SOME STABLY FREE MODULES WHICH ARE NOT SELF DUAL

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Abstract. We give some examples of stably free modules which are not self dual.

1. Introduction

Let R be a commutative ring. Following Lam [7] we say that a module P is stably free of rank n and type t if $P \oplus R^t \approx R^{r+t}$. Lam observes that such a module is self–dual, i.e. $P \approx P^* = \mathrm{Hom}(P, R)$, if $r \leq 2$ or if r is odd and $t = 1$ (or if $t = 0$ when P is free). This leads to the question of whether there are any other cases. I will give examples to show that this is not the case.

We will use the ring $A_n = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_n]/(\sum x_i^2 + \sum y_i^2 - 1)$ of complex polynomial functions on the $2n+1$–sphere $S^{2n+1} = \{(x, y) \in \mathbb{R}^{2n} | \sum x_i^2 + \sum y_i^2 = 1\}$. This is a subring of $\mathbb{C}(S^{2n+1})$, the ring of continuous complex functions on S^{2n+1}. As usual we write $z_\nu = x_\nu + iy_\nu$ and $\bar{z}_\nu = x_\nu - iy_\nu$ so $A_n = \mathbb{C}[z_0, \dots, z_n, \bar{z}_0, \dots, \bar{z}_n]/(\sum z_\nu \bar{z}_\nu - 1)$.

If P is stably free of type 1 and rank n then $P \oplus R \approx R^{n+1}$ and $(0, 1) \in P \oplus R$ corresponds to the unimodular row $(a_0, \dots, a_n)$ in R^{n+1}. We write $P = P(a_0, \dots, a_n) = R^{n+1}/R(a_0, \dots, a_n)$. If necessary we specify R by writing $P_R(a_0, \dots, a_n)$. In [7] the notation $P = P(a_0, \dots, a_n)$ is used for the kernel of $R^{n+1} \xrightarrow{a_0, \dots, a_n} R$ which is the dual of the module $P(a_0, \dots, a_n)$ as defined here [7] (or see Proposition 2.2). For the results considered here I prefer the notation given above since it makes the statement of Proposition 7.3 more natural. I will also denote the action of $\sigma \in GL_{n+1}(R)$ on the row $a = (a_0, \dots, a_n)$ by $\sigma a = a\sigma^T$ in order to be able to write this action as left multiplication conforming to topological usage. This is equivalent to regarding unimodular "rows" as column vectors.

Let $e_\nu > 0$ for all ν. Suslin [9] has shown that $P(a_0^{e_o}, \dots, a_n^{e_n})$ is free if $e_0 \dots e_n \equiv 0 \mod n!$. while in [11] it was shown that $P(a_0^{e_o}, \dots, a_n^{e_n})$ over A_n is not free if $e_0 \dots e_n \not\equiv 0 \mod n!$. The following is our main result here.

Theorem 1.1. *Let R be any subring of $\mathbb{C}(S^{2n+1})$ which contains A_n. Let n be even and let $e_\nu > 0$ for all ν. Let $P = P_R(z_0^{e_o}, \dots, z_n^{e_n})$. If $P \approx P^*$ then $2e_0 \dots e_n \equiv 0$ mod $n!$.*

It follows that the above cases give the only values of (r, t) for which stably free modules of rank r and type t are self–dual.

Corollary 1.2. *Let R be as in the theorem. Suppose either*

(1) *r is even, $r \geq 4$, and $t \geq 1$ or*
(2) *r is odd, $r \geq 3$, and $t \geq 2$.*

Then there is a stably free R–module of rank r and type t which is not self–dual.

Proof. It is sufficient to consider the cases $t = 1$ for n even and $t = 2$ for n odd since if P is stably free of type t it is also stably free of type s for any $s \geq t$. If $n \geq 4$ is even then $P(z_0, z_1, \dots, z_n)$ is not self dual since $2 \not\equiv 0 \mod n!$. The same is true of $P = P(z_0^2, z_1, \dots, z_n)$ since $4 \not\equiv 0 \mod n!$. In [7] it is shown that $P = R \oplus Q$ for some Q, using the fact that z_0^2, z_1, z_2 is completable. This Q has odd rank $r = n - 1 \geq 3$ and type 2 but $Q \not\approx Q^*$ otherwise P would be isomorphic to P^*. □

Remark 1.3. I do not know if $2e_0 \dots e_n \equiv 0 \mod n!$ implies that $P(a_0^{e_o}, \dots, a_n^{e_n})$ is self dual over any commutative ring.

The proof of the theorem will be topological. I will give two such proofs, one using fairly elementary homotopy theory (except for Bott's calculations of the homotopy groups of some unitary groups), and the other using vector bundles (also using Bott's calculations). A similar proof, using the classification of vector bundles on a sphere by clutching functions, was found independently by Nori. These proofs also give additional information on the possibility of a symplectic structure on P. This gives the following additional result.

Theorem 1.4. *Let R be any subring of $\mathbb{C}(S^{2n+1})$ which contains A_n and let $e_\nu > 0$ for all ν. If $n \equiv 0 \mod 4$ then $P = P_R(z_0^{e_o}, \dots, z_n^{e_n})$ does not have a symplectic structure unless $e_0 \dots e_n \equiv 0 \mod n!$ so that P is free.*

2. Well–known Facts

I will recall here some well–known facts used in the proofs. We begin by recalling some standard results on projective modules defined by unimodular rows. For $x \in R^{n+1}$ let $P(x) = R^{n+1}/Rx$ and let $Q(x) = \{z \in R^{n+1} \mid z \cdot x = 0\}$ where $z \cdot x = \sum_0^n z_i x_i$. Note that in [7] $P(x)$ is used for what is here written $Q(x)$.

Lemma 2.1. *If $x \cdot y = 1$ then $R^{n+1} = Rx \oplus Q(y) = Ry \oplus Q(x)$. Therefore $P(x) \approx Q(y)$ and $P(y) \approx Q(x)$.*

Proof. If $z \in R^{n+1}$ write $z = rx + z'$ where $r = z \cdot y$. Then $z' \cdot y = 0$ and, conversely, $z' \cdot y = 0$ implies $r = z \cdot y$ so the decomposition is unique. □

Proposition 2.2. *Let R be a commutative ring and let $\sum_0^n x_i y_i = 1$ in R. Then $P(x_0, \dots, x_n)^* \approx P(y_0, \dots, y_n)$.*

Proof. For sequences $u_0, \dots, u_n$ and $v_0, \dots, v_n$ let $u \cdot v$ denote $\sum_0^n u_i v_i$ as above. The bilinear form $(u, v) = u \cdot v - (u \cdot y)(v \cdot x)$ on $R^{n+1} \times R^{n+1}$ satisfies $(x, -) = 0$ and $(-, y) = 0$ and therefore induces a pairing $P(x) \times P(y) \to R$. This gives us a map $P(y) \to P(x)^*$. It is injective since if $(u, v) = 0$ for all v then, writing $u = rx + u'$ with $u' \cdot y = 0$ as in Lemma 2.1, we have $(u, v) = (u', v) = u' \cdot v = 0$ for all v and therefore $u' = 0$. To see that the map is onto let $f : P(x) \to R$ and regard f as a map $R^{n+1} \to R$ with $f(x) = 0$. Then $f(u) = u \cdot v$ for some v. Since $x \cdot v = f(x) = 0$, $f(u) = (u, v)$ as required. □

Proposition 2.3. *Let R be a commutative ring and let $a = (a_0, \dots, a_n)$ and $b = (b_0, \dots, b_n)$ be unimodular rows over R.Then $P(a_0, \dots, a_n)$ is isomorphic to $P(b_0, \dots, b_n)$ if and only if there is an element σ in $GL_{n+1}(R)$ such that $\sigma a = b$.*

Proof. If σ exists it gives an automorphism of R^{n+1} taking a to b and therefore inducing an isomorphism of the quotients $P(a)$ and $P(b)$. Conversely, given such an isomorphism, we have $Q(a) \approx Q(b)$ by Lemma 2.1. Write $R^{n+1} = Q(b) \oplus Ra = Q(a) \oplus Rb$, let σ map $Q(b)$ isomorphically to $Q(a)$, and send a to b. □

We next recall some facts from topology. As usual S^n denotes the n–sphere $\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum x_i^2 = 1\}$, and $U(n)$ denotes the unitary group. The notation $\mathbb{C}(X)$ will denote the ring of continuous complex functions on a topological space X. We write $[f]$ to denote the homotopy class of f.

Lemma 2.4. *Let X and Y be topological spaces with base points x_0 and y_0. Let $i : X \to X \times Y$ by $i(x) = (x, y_0)$ and let $j : Y \to X \times Y$ by $j(x) = (x_0, y)$. Then, for $n \geq 2$, $(i_*, j_*) : \pi_n(X, x_0) \oplus \pi_n(Y, y_0) \to \pi_n(X \times Y, (x_0, y_0))$ is an isomorphism. If $f : (S^n, s_0) \to (X, x_0)$ and $g : (S^n, s_0) \to (Y, y_0)$ then $([f], [g])$ in $\pi_n(X, x_0) \oplus \pi_n(Y, y_0)$ maps to $[(f, g)]$ in $\pi_n(X \times Y, (x_0, y_0))$*

Proof. It is obvious that the projections give an isomorphism $\pi_n(X \times Y, (x_0, y_0)) \to \pi_n(X, x_0) \oplus \pi_n(Y, y_0)$ and the composition of this with $(i_*, j_*) : \pi_n(X, x_0) \oplus \pi_n(Y, y_0) \to \pi_n(X \times Y, (x_0, y_0))$ is clearly the identity. Under this composition, $([f], [g])$ maps to $([f], [g])$ in $\pi_n(X, x_0) \times \pi_n(Y, y_0)$ and so does $[(f, g)]$. □

Recall that $U(n) \hookrightarrow GL_n(\mathbb{C})$ is a homotopy equivalence. In fact $GL_n(\mathbb{C})$ is homeomorphic to $U(n) \times \mathbb{R}^{n^2}$ [4, Ch. I,§V, Prop. 3]. I will therefore just state the following results for $U(n)$.

Proposition 2.5 ([8, Th. 25.2])**.** *If $i < 2n$, $\pi_i(U(n)) = \pi_i(U(n+1))$.*

This follows immediately from the homotopy sequence of the fibration $U(n) \to U(n+1) \to S^{2n+1}$. It follows that $\pi_i(U(n))$ is independent of n for $i < 2n$ As usual I will write $\pi_i(U)$ for any $\pi_i(U(n))$ with $i < 2n$.

Theorem 2.6 (Bott [2][3])**.**

1. *$\pi_i(U)$ is 0 for $i \geq 0$ even, and $\mathbb{Z}$ for $i \geq 0$ odd.*
2. *For $n \geq 0$, $\pi_{2n}(U(n)) = \mathbb{Z}/n!\,\mathbb{Z}$.*

Note that the homotopy sequence of the bundle $U(n) \to U(n+1) \to S^{2n+1}$ gives

$$\mathbb{Z} = \pi_{2n+1}(S^{2n+1}) \xrightarrow{\partial} \pi_{2n}(U(n)) \to \pi_{2n}(U(n+1)) = \pi_{2n}(U) = 0$$

showing that $\pi_{2n+1}(S^{2n+1}) = \mathbb{Z} \xrightarrow{\partial} \pi_{2n}(U(n))$ is onto. Let ι be the homotopy class of the identity map of S^{2n+1}. We choose the element $\partial\iota$ as the generator of $\pi_{2n}(U(n))$.

To conclude this section I will give some well–known results on the degree of some mappings of spheres. I will write $\operatorname{sgn} x = +, 0, -$ if $x > 0, x = 0, x < 0$.

Lemma 2.7. *Let $f, g : S^n \to S^n$. Write $f(x) = (f_1(x), \dots, f_n(x))$ and similarly for g. Suppose that $\operatorname{sgn} f_i(x) = \operatorname{sgn} g_i(x)$ for all x and i. Then $\deg f = \deg g$. In particular, this is true if $f_i(x) = r_i(x) g_i(x)$ with $r_i(x) > 0$ for all x and i.*

Proof. Let $h_t(x) = tf(x) + (1-t)g(x)$. This is never 0 for $0 \leq t \leq 1$ so $t \mapsto h_t(x)/\|h_t(x)\|$ gives a homotopy between f and g. □

Lemma 2.8. *Let $f : S^n \to S^n$ send $(x_0, \dots, x_n)$ to $(y_0, \dots, y_n)$. Suppose that $\operatorname{sgn} y_i = \operatorname{sgn} x_i$ for $i > r$ and all $x \in S^n$. Then $S^r = \{x \in S^n | x_i = 0 \text{ for } i > r\}$ is stable under f and $\deg f = \deg(f|S^r)$*

Proof. By induction on $n-r$ it will suffice to consider the case where $r = n-1$ Let $H^+ = \{x|x_n \geq 0\}$ and $H^- = \{x|x_n \leq 0\}$ be the two hemispheres. Then $S^n = H^+ \cup H^-$ and $S^{n-1} = H^+ \cap H^-$. Since H^+ and H$^-$ are contractible, the Mayer–Vietoris sequence gives us an isomorphism $H_n(S^n) \xrightarrow{\approx} H_{n-1}(S^{n-1})$. Since all 4 spaces are stable under f, naturality gives us a commutative diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\approx} & H_{n-1}(S^{n-1}) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ H_n(S^n) & \xrightarrow{\approx} & H_{n-1}(S^{n-1}) \end{array}$$

and the result follows immediately. □

Corollary 2.9. *Let $f : S^n \to S^n$ by $f(x_0, \dots, x_n) = (\epsilon_0 x_0, \dots, \epsilon_n x_n)$ where each ϵ_i is 1 or -1. Then $\deg f = \prod \epsilon_i$.*

Proof. This follows from the lemma with $r = 0$ if $\epsilon_i = 1$ for $i \geq 1$. The given map is a composition of maps of this form (re–indexed) and the degree of a composition is the product of the degrees of the factors. □

Corollary 2.10. *Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere. Let $m_i > 0$ be integers for $i = 0, \dots, n$. Let $f : S^{2n+1} \to S^{2n+1}$ by*

$$f(z_0, \dots, z_n) = \frac{(z_0^{m_0}, \dots, z_n^{m_n})}{\|(z_0^{m_0}, \dots, z_n^{m_n})\|}$$

and let $g : S^{2n+1} \to S^{2n+1}$ by

$$g(z_0, \dots, z_n) = \frac{(\bar{z}_0^{m_0}, \dots, \bar{z}_n^{m_n})}{\|(\bar{z}_0^{m_0}, \dots, \bar{z}_n^{m_n})\|}$$

Then $\deg f = \prod m_i$ and $\deg g = (-1)^{n+1} \prod m_i$.

Proof. If $m_i = 1$ for $i \geq 1$, the formula for $\deg f$ follows from the lemma and the fact that the map $z \mapsto z^m$ on $S^1 \subset \mathbb{C}$ has degree m. The general case follows since f is a composition of such maps (after re–indexing them), up to positive factors r_i as in Lemma 2.7. The map g is the composition of f and complex conjugation $c : S^{2n+1} \to S^{2n+1}$ by $c(z_0, \dots, z_n) = (\bar{z}_0, \dots, \bar{z}_n)$. In terms of real coordinates $z_\nu = x_\nu + iy_\nu$, each x_ν maps to x_ν while each y_ν maps to $-y_\nu$. Therefore c has degree $(-1)^{n+1}$ by Corollary 2.9. □

3. Unimodular rows

Let $f_0, \dots, f_n$ lie in $\mathbb{C}(S^{2n+1})$. These functions define a map $f : S^{2n+1} \to \mathbb{C}^{n+1}$ and the row $f = (f_0, \dots, f_n)$ will be unimodular if and only if the image of f does not contain 0 (so that $\sum f_\nu g_\nu = 1$ where $g_\nu = \bar{f}_\nu / \|f\|$ with $\|f\| = \sqrt{\sum |f_\nu|^2}$). In this case I will write $P(f)$ for $P(f_0, \dots, f_n)$ over the ring $\mathbb{C}(S^{2n+1})$. Since $\mathbb{C}^{n+1} - \{0\}$ has the homotopy type of S^{n+1}, a unimodular row $f : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ has a degree $\deg f$ which is the degree of $S^{2n+1} \to S^{2n+1}$ by $x \mapsto f(x)/\|f(x)\|$.

Lemma 3.1. *Choose $1 \in GL_{n+1}(\mathbb{C})$ as base point and choose a base point $e \in \mathbb{C}^{n+1} - \{0\}$. Define $\pi : GL_{n+1}(\mathbb{C}) \to \mathbb{C}^{n+1} - \{0\}$ by $\sigma \mapsto \sigma e$ Let $f : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ and $\sigma : S^{2n+1} \to GL_{n+1}(\mathbb{C})$ preserve base points. Then $[\sigma f] = [\pi \circ \sigma] + [f]$ in $\pi_{2n+1}(\mathbb{C}^{n+1} - \{0\})$*

Proof. The composition of (σ, f) with the map

$$GL_{n+1}(\mathbb{C}) \times (\mathbb{C}^{n+1} - \{0\}) \to \mathbb{C}^{n+1} - \{0\}$$

is σf. Passing to homotopy classes and using Lemma 2.4 we see that under the maps

$$\pi_{2n+1}(GL_{n+1}(\mathbb{C}), 1) \oplus \pi_{2n+1}(\mathbb{C}^{n+1} - \{0\}, e) \to \pi_{2n+1}(GL_{n+1}(\mathbb{C}) \times (\mathbb{C}^{n+1} - \{0\}), (1, e))$$
$$\to \pi_{2n+1}(\mathbb{C}^{n+1} - \{0\}, e),$$

$([\sigma], [f])$ maps to $[(f, \sigma)]$ and then to $[\sigma f]$. Now $([\sigma], [f])$ is the sum of $([\sigma], 0)$ and $(0, [f])$. These map to $[\pi \circ \sigma]$ and $[f]$ respectively. □

Proposition 3.2. *Let $f, g : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$. If $P(f) \approx P(g)$ then $\deg f \equiv \deg g \mod n!$.*

Proof. By Proposition 2.3 there is an element σ in $GL_{n+1}(\mathbb{C}(S^{2n+1}))$ such that $g = \sigma f$. Regard f and g as maps $S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ and σ as a map $S^{2n+1} \to GL_{n+1}(\mathbb{C})$. Let s_0 be the base point of S^{2n+1} and let $\sigma_0 = \sigma(s_0)$. Since $GL_{n+1}(\mathbb{C})$ is connected, $\sigma_0 f$ is homotopic to f so we can replace f by $\sigma_0 f$ and σ by $\sigma\sigma_0^{-1}$. With these new values we now have $\sigma(s_0) = 1$ and therefore $f(s_0) = g(s_0) = e$ say. By replacing f and g by λf and λg where λ is a positive constant we can assume that $\|e\| = 1$. Note that $f \simeq \lambda f$ and similarly for g. Since $g = \sigma f$, Lemma 3.1 shows that $[g] = [\sigma f] = [\pi \circ \sigma] + [f]$ so it will suffice to show that $[\pi \circ \sigma]$ lies in $n!\, \pi_{2n+1}(\mathbb{C}^{n+1} - \{0\}) = n!\, \mathbb{Z}$. In the diagram

$$\begin{array}{ccc} U(n+1) & \longrightarrow & S^{2n+1} \\ \big\downarrow & & \big\downarrow \\ GL_{n+1}(\mathbb{C}) & \longrightarrow & \mathbb{C}^{n+1} - \{0\} \end{array}$$

the vertical arrows are homotopy equivalences and the horizontal arrows send σ to σe. Therefore the image of $\pi_* : \pi_{2n+1}(GL_{n+1}(\mathbb{C})) \to \pi_{2n+1}(\mathbb{C}^{n+1} - \{0\})$ is the same as that of $\pi_* : \pi_{2n+1} U(n+1) \to \pi_{2n+1}(S^{2n+1})$ This map occurs in the homotopy sequence of the fibration $U(n) \to U(n+1) \to S^{2n+1}$ as

$$\cdots \to \pi_{2n+1}(U(n+1)) \to \pi_{2n+1}(S^{2n+1}) \to \pi_{2n}(U(n)) \to \pi_{2n}(U(n+1)) \ldots$$

Since $\pi_{2n+1}(S^{2n+1}) = \mathbb{Z}$ and, by Theorem 2.6, $\pi_{2n}(U(n)) = \mathbb{Z}/n!\,\mathbb{Z}$ and $\pi_{2n}(U(n+1)) = 0$, we see that the image of $\pi_* : \pi_{2n+1} U(n+1) \to \pi_{2n+1}(S^{2n+1})$ is $n!\,\mathbb{Z}$. Therefore $[\pi \circ \sigma]$ lies in $n!\,\mathbb{Z}$ so $[f] \equiv [g] \mod n!$. □

Lemma 3.3. *Let $f_0, \ldots, f_n$ and $g_0, \ldots, g_n$ in $\mathbb{C}(S^{2n+1})$ satisfy $\sum_0^n f_i g_i = 1$. Let $f, g : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ be the maps defined by these functions. Then $\deg f = (-1)^{n+1} \deg g$.*

Proof. Let $r = \|f\|^2 = \sum f_i \bar{f}_i$. Define $h_i(x, t) = t g_i + (1 - t)\bar{f}_i / r$. Then $\sum f_i h_i = 1$ so $h : X \times I \to \mathbb{C} - \{0\}$ defines a homotopy between g and $\bar{f}/r$. Therefore $\deg g = \deg \bar{f}/r = \deg \bar{f}$. Now $\bar{f}$ is the composition of f with the map $(z_0, \ldots, z_n) \mapsto (\bar{z}_0, \ldots, \bar{z}_n)$ which has degree $(-1)^{n+1}$ by Corollary 2.10. □

Proposition 3.4. *Let $f_0, \ldots, f_n$ and $g_0, \ldots, g_n$ be unimodular rows in $\mathbb{C}(S^{2n+1})$ and let $f, g : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ be the maps defined by these functions. If $P(f)^* \approx P(g)$ then $\deg f \equiv (-1)^{n+1} \deg g \mod n!$.*

Proof. Let $\sum_0^n f_i h_i = 1$. Then $P(f)^* \approx P(h)$ by Lemma 2.2. By Proposition 3.2, $\deg g \equiv \deg h \mod n!$. But by Lemma 3.3, $\deg f = (-1)^{n+1} \deg h$. □

Corollary 3.5. *Let $f_0, \dots, f_n$ be a unimodular row in $\mathbb{C}(S^{2n+1})$. If n is even and $P(f_0, \dots, f_n)$ is self–dual then $2 \deg f \equiv 0 \mod n!$.*

Proof of Theorem 1.1. It will suffice to prove the theorem for the ring $\mathbb{C}(S^{2n+1})$. Since $f = (z_0^{e_o}, \dots, z_n^{e_n})$ has degree $\prod e_i$ by Corollary 2.10, the theorem follows from the proposition. □

Remark 3.6. Suppose n is even and $\sum_0^n f_i g_i = 1$ over $\mathbb{C}(S^{2n+1})$. If $P(f)$ is self dual then $P(f) \approx P(g)$ so that $g = \sigma f$ for some σ in $GL_{n+1}(\mathbb{C}(S^{2n+1}))$. However, it is not possible to find such a σ in the elementary subgroup $E_{n+1}(\mathbb{C}(S^{2n+1}))$ unless $\deg f = 0$ since if $g = \sigma f$ with σ in $E_{n+1}(\mathbb{C}(S^{2n+1}))$ then f is homotopic to g: If $\sigma = e_{i_1 j_1}(a_1) \dots e_{i_r j_r}(a_r)$ let $\sigma(t) = e_{i_1 j_1}(ta_1) \dots e_{i_r j_r}(ta_r)$. Then $t \mapsto \sigma(t) f$ where t goes from 1 to 0 gives the required homotopy. It follows that $\deg f = \deg g$ but $\deg f = -\deg g$ for n even by Lemma 3.3

4. More well–known results

The complex symplectic group $Sp_{2n}(\mathbb{C})$ is defined to be the subgroup of $GL_{2n}(\mathbb{C})$ consisting of all M such that $MJM^T = J$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Some authors use the notation $Sp(n, \mathbb{C})$ for this group. On page 1 of [4], Chevalley points out that $Sp(n)$ and $Sp_{2n}(\mathbb{C})$ are related in the same way as $U(n)$ and $GL_n(\mathbb{C})$ but does not give the easy direct proof of this which is very similar to his treatment of $O(n)$ and $O_n(\mathbb{C})$. For the reader's convenience I will give the proof here.

Proposition 4.1. *The group $Sp_{2n}(\mathbb{C})$ is homeomorphic to $Sp(n) \times \mathbb{C}^{2n^2+n}$.*

Proof. By [4, Ch. I,§V], there is an homeomorphism $U(2n) \times H \to GL_{2n}(\mathbb{C})$ where H is the vector space of Hermitian $2n \times 2n$ matrices. There are two possibilities for the map, $(\sigma, \beta) \mapsto \sigma \exp(\beta)$ or $(\sigma, \beta) \mapsto \exp(\beta)\sigma$. Suppose that $\exp(\beta)\sigma$ lies in $Sp_{2n}(\mathbb{C})$. Then $\exp(\beta)\sigma J \sigma^T \exp(\beta^T) = J$ so $\exp(\beta)\sigma J \sigma^T = J \exp(-\beta^T) = \exp(-J\beta^T J^T)J$ since $J^T = J^{-1} = -J$. Since J lies in $U(2n)$, it follows that $\sigma J \sigma^T = J$ and $\beta = -J\beta^T J^{-1}$ showing that σ lies in $U(2n) \cap Sp_{2n}(\mathbb{C}) = Sp(n)$ and β lies in the set $S = \{\beta \mid \beta J + J\beta^T = 0\}$ which is a vector space over $\mathbb{C}$ of dimension $2n^2 + n$ [4, Ch. I,§VIII]. Conversely, if σ lies in $Sp(n)$ and $\beta J + J\beta^T = 0$ then $\sigma \in Sp_{2n}(\mathbb{C})$ and $\exp(\beta) \in Sp_{2n}(\mathbb{C})$ since $\exp(\beta) J \exp(\beta^T) = \exp(\beta)\exp(J\beta^T J^{-1})J = \exp(\beta)\exp(-\beta)J = J$. □

Proposition 4.2. *There is a fiber bundle $Sp_{2m}(\mathbb{C}) \to GL_{2m}(\mathbb{C}) \to W_{2m}$ where W_{2m} is the space of invertible alternating $2m \times 2m$ matrices over $\mathbb{C}$.*

Proof. $GL_{2m}(\mathbb{C})$ acts on W_{2m} by $\sigma \circ M = \sigma M \sigma^T$. This is transitive since any element M of W_{2m} can be sent to J by a suitable σ. This follows from the fact that the symplectic space defined by M has a symplectic base [1, Th. 3.7]. The matrix associated to such a base is J. The isotropy subgroup of $J \in W_{2m}$ is $Sp_{2m}(\mathbb{C})$ by definition. Therefore, all that is required to show that our map is a fiber bundle is a local section in the neighborhood of J [8, §7.4]. A neighborhood of 1 in $GL_{2m}(\mathbb{C})$ is given by $\{1 + X\}$ where $X \in M_{2m}(\mathbb{C})$ is small. Our map sends $1 + X$ to $J + XJ + JX^T + XJX^T$ so the map of tangent spaces is $X \mapsto XJ + JX^T$. The kernel of this map is the space S considered in the proof of Proposition 4.1 which has dimension $2m^2 + m$ over $\mathbb{C}$. Therefore the image has dimension $2m^2 - m$. Since this is the dimension of W_{2m} the map of tangent spaces is onto and the implicit function theorem gives us the required local section. □

Proposition 4.3. *If $i \le 4n+1$, $\pi_i(Sp(n)) = \pi_i(Sp(n+1))$.*

This follows immediately from the homotopy sequence of the fibration $Sp(n) \to Sp(n+1) \to S^{4n+3}$. It follows that $\pi_i(Sp(n))$ is independent of n for $i \le 4n+1$ As usual I will write $\pi_i(Sp)$ for any $\pi_i(Sp(n))$ with $i \le 4n+1$.

Theorem 4.4 (Bott [3])**.** *For $i \ge 0$ and $i \equiv 0,1,2,3,4,5,6,7 \mod 8$, we have $\pi_i(Sp) = 0,\ 0,\ 0,\ \mathbb{Z},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z},\ 0,\ \mathbb{Z}$.*

5. The symplectic case

We now turn to the proof of Theorem 1.4. As in the previous section it is sufficient to prove the theorem for the ring $\mathbb{C}(S^{2n+1})$. I will prove a somewhat more general result.

Theorem 5.1. *Let $f : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$. If $n \equiv 0 \mod 4$ and $P(f)$ has a symplectic structure then $\deg f \equiv 0 \mod n!$.*

In [5] Krusemeyer defines a unimodular row $a_0, \dots, a_n$ of odd length to be skewly completable if there is an invertible alternating matrix with first row $0, a_0, \dots, a_n$. The following lemma gives a criterion for a stably free module of type 1 to have a symplectic structure.

Lemma 5.2 ([6])**.** *Let $a_0, \dots, a_n$ be a unimodular row with n even. Then $P(a_0, \dots, a_n)$ has a symplectic structure if and only if the row $a_0, \dots, a_n$ is skewly completable.*

Proof. Let $a \cdot b = \sum_0^n a_i b_i = 1$. If $a_0, \dots, a_n$ is skewly completable then R^{n+2} has a symplectic structure and a base $f, e_0, \dots, e_n$ with $< f, e_i >= a_i$. Let $e = \sum_0^n b_i e_i$. Then $< f, e >= 1$ so $H = Rf \oplus Re$ is symplectic. Therefore $R^{n+1} = H \perp Q$ where Q is also symplectic. Now $Q = R^{n+2}/H = R^{n+1}/Re = P(b)$ so $P(b)$ is symplectic and therefore so is $P(b)^* \approx P(a)$. Conversely, if $P(a)$ is symplectic so is $P(a)^* \approx P(b) \approx Q(a)$. Now $R^{n+1} = Rb \oplus Q(a)$ by Lemma 2.1. Let $e_0, \dots, e_n$ be the base of R^{n+1}. Then $e_i - a_i b$ lies in $Q(a)$ since $a \cdot (e_i - a_i b) = 0$. Let $H = Rf \oplus Re$ be as above with $< f, e >= 1$. Form the symplectic space $H \perp Q(a)$ and identify it with $R^{n+2} = Rf \oplus R^{n+1}$ by identifying e with b. Since $< f, Q(a) >= 0$ we have $< f, e_i >=< f, a_i b >=< f, a_i e >= a_i$ as required. □

Suppose now that $P = P(f_0, \dots, f_n)$ over $\mathbb{C}(S^{2n+1})$ has a symplectic structure. Then, by Lemma 5.2, the map $f = (f_0, \dots, f_n) : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ factors as $f : S^{2n+1} \to W_{n+2} \to \mathbb{C}^{n+1} - \{0\}$ where the right hand map sends (a_{ij}) to $(a_{01}, \dots, a_{0n+1})$ omitting the term $a_{00} = 0$. By Proposition 4.2 there is an exact homotopy sequence

$$\dots \to \pi_{2n+1}(GL_{n+2}(\mathbb{C})) \to \pi_{2n+1}(W_{n+2}) \to \pi_{2n}(Sp_{n+2}(\mathbb{C})) \to \dots$$

By Proposition 4.1, $Sp_{n+2}(\mathbb{C})$ has the homotopy type of $Sp(n/2+1)$. By Proposition 4.3 and Theorem 4.4, $\pi_{2n}(Sp_(n/2+1)) = 0$ for $n \equiv 0 \mod 4$ so the map $\pi_{2n+1}(GL_{n+2}(\mathbb{C})) \to \pi_{2n+1}(W_{n+2})$ is onto. We can replace $GL_{n+2}(\mathbb{C})$ by $U(n+2)$ which has the same homotopy type. By Proposition 2.5, $\pi_{2n+1}(U(n+1)) \to \pi_{2n+1}(U(n+2))$ is an isomorphism so $\pi_{2n+1}(U(n+1))$ maps onto $\pi_{2n+1}(W_{n+2})$. We identify $U(n+1)$ with the subgroup $\left(\begin{smallmatrix} 1 & 0 \\ 0 & U(n+1) \end{smallmatrix}\right)$ of $U(n+2)$. Write $J = \left(\begin{smallmatrix} 0 & e \\ 0 & * \end{smallmatrix}\right)$

with $e = (0, \dots, 0, 1, 0, \dots, 0)$. The map $U(n+1) \to U(n+2) \to W_{n+2}$ then sends σ to

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 0 & e \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma^T \end{pmatrix} = \begin{pmatrix} 0 & e\sigma^T \\ 0 & * \end{pmatrix}$$

which maps to $e\sigma^T = \sigma e$ in $\mathbb{C}^{n+1} - \{0\}$. This map $U(n+1) \to \mathbb{C}^{n+1} - \{0\}$ induces the map $\pi_* : \pi_{2n+1}U(n+1) \to \pi_{2n+1}(S^{2n+1})$ considered at the end of the proof of Proposition 3.2 where it was shown that the image of this map is $n!\,\mathbb{Z} \subset \mathbb{Z} = \pi_{2n+1}(S^{2n+1})$. Since $[f]$ lies in this image $\deg f \equiv 0 \mod n!$.

6. A converse theorem

I do not know if $2e_0 \dots e_n \equiv 0 \mod n!$ implies that $P(a_0^{e_o}, \dots, a_n^{e_n})$ is self dual over any commutative ring or even over the ring $A_n = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_n]/(\sum x_i y_i - 1)$. I will show here that this is the case over the ring $\mathbb{C}(S^{2n+1})$ and even over the localized ring $A_n[S^{-1}]$ where S is the set of elements of A_n which have no zeros on S^{2n+1}.

Let X be a compact Hausdorff space and let $\mathbb{C}(X)$ be the ring of continuous complex functions on X. If $f = (f_0, \dots, f_n)$ and $g = (g_0, \dots, g_n)$ are rows of elements of $\mathbb{C}(X)$ (or if they are elements of $\mathbb{C}^{n+1}$), I will write $(f, g) = \sum_0^n f_\nu \bar{g}_\nu$ and $\|f\|^2 = (f, f)$. Note that this is a function on X and not a global Banach algebra type norm. If R is a subring of $\mathbb{C}(X)$ I will say that f has coordinates in R if all f_i lie in R.

Lemma 6.1. *Let X be a compact Hausdorff space. Let $R \subseteq \mathbb{C}(X)$ be a $\mathbb{C}$–subalgebra which is closed under complex conjugation and has the property that if $r \in R$ is never 0 on X then r^{-1} is in R. Let $f, g : X \to \mathbb{C}^{n+1} - \{0\}$ have coordinates in R. If $\|f - g\| < \|f\|$ then $P_R(f) \approx P_R(g)$.*

Proof. For x, y, z in $\mathbb{C}^{n+1}$ with $x \neq 0$, define

$$E(x, y)z = z + \frac{(z, x)}{\|x\|^2}(y - x)$$

Then $E(x, y)x = y$ and $E(x, y)z = z$ if $(z, x) = 0$. Write $\mathbb{C}^{n+1} = \mathbb{C}x \oplus \{z \mid (z, x) = 0\}$. In this decomposition we have

$$y = \frac{(y, x)}{\|x\|^2}x + (y - \frac{(y, x)}{\|x\|^2}x).$$

It follows that $\det E(x, y) = (y, x)/\|x\|^2$. The matrix of $E(f, g)$ has entries

$$E(f, g)_{ij} = \delta_{ij} + (g_i - f_i)\frac{\bar{f}_j}{\|f\|^2}.$$

These lie in the ring R by the hypothesis since $\|f\|$ is never zero. Now $|(f, g) - \|f\|^2| = |(f, g - f)| \leq \|f\|\|g - f\|$ so $|(f, g)| \geq \|f\|^2 - \|f\|\|g - f\|$. If $\|f - g\| < \|f\|$ this implies that $|(f, g)| \geq \|f\|(\|f\| - \|f - g\|) > 0$ Therefore $\det E(f, g) = (g, f)/\|f\|^2$ is never 0 so $E(f, g)$ lies in $GL_{n+1}(R)$. Since $E(f, g)f = g$, this implies that $P(f) \approx P(g)$ by Proposition 2.3. □

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Proposition 6.2. *Let X be a compact Hausdorff space. Let $R \subseteq \mathbb{C}(X)$ be a $\mathbb{C}$–subalgebra which separates the points of X, is closed under complex conjugation,* *and has the property that if $r \in R$ is never 0 on X then r^{-1} is in R. Let $f, g : X \to \mathbb{C}^{n+1} - \{0\}$ have coordinates in R. If $f \simeq g$ then $P_R(f) \approx P_R(g)$.*

Proof. Let $h : X \times I \to \mathbb{C}^{n+1} - \{0\}$ be a homotopy between f and g. Since X is compact we can find a constant m such that $\|h(x,t)\| > m > 0$ for all (x,t) in $X \times I$. By compactness we can find $0 = t_o < t_1 < \cdots < t_N = 1$ in I such that if $h_i(x) = h(x, t_i)$ then $\|h_i(x) - h_{i+1}(x)\| < m/4$ for all $x \in X$. By the Stone–Weierstrass theorem we can find maps $f_i : X \to \mathbb{C}^{n+1} - \{0\}$ with coordinates in R such that $\|h_i - f_i\| < m/4$. We choose $f_0 = h_0 = f$ and $f_N = h_N = g$. Now $\|f_i\| \geq \|h_i\| - \|h_i - f_i\| > 3m/4$ while $\|f_i - f_{i+1}\| < \|f_i - h_i\| + \|f_{i+1} - h_{i+1}\| + \|h_i - h_{i+1}\| < 3m/4$. Therefore $\|f_i - f_{i+1}\| < \|f_i\|$ so $P(f_i) \approx P(f_{i+1})$ by Lemma 6.1. Therefore $P(f) = P(f_0) \approx P(f_N) = P(g)$. □

Theorem 6.3. *Let $R \subseteq \mathbb{C}(S^{2n+1})$ be a $\mathbb{C}$–subalgebra which separates the points of S^{2n+1}, is closed under complex conjugation, and has the property that if $r \in R$ is never 0 on S^{2n+1} then r^{-1} is in R. Let $f, g : S^{2n+1} \to \mathbb{C}^{n+1} - \{0\}$ have coordinates in R. Then $P_R(f) \approx P_R(g)$ if and only if $\deg f \equiv \deg g \mod n!$.*

Proof. The "only if" statement is proved in Proposition 3.2. For the converse we first replace f by ρf for a fixed ρ in $GL_{n+1}(\mathbb{C})$ to make f preserve base points: $f(s_0) = e$, and similarly for g. We then use the fact that the image of $\pi_* : \pi_{2n+1}(GL_{n+1}(\mathbb{C})) \to \pi_{2n+1}(\mathbb{C}^{n+1} - \{0\}) = \mathbb{Z}$ is $n!\,\mathbb{Z}$. Therefore we can find $\tau : S^{2n+1} \to GL_{n+1}(\mathbb{C})$ with $\tau(s_0) = 1$ and $[\pi \circ \tau] = [g] - [f]$ in $\pi_{2n+1}(\mathbb{C}^{n+1} - \{0\})$. By the Stone–Weierstrass theorem we can find σ in $M_{n+1}(R)$ very close to τ. Note that $\sigma(s_0)$ is very close to $\tau(s_0) = 1$ so after replacing σ by $\sigma(s_0)^{-1}\sigma$ we can assume that σ also preserves the base point: $\sigma(s_0) = 1$. Now $t\sigma + (1-t)\tau$ will also be very close to τ for $0 \leq t \leq 1$ and therefore $\det(t\sigma + (1-t)\tau)$ will be very close to $\det \tau$ and, in particular, will have no zeros on S^{2n+1}. Therefore σ will lie in $GL_{n+1}(R)$ by our hypothesis on R and $t\sigma + (1-t)\tau$ will lie in $GL_{n+1}(\mathbb{C}(S^{2n+1}))$. Now $t \mapsto t\sigma + (1-t)\tau$ gives a homotopy of τ with σ so $[\pi \circ \sigma] = [\pi \circ \tau] = [g] - [f]$ in $\pi_{2n+1}(\mathbb{C}^{n+1} - \{0\})$. By Lemma 3.1 we have $[\sigma f] = [g]$ so $\sigma f \simeq g$ and therefore $P(\sigma f) \approx P(g)$ by Proposition 6.2. Since $P(\sigma f) \approx P(f)$ by Proposition 2.3, it follows that $P(f) \approx P(g)$. □

Corollary 6.4. *Let n be even and let R be any subring of $\mathbb{C}(S^{2n+1})$ which contains the localized ring $A_n[S^{-1}]$ where S is the set of elements of A_n which have no zeros on S^{2n+1}. Let $e_0, \ldots, e_n$ be positive integers. Then $P_R(z_0^{e_o}, \ldots, z_n^{e_n})$ is self dual if and only if $2e_0 \ldots e_n \equiv 0 \mod n!$.*

Proof. The "only if" part follows from Theorem 1.1. For the converse it is sufficient to consider the ring $A_n[S^{-1}]$ which satisfies the hypotheses of Theorem 6.3. Let $f = (z_0^{e_o}, \ldots, z_n^{e_n})$ and let $\sum f_i g_i = 1$. By Lemma 2.2 $P_R(z_0^{e_o}, \ldots, z_n^{e_n})^* \approx P(g)$ and by Lemma 3.3, $\deg f = -\deg g$ since n is even. Since $2\deg f = 2e_0 \ldots e_n \equiv 0 \mod n!$, $\deg f \equiv \deg g \mod n!$ so by Theorem 6.3, $P(f) \approx P(g) \approx P(f)^*$. □

7. Vector Bundles

In this section I will give an alternative proof of the main results using vector bundles. Let X be a compact Hausdorff space. By [10], isomorphism classes of finitely generated projective modules over the ring $\mathbb{C}(X)$ of continuous complex functions on X are in 1–1 correspondence with isomorphism classes of complex vector bundles on X. This correspondence is obtained by associating to each complex vector bundle E its module of sections $\Gamma(X, E)$. The most general form of this correspondence was found by Vaserstein [13]. We will only need the case in which X is a compact Hausdorff space here. Let $\mathcal{O}_X = \mathcal{O}$ be the trivial bundle $X \times \mathbb{C}$. Note that $\Gamma(X, \mathcal{O}) = \mathbb{C}(X)$.

If $f_0, \ldots, f_n$ is a unimodular row in $\mathbb{C}(X)$ we can construct the vector bundle $E(f_0, \ldots, f_n)$ corresponding to the module $P(f_0, \ldots, f_n)$ as follows. Let $E(f_0, \ldots, f_n) = E_X(f_0, \ldots, f_n)$ be the cokernel in

$$0 \to \mathcal{O} \xrightarrow{f_0,\ldots,f_n} \mathcal{O}^{n+1} \to E(f_0, \ldots, f_n) \to 0.$$

It is a vector bundle by [10, Prop. 1]. Since the sequence splits [10, Prop. 2] we can apply Γ getting

$$0 \to \Gamma(X, \mathcal{O}) \xrightarrow{f_0,\ldots,f_n} \Gamma(X, \mathcal{O}^{n+1}) \to \Gamma(X, E(f_0, \ldots, f_n)) \to 0.$$

Since $\Gamma(X, \mathcal{O}) = \mathbb{C}(X)$, this shows that $\Gamma(X, E(f_0, \ldots, f_n)) = P(f_0, \ldots, f_n)$ over $\mathbb{C}(X)$.

Lemma 7.1. *Let $g : X \to Y$ be a map of compact Hausdorff spaces and let $f_0, \ldots, f_n$ lie in $\mathbb{C}(Y)$. Then $g^*E_Y(f_0, \ldots, f_n) = E_X(f_0 g, \ldots, f_n g)$*

Proof. Since g^* is an exact functor we get

$$0 \to g^*\mathcal{O}_Y \xrightarrow{g^*(f_0),\ldots,g^*(f_n)} g^*\mathcal{O}_Y^{n+1} \to g^*E_Y(f_0, \ldots, f_n) \to 0.$$

Since $g^*\mathcal{O}_Y = \mathcal{O}_X$ and it is easy to check that $f : \mathcal{O}_Y \to \mathcal{O}_Y$ induces $g^*(f) = f \circ g : \mathcal{O}_X \to \mathcal{O}_X$, the result follows. □

By [8, 8.2], two vector bundles are isomorphic if and only if their associated principal bundles are isomorphic. Principal bundles $G \to P \to X$ with group G are classified by $[X, BG]$ the set of homotopy classes of maps of X into the classifying space BG [8, 19.3]. The bundle E with fiber F is recovered from the principal bundle by forming $E = P \times_G F$.

For vector bundles over a sphere S^n there is an alternative classification [8, 18.6]. If G is pathwise connected, bundles over S^n with group G are classified by $\pi_{n-1}(G)$. The classifying element is $\partial\iota$ where ι is the homotopy class of the identity map of S^n and ∂ is the boundary map in the homotopy sequence

$$\cdots \to \pi_n(P) \to \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(G) \to \pi_{n-1}(P)$$

of the associated principal bundle $G \to P \to S^n$. The element $\partial\iota$ is the homotopy class of the characteristic map defined in [8, §18].

For any Lie group G there is a universal bundle $G \to EG \to BG$ where BG is the classifying space and EG is contractible. Since $\pi_i(EG) = 0$, the homotopy sequence of the universal bundle shows that $\pi_n(BG) \xrightarrow{\partial} \pi_{n-1}(G)$ is an isomorphism.

Lemma 7.2. *This isomorphism sends the classifying element of a bundle with group G over S^n to the homotopy class of the characteristic map.*

Proof. The classifying map $f : S^n \to BG$ induces a bundle map

$$\begin{array}{ccccc} G & \longrightarrow & P & \longrightarrow & S^n \\ \Big\| & & \Big\downarrow & & \Big\downarrow f \\ G & \longrightarrow & EG & \longrightarrow & BG \end{array}$$

and the homotopy sequence gives

$$\begin{array}{ccc} \pi_n(S^n) & \longrightarrow & \pi_{n-1}(G) \\ \downarrow & & \| \\ \pi_n(BG) & \longrightarrow & \pi_{n-1}(G) \end{array}$$

The result follows since ι maps to the class of the characteristic map in $\pi_{n-1}(G)$ and to the classifying element in $\pi_n(BG)$ □

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ with coordinates $z_0, \dots, z_n$. These form a unimodular row since $\sum z_\nu \bar{z}_\nu = 1$.

Proposition 7.3. *The associated principal bundle of $E(z_0, \dots, z_n)$ is the bundle $U(n) \to U(n+1) \to S^{2n+1}$ corresponding to the identification $S^{2n+1} = U(n+1)/U(n)$.*

Proof. We have to show that $E(z_0, \dots, z_n) \cong U(n+1) \times_{U(n)} \mathbb{C}^n$. We identify $U(n)$ with the subgroup $\left(\begin{smallmatrix} 1 & 0 \\ 0 & U(n) \end{smallmatrix}\right)$ of $U(n+1)$. The map $U(n+1) \to S^{2n+1}$ sends $\sigma \in U(n+1)$ to the first column $\bar{\sigma} = \mathrm{Col}_1(\sigma)$ of σ. Write $\mathbb{C}^{n+1} = \mathbb{C} \oplus \mathbb{C}^n$ so that $U(n)$ acts as the identity on $\mathbb{C}$ and as usual on $\mathbb{C}^n$. The exact sequence

$$0 \to \mathbb{C} \to \mathbb{C}^{n+1} \to \mathbb{C}^n \to 0$$

of $U(n)$–modules induces an exact sequence

$$(1) \quad 0 \to U(n+1) \times_{U(n)} \mathbb{C} \to U(n+1) \times_{U(n)} \mathbb{C}^{n+1} \to U(n+1) \times_{U(n)} \mathbb{C}^n \to 0.$$

Now $U(n+1) \times_{U(n)} \mathbb{C} = S^{2n+1} \times \mathbb{C}$ by the map sending (σ, w) to $(\bar{\sigma}, w)$ and $U(n+1) \times_{U(n)} \mathbb{C}^{n+1} = S^{2n+1} \times \mathbb{C}^{n+1}$ by the map sending (σ, w) to $(\bar{\sigma}, \sigma w)$ which has inverse $(s, w) \mapsto (\sigma, \sigma^{-1} w)$ for any σ such that $\bar{\sigma} = s$. The continuity is easily checked using the fact that $U(n+1) \to S^{2n+1}$ has a local cross section. The map $U(n+1) \times_{U(n)} \mathbb{C} \to U(n+1) \times_{U(n)} \mathbb{C}^{n+1}$ sends (s, w) to $(\bar{\sigma}, \sigma w)$ where $\bar{\sigma} = s$. Now $w \in \mathbb{C}$ maps to the column vector $(w, 0, \dots, 0)^T$ in $\mathbb{C}^{n+1}$. Also we have $s = \bar{\sigma} = \mathrm{Col}_1(\sigma)$ so (s, w) maps to (s, w') where, if $s = (z_0, \dots, z_n)$,

$$w' = \begin{pmatrix} z_0 & \\ 0 & \\ \vdots & \bigstar \\ z_n & \end{pmatrix} \begin{pmatrix} w \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} z_0 w \\ \vdots \\ z_n w \end{pmatrix}$$

Therefore the exact sequence (1) is

$$0 \to S^{2n+1} \times \mathbb{C} \xrightarrow{z_0, \dots, z_n} S^{2n+1} \times \mathbb{C}^{n+1} \to U(n+1) \times_{U(n)} \mathbb{C}^n \to 0.$$

showing that $U(n+1) \times_{U(n)} \mathbb{C}^n = E(z_0, \dots, z_n)$. □

The complex vector bundles of rank n on S^{2n+1} are classified by the elements of $\pi_{2n+1}(BU(n)) = \pi_{2n}(U(n))$ which is $\mathbb{Z}/n!\,\mathbb{Z}$ by Theorem 2.6(2). If E is a vector bundle of rank n on S^{2n+1} I will write $\mathrm{cl}(E)$ for its class in $\mathbb{Z}/n!\,\mathbb{Z}$.

Corollary 7.4. *Let $E = E(z_0, \dots, z_n)$ be the vector bundle corresponding to the projective module $P(z_0, \dots, z_n)$ over $\mathbb{C}(S^{2n+1})$. Then $\mathrm{cl}(E) = 1 \mod n!$.*

Proof. The characteristic map for the associated principal bundle is $\partial\iota$ for the bundle $U(n) \to U(n+1) \to S^{2n+1}$. This is the element we have chosen as the generator of $\pi_{2n}U(n) = \mathbb{Z}/n!\,\mathbb{Z}$ $\square$

Theorem 7.5. *Let $e_i > 0$ be integers for $i = 0, \dots, n$. Then*

(1) $\mathrm{cl}(E(z_0^{e_o}, \dots, z_n^{e_n})) = e_0 \cdots e_n \mod n!$ *and*
(2) $\mathrm{cl}(E(\bar{z}_0^{e_o}, \dots, \bar{z}_n^{e_n})) = (-1)^{n+1} e_0 \cdots e_n \mod n!$.

Proof. Let $f : S^{2n+1} \to S^{2n+1}$ by

$$f(z_0, \dots, z_n) = \frac{(z_0^{e_0}, \dots, z_n^{e_n})}{r}$$

where $r = \|(z_0^{e_0}, \dots, z_n^{e_n})\|$. By Lemma 7.1, $f^*E(z_0, \dots, z_n) = E(z_0^{e_o}/r, \dots, z_n^{e_n}/r) \approx E(z_0^{e_o}, \dots, z_n^{e_n})$. The classifying map for this bundle is $S^{2n+1} \xrightarrow{f} S^{2n+1} \xrightarrow{h} BU(n)$ where h is the classifying map for $E(z_0, \dots, z_n)$. This sends ι to $h_*(f_*(\iota))$ but $f_*(\iota) = e_0 \cdots e_n \iota$ by Lemma 2.10 and $h_*(\iota) = 1 \mod n!$ by Corollary 7.4 so the classifying element for $E(z_0^{e_o}, \dots, z_n^{e_n})$ is $e_0 \cdots e_n \mod n!$. For $E(\bar{z}_0^{e_o}, \dots, \bar{z}_n^{e_n})$ the same argument shows the classifying map is hg where g is as in Lemma 2.10 and has degree $(-1)^{n+1} e_0 \cdots e_n$ and the same argument applies. $\square$

Lemma 7.6. *Let $f_0, \dots, f_n$ be a unimodular row over $\mathbb{C}(S^{2n+1})$. Then $P(f_0, \dots, f_n)^* \approx P(\bar{f}_0, \dots, \bar{f}_n)$*

Proof. $\|f\|$ is never 0 so by Proposition 2.2, $P(f_0, \dots, f_n)^* \approx P(\bar{f}_0/\|f\|, \dots, \bar{f}_n/\|f\|) \approx P(\bar{f}_0, \dots, \bar{f}_n)$. $\square$

Proof of Theorem 1.1. Let n be even and let $e_\nu > 0$ for all ν. Let $P = P(z_0^{e_o}, \dots, z_n^{e_n})$ over A_n. If $P \approx P^*$ then the same is true over $\mathbb{C}(S^{2n+1})$ which contains A_n. By Lemma 7.6 $P(z_0^{e_0}, \dots, z_n^{e_n}) \approx P(\bar{z}_0^{e_0}, \dots, \bar{z}_n^{e_n})$ so that $E(z_0^{e_0}, \dots, z_n^{e_n}) \approx E(\bar{z}_0^{e_0}, \dots, \bar{z}_n^{e_n})$. Therefore $\mathrm{cl}(E(z_0^{e_0}, \dots, z_n^{e_n})) = \mathrm{cl}(E(\bar{z}_0^{e_0}, \dots, \bar{z}_n^{e_n}))$. By Theorem 7.5, this implies that $e_0 \dots e_n \equiv (-1)^{n+1} e_0 \dots e_n \mod n!$. Since n is even, this is equivalent to $2e_0 \dots e_n \equiv 0 \mod n!$. $\square$

Corollary 6.4 can be deduced from the above by applying the results of [12, Th. 11.1] to prove it over the ring $A_n[S^{-1}]$. The general case is an immediate consequence of this.

Proof of Theorem 1.4. If $P(z_0^{e_o}, \dots, z_n^{e_n})$ has a symplectic structure so does $E(z_0^{e_o}, \dots, z_n^{e_n})$ so the group of the bundle can be reduced to the complex symplectic group $Sp_n(\mathbb{C})$. This group has the same homotopy type as $Sp(n/2)$ so $\mathrm{cl}(E(z_0^{e_0}, \dots, z_n^{e_n}))$ will lie in the image of $\pi_{2n}(Sp(n/2)) \to \pi_{2n}(U(n))$. Since $n \equiv 0 \mod 4$, $\pi_{2n}(Sp(n/2)) = 0$ by Theorem 4.4. Therefore the class of our bundle must be 0. $\square$

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