EXCERPT FROM ON SOME ACTIONS OF STABLY ELEMENTARY MATRICES ON ALTERNATING MATRICES

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Abstract. This is an excerpt from a paper still in preparation. We show that there are examples of 2–stably elementary 3×3 matrices over a 4 dimensional ring, whose first row is not completable to an elementary matrix. We also give examples of 2–stably elementary matrices ρ of size $(2n-1)$, with $(1 \perp \rho) \notin E_{2n}(A)Sp_{2n}(A)$.

1. Introduction

The leit motif of this paper is the following beautiful lemma of L.N. Vaserstein: Let ρ be an odd sized $(2n-1)$, $n > 1$, invertible matrix which is such that $1 \perp \rho$ is an elementary matrix (i.e. a product of elementary generators $e_{ij}(\lambda)$, $i \neq j$). Then the first row $e_1\rho$ of ρ can be completed to an elementary matrix. Let $\psi_n \in \mathrm{SL}_{2n}(\mathbb{Z})$ denote the standard alternating matrix of Pfaffian 1 obtained by taking a direct sum of n copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. One can even show above that $(1 \perp \rho)$ acts like an elementary matrix of size $(2n-1)$ on ψ_n, i.e.

$$(1 \perp \rho)^\top \psi_n (1 \perp \rho) = (1 \perp \epsilon)^\top \psi_n (1 \perp \epsilon),$$

for some $\epsilon \in E_{2n-1}(R)$. We show that there are examples of 2-stably elementary matrices ρ, of size 3×3, over a 4 dimensional ring A, whose first row is not completable to an elementary matrix. The main counter examples in this paper show that there is a commutative ring of dimension $2n-1$, and a 2-stably elementary matrix ρ of size $(2n-1)$, with $(1 \perp \rho) \notin E_{2n}(A)Sp_{2n}(A)$, for $(2n-1) \equiv 3$ modulo 8. In particular, this means, that $(1 \perp \rho)$ does not act like an elementary matrix, in general.

2. Some basic results from topology.

We recall here a few simple facts from topology. As usual if X and Y are topological spaces we let $[X, Y]$ be the set of homotopy classes of maps from X to Y and let $[(X, x_0), (Y, y_0)]$ be the set of homotopy classes preserving the base points. The following well–known result shows that we do not have to worry about base points in the cases to be considered.

Lemma 2.1. *Let X be a finite simplicial complex. If Y is simply connected or is a path connected topological group then $[(X, x_0), (Y, y_0)] \to [X, Y]$ is an isomorphism.*

Since this is usually only proved for the case where X is a sphere, using the action of the fundamental group on the higher homotopy groups, we will give a proof here.

Proof. To show the map is onto, suppose $f : X \to Y$ is given. Join $f(x_0)$ to y_0 by a path C and define a homotopy h of $f|\{x_0\}$ by $h_t(x_0) = C(t)$. Extend this to a homotopy H of f by the homotopy extension theorem [9, Ch. I, Prop. 9.2]. Then $f = H_0$ is homotopic to H_1 which preserves the base points. Therefore the map is onto.

To show the map is injective, let $f, g : (X, x_0) \to (Y, y_0)$ be homotopic as maps $X \to Y$ by $h : X \times I \to Y$. If Y is a topological group let $H_t(x) = y_0 h_t(x_0)^{-1} h_t(x)$. Then $H : f \simeq g$ preserving the base point. This argument is taken directly from [9, Ch. IV, Prop. 16.10].

If Y is simply connected and $h : f \simeq g$ as above, then $t \mapsto h_t(x_0)$ is a loop at y_0 and is therefore contractible. Let σ_s, $0 \le s \le 1$, be a homotopy between this loop and the constant map y_0 so $\sigma_0(t) = h_t(x_0)$, $\sigma_1(t) = y_0$, and $\sigma_s(0) = \sigma_s(1) = y_0$ Define a homotopy k_s of $h|(X \times 0) \cup (x_0 \times I) \cup (X \times 1)$ by $k_s|(X \times 0) = f$, $k_s|(x_0 \times I) = \sigma_s$, and $k_s|(X \times 1) = g$. Extend this to a homotopy H_s of h. Then $H_1 : X \times I \to Y$ is a homotopy between f and g which preserves the base point. □

In particular, $[S^m, Y] = \pi_m(Y)$ if Y is as in the lemma. The lemma holds more generally for a CW complex X since the homotopy extension theorem holds for such complexes [8, Ch. VII, Th. 1.4].

If X is a topological space let $C(X, \mathbb{R})$ be the ring of continuous real functions on X and let $C(X, \mathbb{C})$ be the ring of continuous complex functions on X. Clearly the space of mappings $X \to \mathrm{SL}_m(\mathbb{R})$ can be identified with $\mathrm{SL}_m(C(X, \mathbb{R}))$ and similarly for $\mathbb{C}$. The following is well–known in the case $A = C(X, \mathbb{R})$ [2, Ch. XIV,§5], [10, §7].

Lemma 2.2. *Let X be a compact Hausdorff space and let A be a subring of $C(X, \mathbb{R})$ containing $\mathbb{R}$ and which is dense in $C(X, \mathbb{R})$. Assume that any a in A with no zeros on X is a unit of A. Then the maps*

$$\mathrm{SL}_n(A)/\,\mathrm{E}_n(A) \to \mathrm{SL}_n(C(X, \mathbb{R}))/\,\mathrm{E}_n(C(X, \mathbb{R})) \to [X, \mathrm{SL}_n(\mathbb{R})]$$

are isomorphisms. A similar statement holds for A in $C(X, \mathbb{C})$.

Proof. It suffices to show that $\mathrm{SL}_n(A)/\,\mathrm{E}_n(A) \to [X, \mathrm{SL}_n(\mathbb{R})]$ is an isomorphism since we can take $A = C(X, \mathbb{R})$. We use the following fact
(*) If $\sigma \in \mathrm{SL}_n(A)$ is sufficiently close to 1 then $\sigma \in \mathrm{E}_n(A)$.
This is proved in [10, Lemma 7.4]. It uses only the property that an element of A sufficiently near 1 is a unit.

The map is well defined since if σ and τ lie in $\mathrm{SL}_n(A)$ and are congruent modulo $\mathrm{E}_n(A)$ then $\sigma = \tau \prod e_{i_k j_k}(f_k)$, so $\sigma_t = \tau \prod e_{i_k j_k}(t f_k)$ gives a homotopy between σ and τ as maps $X \to \mathrm{SL}_n(\mathbb{R})$..

For the ontoness, let $\sigma : X \to \mathrm{SL}_n(\mathbb{R})$ and write $\sigma = (f_{ij})$ with entries f_{ij} in $C(X, \mathbb{R})$. Let $\rho = (g_{ij})$ with entries g_{ij} in A very close to f_{ij} and let $\rho_t = (1-t)\sigma + t\rho$. This is still very close to σ for $0 \le t \le 1$ so $d_t = \det \rho_t$ is very close to $\det \sigma = 1$ and therefore is a unit of A. Let $\tau_t = \mathrm{diag}(d_t^{-1}, 1, \ldots, 1)\rho_t$. This lies in $SL_n(C(X, \mathbb{R}))$ and gives a homotopy of σ with $\tau = \tau_1 = \mathrm{diag}(d_1^{-1}, 1, \ldots, 1)\rho$ which lies in $\mathrm{SL}_n(A)$.

For the injectivity, suppose σ and τ lie in $\mathrm{SL}_n(A)$ and are homotopic when considered as maps $X \to \mathrm{SL}_n(\mathbb{R})$. Let $h_t : X \to \mathrm{SL}_n(\mathbb{R})$ be a homotopy between them. Since X is compact we can find a sequence $0 = t_0 < t_1 < \cdots < t_N = 1$ such that if $h_i = h_{t_i}$, the matrices $h_i^{-1} h_{i+1}$ are arbitrarily close to 1 so that, by (*), each of these matrices lies in $\mathrm{E}_n(C(X, \mathbb{R}))$. Therefore $\tau = \sigma\epsilon$ where $\epsilon = \prod_0^N (h_i^{-1} h_{i+1})$

lies in $\mathrm{E}_n(C(X,\mathbb{R}))$. Write $\epsilon = \prod e_{i_k j_k}(f_k)$. Choose g_k in A very close to f_k. Then $\eta = \prod e_{i_k j_k}(g_k)$ is very close to ϵ. Now $\tau^{-1}\sigma\eta$ is very close to $\tau^{-1}\sigma\epsilon = 1$ and therefore lies in $\mathrm{E}_n(A)$ and hence so does $\tau^{-1}\sigma$. □

The following is a special case of a result of Vaserstein [13]. Recall that the stable dimension $\operatorname{sdim} A$ is defined to be one less that the stable range of A. Bass' stability theorem says that $\operatorname{sdim} A \leq \dim A$ for commutative noetherian rings A.

Lemma 2.3 (Vaserstein). *Let X be a compact Hausdorff space and let A be a subring of $C(X,\mathbb{R})$ containing $\mathbb{R}$. Assume that any a in A with no zeros on X is a unit of A. Then* $\operatorname{sdim} \mathbb{C}\otimes_{\mathbb{R}} A \leq [\frac{1}{2}\operatorname{sdim} A]$

Vaserstein notes that under the hypothesis a row $(c_1,\dots,c_n)$ is unimodular if and only if $\sum |c_n|^2$ is never 0 and hence is a unit. Write $c_n = a_n + ib_n$. Then $(a_1, b_1, \dots, a_{n-1}, b_{n-1}, |c_n|^2)$ is unimodular so if $2n-1 \geq 2 + \operatorname{sdim} A$ we can make $(a_1, b_1, \dots, a_{n-1}, b_{n-1})$ unimodular by adding multiples of $|c_n|^2$ to its entries. Therefore the same is true for $(c_1,\dots,c_{n-1})$. The result follows since the condition on n is equivalent to $n \geq 2 + [\frac{1}{2}\operatorname{sdim} A]$.

We conclude this section by recording the following classical result.

Lemma 2.4. *The following inclusions are homotopy equivalences.*

(1) $\mathrm{SU}(m) \subset \mathrm{SL}_m(\mathbb{C})$
(2) $\mathrm{Sp}(m) \subset \mathrm{Sp}_{2m}(\mathbb{C})$
(3) $\mathrm{SO}(m) \subset \mathrm{SL}_m(\mathbb{R})$
(4) $\mathrm{U}(m) \subset \mathrm{Sp}_{2m}(\mathbb{R})$

In fact, in each case, the larger group is the product of the smaller group with a Euclidean space.

Items (1) and (2) are given in [6, Ch. VI, §X]. For (3) and(4) we can use [7, Ch. IX, Lemma 4.3] from which (3) is clear. For (4) we also use [7, Ch. IX, Lemma 4.1(c)].

The inclusion $\mathrm{U}(m) \subset \mathrm{Sp}_{2m}(\mathbb{R})$ in (4) is obtained by observing that $\mathrm{U}(m)$ preserves the (real) alternating form $\Im(z,w)$. Similarly, the inclusion $\mathrm{U}(m) \subset \mathrm{SO}(2m)$ is obtained by observing that $\mathrm{U}(m)$ preserves $\Re(z,w)$ which is the usual inner product on $\mathbb{R}^{2m}$. Therefore we have a commutative diagram

$$\begin{array}{ccc} \mathrm{U}(m) & \hookrightarrow & \mathrm{SO}(2m) \\ \downarrow & & \downarrow \\ \mathrm{Sp}_{2m}(\mathbb{R}) & \hookrightarrow & \mathrm{SL}_{2m}(\mathbb{R}) \end{array}$$

showing that, up to homotopy, $\mathrm{Sp}_{2m}(\mathbb{R}) \to \mathrm{SL}_{2m}(\mathbb{R})$ looks just like $\mathrm{U}(m) \to \mathrm{SO}(2m)$.

3. The Main Counterexample

Let A be a noetherian ring. Let ρ be an element of $\mathrm{SL}_{2n-1}(A)$ which is 2–stably elementary i.e. $I_2 \perp \rho \in \mathrm{E}_{2n+1}(A)$. If $\dim A \leq 2n-2$ then $\mathrm{SL}_{2n}(A)/\mathrm{E}_{2n}(A) \to \mathrm{SK}_1(A)$ is an isomorphism by Vaserstein's stability theorem [12] and therefore $1 \perp \rho \in \mathrm{E}_{2n}(A)$. Therefore, in the following counterexample, the dimension of A is as small as possible.

The condition $(1 \perp \rho)^{\top}\psi_n(1\perp\rho) = (1\perp\epsilon)^{\top}\psi_n(1\perp\epsilon)$ mentioned above is equivalent to $\rho\epsilon^{-1} \in \mathrm{Sp}_n(A)$ or $\rho \in \mathrm{E}_{2n-1}(A)\,\mathrm{Sp}_{2n}(A)$

Theorem 3.1. *If $2n-1 \equiv 3 \mod 8$, there is an affine domain A over $\mathbb{R}$ of dimension $2n-1$ and a 2–stably elementary element $\rho \in \mathrm{SL}_{2n-1}(A)$ such that $(1 \perp \rho) \notin \mathrm{E}_{2n}(A)Sp_{2n}(A)$. Moreover, if $n = 2$, we can choose ρ in such a way that its first row in not completable to an elementary matrix or, equivalently, $e_1\rho$ is not elementarily equivalent to e_1*

We begin by offering a simpler example which has a weaker conclusion but works for all $2n-1 \geq 3$. In the following theorem we have replaced the condition $\dim A = 2n-1$ by $\operatorname{sdim} A = 2n$ and have replaced $\mathbb{R}$ by $\mathbb{C}$.

Theorem 3.2. *If $n \geq 2$ there is a regular domain A, essentially of finite type over $\mathbb{C}$, of stable dimension $\leq 2n$, and a 2–stably elementary element $\rho \in \mathrm{SL}_{2n-1}(A)$ such that $(1 \perp \rho) \notin \mathrm{E}_{2n}(A)Sp_{2n}(A)$. Moreover, if $n = 2$, we can choose ρ in such a way that its first row in not completable to an elementary matrix or, equivalently, $e_1\rho$ is not elementarily equivalent to e_1*

The proof of these theorems is topological.

We begin with the proof of Theorem 3.2 which is considerably simpler. We construct the example using a subring A of $C(S^{4n}, \mathbb{C})$, the ring of continuous complex functions on the $4n$–sphere. We assume that A satisfies the conditions of Lemma 2.2.

The problem can be rephrased as follows: Find ρ in $\mathrm{SL}_{2n-1}(A)/\mathrm{E}_{2n-1}(A)$ whose image in $\mathrm{SL}_{2n+1}(A)/\mathrm{E}_{2n+1}(A)$ is 0 and whose image in $\mathrm{SL}_{2n}(A)/\mathrm{E}_{2n}(A)$ does not lie in the image of $\mathrm{Sp}_{2n}(A)$. By Lemma 2.2 and Lemma 2.1, we have $\mathrm{SL}_m(A)/\mathrm{E}_m(A) = [S^{4n}, \mathrm{SL}_m(\mathbb{C})] = \pi_{4n}(\mathrm{SL}_m(\mathbb{C}))$. We can replace $\mathrm{SL}_m(\mathbb{C})$ by $\mathrm{SU}(m)$ which has the same homotopy type by Lemma 2.4. The map $\mathrm{Sp}_{2n}(A) \to \mathrm{SL}_{2n}(A)/\mathrm{E}_{2n}(A) = [S^{4n}, \mathrm{SL}_{2n}(\mathbb{C})]$ factors as

$$\mathrm{Sp}_{2n}(A) \to \mathrm{Sp}_{2n}(C(S^{4n}, \mathbb{C})) \to [S^{4n}, \mathrm{Sp}_{2n}(\mathbb{C})] = \pi_{4n}(\mathrm{Sp}_{2n}(\mathbb{C})) \to \pi_{4n}(\mathrm{SL}_{2n}(\mathbb{C}))$$

so it will suffice to find a ρ whose image in $\pi_{4n}(\mathrm{SL}_{2n}(\mathbb{C}))$ is not in the image of $\pi_{4n}(\mathrm{Sp}_{2n}(\mathbb{C}))$. We can replace $\mathrm{Sp}_{2n}(\mathbb{C})$ by $\mathrm{Sp}(n)$ which has the same homotopy type as $\mathrm{Sp}_{2n}(\mathbb{C})$ by Lemma 2.4 and similarly replace $\mathrm{SL}_{2n}(\mathbb{C})$ by $\mathrm{SU}(2n)$.

Our problem now appears as follows. Consider the diagram

$$\begin{array}{ccccc} \pi_{4n}(\mathrm{SU}(2n-1)) & \longrightarrow & \pi_{4n}(\mathrm{SU}(2n)) & \longrightarrow & \pi_{4n}(\mathrm{SU}(2n+1)) \\ & & \uparrow & & \\ & & \pi_{4n}(\mathrm{Sp}(n)) & & \end{array}$$

and find ρ in $\pi_{4n}(\mathrm{SU}(2n-1))$ whose image in $\pi_{4n}(\mathrm{SU}(2n+1))$ is 0 and whose image in $\pi_{4n}(\mathrm{SU}(2n))$ does not lies in the image of $\pi_{4n}(\mathrm{Sp}(n))$. The first part is no problem since $\pi_{4n}(\mathrm{SU}(2n+1))$ is stable and is 0 by Bott's calculations [4]. By [3], $\pi_{4n}(\mathrm{SU}(2n)) = \mathbb{Z}/(2n)!\mathbb{Z}$. The homotopy sequence of the fibration $\mathrm{SU}(2n-1) \to \mathrm{SU}(2n) \to S^{4n-1}$ gives us

$$\pi_{4n}(\mathrm{SU}(2n-1)) \to \pi_{4n}(\mathrm{SU}(2n)) \to \pi_{4n}(S^{4n-1}) = \mathbb{Z}/2\mathbb{Z}$$

Therefore the image of $\pi_{4n}(\mathrm{SU}(2n-1)) \to \pi_{4n}(\mathrm{SU}(2n))$ has order either $(2n)!$ or $(2n)!/2$ and is therefore greater than 2. But $\pi_{4n}(\mathrm{Sp}(n))$ is stable and is either 0 or $\mathbb{Z}/2\mathbb{Z}$ by [4]. This shows that the required ρ exists. For example, we could take ρ to have order 3.

For the last part of the theorem we want to choose ρ so that $\mathrm{Row}_1(\rho) = e_1\rho$ is not elementarily completable or, equivalently such that $e_1\rho \neq e_1\epsilon$ for any $\epsilon \in \mathrm{E}_{2n-1}(A)$ In other words, we want to find a ρ in $\mathrm{SL}_{2n-1}(A)/\mathrm{E}_{2n-1}(A)$ whose image in $\mathrm{SL}_{2n+1}(A)/\mathrm{E}_{2n+1}(A)$ is trivial and such that the image $\mathrm{Row}_1(\rho)$ in $\mathrm{Um}_{2n-1}(A)/\mathrm{E}_{2n-1}(A)$ is non–trivial. As above we can restate this as follows: Find ρ in $\pi_{4n}(\mathrm{SU}(2n-1))$ whose image in $\pi_{4n}(\mathrm{SU}(2n+1))$ is 0 and whose image in $\pi_{4n}(S^{4n-3})$ is non–zero. We use here the fact that $\mathrm{Row}_1 : \mathrm{SL}_{2n-1}(\mathbb{C}) \to \mathbb{C}^{2n-1} - \{0\}$ restricts to $\mathrm{SU}(2n-1) \to S^{4n-3}$. The first part is clear since $\pi_{4n}(\mathrm{SU}(2n+1)) = 0$. The homotopy sequence of the fibration $\mathrm{SU}(2n-2) \to \mathrm{SU}(2n-1) \to S^{4n-3}$ gives $\pi_{4n}(\mathrm{SU}(2n-2)) \to \pi_{4n}(\mathrm{SU}(2n-1)) \to \pi_{4n}(S^{4n-3}) \to \pi_{4n-1}(\mathrm{SU}(2n-2))$. We restrict to the case $n = 2$ where the required homotopy groups are well–known. Since $SU(2) \cong S^3$, our sequence becomes $\pi_8(S^3) \to \pi_8(\mathrm{SU}(3)) \to \pi_8(S^5) \to \pi_7(S^3)$. Now $\pi_7(S^3) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_8(S^5) = \mathbb{Z}/24\mathbb{Z}$ [9, Ch. XI, Th. 16.4, Th. 17.1]. It follows that there is a ρ in $\pi_8(\mathrm{SU}(3))$ mapping to an element of order 3 in $\pi_8(S^5)$ as required. Since $\pi_8(S^3) = \mathbb{Z}/2\mathbb{Z}$ [9, Ch. XI, §18], any ρ of order 3 in $\pi_8(\mathrm{SU}(3))$ will do.

Finally, we choose A as follows. Consider S^{4n} as the set of $x \in \mathbb{R}^{4n+1}$ satisfying $\sum_0^{4n} x_i^2 = 1$. Let $B = \mathbb{R}[x_0, \ldots, x_{4n}]/(\sum_0^{4n} x_i^2 - 1)$ be the ring of real polynomial functions on S^{4n}. Let $S \subset B$ be the set of such functions with no zeros on S^{4n} and let $C = B_S = B[S^{-1}]$. Let $A = \mathbb{C} \otimes_{\mathbb{R}} C = \mathbb{C}[x_0, \ldots, n_{4n}]/(\sum_0^{4n} x_i^2 - 1)[S^{-1}]$. Then A is dense in $C(S^{4n}, \mathbb{C})$ by the Stone–Weierstrass theorem. It is clearly regular and $\operatorname{sdim} A \leq \frac{1}{2} \operatorname{sdim} C \leq 2n$ by Lemma 2.3.

Proof of Theorem 3.1. For this proof we use homotopy groups with coefficients [5]. Let Y be a Moore space $Y = e^m \cup_d S^{m-1}$. The group $\pi_m(X, x_0; \mathbb{Z}/d)$ is the group of homotopy classes of basepoint preserving maps $(Y, y_0) \to (X, x_0)$. As above we will ignore the basepoints using Lemma 2.1. The homotopy sequence of a fibration also holds for these groups which are related to the ordinary homotopy groups by an exact sequence

$$(1) \qquad \cdots \to \pi_m(X) \xrightarrow{d} \pi_m(X) \to \pi_m(X, \mathbb{Z}/d) \to \pi_{m-1}(X) \xrightarrow{d} \pi_{m-1}(X) \to \ldots$$

For the proof of theorem 3.1, we let $d = 2$ and let $Y = e^{2n-1} \cup_2 S^{2n-2}$. Let $A \subseteq C(Y, \mathbb{R})$ be as in Lemma 2.2. As above it will suffice to find ρ in $[Y, \mathrm{SO}(2n-1)] = \pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2)$ such that ρ maps to 0 in $\pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2)$ and such that the image of rho in $\pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2)$ does not lie in the image of $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2)$ (using Lemma 2.4).

Let $T : S^{2n-1} \to \mathrm{SO}(2n)$ be the characteristic map of the fibration

$$(2) \qquad \mathrm{SO}(2n) \to \mathrm{SO}(2n+1) \to S^{2n}$$

defined in [11, §23.2]. Let π be the projection in the bundle

$$(3) \qquad \mathrm{SO}(2n-1) \to \mathrm{SO}(2n) \xrightarrow{\pi} S^{2n-1}$$

and let $\Sigma : \pi_{2n-1}(S^{2n-1}) \to \pi_{2n}(S^{2n})$ be the suspension. The diagram

$$(4) \qquad \begin{array}{ccc} \pi_{2n}(S^{2n}) & \xrightarrow{\partial} & \pi_{2n-1}(\mathrm{SO}(2n)) \\ \Sigma \uparrow \approx & \overset{T_*}{\nearrow} & \downarrow \pi_* \\ \pi_{2n-1}(S^{2n-1}) & \xrightarrow{2} & \pi_{2n-1}(S^{2n-1}) \end{array}$$

commutes since the northwest triangle commutes by [11, §23.2] and the southeast triangle commutes by [11, Th. 23.4]. It follows that in the diagram

$$
\begin{array}{ccccc}
\pi_{2n}(S^{2n}) & \xrightarrow{\partial} & \pi_{2n-1}(\mathrm{SO}(2n)) & \xrightarrow{\pi_*} & \pi_{2n-1}(S^{2n-1}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{2n}(S^{2n}, \mathbb{Z}/2) & \xrightarrow{\partial} & \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) & \xrightarrow{\pi_*} & \pi_{2n-1}(S^{2n-1}, \mathbb{Z}/2)
\end{array}
\tag{5}
$$

the composition of the upper maps is multiplication by 2 so the same is true of the composition of the lower maps which is therefore 0. The generator ι of $\pi_{2n}(S^{2n}, \mathbb{Z}/2)$ maps to an element $\alpha = \partial\iota$ of $\pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2)$, and $\pi_*\alpha = 0$ in $\pi_{2n-1}(S^{2n-1}, \mathbb{Z}/2)$. The homotopy sequence

$$\cdots \to \pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2) \to \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) \to \pi_{2n-1}(S^{2n-1}, \mathbb{Z}/2) \to \dots$$

of the fibration (3) shows that α is the image of an element ρ of $\pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2)$. The homotopy sequence

$$\cdots \to \pi_{2n}(S^{2n}, \mathbb{Z}/2) \to \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) \to \pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2) \to \dots$$

of the fibration (2) shows that α $(= \partial\iota)$, and therefore also ρ, maps to 0 in $\pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2)$. It remains to show that α does not lie in the image of $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2) \to \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2)$ if $2n-1 \equiv 3 \mod 8$.

The homotopy sequence of the fibration $\mathrm{U} \to \mathrm{SO} \to \mathrm{U}/\mathrm{SO}$ and Bott's calculations [4, 1.7] show that $\pi_m(\mathrm{U}) \to \pi_m(\mathrm{SO})$ is onto for $m \equiv 3 \mod 8$ and is therefore an isomorphism since both groups are $\mathbb{Z}$ [4]. Since $\pi_{2n-1}(\mathrm{U}) = \mathbb{Z}$ for $n \geq 2$ and $\pi_{2n-2}(\mathrm{U}) = 0$, (1) shows that $\pi_{2n-1}(\mathrm{U}, \mathbb{Z}/2) = \pi_{2n-1}(\mathrm{U})/2$. Moreover, the map from (1) for $\mathrm{U}(n)$ to (1) for U is an isomorphism in these dimensions so $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2) \to \pi_{2n-1}(\mathrm{U}, \mathbb{Z}/2)$ is an isomorphism. Consider the diagram

$$
\begin{array}{ccccccc}
\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2) & \xrightarrow{\approx} & \pi_{2n-1}(\mathrm{U}, \mathbb{Z}/2) & \xleftarrow{\approx} & \pi_{2n-1}(\mathrm{U})/2 & = & \mathbb{Z}/2 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\approx} & & \\
\pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) & \longrightarrow & \pi_{2n-1}(\mathrm{SO}, \mathbb{Z}/2) & \hookleftarrow & \pi_{2n-1}(\mathrm{SO})/2 & &
\end{array}
$$

Since α maps to 0 in $\pi_{2n-1}(\mathrm{SO}, \mathbb{Z}/2)$, we see that if α lies in the image of $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2)$, then $\alpha = 0$ so we need to show that α is non–trivial.

The homotopy sequence of the fibration (2) gives us

$$0 \to \pi_{2n}(S^{2n}) \xrightarrow{\partial} \pi_{2n-1}(\mathrm{SO}(2n)) \to \pi_{2n-1}(\mathrm{SO}(2n+1)) \to 0 \tag{6}$$

since diagram (4) shows that ∂ is injective and $\pi_{2n-1}(S^{2n}) = 0$. Now $\pi_{2n-1}(\mathrm{SO}(2n+1)) = \pi_{2n-1}(\mathrm{SO}) = \mathbb{Z}$ if $2n-1 \equiv 3 \mod 8$ by Bott's calculations [4] so (6) splits and we get a diagram in which the top map is a split monomorphism

$$
\begin{array}{ccc}
\pi_{2n}(S^{2n})/2 & \longrightarrow & \pi_{2n-1}(\mathrm{SO}(2n))/2 \\
{\scriptstyle\approx}\downarrow & & \downarrow \\
\pi_{2n}(S^{2n}, \mathbb{Z}/2) & \longrightarrow & \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2).
\end{array}
$$

The right hand vertical map is injective by (1). The generator of $\pi_{2n}(S^{2n})/2$ maps to ι which maps to α so it follows that $\alpha \neq 0$.

For the last statement we want, as above, to find ρ in $\pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2)$ such that ρ maps to 0 in $\pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2)$ but whose image in $\pi_{2n-1}(S^{2n-2}, \mathbb{Z}/2)$ (which is represented by $\mathrm{Row}_1(\rho) = \pi_*(\rho)$) is non–zero. This map occurs in the homotopy sequence

$$(7)\qquad \cdots \to \pi_{2n-1}(\mathrm{SO}(2n-2), \mathbb{Z}/2) \to \pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2) \to \\ \pi_{2n-1}(S^{2n-2}, \mathbb{Z}/2) \to \pi_{2n-2}(\mathrm{SO}(2n-2), \mathbb{Z}/2) \to \dots$$

of $\mathrm{SO}(2n-2) \to \mathrm{SO}(2n-1) \to S^{2n-2}$. As above we consider the case $n = 2$ where this homotopy sequence becomes

$$0 = \pi_3(\mathrm{SO}(2), \mathbb{Z}/2) \to \pi_3(\mathrm{SO}(3), \mathbb{Z}/2) \xrightarrow{\pi_*} \pi_3(S^2, \mathbb{Z}/2) \to \pi_2(\mathrm{SO}(2), \mathbb{Z}/2) = 0$$

so π_* is an isomorphism and any non–zero ρ will have a non–zero image.

This concludes the proof except for the choice of A which can be done as follows.

Let $E^{2n-1} = \{x \in \mathbb{R}^{2n-1} \mid \|x\| = \sum x_i^2 \le 1\}$. Map the boundary $S^{2n-2} = \{x \in \mathbb{R}^{2n-1} \mid \|x\| = \sum x_i^2 = 1\}$ to S^{2n-2} by a map η of degree 2 and let $Y = E^{2n-1} \cup_2 S^{2n-2}$ be the quotient of $E^{2n-1} \sqcup S^{2n-2}$ obtained by identifying x with ηx for points x on the boundary. If we think of $\mathbb{R}^{2n-1}$ as $\mathbb{C} \times \mathbb{R}^{2n-3}$ we can choose $\eta(z, x_3, \dots, x_{2n-1}) = (z^2, x_3, \dots, x_{2n-1})$. In real coordinates $\eta(x_1, x_2, x_3, \dots, x_{2n-1}) = (x_1^2 - x_2^2, 2x_1x_2, x_3, \dots, x_{2n-1})$. Therefore Y is the quotient of E^{2n-1} obtained by identifying $(x_1, x_2, x_3, \dots, x_{2n-1})$ with $(-x_1, -x_2, x_3, \dots, x_{2n-1})$ for points on the boundary S^{2n-2}. Let B be the sub–$\mathbb{R}$–algebra of $C(E^{2n-1}, \mathbb{R})$ generated by $x_3, \dots, x_{2n-1}$, x_1^2, x_2^2, x_1x_2, $(1 - \sum x_i^2)x_1$, and $(1 - \sum x_i^2)x_2$. These functions all factor through Y so we can regard B as a subalgebra of $C(Y, \mathbb{R})$. It separates points of Y and so is dense in $C(Y, \mathbb{R})$ by the Stone–Weierstrass Theorem. Let S be the set of elements of B with no zero on Y and define $A = B_S$. This clearly has the rquired properties. We can then replace A by B_s with $s \in S$ chosen so that ρ is in $\mathrm{SL}_{2n-1}(A)$ and is 2–stably elementary. □

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