CORRECTION TO: VECTOR BUNDLES AND PROJECTIVE MODULES

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Abstract. This corrects an error in the statement of Theorem 6 of my paper Vector Bundles and Projective Modules

Example 4 of [5] concerns the following question: If a projective module becomes free after extending the ground field (say from $\mathbb{R}$ to $\mathbb{C}$) is it stably free? The example considered was the coordinate ring of the n–sphere $A_n = \mathbb{R}[x_0, \dots, x_n]/(\sum_0^n x_i^2 - 1)$ and Theorem 6 claimed that there was an example over this ring for $n = 4$. This is incorrect. The error is in the assertion that $\mathbb{C} \otimes A \approx P \oplus P'$ in line -5 on page 276. I had originally worked out this example for the case of the 2–sphere mentioned in lines 5 and 6 on page 277. In this case, the argument is correct. If P is a $\mathbb{C} \otimes_{mathbbR} \Lambda$–module, then $\mathbb{C} \otimes_{mathbbR} P \approx P \oplus P'$ where P' is the conjugate module defined by letting $\mathbb{C} \otimes_{mathbbR} \Lambda$ act on P via the automorphism $\mathbb{C} \otimes_{mathbbR} \Lambda \to \mathbb{C} \otimes_{mathbbR} \Lambda$ sending $z \otimes a$ to $\bar{z} \otimes a$ so the assertion $\mathbb{C} \otimes A \approx P \oplus P'$ is correct in this case. When writing this up, it struck me that the argument could be extended to the case of the 4–sphere by replacing $\mathbb{C}$ by the quaternions $\mathbb{H}$, but I failed to check the details with sufficient care and the argument runs afoul of the non–commutativity of the quaternions. Here P is an $\mathbb{H} \otimes_{mathbbR} \Lambda$–module. To find $\mathbb{C} \otimes_{mathbbR} P$ we can regard P as a $\mathbb{C} \otimes_{mathbbR} \Lambda$–module via the inclusion $\mathbb{C} \otimes_{mathbbR} \Lambda \subset \mathbb{H} \otimes_{mathbbR} \Lambda$. The automorphism sending $z \otimes a$ to $\bar{z} \otimes a$ extends to an automorphism of $\mathbb{H} \otimes_{mathbbR} \Lambda$ sending $q \otimes a$ to $jqj^{-1} \otimes a$ but this sends j to j not $-j$ and so does not agree with the definition of P' in [5]. If, as in [5], we try the map sending $q \otimes a$ to $\bar{q} \otimes a$, we see that this is an anti–automorphism making the resulting module P' a right module so again this does not produce the module P' used in [5] and the statement that P' is the conjugate module of P on page 276, line -9 is incorrect.

The correct version of the theorem can be easily deduced from the results of [6]. The question can be reformulated in K-theoretic terms as follows: Let A be an $\mathbb{R}$–algebra. The base change map $\mathrm{bch} : K_0(A) \to K_0(\mathbb{C} \otimes_{\mathbb{R}} A)$ is defined by sending $[M]$ to $\mathbb{C} \otimes_{\mathbb{R}} M$. The question then is whether this map is injective. Here is the correct answer for $A = A_n$.

Theorem 0.1. *The base change map $K_0(A_n) \to K_0(\mathbb{C} \otimes_{\mathbb{R}} A_n)$ is not injective if and only if $n \equiv 1, 2 \pmod 8$.*

Proof. The restriction map $\mathrm{res} : K_0(\mathbb{C} \otimes_{\mathbb{R}} A) \to K_0(A)$ is defined by sending $[M]$ to $[M]$ i.e. we forget the complex structure. The composition $\mathrm{res}\,\mathrm{bch}$ is multiplication by 2 since $\mathbb{C} \otimes_{\mathbb{R}} M$ is isomorphic to $M \oplus M$ as an A–module. Therefore, the kernel of bch is annihilated by 2. It is well–known that $K_0(\mathbb{C} \otimes_{\mathbb{R}} A_n)$ is 0 for n odd and $\mathbb{Z}$ for n even [2, 3, 4] or [7, Theorem 10.2]. Therefore the kernel of bch for $A = A_n$ is equal to the torsion submodule of $K_0(A_n)$. Since $K_0(A_n) = \mathbb{Z} \oplus \widetilde{K}_0(A_n)$ this is also the torsion submodule of $\widetilde{K}_0(A_n)$. By [6], $\widetilde{K}_0(A_n)$ is the same as $\widetilde{K}^0(S^n)$ which is periodic with period 8, the first 8 values, beginning with $n = 1$ being $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$ [1, Table 2]. Therefore $K_0(A_n)$ has non–trivial torsion if and only if $n \equiv 1, 2 \pmod 8$. □

We can also determine the maps bch and res explicitly for A_n. The summands of $K_0(A_n) = \mathbb{Z} \oplus \widetilde{K}_0(A_n)$ are clearly stable under these maps and on the summand $\mathbb{Z}$ bch = 1 and res = 2. On the other summand the maps are as follows.

Theorem 0.2. *The maps* $\widetilde{K}_0(A_n) \xrightarrow{\text{bch}} \widetilde{K}_0(\mathbb{C} \otimes_{\mathbb{R}} A_n) \xrightarrow{\text{res}} \widetilde{K}_0(A_n)$ *are* 0 *if* n *is odd or* $n \equiv 6 \pmod 8$*. Otherwise they are as follows for appropriate choices of generators.*

(1) $\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ *if* $n \equiv 2 \pmod 8$.
(2) $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}$ *if* $n \equiv 4 \pmod 8$.
(3) $\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ *if* $n \equiv 0 \pmod 8$.

Proof. If n is odd, $\widetilde{K}_0(\mathbb{C} \otimes_{\mathbb{R}} A_n) = 0$ while $\widetilde{K}_0(A_n) = 0$ if $n \equiv 6 \pmod 8$. Let $q_n = \sum_0^n x_i^2$ as a quadratic form over $\mathbb{R}$ and let $C(q_n)$ be the Clifford algebra of q_n. The group $\mathrm{ABS}(q_n)$ is defined to be the cokernel of $C(q_n \perp 1) \to C(q_n)$. This is the same as the group A_n of [1] as noted in [7, page 457]. It was shown in [6] that the map $\mathrm{ABS}(q_n) \to \widetilde{K}_0(A_n)$ is an isomorphism. The same is true for the complex analogue $\mathrm{ABS}_{\mathbb{C}}(q_n) \to \widetilde{K}_0(\mathbb{C} \otimes_{\mathbb{R}} A_n)$ as was observed much earlier by Fossum [3]. These maps clearly commute with bch and res so it is enough to prove the theorem for the maps $\mathrm{ABS}(q_n) \xrightarrow{\text{bch}} \mathrm{ABS}_{\mathbb{C}}(q_n) \xrightarrow{\text{res}} \mathrm{ABS}(q_n)$. By [7,] $\mathrm{ABS}(q)$ is generated by any simple $C(q)$–module. From [1, Table 1] where $C(q_n)$ is denoted C'_{n+1} we see that for $n \equiv 2 \pmod 8$ we have $C(q_n) = M_m(\mathbb{C})$ an $m \times m$–matrix algebra over $\mathbb{C}$ for an appropriate m and $C_{\mathbb{C}}(q_n) = M_m(\mathbb{C}) \times M_m(\mathbb{C})$. Therefore a simple $C_{\mathbb{C}}(q_n)$–module restricts to a simple $C(q_n)$–module showing that the map res is onto. If $n \equiv 4 \pmod 8$, then $C(q_n) = M_m(\mathbb{H}) \times M_m(\mathbb{H})$ and $C_{\mathbb{C}}(q_n) = M_{2m}(\mathbb{C}) \times M_{2m}(\mathbb{C})$. Here a simple $C_{\mathbb{C}}(q_n)$–module again restricts to a simple $C(q_n)$–module as one sees by comparing dimensions so res = 1. If $n \equiv 0 \pmod 8$, then $C(q_n) = M_m(\mathbb{R}) \times M_m(\mathbb{R})$ and $C_{\mathbb{C}}(q_n) = M_m(\mathbb{C}) \times M_m(\mathbb{C})$. Here a simple $C(q_n)$–module extends to a simple $C_{\mathbb{C}}(q_n)$–module so bch = 1. □

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