EXCERPT FROM
A REGENERATIVE PROPERTY OF A FIBRE OF INVERTIBLE ALTERNATING MATRICES

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Abstract. This is an excerpt from a paper still in preparation. It gives an example of a polynomial unimodular row of length 3 which is not elementarily completable.

1. Introduction

Let R be a noetherian local ring of dimension 3 with $\frac{1}{2} \in R$ and let $v(X) = (v_1(X), v_2(X), v_3(X))$ be a unimodular row over $R[X]$. According to [6] $v(X)\alpha = v(0)$ for some α in $GL_3(R[X])$. We will give an example to show that it is not always possible to choose such an α in $E_3(R[X])$.

2. Non–triviality of $W_E(R[x])$

Following H. Bass in [1], $NF(A)$ means $\ker(F(A[X]) \to F(A))$ for any functor F.

Defiine an involution on $K_0(A)$ and $K_1(A)$ by $\alpha = [P] \mapsto \alpha^* = [P^*]$, where $P^* = \mathrm{Hom}_A(P, A)$ and $\alpha = [P, f] \mapsto \alpha^* = [P^*, f^*]$ on K_1 using Bass' categorical definition of K_1. On K_1 this is just the involution induced by transposition since K_1 is generated by $[A^n, f]$ and $f^* = f^\top$ if $(A^n)^*$ is identified with A^n.

Lemma 2.1. *Let A be a commutative ring with identity. Suppose that $NW_E(A) = 0$. Then $\alpha^* + \alpha = 0$ for $\alpha \in NSK_1(A)$.*

Proof. Let $\alpha = [M]$, $M \in SL(A[X])$. Using $M(0)^{-1}M$ for M, we can assume that $M(0) = I$. Let ψ be the standard hyperbolic matrix of Pfaffian 1 obtained by taking a direct sum of copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $M\psi M^\top$ has Pfaffian 1 so $[M\psi M^\top] \in NW_E(A) = 0$ and it maps to $\alpha^* + \alpha = 0$ under the canonical map $NW_E(A) \to NSK_1(A)$ sending $[N] \to [N]$. □

If $A \to B$ then $K_0(A) \to K_0(B)$ and $K_1(A) \to K_1(B)$ preserve $\alpha \to \alpha^*$ since $B \otimes_A P^* = B \otimes_A \mathrm{Hom}(P, A) \xrightarrow{\approx} \mathrm{Hom}_B(B \otimes_A P, B) = (B \otimes_A P)^*$.

Lemma 2.2. *Let A be a commutative ring with identity. Suppose that $NSK_1(A[t, t^{-1}])$ satisfies $\alpha^* + \alpha = 0$. Then so does $NK_0(A)$.*

Proof. The Fundamental Theorem [1, Ch. XII,7.4] gives an injection $h : K_0(A) \to K_1(A[t,t^{-1}])$ by $[P] \mapsto [P[t,t^{-1}],t]$. Clearly $h(\alpha^*) = h(\alpha)^*$. By [1, Ch. XII,7.8]

$$\begin{array}{ccc} K_0(A) & \xrightarrow{h} & K_1(A[t,t^{-1}]) \\ \downarrow & & \downarrow \\ H_0(A) & \longrightarrow & U(A[t,t^{-1}]) \end{array}$$

commutes. Taking kernels we get an injection $\widetilde{K}_0(A) \to SK_1(A[t,t^{-1}])$. Applying N gives $NK_0(A) \to NSK_1(A[t,t^{-1}])$ since $NH_0 = 0$ so $N\widetilde{K}_0 \xrightarrow{\approx} NK_0$. □

Corollary 2.3. *Let A be a commutative ring with identity. Suppose that $NW_E(A[t,t^{-1}]) = 0$. Then $\alpha^* + \alpha = 0$ for $\alpha \in NK_0(A)$.*

Lemma 2.4. *Let $A = A_0 \oplus A_1 \oplus \dots$ be a positively graded ring with $A_0 = k$, a field. If $NK_0(A)$ satisfies $\alpha^* + \alpha = 0$, so does $\widetilde{K}_0(A)$.*

Proof. We use the Weibel homotopy trick in a minor variation of the argument that $NK_0(A) = 0 \Longrightarrow \widetilde{K}_0(A) = 0$ for such A. Let i be the inclusion map $A \hookrightarrow A[Z]$, and let $e_0, e_1 : A[Z] \to A$ by $Z \mapsto 0, 1$. Then $e_0 \circ i = e_0 \circ i$, $NK_0(A) = \ker(e_{0*} = K_0(e_0) : K_0(A[Z]) \to K_0(A))$, and $K_0(A[Z]) = K_0(A) \oplus NK_0(A)$. Now $e_{1*} - e_{0*}$ is 0 on $K_0(A)$ so $\mathrm{im}(e_{1*} - e_{0*}) = (e_{1*} - e_{0*})NK_0(A) = e_{1*}NK_0(A)$ and therefore $\alpha \in \mathrm{im}(e_{1*} - e_{0*})$ satisfies $\alpha^* + \alpha = 0$. Let $w : A \to A[Z]$ by $w(a) = aZ^n$ for $a \in A_n$. Then $e_1 w = 1$ while $e_0 w$ is $A \to k \to A$. If $\alpha \in \widetilde{K}_0(A)$ then $e_{0*}(\alpha) = 0$ while $e_{1*}(\alpha) = \alpha$ so $\alpha = (e_{1*} - e_{0*})w(\alpha)$ satisfies $\alpha^* + \alpha = 0$. □

Corollary 2.5. *Let A be a commutative ring with identity and let $B = A[t,t^{-1}]$. Suppose that $NW_E(B) = 0$. If A is as in Lemma 2.4 then $\alpha^* + \alpha = 0$ for $\alpha \in \widetilde{K}_0(A)$. The same conclusion holds if A is reduced and $NW_E(B_{\mathfrak{m}}) = 0$ for all maximal ideals $\mathfrak{m}$ of B.*

The first part follows immediately from the above results. For the last statement Lemma 2.1 shows that $\alpha^* + \alpha = 0$ for all $\alpha \in NSK_1(B_{\mathfrak{m}})$ and all maximal ideals $\mathfrak{m}$ of B. By Vorst's localization theorem [9, Cor. 1.9(iii)] $NSK_1(B)$ embeds in $\prod_{\mathfrak{m}} NSK_1(B_{\mathfrak{m}})$ so $NSK_1(B)$ also satisfies $\alpha^* + \alpha = 0$.

Lemma 2.6. *Let R be a 1–dimensional domain. Then $SK_1(R)$ satisfies $\alpha^* + \alpha = 0$.*

Proof. By [1, Ch. VI 2.3] $SL_2(R) \to SK_1(R)$ is onto and factors through a Mennicke symbol. (We only need the easy part of [1, Ch. VII 12.3]. The hard part is that the Mennicke relations [1, Ch. VII 1] give a presentation of $SK_1(R)$.) Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(R)$ map to $\alpha \in SK_1(R)$. Then $\alpha = \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)$ Since $M^\top = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ maps to α^*, we have $\alpha^* = \left(\begin{smallmatrix} c \\ a \end{smallmatrix}\right)$ so $\alpha^* + \alpha$ corresponds to $\left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\left(\begin{smallmatrix} c \\ a \end{smallmatrix}\right)$ (writing the operation in $SK_1(R)$ as multiplication as in [1]). By the Mennicke relations

$$\begin{pmatrix} b \\ a \end{pmatrix}\begin{pmatrix} c \\ a \end{pmatrix} = \begin{pmatrix} ad-1 \\ a \end{pmatrix} = \begin{pmatrix} -1 \\ a \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 1,$$

so $\alpha^* + \alpha = 0$. □

Lemma 2.7. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & \overline{A} \end{array}$$

be a Milnor patching diagram giving a Mayer–Vietoris sequence

$$\cdots \to K_1(\overline{A}) \xrightarrow{\partial} K_0(A) \to \ldots.$$

Then $\partial(\alpha^*) = -\partial(\alpha)^*$.

Proof. If $\alpha \in GL_n(\overline{A})$ then $\partial(\alpha) = [P(\alpha)] - [A^n]$ where

$$\begin{array}{ccccc} P(\alpha) & & \longrightarrow & & A_1^n \\ \downarrow & & & & \downarrow \\ A_2^n & \longrightarrow & \overline{A}^n & \xrightarrow[\approx]{\alpha} & \overline{A}^n. \end{array}$$

is a pullback diagram. We claim that $P(\alpha)^* = P(\alpha^{*-1})$. Then $\partial(\alpha^{*-1}) = [P(\alpha)^*] - [A^n]$ so $-\partial(\alpha^*) = \partial(\alpha^{*-1}) = \partial(\alpha)^*$.

Now $P \approx P(\alpha)$ if and only if there are isomorphisms θ_1, θ_2 so that $\alpha = \overline{\theta_1\theta_2}^{-1}$ in the diagram

$$\begin{array}{ccccc} P & \longrightarrow & A_1 \otimes_A P & \xrightarrow[\approx]{\theta_1} & A_1^n \\ \downarrow & & \downarrow & & \downarrow \\ A_2 \otimes_A P & \longrightarrow & \overline{A} \otimes_A P & \xrightarrow[\approx]{\overline{\theta_1}} & \overline{A}^n \\ {\scriptstyle \theta_2}\downarrow\approx & & {\scriptstyle \overline{\theta_2}}\downarrow\approx & \nearrow{\scriptstyle \alpha} & \\ A_2^n & \longrightarrow & \overline{A}^n & & \end{array}$$

Since P is projective,

$$A_i \otimes_A P^* = A_i \otimes_A \operatorname{Hom}_A(P, A) = \operatorname{Hom}_{A_i}(A_i \otimes_A P, A_i) \xleftarrow[\theta_i^*]{\approx} \operatorname{Hom}_{A_i}(A_i^n, A_i) = A_i^n.$$

Using this we get the diagram

$$\begin{array}{ccccc} P^* & \longrightarrow & A_1 \otimes_A P^* & \xrightarrow[\approx]{\theta_1^{*-1}} & A_1^n \\ \downarrow & & \downarrow & & \downarrow \\ A_2 \otimes_A P^* & \longrightarrow & \overline{A} \otimes_A P^* & \xrightarrow[\approx]{\overline{\theta_1^{*-1}}} & \overline{A}^n \\ {\scriptstyle \theta_2^{*-1}}\downarrow\approx & & {\scriptstyle \overline{\theta_2}^{*-1}}\downarrow\approx & \nearrow{\scriptstyle \beta} & \\ A_2^n & \longrightarrow & \overline{A}^n & & \end{array}$$

So $P^* = P(\beta)$ where $\beta = \overline{\theta_1}^{*-1}\overline{\theta_2}^{*} = \operatorname{Hom}(\overline{\theta_1}^{-1}, 1)\operatorname{Hom}(\overline{\theta_2}, 1) = \operatorname{Hom}(\overline{\theta_2\theta_1}^{-1}, 1) = \operatorname{Hom}(\alpha^{-1}, 1) = \alpha^{*-1}$. □

The example is derived from an example of Bloch and Murthy of a UFD with $NK_0(R) \neq 0$ (see [2]), following an exposition by Murthy's student Hongnian Li. The example is the ring

$$R = k[X,Y,Z]/(Z^7 - X^2 - Y^3)$$

where k is $\mathbb{C}$ or any sufficently large field of characteristic not 2.

The first step is to blow up R by letting $W = X/Z^2$ and $T = Y/Z^2$ giving a ring $A = k[W,Z,T]/(W^2 - Z^2(W - T^3))$. This ring is no longer normal. Its normalization is $B = k[S,T]$ where $W = S(S^2 + T^3)$, $T = T$, and $Z = S^2 + T^3$. The conductor is $C = B(S^2 + T^3)$ so we get a Milnor square

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ A/C & \longrightarrow & B/C \end{array}$$

with $A/C = k[t]$ and $B/C = k[u^2, u^3]$ where $S \mapsto u^3$ and $T \mapsto u^2$. The Mayer–Vietoris sequence gives $\partial : SK_1(k[u^2,u^3]) \xrightarrow{\approx} SK_0(A)$. In [4], M. Krusemeyer has shown that $SK_1(k[u^2,u^3]) \to \Omega_{k/\mathbb{Z}}$ is onto so, for $k = \mathbb{C}$, $SK_0(A)$ is huge and not of exponent 2. The same result holds for R by the following lemma.

Lemma 2.8. *$SK_0(R) \to SK_0(A)$ is onto.*

By lemma 2.6, $SK_1(A/C)$ satisfies $\alpha^* + \alpha = 0$. By Lemma 2.7, $SK_0(A)$ satisfies $\alpha^* = \alpha$. Since it is not of exponent 2, it cannot satisfy $\alpha^* + \alpha = 0$. The same is then true of $SK_0(R)$ which maps onto $SK_0(A)$.

Now A can be graded by $\deg T = 2$, $\deg Z = 6$, and $\deg W = 9$ and R can be graded by $\deg X = 21$, $\deg Y = 14$, and $\deg Z = 6$. Therefore, by Corollary 2.5 we see that if $B = A[t,t^{-1}]$ or $B = R[t,t^{-1}]$ then for some maximal ideal $\mathfrak{m}$ we have $NW_E(B_\mathfrak{m}) \neq 0$. Let $D = B_\mathfrak{m}$, a local ring of dimension 3. We will construct the required unimodular row over D.

Since $NW_E(D) \neq 0$ we have $W_E(D[X]) \neq 0$. By [7, Theorem 2.6], $E_r(D[X])$ is transitive on $Um_r(D[X])$ for $r \geq 5$ so by [8, Th. 5.2(b)], $Um_3(D[X])/E_3(D[X]) \to W_E(D[X])$ is onto. Therefore there is some $v = (v_1, v_2, v_3) \in Um_3(D[X])$ which cannot be transformed into e_1 by an elementary transformation. Since D is local, $v(0) \sim_E e_1$ so v cannot be transformed into $v(0)$ by an elementary transformation.

Finally we sketch a proof of Lemma 2.8 for completeness following Li's exposition. It is easy to see that R is a UFD using [5], since Z is a prime element and $R_Z = k[U,T]_{U^2+T^3}$ (with $U = W/Z$) is a UFD. It is also clear that $R_Z = A_Z$ and this is regular. Suppose that ξ is an element of $SK_0(A)$ and write $\xi = [P] - [A^2]$ where P is projective of rank 2 and $\Lambda^2 P = A$. Since $A/(Z)$ is 1–dimensional, P/ZP will have a unimodular element and therefore there is an epimorphism $P/ZP \xrightarrow{f} A/(Z)$. The element $(f, Z) \in P^* \oplus A$ is unimodular so by [3] we can find an element g in P^* such that $\mathcal{O}_{P^*}(f + Zg)$ has height ≥ 2. Replace f by $f + Zg$ and write $I = \mathcal{O}_{P^*}(f) = f(P)$ If $I = A$ then P is free and we are done. Suppose $\mathrm{ht}(I) = 2$. If $\mathfrak{m}$ is a maximal ideal containing I then $I_\mathfrak{m}$ has height 2 and is generated by two elements $I_\mathfrak{m} = (a,b)$ since $P_\mathfrak{m} = A_\mathfrak{m}^2$ maps onto it. Since $A_Z = R_Z$, $A_\mathfrak{m}$ is a UFD, this implies that a, b is a regular sequence. Using this and localizing we see that the Koszul resolution

$$0 \to \Lambda^2 P \xrightarrow{\varphi} P \xrightarrow{f} I \to 0$$

(where $\varphi(p \wedge q) = f(p)q - f(q)p$) is exact. Since $\Lambda^2 P = A$, this shows that $\xi = -[A/I]$ in $K_0(A)$. Let $J = R \cap I$. Since I has height 2, A/I is finite over k and therefore over R/J. Since Z is a unit in A/I it will also be a unit in R/J so $J + RZ = R$. This implies that if $\mathfrak{n}$ is a maximal ideal of R containing Z then $J_\mathfrak{n} = R_\mathfrak{n}$ while if $Z \notin \mathfrak{n}$ then $R_\mathfrak{n}$ is regular. Therefore R/J has finite projective dimension and defines an element $\eta \in \widetilde{K}_0(R)$. Since R is a UFD, $\operatorname{Pic} R = 0$ so η lies in $SK_0(R)$. The image of η in $SK_0(A)$ is given by $\sum(-1)^i[\operatorname{Tor}_i^R(A, R/J)]$. If $Z \notin \mathfrak{n}$ then $R_\mathfrak{n} = A_\mathfrak{n}$ while if $Z \in \mathfrak{n}$ then $J_\mathfrak{n} = R_\mathfrak{n}$. Therefore $\operatorname{Tor}_i^R(A, R/J) = 0$ for $i > 0$ and the image of η is $[\operatorname{Tor}_0^R(A, R/J)] = [A/JA]$. Now $JA = I$ since this holds when Z is inverted and if $\mathfrak{n}$ contains Z then $J_\mathfrak{n} = R_\mathfrak{n}$. Therefore $-\eta$ maps to ξ proving the lemma.

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