda^!$ (index in $\mathcal{C}_\lambda$)

2) $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ & ker has filtr. w. quot-s $\Delta(\mu), \mu > \lambda$.
Example: $\mathcal{O}$: $\Lambda = \mathbb{Z}^n$ - wt. lattice for $\mathfrak{gl}_n$, order as before
$\Delta(\lambda)$ - Verma ($\pi_\lambda^!$(indec) has univ. property; $\mathcal{C}_{\leq\lambda}$ - all modules w. wts $\leq \lambda$).

5.2) Stand. stratif. cat-s: $\Lambda \xrightarrow{\sim} \mathrm{Irr}\,\mathcal{C}$, now equip Λ w. pre-order (equiv. classes may have ≥ 1 el-t), $\Xi = \Lambda/\sim$
$\xi \in \Xi \rightsquigarrow \mathcal{C}_{\leq\xi}, \mathcal{C}_{<\xi}, \mathcal{C}_\xi, \pi_\xi, \pi_\xi^! : \mathcal{C}_\xi \to \mathcal{C}_{\leq \xi}$.
Axioms: 1) $\pi_\xi^!$ is exact.
$\lambda \mapsto \xi$, $\Delta(\lambda) = \pi_\xi^!$(proj. cover in $\mathcal{C}_\xi$ of simple in $\mathcal{C}_\xi$ corresp. to λ)
2) As before
Opposite cases: part. order $\rightsquigarrow$ h.wt. cat-y
pre-order w. single equiv. class $\rightsquigarrow$ no addit. str-re
Rem: stand. stratif. str-re on $\mathcal{C} \rightsquigarrow \mathrm{gr}\,\mathcal{C} = \bigoplus_\xi \mathcal{C}_\xi$
5.3) $\mathcal{C}_1, \mathcal{C}_2$ - $\mathfrak{g}(I)$-cat-ns
(prelim.) Def (of $\mathcal{C}_1 \otimes \mathcal{C}_2$) - stand. stratif. cat-y + cat-l $\mathfrak{g}(I)$-action s.t.
1) $\Xi = \{(\mu_1, \mu_2) \mid \mu_i$ - wt of $[\mathcal{C}_i]\}$, $(\mu_1, \mu_2) \geq (\mu_1', \mu_2')$ if $\mu_1 + \mu_2 = \mu_1' + \mu_2'$ & $\mu_1 \geq \mu_1'\}$
2) $\mathrm{gr}\,\mathcal{C} \xrightarrow{\sim} \mathcal{C}_1 \boxtimes \mathcal{C}_2$.
3) write $\Delta : \mathrm{gr}\,\mathcal{C} \to \mathcal{C}$ for $\bigoplus \pi_\xi^!$
cat-n of $e_i(v_1 \otimes v_2) = e_i v_1 \otimes v_2 + v_1 \otimes e_i v_2$
$E_i \Delta(M_1 \boxtimes M_2)$ has filtr-n w. quot-s $\Delta(E_i M_1 \boxtimes M_2)$ (exer: sub)
$\Delta(M_1 \boxtimes E_i M_2)$ (quot-t)
Simil. for $F_i \Delta(M_1 \boxtimes M_2)$
Simil. def-n for any finite number of cat-s $\mathcal{C}_i$.
Problem: do not know whether exists/is unique except
$\mathcal{C}_1 \ldots \mathcal{C}_n$ - min cat-ns
Existence (constr-n): Webster
Uniqueness: I.L. - Webster

or rather the sum of contents

Comments: (1) $[M_\lambda] \in [Rep(\mathbb{F}G_n)]^\beta$, $\beta = cont(\lambda) \mod p$

$\mathbb{Z}$-form on $EM_\lambda = Res_n^{n-1}(\mathbb{Z}\text{-form } M_\lambda)$

So $[E][M_\lambda] = \bigoplus_\mu [M_\mu]$; since $[E_i][M_\lambda]$ = proj-n of $[E][M_\lambda]$ to $[Res(\mathbb{F}G_n)]^{\beta-i}$

f-la follows

F-la for $[F_i][M_\lambda]$ similar

(2) In def-n of cat-C action one needs to fix one adjointness, say $Id \xrightarrow[\eta]{} FE$ & $EF \xrightarrow[\varepsilon]{} Id$

Also one can give def-n for action on additive cat-s (say $\mathcal{C}$ as before, take $\mathcal{C}$-proj)