ON A THEOREM OF MOHAN KUMAR AND NORI

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Abstract. A celebrated theorem of Suslin asserts that a unimodular row $x_1^{m_1}, \ldots, x_n^{m_n}$ is completable if $m_1 \cdots m_n$ is divisible by $(n-1)!$. Examples due to Suslin and to Mohan Kumar and Nori show that this result is the best possible in all characteristics. We give a new version of the proof of Mohan Kumar and Nori which avoids the need to use Grothendieck's Riemann–Roch theorem or other deep results of algebraic geometry. We also adapt the proof to give examples of stably free modules which are not self dual in all characteristics.

1. Introduction

Let R be a commutative ring and let $(a_1, \ldots, a_n)$ be a unimodular row over R. We write $P(a_1, \ldots, a_n)$ for the kernel of $R^n \xrightarrow{a_1,\ldots,a_n} R$. This is dual to the definition used in [18] and the early sections of [16] but the present usage agrees with that of [8] and of [16, §17]. If necessary we specify R by writing $P_R(a_1, \ldots, a_n)$. Suslin [12] has shown that if $m_\nu > 0$ for all ν then $P(a_1^{m_1}, \ldots, a_n^{m_n})$ is free if $m_1 \ldots m_n \equiv 0 \mod (n-1)!$. while in [15] it was shown that $P(x_1^{m_1}, \ldots, x_n^{m_n})$ over

$$A_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum x_i y_i - 1)$$

is not free if $m_1 \ldots m_n \not\equiv 0 \mod (n-1)!$. Examples due to Suslin himself [13] [14] and, independently, to Mohan Kumar and Nori [16, §17], have extended this result to all characteristics as follows.

Theorem 1.1. *Let R be any non–zero commutative ring and let*

$$A = R[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum x_i y_i - 1).$$

Let $m_\nu > 0$ for all $\nu = 1, \ldots, n$. If $P_A(x_1^{m_1}, \ldots, x_n^{m_n})$ is free over A then $m_1 \ldots m_n \equiv 0 \mod (n-1)!$.

If P is free over A then $P \otimes_R k$ will be free over

$$k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum x_i y_i - 1)$$

where $k = R/\mathfrak{m}$ is a field. Therefore it is sufficient to consider the case where R is a field.

The proofs given by Suslin and by Mohan Kumar and Nori are quite different. Suslin's proof makes use of his theorem that $SK_1(A) = \mathbb{Z}$ is generated by a matrix with first row $x_1, x_2, x_3^2, \ldots, x_n^{n-1}$ while the proof given by Mohan Kumar and Nori makes use of Chern classes and uses a consequence of Grothendieck's Riemann–Roch theorem. [1] The main purpose of the present paper is to give a more elementary proof avoiding the use of this difficult result. Hopefully this will

[1]Unfortunately a sign $(-1)^{i-1}$ was omitted in both equations in the statement of this result in [16, Theorem 13.2].

make this proof accessible to a wider audience. We will, however, use the relations satisfied by the Adams operations. This requires the splitting principle so some basic results on schemes are still needed. I will also show how to adapt the proof of the theorem to extend the main result of [18] to all characteristics. This can also be proved using Suslin's methods as Ravi Rao has shown in [11].

Theorem 1.2. *Let R and A be as in the previous theorem Let n be odd and let $m_\nu > 0$ for all ν. Let $P = P_A(x_1^{m_1}, \dots, x_n^{m_n})$. If $P \approx P^* = \mathrm{Hom}(P, A)$ then $2m_1 \dots m_n \equiv 0 \mod (n-1)!$.*

In [18] this was proved for $R = \mathbb{C}$. We refer to [18] for further discussion of this result. As above it will suffice to prove the theorem for the case where R is a field.

2. λ, γ, and ψ Operations

I will assume familiarity with classical algebraic K–theory but will begin by recalling some standard results on Grothendieck's λ and γ operations and Adams' ψ operations. Standard references for this material are [4], [6], and [10] as well as [1] which considers the topological case.

Let R be a commutative ring and let P be a finitely generated projective R–module. Let $\Lambda^n(P)$ be its n–th exterior power and let $\lambda^n(P) = [\Lambda^n(P)]$ in $K_0(R)$. We set $\Lambda^0(P) = R$ as usual. Since $\Lambda(P \oplus Q) = \Lambda(P) \otimes \Lambda(Q)$ as graded rings we have

$$\Lambda^n(P \oplus Q) = \bigoplus_{p+q=n} \Lambda^p(P) \otimes \Lambda^q(Q).$$

and therefore $\lambda^n(P \oplus Q) = \sum_{p+q=n} \lambda^p(P)\lambda^q(Q)$ Let $\lambda_t(P)$ be the formal power series $\sum_{n=0}^\infty \lambda^n(P)t^n$ in $K_0(R)[[t]]$. Then $\lambda_t(P \oplus Q) = \lambda_t(P)\lambda_t(Q)$ so λ_t is an additive function with values in the abelian group $1 + tK_0(R)[[t]]$ and hence factors through $K_0(R)$ giving a homomorphism $\lambda_t : K_0(R) \to 1 + tK_0(R)[[t]]$. We write $\lambda_t(x) = \sum_{n=0}^\infty \lambda^n(x)t^n$. It follows that $\lambda^n(x + y) = \sum_{p+q=n} \lambda^p(x)\lambda^q(y)$ for all x and y in $K_0(R)$.

In addition to the λ^n, Grothendieck also defines new operations on K_0 by letting $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x)$ and writing $\gamma_t(x) = \sum_{n=0}^\infty \gamma^n(x)t^n$. In particular, $\gamma^0(x) = 1$, $\gamma^1(x) = \lambda^1(x) = x$, and $\gamma^n(x + y) = \sum_{p+q=n} \gamma^p(x)\gamma^q(y)$ for all x and y in $K_0(R)$.

It is not hard to express the relation between the λ^n and the γ^n in the form of finite sums.

Lemma 2.1. *For $n \geq 1$,*

(1) $\gamma^n(x) = \sum_{p=0}^{n-1} \binom{n-1}{p} \lambda^{p+1}(x)$
(2) $\lambda^n(x) = \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^{n-1-q} \gamma^{q+1}(x)$.

Proof. We have

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \sum_{m=1}^\infty \lambda^m(x) \frac{t^m}{(1-t)^m} = 1 + \sum_{m=1}^\infty \sum_{k=0}^\infty \lambda^m(x) t^m \binom{-m}{k} (-1)^k t^k.$$

Now, for $m \geq 1$,

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k} = (-1)^k \binom{m+k-1}{m-1}$$

so

$$\gamma_t(x) = 1 + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\binom{m+k-1}{m-1}\lambda^m(x)t^{m+k}$$

and therefore, for $n \geq 1$,

$$\gamma^n(x) = \sum_{m=1}^{n}\binom{n-1}{m-1}\lambda^m(x) = \sum_{p=0}^{n-1}\binom{n-1}{p}\lambda^{p+1}(x).$$

A similar argument applies to $\lambda_t(x) = \gamma_{\frac{t}{1+t}}(x)$ □

Corollary 2.2. *Let $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ be finite sequences of integers. Either of the equivalent conditions*

(1) $\sum_1^N a_k t(1+t)^{k-1} = \sum_1^N b_k t^k$
(2) $\sum_1^N a_k t^k = \sum_1^N b_k t(t-1)^{k-1}$

implies $\sum_1^N a_k \gamma^k = \sum_1^N b_k \lambda^k$.

Proof. The first condition implies the second by substituting $t-1$ for t while the second implies the first by substituting $t+1$ for t. If the first condition holds then

$$\sum_1^N a_k t(1+t)^{k-1} = \sum_{k=1}^{N}\sum_{p=0}^{k-1} a_k \binom{k-1}{p} t^{p+1}$$

so $b_m = \sum_{k=m}^N a_k \binom{k-1}{m-1}$ and therefore

$$\sum_1^N a_k\gamma^k = \sum_1^N a_k \sum_{m=1}^{k}\binom{k-1}{m-1}\lambda^m = \sum_{m=1}^{N} b_m\lambda^m.$$

□

Remark 2.3. This result can be reinterpreted as follows: The coefficient of λ^k in Lemma 2.1(1) is the coefficient of t^k in $t(1+t)^{n-1}$ and the coefficient of γ^k in (2) is the coefficient of t^k in $t(t-1)^{n-1}$. We can write this symbolically as $\gamma^n = \lambda(1+\lambda)^{n-1}$ and $\lambda^n = \gamma(\gamma-1)^{n-1}$ for $n \geq 1$ with the convention that a k–th power of the symbol λ on the right hand side should be replaced by the operator λ^k and similarly for γ. Therefore a linear combination of the λ^k can be converted to a linear combination of the γ^k by replacing each λ^n by $t(1+t)^{n-1}$, collecting powers of t, and replacing each t^n by γ^n.

For simplicity, I will assume in the remainder of this section that the ring R is connected so that each finitely generated projective module P has a well–defined rank $\operatorname{rk} P$. This induces a map $\epsilon : K_0(R) \to \mathbb{Z}$ with kernel $\widetilde{K}_0(R)$. If $\operatorname{rk} P = n$ then $\operatorname{rk}\Lambda^k(P) = \binom{n}{k}$. In particular $\Lambda^k(P) = 0$ for $k > n$ and $\Lambda^n(P)$ is invertible, i.e. a rank 1 projective, so $\lambda^n(P)$ is a unit in $K_0(R)$. We define $\epsilon\lambda_t(x) = \sum_{n=0}^{\infty}\epsilon\lambda^n(x)t^n$.

Lemma 2.4. $\epsilon\lambda_t(x) = (1+t)^{\epsilon(x)}$.

Proof. This is clear if $x = [P]$. The general case follows since both sides are homomorphisms from $K_0(R)$ to $1 + t\mathbb{Z}[[t]]$ □

Corollary 2.5. $\epsilon\gamma_t(x) = (1-t)^{-\epsilon(x)}$.

This follows by substituting $t/(1-t)$ for t in Lemma 2.4.

Corollary 2.6. *If x lies in $\widetilde{K}_0(R)$ then $\lambda^n(x)$ and $\gamma^n(x)$ also lie in $\widetilde{K}_0(R)$ for $n > 0$.*

Lemma 2.7. *If $x = [P]$ where $\operatorname{rk} P = 1$, then $\lambda_t(x) = 1 + tx$, and $\gamma_t(x) = (1 + t(x-1))/(1-t)$.*

This is clear from the definitions.

Corollary 2.8. *If $x = [P]$ where $\operatorname{rk} P = 1$, then*
(1) *$\lambda^0(x) = 1$, $\lambda^1(x) = x$, and $\lambda^n(x) = 0$ if $n > 1$*
(2) *$\gamma^0(x) = 1$, and $\gamma^n(x) = x$ if $n \geq 1$.*

Lemma 2.9. *Let $x = [P] - [R^n] \in \widetilde{K}_0(R)$. Then $\gamma^i(x) = 0$ for $i > n$ and $\gamma^n(x) = (-1)^n \lambda_{-1}(P) = \sum_{i=0}^n (-1)^{n-i} \lambda^i(P)$.*

Proof.

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(P)/\lambda_{\frac{t}{1-t}}(R)^n = \sum_0^n \lambda^i(P)\frac{t^i}{(1-t)^i}\Big/\frac{1}{(1-t)^n} = \sum_0^n \lambda^i(P)t^i(1-t)^{n-i}$$

□

Corollary 2.10. *$\widetilde{K}_0(R)$ is a nil ideal of $K_0(R)$.*

Proof. If $x \in \widetilde{K}_0(R)$ then $\gamma_t(x)$ and $\gamma_t(-x)$ are polynomials in t whose product is $\gamma_t(0) = 1$ but in a reduced ring a polynomial which divides 1 must be a constant. Since $\gamma^1(x) = \lambda^1(x) = x$, x must be nilpotent. More details may be found in [1, Cor. 3.1.6]. □

The Adams operations are defined by $\psi^0(x) = \epsilon(x)$ and

$$\psi_t(x) = \sum_{n=0}^\infty \psi^n(x)t^n = \psi^0(x) - t\lambda_{-t}(x)^{-1}\frac{d}{dt}\lambda_{-t}(x).$$

The right hand side is usually written as $\psi^0(x) - t\frac{d}{dt}\log \lambda_{-t}(x)$ with the observation that although log introduces denominators, the operation $\frac{d}{dt}\log$ does not.

These operations are additive since $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$ and therefore differentiating and dividing by $\lambda_t(x+y)$ gives $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$.

Lemma 2.11. *$\epsilon\psi_t(x) = \epsilon(x)(1-t)^{-1}$ and therefore $\epsilon\psi^n(x) = \epsilon(x)$ for all n.*

Proof.

$$\epsilon\psi_t(x) = \sum_0^\infty \epsilon(\psi^n(x))t^n = \psi^0(x) - t\epsilon(\lambda_{-t}(x))^{-1}\frac{d}{dt}\epsilon(\lambda_{-t}(x)).$$

The result now follows immediately from Lemma 2.4 and the fact that $\psi^0 = \epsilon$. □

Applying the definition to the rank one case gives us the following result.

Lemma 2.12. *If $x = [P]$ where $\operatorname{rk} P = 1$, then $\psi_t(x) = (1-tx)^{-1}$ so $\psi^0(x) = 1$, and $\psi^n(x) = x^n$ if $n \geq 1$.*

The Adams operations behave particularly well with respect to products and composition. The analogous formulas for the λ and γ operations are given by universal polynomials with coefficients in $\mathbb{Z}$ but these seem to be very complicated and apparently have never been written down explicitly. The following theorem of Adams is the only deep result which we will need. In fact, only (2) is required.

Theorem 2.13 (Adams)**.**

(1) $\psi^n(xy) = \psi^n(x)\psi^n(y)$.
(2) $\psi^p(\psi^q(x)) = \psi^{pq}(x)$.

Proof. According to the splitting principle [4], [5], [6] [10] we can embed $K_0(R)$ in $K_0(X)$ for a scheme X preserving all operations and such that any finite number of specified elements become sums of rank one elements and negatives of such elements. Therefore it is enough to check the relations for elements x and y of this form. This is immediate by Lemma 2.12. □

3. A very special case

In this section we prove some facts applying to the rather special case where $\widetilde{K}_0(R)^2 = 0$ and where $K_0(R)$ is torsion free. As above we assume R is connected so $\epsilon : K_0(R) \to \mathbb{Z}$ is defined.

Lemma 3.1. *If $\widetilde{K}_0(R)^2 = 0$ then $\lambda^n(x+y) = \lambda^n(x) + \lambda^n(y)$ and $\gamma^n(x+y) = \gamma^n(x) + \gamma^n(y)$ for $x, y \in \widetilde{K}_0(R)$ and $n > 0$.*

Proof. Write $\lambda_t(x) = 1 + f_t(x)$ and similarly for y. By Lemma 2.6, $f_t(x)$ and $f_t(y)$ have all coefficients in $\widetilde{K}_0(R)$ so $f_t(x)f_t(y) = 0$. Therefore $1 + f_t(x+y) = (1 + f_t(x))(1 + f_t(y)) = 1 + f_t(x) + f_t(y)$. A similar argument applies to γ. □

Lemma 3.2. *If $\widetilde{K}_0(R)^2 = 0$ then $\psi^n(x) = (-1)^{n+1} n\lambda^n(x)$ for $x \in \widetilde{K}_0(R)$.*

Proof. Let $f_t(x) = \lambda_t(x) - 1 = \sum_1^\infty \lambda^n(x)t^n$ as above. Since $f_t(x)$ has coefficients in $\widetilde{K}_0(R)$ we have $\lambda_{-t}(x)^{-1} = (1 + f_{-t}(x))^{-1} = 1 - f_{-t}(x)$. Therefore

$$\psi_t(x) = 0 - t(1 - f_{-t}(x))\frac{d}{dt}(1 + f_{-t}(x)) = -t\frac{d}{dt}f_{-t}(x) = -t\frac{d}{dt}\sum_1^\infty (-1)^n\lambda^n(x)t^n$$

$$= -\sum_1^\infty (-1)^n \lambda^n(x) n t^n$$

□

Corollary 3.3. *Suppose $\widetilde{K}_0(R)^2 = 0$ and $K_0(R)$ is torsion free. If $x \in \widetilde{K}_0(R)$ then $\lambda^p(\lambda^q(x)) = (-1)^{(p+1)(q+1)}\lambda^{pq}(x)$ for $p, q > 0$.*

Proof. Since $\psi^p(\psi^q(x)) = \psi^{pq}(x)$ by Theorem 2.13 we have

$$(-1)^{p+1}p\lambda^p((-1)^{q+1}q\lambda^q(x)) = (-1)^{pq+1}pq\lambda^{pq}(x).$$

Since $p, q > 0$ and $K_0(R)$ is torsion free we can divide this by pq. □

Proposition 3.4. *Suppose $\widetilde{K}_0(R)^2 = 0$ and $K_0(R)$ is torsion free. If $x \in \widetilde{K}_0(R)$ then*

$$\gamma^n(\gamma^n(x)) = (-1)^{n-1}(n-1)!\gamma^n(x) + \sum_{k=n+1}^\infty a_k\gamma^k(x)$$

for some integers a_k.

Proof. By Lemma 2.1, Corollary 2.6, Lemma 3.1, and Corollary 3.3 we have

$$\gamma^n(\gamma^n(x)) = \sum_{p=0}^{n-1} \binom{n-1}{p} \lambda^{p+1}(\gamma^n(x)) = \sum_{p=0}^{n-1} \binom{n-1}{p} \sum_{q=0}^{n-1} \binom{n-1}{q} \lambda^{p+1}(\lambda^{q+1}(x))$$
$$= \sum_{p=0}^{n-1} \binom{n-1}{p} \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^{pq} \lambda^{(p+1)(q+1)}$$

To express this in terms of the γ^k we use Corollary 2.2. Let

$$S = \sum_{p=0}^{n-1} \binom{n-1}{p} \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^{pq} t(t-1)^{(p+1)(q+1)-1}$$
$$= \sum_{p=0}^{n-1} \binom{n-1}{p} \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^{p+q} t(1-t)^{pq+p+q}$$
$$= \sum_{p=0}^{n-1} \binom{n-1}{p} \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^{p+q} t(1-t)^{(p+1)q}(1-t)^p$$
$$= \sum_{p=0}^{n-1} \binom{n-1}{p} (-1)^p t(1-t)^p \{1-(1-t)^{p+1}\}^{n-1}$$

Now $1-(1-t)^{p+1} = (p+1)t + O(t^2)$ so

$$S = \sum_{p=0}^{n-1} \binom{n-1}{p} (-1)^p (p+1)^{n-1} t^n + O(t^{n+1}).$$

The required result now follows from Corollary 2.2 and the following lemma. □

Lemma 3.5. $\sum_{p=0}^{n-1} \binom{n-1}{p} (-1)^p (p+1)^{n-1} = (-1)^{n-1}(n-1)!$.

Proof. Let

$$f(t) = \sum_{p=0}^{n-1} \binom{n-1}{p} (-1)^p e^{(p+1)t} = e^t(1-e^t)^{n-1} = (-1)^{n-1} t^{n-1} + O(t^n).$$

The lemma follows by differentiating $n-1$ times and setting $t = 0$. □

Theorem 3.6. *Let R be a connected commutative ring with $K_0(R) = \mathbb{Z} \oplus \mathbb{Z}$. Let P be a finitely generated projective R–module with $\operatorname{rk} P = n$. Then $\lambda_{-1}(P) \equiv 0 \mod (n-1)!$.*

Proof. Let $\widetilde{K}_0(R) = \mathbb{Z}\xi$. Then $\xi^2 = r\xi$ for some $r \in \mathbb{Z}$ and therefore $\xi^m = r^{m-1}\xi$ for $m > 0$. By Corollary 2.10, ξ is nilpotent so $r = 0$ and $\xi^2 = 0$. Therefore $\widetilde{K}_0(R)^2 = 0$ and $K_0(R)$ is clearly torsion free. Let $x = [P] - [R^n]$. By Lemma 2.9, $\gamma^n(x) = (-1)^n \lambda_{-1}(P)$ so we have to show that $\gamma^n(x) \equiv 0 \mod (n-1)!$. By Corollary 2.6, $\gamma^n(x)$ lies in $\widetilde{K}_0(R)$ and therefore $\gamma^n(x) = k\xi$ for some $k \in \mathbb{Z}$. If $k = 0$ we are done. Assume $k \neq 0$. By Lemma 2.9, $\gamma^i(x) = 0$ for $i > n$ so by Proposition 3.4, $\gamma^n(\gamma^n(x)) = (-1)^{n-1}(n-1)!\gamma^n(x)$. By Lemma 3.1, $\gamma^n(\gamma^n(x)) = \gamma^n(k\xi) = k\gamma^n(\xi)$. Therefore $k\gamma^n(\xi) = (-1)^{n-1}(n-1)!\gamma^n(x) = (-1)^{n-1}(n-1)!k\xi$. Since we are assuming $k \neq 0$ it follows that $\gamma^n(\xi) = (-1)^{n-1}(n-1)!\xi$. If $x = m\xi$, then $\gamma^n(x) = m\gamma^n(\xi) = (-1)^{n-1}(n-1)!m\xi \equiv 0 \mod (n-1)!$ □

Next we recall some standard facts about Cohen–Macaulay rings. If R is a noetherian local ring and $x \in \mathfrak{m}_R$ then $\dim R - 1 \leq \dim R/(x) \leq \dim R$ and $\dim R/(x) = \dim R - 1$ if x is regular. It follows by induction on n that if $x_1, \dots, x_n \in \mathfrak{m}_R$ then $\dim R - n \leq \dim R/(x_1, \dots, x_n) \leq \dim R$ and also that $\dim R/(x_1, \dots, x_n) = \dim R - n$ if $x_1, \dots, x_n$ is a regular sequence.

Lemma 3.7. *If R is a Cohen–Macaulay local ring and $x_1, \dots, x_n \in \mathfrak{m}_R$ then $\dim R/(x_1, \dots, x_n) = \dim R - n$ if and only if $x_1, \dots, x_n$ is a regular sequence. If so, $R/(x_1, \dots, x_n)$ is again Cohen–Macaulay.*

This is proved for $n = 1$ in [17, Lemma 8.6] and the general case follows by induction. A more general version is given in [3, Th. 2.1.2(c)].

Corollary 3.8. *Let R be a Cohen–Macaulay local ring and let $I = (x_1, \dots, x_n)$ be an ideal with $\operatorname{ht} I \geq n$. Then $x_1, \dots, x_n$ is a regular sequence.*

Proof.

$$\dim R/(x_1, \dots, x_n) = \dim R/I \leq \dim R - \operatorname{ht} I \leq \dim R - n.$$

Since $\dim R/(x_1, \dots, x_n) \geq \dim R - n$, we have $\dim R/(x_1, \dots, x_n) = \dim R - n$ and the lemma shows that $x_1, \dots, x_n$ is a regular sequence. $\square$

Let R be a commutative ring, let P be an R–module and let $f : P \to R$. The Koszul complex $\operatorname{Kosz}(f)$ of f is defined to be the exterior algebra $\Lambda(P)$ with its usual grading and with $d : \Lambda^{k+1}(P) \to \Lambda^k(P)$ by $d(p_0 \wedge \dots \wedge p_k) = \sum_{m=0}^{k} (-1)^m f(p_m) p_0 \wedge \dots \wedge \widehat{p_m} \wedge \dots \wedge p_k$. This has the augmentation $\epsilon : \Lambda^0(P) = R \to R/\operatorname{im} f$.

Lemma 3.9. *Let R be a Cohen–Macaulay ring. Let I be an ideal of R with $\operatorname{ht} I \geq n$ and let P be a finitely generated projective R–module of rank $\leq n$. If there is an epimorphism $f : P \twoheadrightarrow I$ then $\operatorname{Kosz}(f)$ is a projective resolution of R/I.*

Proof. It is sufficient to check this locally. After locallizing at a prime ideal, P will become free with base $e_1, \dots, e_r$ mapping to a set of generators $a_1, \dots, a_r$ of I and the Koszul complex localizes to the usual Koszul complex $K(a_1, \dots, a_r)$. If I localizes to R, $a_1, \dots, a_r$ will be a unimodular row so $K(a_1, \dots, a_r)$ will be exact and therefore a resolution of $R/I = 0$ (locally). Otherwise the localization of I will be a proper ideal of height at least n so $r \geq n$ but since $\operatorname{rk} P \leq n$, $r \leq n$ and therefore $r = n$. By Corollary 3.8, $a_1, \dots, a_n$ will be a regular sequence and therefore the Koszul complex will be a resolution of R/I. $\square$

The following is the main result of this section.

Corollary 3.10. *Let R be a Cohen–Macaulay ring with $K_0(R) = \mathbb{Z} \oplus \mathbb{Z}$. Let I be an ideal of R with $\operatorname{ht} I \geq n$ and let P be a finitely generated projective R–module of rank n. If there is an epimorphism $f : P \twoheadrightarrow I$ then R/I has finite projective dimension so $[R/I]$ is defined in $K_0(R)$ and $[R/I] \equiv 0 \mod (n-1)!$ in $K_0(R)$.*

Proof. The previous lemma shows that R/I has finite projective dimension and that $\lambda_{-1}(P) = [R/I]$ in $K_0(R)$. The final statement follows from Theorem 3.6. $\square$

4. A Patching Lemma

The remainder of the proof is essentially the same as the original proof of Mohan Kumar and Nori. I will give here a slightly more general version of one of their lemmas which will also be useful in proving Theorem 1.2. We use the notation R_s for the localization $R[s^{-1}]$.

Lemma 4.1. *Let R be a commutative ring and let M be a finitely generated R–module. Let $R = Ra + Rb$ and let P and Q be finitely generated projective of rank n over R_a and R_b respectively. Suppose we have resolutions*

$$0 \to L \to P \to M_a \to 0$$

and

$$0 \to N \to Q \to M_b \to 0$$

over R_a and R_b. If

$$0 \to L_b \to P_b \to M_{ab} \to 0$$

and

$$0 \to N_a \to Q_a \to M_{ab} \to 0$$

split (e.g. if M_{ab} is projective over R_{ab}) and if $L_b \approx N_a$ over R_{ab}, then there is a finitely generated projective R–module S of rank n with an epimorphism $S \twoheadrightarrow M$.

Proof. Since $P_b \approx L_b \oplus M_{ab}$ and $Q_a \approx N_a \oplus M_{ab}$ we can use the isomorphism $L_b \approx N_a$ to get $P_b \approx Q_a$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_b \arrow[r] \arrow[d, "\approx"] & P_b \arrow[r] \arrow[d, "\approx"] & M_{ab} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_a \arrow[r] & Q_a \arrow[r] & M_{ab} \arrow[r] & 0
\end{tikzcd}$$

From this we get

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & Q_a \arrow[r, "\approx"] \arrow[d] & P_b \arrow[d] & P \arrow[l] \arrow[d] \\
M_b \arrow[r] & M_{ab} \arrow[r, equal] & M_{ab} & M_a \arrow[l]
\end{tikzcd}$$

The pullback S of the upper line maps to the pullback M of the bottom line. By localizing we see that $S_a \approx P$ and $S_b \approx Q$ showing that S is projective of rank n. The map $S \to M$ is onto since it localizes to $S_a = P \to M_a$ and $S_b = Q \to M_b$. $\square$

5. A Useful Ring

The proof of Mohan Kumar and Nori makes use of the following auxiliary ring. Let $B = B_n = k[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum x_i y_i - z(1-z))$ where k is a field. We have $B_0 = k \times k$ and B_n is a domain for $n > 0$ since the polynomial $f = \sum x_i y_i - z(1-z)$ is irreducible. Also x_i and y_i are non–zero in B_n since f does not divide x_i or y_i. As above, we use the notation R_s for the localization $R[s^{-1}]$.

Lemma 5.1. *Let $B = B_n$.*

1. $B_z = k[x_1, \ldots, x_n, u_1, \ldots, u_n]_{1-\sum x_i u_i}$ *where* $u_i = y_i/z$.
2. $B_{1-z} = k[x_1, \ldots, x_n, v_1, \ldots, v_n]_{1-\sum x_i v_i}$ *where* $v_i = y_i/(1-z)$.

Proof. Letting $u_i = y_i/z$ we can write

$$B_z = k[x_1, \ldots, x_n, u_1, \ldots, u_n, z, z^{-1}]/(\sum x_i u_i - (1-z)).$$

Therefore $z = 1 - \sum x_i u_i$ can be eliminated provided we ensure its invertibility. A similar argument applies to (2) where $z = \sum x_i v_i$. $\square$

Corollary 5.2. *B_n is a regular domain for $n \geq 1$.*

Proof. It is sufficient to prove the regularity locally so it is enough to show that B_z and B_{1-z} are regular. This is clear from the lemma. $\square$

Remark 5.3. The fact that B_n is a domain for $n \geq 1$ can also be deduced from the lemma using the following elementary result.

Lemma 5.4. *Let R be a commutative ring and let $R = \sum Rs_i$. If all R_{s_i} are domains and if s_1 maps to a non–zero element of R_{s_i} for all i, then R is a domain.*

Proof. We claim that $R \to R_{s_1}$ is injective. It is enough to prove this locally by considering the maps $R_{s_i} \to R_{s_1 s_i}$. These are injective since we are localizing a domain R_{s_i} at a non–zero element s_1. $\square$

The following calculation is due to J. P. Jouanolou [7] who also considered the case of higher K–theory. The proof here uses only classical K–theory.

Proposition 5.5. $\widetilde{K}_0(B_n) = \mathbb{Z}$ *generated by* B_n/I_n *where* $I_n = (x_1, \ldots, x_n, z)$.

For the proof we will use the following localization sequence, a special case of [2, Ch. IX, Th. 6.5].

Lemma 5.6. *Let R be a commutative regular ring and let s be a non–zero divisor in R. Then the sequence*

$$K_1(R) \to K_1(R_s) \xrightarrow{\partial} G_0(R/(s)) \to K_0(R) \to K_0(R_s) \to 0$$

is exact and the map ∂ is given by sending $\alpha \in M_n(R_s)$ to $[\mathrm{ckr}(s^m\alpha)] - [R^n/s^m R^n]$ for any m such that $s^m\alpha$ lies in $M_n(R)$.

Proof. The standard localization sequence [2, Ch. IX, Th. 6.3] has the form

$$K_1(R) \to K_1(R_s) \xrightarrow{\partial} K_0(\mathcal{H}) \to K_0(R) \to K_0(R_s)$$

where the map ∂ is as in the lemma and where $\mathcal{H}$ is the category of finitely generated R–modules M such that $M_s = 0$ and $pd_R(M) < \infty$. Since R is regular the second condition is always satisfied so $K_0(\mathcal{H})$ is K_0 of the category of finitely generated R–modules M with $M_s = 0$. By devissage [2, Ch. VIII, Th. 3.3] this is equal to $G_0(R/(s))$. We can add a zero on the right since $K_0(R) = G_0(R)$ and similarly for R_s. $\square$

If $R/(s)$ is also regular we can replace $G_0(R/(s))$ by $K_0(R/(s))$.

Proof of Proposition 5.5. The result clearly holds for $B_0 = k \times k$. We use induction on n. We apply Lemma 5.6 with $R = B_n$ and $s = x_n$ getting

$$K_1(B_n) \to K_1((B_n)_{x_n}) \to K_0(B_n/(x_n)) \to K_0(B_n) \to K_0((B_n)_{x_n}) \to 0$$

Note that $B_n/(x_n) = B_{n-1}[y_n]$ which is also regular. Now

$$(B_n)_{x_n} = k[x_1, \ldots, x_n, x_n^{-1}, y_1, \ldots, y_{n-1}, z]$$

so by standard K-theoretic calculations [2, Ch. XII] we have $K_0((B_n)_{x_n}) = \mathbb{Z}$ and $K_1((B_n)_{x_n}) = k^* \times \mathbb{Z}$ where the $\mathbb{Z}$ is generated by $x_n \in U((B_n)_{x_n})$. It follows that $\ker[K_0(B_n) \to K_0((B_n)_{x_n})]$ is just $\widetilde{K}_0(B_n)$ and, since k^* is in the image of $K_1(B_n)$, the image of $K_1((B_n)_{x_n}) \to K_0(B_n/(x_n))$ is generated by the image of x_n which is $[B_n/(x_n)]$ and therefore the cokernel of $K_1((B_n)_{x_n}) \to K_0(B_n/(x_n))$ is $\widetilde{K}_0(B_n/(x_n))$. It follows that $\widetilde{K}_0(B_n/(x_n)) \xrightarrow{\approx} \widetilde{K}_0(B_n)$. Now since $B_n/(x_n) =$

$B_{n-1}[y_n]$ and B_{n-1} is regular, we have $K_0(B_{n-1}) \xrightarrow{\approx} K_0(B_n/(x_n))$. This takes the generator $[B_{n-1}/I_{n-1}]$ of $\widetilde{K}_0(B_{n-1})$ to

$$[B_{n-1}/I_{n-1} \otimes_{B_{n-1}} B_n/(x_n)] = [B_n/(x_n, I_{n-1})] = [B_n/(x_1, \dots, x_{n-1}, z, x_n)]$$

This is just $[B_n/I_n]$ so this element generates $\widetilde{K}_0(B_n)$. □

Next we investigate some useful ideals of B_n. The following simple observation will be useful.

Lemma 5.7. *If I is an ideal of a commutative ring R and if $b \in R$ is a unit modulo I then $ab \in I$ implies $a \in I$.*

The proof is immediate: If $bc \equiv 1 \mod I$ then $a \equiv abc \equiv 0 \mod I$.

Lemma 5.8. *If $N \geq \sum m_i$ then $z^N(1-z)^N$ lies in the ideals $(x_1^{m_1}, \dots, x_n^{m_n})$ and $(y_1^{m_1}, \dots, y_n^{m_n})$.*

Proof. If $N \geq \sum m_i$, then in $z^N(1-z)^N = (\sum x_i y_i)^N$ each term in the expansion of $(\sum x_i y_i)^N$ will contain a power at least m_i of x_i for some i. Therefore $z^N(1-z)^N$ lies in $(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})$ and similarly it lies in $(y_1^{m_1}, \dots, y_n^{m_n})$ □

Lemma 5.9. *The ideal $(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}, z^N)$ of B_n is independent of N for $N \geq \sum m_i$. Also $(x_1, x_2, \dots, x_n, z^N) = (x_1, x_2, \dots, x_n, z)$ for all $N \geq 1$.*

Proof. Suppose $M \geq \sum m_i$. Then, by Lemma 5.8, $z^M(1-z)^M$ lies in the ideal $(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}, z^N)$ and therefore, by Lemma 5.7 z^M lies in this ideal. The same applies with M and N interchanged. For the last statement, the same argument applies with $M, N \geq 1$ since $z(1-z) = \sum x_i y_i$ so $z(1-z)$ lies in $(x_1, x_2, \dots, x_n, z^N)$. □

Define $J_{m_1, m_2, \dots, m_n}$ to be the ideal considered in Lemma 5.9. In particular, $J_{1,1,\dots,1} = I$, the ideal considered in Proposition 5.5.

Lemma 5.10. *$J_{m_1, m_2, \dots, m_n}/J_{m_1+1, m_2, \dots, m_n} \approx B_n/J_{1, m_2, \dots, m_n}$ and similarly for each m_i.*

Corollary 5.11. *$[B_n/J_{m_1, m_2, \dots, m_n}] = m_1 \cdots m_n [B_n/I]$ in $K_0(B_n)$.*

This follows by induction on the m_i since $J_{1,\dots,1} = I$.
For the proof of the lemma we use the following.

Lemma 5.12. *Let I be an ideal of a commutative ring R such that*

$$R/I \approx A[X]/(X^{m+1} f)$$

where $A[X]$ is a polynomial ring in one variable and $f = f(X) \in A[X]$. Let $x \in R$ map to X modulo I. Then

$$(x^m, I)/(x^{m+1}, I) \approx R/(x, I).$$

Proof. $(x^m, I)/(x^{m+1}, I)$ is generated by x^m and $(x, I)x^m \subseteq (x^{m+1}, I)$. We must show that $rx^m \in (x^{m+1}, I)$ implies $r \in (x, I)$. Now $rx^m = ax^{m+1} + i$ implies $tx^m \in I$ where $t = r - ax$ and we need to show that t lies in (x, I). Let $\bar{t} = t \mod I$. Then $\bar{t}X^m = 0$ in $A[X]/(X^{m+1} f)$ but in $A[X]/(X^{m+1} f)$ the annihilator of X^m lies in (X) since $hX^m \in (X^{m+1} f)$ implies $h \in (Xf)$. Therefore $\bar{t}$ lies in (X) in R/I and hence t lies in (x, I) as required. □

Proof of Lemma 5.10. Since $J_{m_1,m_2,\dots,m_n} = (x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}, z^N)$ is independent of N for large N we can choose $N > \sum m_i$. We apply the lemma with $R = B_n$, $x = x_1$, $m = m_1$, and $I = (x_2^{m_2}, \dots, x_n^{m_n}, z^N)$. In $B_n/(z^N)$, $1-z$ is a unit so by Lemma 5.1

$$R/I = R_{1-z}/I_{1-z} = k[x_1, \dots, x_n, v_1, \dots, v_n]/(x_2^{m_2}, \dots, x_n^{m_n}, (\sum x_i v_i)^N)$$

where we have substituted $\sum x_i v_i$ for z. Let

$$A = k[x_2, \dots, x_n, v_1, \dots, v_n]/(x_2^{m_2}, \dots, x_n^{m_n}).$$

Then $R/I = A[x_1]/((\sum x_i v_i)^N)$. Since $N > \sum m_i$, if we multiply out $(\sum x_i v_i)^N$, each term will be divisible by some $x_i^{m_i+1}$. Except for $i = 1$ all such terms will be 0 in $A[x_1]$. Therefore, in $A[x_1]$, $(\sum x_i v_i)^N$ will have the form $x_1^{m_1+1} f(x_1)$. Therefore Lemma 5.12 applies and finishes the proof. □

6. Proof of Theorem 1.1

Let $B = B_n$ and let $J = J_{m_1,m_2,\dots,m_n}$. Since $J = (x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}, z^N)$ we have $J_z = B_z$. Since $z^N(1-z)^N$ lies in $(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})$ by Lemma 5.8, Lemma 5.7 shows that $J_{1-z} = (x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})B_{1-z}$. Map B_{1-z}^n with base e_i to J_{1-z} by sending e_i to $x_i^{m_i}$ and map B_z to $J_z = B_z$ by sending e_1 to 1 and e_i to 0 for $i > 1$. We get short exact sequences

$$0 \to L \to B_{1-z}^n \to J_{1-z} \to 0$$

and

$$0 \to N \to B_z^n \to J_z \to 0.$$

These split when localized to $B_{z(1-z)}$ since $J_{z(1-z)} = B_{z(1-z)}$. Since

$$J_{z(1-z)} = (x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})B_{z(1-z)} = B_{z(1-z)}$$

we see that $L_z = P(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})$ over $B_{z(1-z)}$. This is induced from the module $P = P(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})$ over the ring

$$A = R[x_0, \dots, x_n, y_0, \dots, y_n]/(\sum x_i y_i - 1)$$

via the map $A \to B_{z(1-z)}$ sending x_i to x_i and y_i to $y_i/z(1-z)$. Since N_{1-z} is free it follows that if P is free then $L_z \approx N_{1-z}$. Therefore Lemma 4.1 applies and shows that there is an epimorphism $Q \twoheadrightarrow J$ where Q is finitely generated projective of rank n. By Corollary 3.10 we see that $[B/J] \equiv 0 \mod (n-1)!$ in $\widetilde{K}_0(B)$. Since $[B/J] = m_1 \cdots m_n [B_n/I]$ by Corollary 5.11 and $\widetilde{K}_0(B) = \mathbb{Z}$ generated by $[B/I]$, it follows that $m_1 \cdots m_n \equiv 0 \mod (n-1)!$ if P is free.

7. A Duality Theorem

The next two sections contain preliminary results needed in the proof of Theorem 1.2. We first recall a standard result.

Lemma 7.1. *Let R be a commutative ring and let $\sum_1^n a_i b_i = 1$ in R. Then $P(a_1, \dots, a_n)^* \approx P(b_1, \dots, b_n)$.*

Proof. We abbreviate $a_1, \dots, a_n$ to a and let $a \cdot b = \sum a_i b_i$. We have $R^n = P(a) \oplus Rb = P(b) \oplus Ra$ because $z \in R^n$ has the form $z = w + rb$ with $a \cdot w = 0$ if and only if $r = z \cdot a$. The bilinear form $(x, y) \mapsto x \cdot y$ induces a pairing $P(a) \times P(b) \to R$ and therefore gives a map $P(b) \to P(a)^*$. This is injective since if y maps to 0 then $y \cdot z = 0$ for all z in $P(a)$. Since $y \cdot b = 0$ it follows that $y \cdot R^n = 0$ so $y = 0$. If $f : P(a) \to R$, extend f to R^n by $f(b) = 0$. Then $f(z) = c \cdot z$ for some c in R^n. Since $f(b) = c \cdot b = 0$, c lies in $P(b)$ and maps to f showing that our map is onto. □

Proposition 7.2. *Let R be a commutative ring and let $\sum_1^n x_i y_i = 1$ in R. Let $m_1, \dots, m_n$ be positive integers. Then $P(x_1^{m_1}, \dots, x_n^{m_n})^* \approx P(y_1^{m_1}, \dots, y_n^{m_n})$.*

Proof. By a theorem of Suslin [12], if $m = m_1 \dots m_n$ then $P(x_1^{m_1}, \dots, x_n^{m_n}) \approx P(x_1^m, x_2, \dots, x_n)$ so it will suffice to treat the case where $m_1 = m$ and $m_i = 1$ for $i > 1$. A new proof of this result of Suslin was given by Mohan Kumar. An account of this proof is given by Mandal in [9, Lemma 5.3.1]. The following proof is based on this idea. We can assume that $n > 2$ since otherwise $P(x_1^m, x_2)$ is free, being stably free of rank 1. Let

$$z = 1 + x_1 y_1 + \cdots + (x_1 y_1)^{m-1}.$$

Then

$$x_1^m y_1^m + z \sum_2^n x_i y_i = x_1^m y_1^m + z(1 - x_1 y_1) = 1$$

so

$$P(x_1^m, x_2, \dots, x_n)^* \approx P(y_1^m, z y_2, \dots, z y_n)$$

by Lemma 7.1. Let

$$z_t = 1 + t x_1 y_1 + \cdots + t^{m-1} (x_1 y_1)^{m-1}.$$

Then $P_t = P(y_1^m, z_t y_2, \dots, z_t y_n)$ over $R[t]$ is extended. To see this we can assume that R is local by Quillen's patching theorem. Therefore one of the y_i is a unit. If y_1 is a unit then P_t is free. Suppose y_2 is a unit. Then by an elementary transformation we can replace $z_t y_3$ by $z_t y_3 - y_2^{-1} y_3 (z_t y_2) = 0$ and therefore P_t is free. It follows that $P_1 = P(y_1^m, z y_2, \dots, z y_n) \approx P(x_1^m, x_2, \dots, x_n)^*$ is isomorphic to $P_0 = P(y_1^m, y_2, \dots, y_n)$ as required. □

8. Automorphisms

We determine some automorphisms of $\widetilde{K}_0(B_n)$ using the following simple fact.

Lemma 8.1. *Let f be an automorphism of a commutative noetherian ring R. Then $f_* : G_0(R) \to G_0(R)$ sends $[R/I]$ to $[R/f(I)]$.*

Proof. More generally let $f : R \to R'$ where R' is finitely generated and flat as an R–module. Then $f_* : G_0(R) \to G_0(R)$ sends $[R/I]$ to $[R' \otimes_R R/I] = [R'/R' f(I)]$. □

Let $n \geq 1$ and let

$$B = B_n = k[x_1, \dots, x_n, y_1, \dots, y_n, z]/(\sum x_i y_i - z(1 - z))$$

as above. Let α_i be the automorphism of B which interchanges x_i and y_i and fixes the remaining x_j and y_j as well as z. Let β be the automorphism of B which sends z to $1 - z$ and fixes the x_i and y_i.

Lemma 8.2. *The α_i and β induce the automorphism $x \mapsto -x$ on $\widetilde{K}_0(B_n)$.*

Proof. Let

$$C = B/(x_2, \dots, x_n) = (k[x_1, y_1, z]/(x_1y_1 - z(1-z)))[y_2, \dots, y_n].$$

Then $C = B_1[y_2, \dots, y_n]$ so C is a domain. Now $C/(x_1) = k[z]/(z(1-z))[y_1, \dots, y_n]$. Since $k[z]/(z(1-z)) = k \times k$ we have

$$C/(x_1) = B/(x_1, \dots, x_n, z) \times B/(x_1, \dots, x_n, 1-z).$$

The exact sequence $0 \to C \xrightarrow{x_1} C \to C/(x_1) \to 0$ shows that $[C/(x_1)] = 0$ in $K_0(B)$ and therefore $[B/(x_1, \dots, x_n, z)] + [B/(x_1, \dots, x_n, 1-z)] = 0$. Since $[B/I] = [B/(x_1, \dots, x_n, z)]$ generates $\widetilde{K}_0(B)$, this shows that β acts as -1. It will suffice to treat the case of α_1. I will write x for x_1 and y for y_1 here. Let $\mathfrak{a} = (x, z)$ and $\mathfrak{b} = (y, z)$ in C. Then $\mathfrak{a} \cap \mathfrak{b} = (z)$. It is sufficient to check this in $C/(z) = (k[x, y]/(xy))[y_2, \dots, y_n]$ where it is clear that $(x) \cap (y) = (xy) = 0$. We have a cartesian diagram

$$\begin{array}{ccc} C/(z) & \longrightarrow & C/\mathfrak{a} \\ \big\downarrow & & \big\downarrow \\ C/\mathfrak{b} & \longrightarrow & C/(\mathfrak{a}+\mathfrak{b}) \end{array}$$

which leads to an exact sequence

$$0 \to C/(z) \to C/\mathfrak{a} \oplus C/\mathfrak{b} \to C/(\mathfrak{a}+\mathfrak{b}) \to 0.$$

Now $[C/(z)] = 0$ and $C/(\mathfrak{a}+\mathfrak{b}) = B/(I, y)$ also has $[B/(I, y)] = 0$ since $B/I = k[y_1, \dots, y_n]$ and $y = y_1$ is regular on it so

$$0 \to B/I \to B/I \to B/(I, y) \to 0$$

is exact. It follows that $[C/\mathfrak{a}] + [C/\mathfrak{b}] = 0$ and $C/\mathfrak{a} = B/I$ while $C/\mathfrak{b} = B/\alpha_1(I)$. $\square$

Corollary 8.3. *Let $\theta = \alpha_1 \dots \alpha_n \beta$. Then θ induces $(-1)^{n-1}$ on $\widetilde{K}_0(B_n)$.*

9. Proof of Theorem 1.2

In this proof we use the ideal

$$J = ((1-z)^N x_1^{m_1}, \dots, (1-z)^N x_n^{m_n}, z^N y_1^{m_1}, \dots, z^N y_n^{m_n})$$

of B_n where N is some integer such that $N \geq \sum m_i$.

Lemma 9.1. *If $M \geq \sum m_i$, then $z^M(1-z)^M \in J$.*

Proof. By Lemma 5.8, $z^M(1-z)^M$ lies in $(x_1^{m_1}, \dots, x_n^{m_n})$ and also in $(y_1^{m_1}, \dots, y_n^{m_n})$. Therefore $z^{M+N}(1-z)^M$ and $z^M(1-z)^{M+N}$ lie in J. Since

$$(z^{M+N}(1-z)^M, z^M(1-z)^{M+N}) = z^M(1-z)^M(z^N, (1-z)^N) = (z^M(1-z)^M),$$

the result follows. $\square$

Lemma 9.2. *$J_z = (y_1^{m_1}, \dots, y_n^{m_n})B_z$ and $J_{1-z} = (x_1^{m_1}, \dots, x_n^{m_n})B_{1-z}$.*

Proof. Since $z^N(1-z)^N$ lies in J, $(1-z)^N$ lies in J_z and the first statement follows immediately. The second statement is proved similarly. $\square$

We therefore get short exact sequences

$$0 \to L \to B_{1-z}^n \to J_{1-z} \to 0$$

and

$$0 \to N \to B_z^n \to J_z \to 0.$$

Since $J_{z(1-z)} = B_{z(1-z)}$ the sequences split after localizing to $B_{z(1-z)}$ and we see that $L_z = P(x_1^{m_1}, \ldots, x_n^{m_n})$ over $B_{z(1-z)}$ and $N_{1-z} = P(y_1^{m_1}, \ldots, y_n^{m_n})$ over $B_{z(1-z)}$. So L_z and N_{1-z} are induced from $P(x_1^{m_1}, \ldots, x_n^{m_n})$ and $P(y_1^{m_1}, \ldots, y_n^{m_n})$ over

$$A = k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum x_i y_i - 1)$$

using the map $A \to B_{z(1-z)}$ sending x_i, y_i to x_i/z, $y_i/(1-z)$. If $P(x_1^{m_1}, \ldots, x_n^{m_n})$ is self dual then $P(x_1^{m_1}, \ldots, x_n^{m_n}) \approx P(y_1^{m_1}, \ldots, y_n^{m_n})$ by Proposition 7.2 and therefore $L_z \approx N_{1-z}$. Therefore Lemma 4.1 gives us an epimorphism $Q \twoheadrightarrow J$ where Q is finitely generated projective of rank n. By Corollary 3.10 we see that $[B/J] \equiv 0 \mod (n-1)!$ in $\widetilde{K}_0(B)$. By Lemma 9.1,

$$B/J = B/(J, z^N) \times B/(J, (1-z)^N)$$

Now $(J, z^N) = ((1-z)^N x_1^{m_1}, \ldots, (1-z)^N x_n^{m_n}, z^N)$. Since $(z^N, (1-z)^N) = B$, it follows that

$$(J, z^N) = (x_1^{m_1}, \ldots, x_n^{m_n}, z^N) = J_{m_1, \ldots, m_n}.$$

Similarly

$$(J, (1-z)^N) = (y_1^{m_1}, \ldots, y_n^{m_n}, (1-z)^N) = \theta J_{m_1, \ldots, m_n}$$

where θ is as in Corollary 8.3. By Corollary 5.11 $[B/J_{m_1, \ldots, m_n}] = m_1 \cdots m_n [B/I]$ in $\widetilde{K}_0(B)$. By Corollary 8.3, $[B/\theta J_{m_1, \ldots, m_n}]$ is $(-1)^{n-1}$ times this so

$$[B/J] = m_1 \cdots m_n (1 + (-1)^{n-1})[B/I]$$

and therefore

$$m_1 \cdots m_n (1 + (-1)^{n-1})[B/I] \equiv 0 \mod (n-1)!.$$

This is vacuous for n even while for n odd it is equivalent to $2 m_1 \cdots m_n \equiv 0 \mod (n-1)!$.

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