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ON THE STRAIGHTENING LAW FOR MINORS OF A MATRIX

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In memory of Gian Carlo Rota

Abstract. We give a simple new proof for the straightening law of Doubilet, Rota, and Stein using a generalization of the Laplace expansion of a determinant.

1. Introduction

The straightening law was proved in [5] by Doubilet, Rota, and Stein generalizing earlier work of Hodge [6]. Since then a number of other proofs have been given [4, 2, 1]. The object of the present paper is to offer yet another proof of this result based on a generalization of the Laplace expansion of a determinant. This proof has the advantage (to some!) of not requiring any significant amount of combinatorics, Young diagrams, etc. On the other hand, for the same reason, it does not show the interesting relations between the straightening law and invariant theory but these are very well covered in the above references and in [3]. For completeness, I have also included a proof of the linear independence of the standard monomials.

2. Laplace Products

Let $X = (x_{ij})$ be an $m \times n$ matrix where $1 \le i \le m$ and $1 \le j \le n$. If $A \subseteq \{1, \dots, m\}$, $B \subseteq \{1, \dots, n\}$, and A and B have the same number of elements we define $X(A|B)$ to be the minor determinant of X with row indices in A and column indices in B. I will usually just write $(A|B)$ for $X(A|B)$ when it is clear what X is. I will write $|A|$ for the number of elements in A. We set $X(A|B) = 0$ if $|A| \neq |B|$.

I will write $\widetilde{A}$ for the complement $\{1, \dots, m\} - A$ and $\widetilde{B}$ for $\{1, \dots, n\} - B$. Also $\sum A$ will denote the sum of the elements of A.

Definition 2.1. If $m = n$, we define the Laplace product $X\{A|B\}$ to be $X\{A|B\} = (-1)^{\sum A + \sum B}(A|B)(\widetilde{A}|\widetilde{B})$.

If X is understood, I will just write $\{A|B\}$ for $X\{A|B\}$. This notation is, of course, for this paper only and is not recommended for general use.

The terminology comes from the Laplace expansion

$$\det X = \sum_{|S|=|B|} \{S|B\} = \sum_{|T|=|A|} \{A|T\} \tag{1}$$

where A and B are fixed.

The following lemma explains the sign in Definition 2.1.

Lemma 2.2. *Let $y_{ij} = x_{ij}$ if (i, j) lies in $A \times B$ or in $\widetilde{A} \times \widetilde{B}$ and let $y_{ij} = 0$ otherwise. Then $X\{A|B\} = \det(y_{ij})$.*

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[1] I would like to thank Darij Grinberg for many corrections to an earlier version of this paper

Proof. Rearrange the rows and columns of $Y = (y_{ij})$ so that those with indices in A and B lie in the upper left hand corner. The resulting matrix has determinant $(A|B)(\widetilde{A}|\widetilde{B})$. The sign of the permutation of rows and columns is $(-1)^{\sum A + \sum B}$ by the next lemma. ☐

Lemma 2.3. *Let $A = \{a_1 < \cdots < a_p\}$ be a subset of $\{1, \ldots, n\}$ and let $\{c_1 < \cdots < c_q\} = \{1, \ldots, n\} - A$. Then the sign of the permutation taking $\{1, \ldots, n\}$ to $\{a_1, \ldots, a_p, c_1, \ldots, c_q\}$ is $(-1)^{\sum(a_i - i)}$.*

Proof. Starting with $\{1, \ldots, n\}$ move a_1 to position 1, then a_2 to position 2, etc., each time keeping the remaining elements in their given order. The number of transpositions used is $\sum(a_i - i)$. ☐

The Laplace expansion (1) gives us a non-trivial relation between the Laplace products of X. This suggests looking for more general relations of the form

$$\sum_i a_i\{S_i|T_i\} = 0 \tag{2}$$

with constant a_i.

Since $\{A|B\}$ is multilinear in the rows of X it will suffice to check a relation (2) for the case in which the rows of X are all of the form $0, \ldots, 0, 1, 0, \ldots, 0$. Since $\{A|B\}$ is also multilinear in the columns of X, all terms of (2) will be 0 unless there is a 1 in each column. Therefore it will suffice to check a relation (2) for the case where X is a permutation matrix, $X = P(\sigma^{-1}) = (\delta_{\sigma i,j})$. To do this we first compute $\{A|B\}$ for this X.

Lemma 2.4. *If $X = (\delta_{\sigma i,j})$ is a permutation matrix $P(\sigma^{-1})$ then $\{A|B\} = \operatorname{sgn}\sigma$ if $\sigma A = B$ and $\{A|B\} = 0$ otherwise.*

This is immediate from Lemma 2.2 and the fact that $\det X$ becomes 0 if any entry 1 is replaced by 0.

Corollary 2.5. *A relation $\sum_i a_i\{S_i|T_i\} = 0$ between Laplace products (with constant a_i) holds if and only if for each σ in $\mathcal{S}_n$ we have $\sum a_i = 0$ over those i with $\sigma S_i = T_i$.*

Theorem 2.6. *For given A and B we have*

$$\sum_{V \subseteq B}\{A|V\} = \sum_{U \supseteq A}\{U|B\}$$

Proof. As above, it will suffice to check this when X is a permutation matrix, $X = P(\sigma^{-1}) = (\delta_{\sigma i,j})$. For this X the right hand side is $\operatorname{sgn}\sigma$ if $\sigma A \subseteq B$ and otherwise is 0. The same is true of the left hand side. ☐

In particular, we recover the Laplace expansion (1) by setting $A = \emptyset$ or by setting $B = \{1, \ldots, n\}$.

We will show later that, for generic X, all linear relations between Laplace products are consequences of those in Theorem 2.6.

Corollary 2.7. *For given A, B, and for $C \subseteq B$ we have*

$$\sum_{C \subseteq V \subseteq B}\{A|V\} = \sum_{\substack{U \supseteq A \\ W \subseteq C}}(-1)^{|W|}\{U|B - W\}$$

Proof. By Theorem 2.6, the right hand side is

$$\sum_{W\subseteq C}(-1)^{|W|}\sum_{V\subseteq B-W}\{A|V\}=\sum_{V\subseteq B}\sum_{W\subseteq C-V}(-1)^{|W|}\{A|V\}$$

which is equal to the left hand side since the inner sum is 0 unless $C \subseteq V$. □

Recall that $\widetilde{A}$ here denotes the complement of A.

Corollary 2.8. *For given A, B,*

$$\sum_{\substack{U\supseteq A\\ W\supseteq B}}(-1)^{|\widetilde{W}|}\{U|W\}=\sum_{V\subseteq B}\{A|\widetilde{V}\}$$

Proof. Set $B = \{1, \dots, n\}$ in Corollary 2.7, getting

$$\sum_{V\supseteq C}\{A|V\}=\sum_{\substack{U\supseteq A\\ W\subseteq C}}(-1)^{|W|}\{U|\widetilde{W}\}$$

Replace C, V, W by $\widetilde{B}$, $\widetilde{V}$, $\widetilde{W}$ where B is now the B given in Corollary 2.8. □

3. Straightening Laplace products

If $S = \{s_1 < \cdots < s_p\}$ and $T = \{t_1 < \cdots < t_q\}$ are subsets of $\{1, \cdots, n\}$, we define a partial ordering $S \leq T$ as in [3] to mean $p \geq q$ and $s_\nu \leq t_\nu$ for all $\nu \leq q$. Equivalently $S \leq T$ if and only if $|S \cap \{1, \cdots, r\}| \geq |T \cap \{1, \cdots, r\}|$ for all $1 \leq r \leq n$. Note that $S \supseteq T$ implies $S \leq T$ and $S \supset T$ implies $S < T$.

As above let $\widetilde{S}$ be the complement $\{1, \dots, n\} - S$. For want of a better terminology, I will say that S is good if $S \leq \widetilde{S}$ and that S is bad otherwise.

The following theorem is our straightening law for Laplace products.

Theorem 3.1. *For any $\{A|B\}$ we have $\{A|B\} = \sum \pm\{A_i|B_i\}$ where $A_i \leq A$, $B_i \leq B$ and A_i and B_i are good.*

Note that some of the (A_i, B_i) may be equal.

Proof. By induction on the set of pairs of subsets of $\{1, \cdots, n\}$ partially ordered by $(A, B) \leq (C, D)$ if $A \leq C$ and $B \leq D$, it is sufficient to prove that if A is bad, then $\{A|B\} = \sum \pm\{A_i|B_i\}$ with $A_i < A$ and $B_i \leq B$, and, similarly, if B is bad then $\{A|B\} = \sum \pm\{A_i|B_i\}$ with $A_i \leq A$ and $B_i < B$. Each statement implies the other by transposing our matrix.

Suppose first that $|A| = |B| < n/2$. Note both A and B are bad in this case. By Corollary 2.8 we have

$$\sum_{\substack{U\supseteq A\\ W\supseteq B}}\pm\{U|W\}=0.$$

since the other side in Corollary 2.8 is 0 because $|A| < n/2 < |\widetilde{V}|$. One term of this sum is $\pm\{A|B\}$ while all other nonzero terms have the form $\pm\{U|W\}$ where $U < A$ and $W < B$.

In the remaining case $|A| = |B| \geq n/2$ we use an argument similar to that of Hodge [6]. Suppose B is bad. Let $B = \{i_1 < \cdots < i_p\}$ and let $\widetilde{B} = \{j_1 < \cdots < j_q\}$. Since B is bad and $q \leq p$, we have $i_\nu > j_\nu$ for some ν which we choose minimal. Let $D = B \cup \{j_1, \dots, j_\nu\}$ and let $C = \{j_1 < \cdots < j_\nu < i_\nu < \cdots < i_p\}$. Apply

Corollary 2.7 with D in place of B. The left hand side is 0 since $|A| = p < |C| = p+1$ so we get

$$\sum_{\substack{U \supseteq A \\ W \subseteq C}} (-1)^{|W|}\{U|D-W\} = 0$$

The term with $U = A$ and $W = \{j_1, \ldots, j_\nu\}$ is $\pm\{A|B\}$. This is the only term of the form $\pm\{U|B\}$ since $|U| = |B|$ and $U \supseteq A$. In the remaining terms we have $U \leq A$ since $U \supseteq A$. In these terms $D - W \neq B$ so if $W \subseteq \{j_1 < \cdots < j_\nu\}$ then $D - W \supset B$ and therefore $D - W < B$. In any case we have $|D-W| = |U| \geq |A| = |B|$. If W contains some i_μ then $D - W$ is obtained from B by removing some (and at least one) of the elements $\{i_\nu < \cdots < i_p\}$ and replacing them by at least as many of the smaller elements $\{j_1 < \cdots < j_\nu\}$. This operation does not decrease the size of the sets $(D-W) \cap \{1, \cdots, r\}$ for $r \geq 1$ so $|(D-W) \cap \{1, \cdots, r\}| \geq |B \cap \{1, \cdots, r\}|$. Therefore $D - W < B$. □

4. The straightening law for minors

We now use a simple trick to generalize Theorem 3.1 to the case of products of any two minors of a rectangular matrix $X = (x_{ij})$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. Recall that $(S|T)$ is the minor of X with row indices in S and column indices in T. We set $(S|T) = 0$ if $|S| \neq |T|$. As above we write $(S', S'') \leq (T', T'')$ if $S' \leq T'$ and $S'' \leq T''$.

The following theorem is the straightening law for minors.

Theorem 4.1. *If $(S', T') \not\leq (S'', T'')$ then*

$$(S'|T')(S''|T'') = \sum \pm(S'_i|T'_i)(S''_i|T''_i)$$

where $(S'_i, T'_i) < (S', T')$ and $(S'_i, T'_i) \leq (S''_i, T''_i)$.

We need two lemmas for the proof. By an order preserving map I mean one satisfying $f(a) \leq f(b)$ if $a \leq b$.

Lemma 4.2. *Let U' and U'' be finite subsets of a totally ordered set and let $k = |U'| + |U''|$. Then there is an order preserving map $f : K = \{1, \ldots, k\} \to U' \cup U''$ and disjoint subsets K' and K'' of K with $K = K' \sqcup K''$ such that*

- **(a):** *f maps K' isomorphically onto U' and K'' isomorphically onto U''*
- **(b):** *If $a, b \in K$, $f(a) = f(b)$, and $a < b$, then $a \in K'$ and $b \in K''$.*

Proof. Let L' and L'' be disjoint sets in $1-1$ correspondence with U' and U''. Let $L = L' \sqcup L''$ and let $f : L \to U' \cup U''$ map L' bijectively onto U' and L'' bijectively onto U''. Define an ordering on L by setting $a < b$ if $f(a) < f(b)$ or if $f(a) = f(b)$ with $a \in L'$ and $b \in L''$. It is easy to check that this defines a total ordering preserved by f. Since L is a totally ordered finite set of order k it is isomorphic to $K = \{1, \ldots, k\}$ and we can substitute K for L letting K' and K'' correspond to L' and L''. (a) is clear and (b) follows from the definition of the ordering/ □

If f is order preserving and is injective on P and Q then $P \leq Q$ implies $f(P) \leq f(Q)$ since the injectivity guarantees that $|f(P)| = |P|$ and similarly for Q.

Lemma 4.3. *In the situation of Lemma 4.2 if P is a subset of K on which f is injective then $P < K'$ implies that $f(P) < f(K') = U'$.*

Proof. It is clear that $f(P) \le f(K')$. We must show that $f(P) \ne f(K')$. Suppose $f(P) = f(K')$. Injectivity shows that $|P| = |f(P)| = |f(K')| = |K'|$. Let $P = \{u_1 < \cdots < u_p\}$ and $K' = \{v_1 < \cdots < v_p\}$. Since f is order preserving and injective on P and on K we have $f(P) = \{f(u_1) < \cdots < f(u_p)\}$ and $f(K') = \{f(v_1) < \cdots < f(v_p)\}$ Now $u_\nu \le v_\nu$ for all ν since $P < K'$ but $f(u_\nu) = f(v_\nu)$ for all ν since $f(P) = f(K')$. Since $P < K'$, $u_\nu = v_\nu$ can't hold for all ν otherwise P and K' would be equal, so for some ν we have $u_\nu < v_\nu$. But since $f(u_\nu) = f(v_\nu)$, Lemma 4.2(b) shows that v_ν must lie in K'' which is a contradiction. $\square$

Proof of Theorem 4.1. We can assume that $|S'| = |T'|$ and $|S''| = |T''|$ since otherwise the left hand side is 0. Let $k = |S'| + |S''| = |T'| + |T''|$ and apply Lemma 4.2 to $U' = S'$ and $U'' = S''$ getting an order preserving map $\varphi : I \to S' \cup S''$ with $I = I' \sqcup I''$(disjoint union), I' mapping isomorphically to S', and I'' mapping isomorphically to S''. Similarly define $\psi : J \to T' \cup T''$ with $J = J' \sqcup J''$. Note that $I = J = \{1, \dots, k\}$. We call them I and J to distinguish their use as row and column indices.

Define a $k \times k$ matrix Y indexed by I and J by setting $y_{ij} = x_{\varphi(i)\psi(j)}$. Then, for $P \subseteq I$ and $Q \subseteq J$, we have $Y(P|Q) = X(\varphi(P)|\psi(Q))$ if $\varphi|P$ and $\psi|Q$ are injective while $Y(P|Q) = 0$ otherwise since two rows or columns will be equal.

By Theorem 3.1 we have $Y\{I'|J'\} = \sum \pm Y\{I'_i|J'_i\}$ which we can write as

$$Y(I'|J')Y(I''|J'') = \sum \pm Y(I'_i|J'_i)Y(I''_i|J''_i) \tag{3}$$

where $I''_i = I - I'_i = \widetilde{I}'_i$ and $J''_i = J - J'_i = \widetilde{J}'_i$. By Theorem 3.1 we see that $I'_i \le I'$, $J'_i \le J'$ and that I'_i and J'_i are good so that $I'_i \le I''_i$ and $J'_i \le J''_i$. By omitting all 0 terms in (3) we can insure that φ is injective on all I'_i and all I''_i and that ψ is injective on all J'_i and J''_i.

Let $S'_i = \varphi(I'_i)$, $S''_i = \varphi(I''_i)$ and similarly for T. Because of the injectivity we can write (3) (with the 0 terms removed) as

$$(S'|T')(S''|T'') = \sum \pm (S'_i|T'_i)(S''_i|T''_i) \tag{4}$$

where $(S'_i, T'_i) \le (S', T')$ and $(S'_i, T'_i) \le (S''_i, T''_i)$.

If I' and J' are good then $I' \le I''$ and $J' \le J''$ which implies that $S' \le S''$ and $T' \le T''$ contrary to the hypothesis. Therefore one of I' and J', say I', must be bad. Since I' is bad, I'_i is good, and $I'_i \le I'$, we have $I'_i < I'$. By Lemma 4.3 it follows that $S'_i < S'$ showing that $(S'_i, T'_i) < (S', T')$. The same argument applies if J' is bad. $\square$

Remark 4.4. Suppose $A = A' \sqcup A''$ where A' and A'' are disjoint and $\varphi : A \to B$ is onto. Suppose φ is injective on A' and on A''. Let $B' = \varphi(A')$ and $B'' = \varphi(A'')$. Then $B' \cap B'' = \{x | |\varphi^{-1}(x)| = 2\}$ is independent of A' and A''. We can think of $B = \varphi(A)$ as a set with multiplicities where the multiplicity of a point x is the order of $\varphi^{-1}(x)$. If we think of B' and B'' as sets with all points of multiplicity 1 then $B' \cup B'' = B$ as sets with multiplicities. Applying this remark to the maps $\varphi : I \to S' \cup S''$ and $\psi : J \to T' \cup T''$ defined in the proof of Theorem 4.1 we see that the sets given by our proof of Theorem 4.1 satisfy $S'_i \cup S''_i = S' \cup S''$ and $T'_i \cup T''_i = T' \cup T''$ as multisets.

5. Standard monomials

We say that a productl $(A_1|B_1)\cdots(A_r|B_r)$ of the minors of a matrix X is a standard monomial if $A_1 \le A_2 \le \cdots \le A_r$ and $B_1 \le B_2 \le \cdots \le B_r$. We regard two standard monomials which only differ by factors of the form $(\emptyset|\emptyset)$ as identical. The following is an easy consequence of Theorem 4.1.

Corollary 5.1. *Any polynomial in the entries of X is a linear combination of standard monomials in the minors of X.*

Proof. Since $x_{ij} = (\{i\}|\{j\})$, it is clear that any such polynomial is a linear combination of products of the minors of X. We show that any product $(A_1|B_1)\cdots(A_r|B_r)$ with r factors is a linear combination of standard monomials with r factors by induction on r and on (A_1, B_1) in the finite partially ordered set of pairs of subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$. By induction on r we can assume that $(A_2, B_2) \le \cdots \le (A_r, B_r)$. If $(A_1, B_1) \le (A_2, B_2)$ or $r = 1$ we are done. If not, Theorem 4.1 shows that $(A_1|B_1)(A_2|B_2) = \sum \pm(C_i|D_i)(P_i|Q_i)$ where $(C_i, D_i) < (A_1, B_1)$ so we are done by induction on (A_1, B_1). $\square$

Remark 5.2. It follows from Remark 4.4 that if we write $(A_1|B_1)\cdots(A_r|B_r)$ as a linear combination of standard monomials $(A_1^{(i)}|B_1^{(i)})\cdots(A_{r_i}^{(i)}|B_{r_i}^{(i)})$ then $\bigcup_j A_j^{(i)} = \bigcup_j A_j$ and $\bigcup_j B_j^{(i)} = \bigcup_j B_j$ for all i, counting multiplicities. In other words the two sides have the same content in the sense of [2].

To conclude, we give a proof of the following theorem which is rather similar to the proof in [4] but which uses no combinatorial constructions. By a generic matrix we mean one whose entries are distinct indeterminates.

Theorem 5.3. *If X is a generic matrix, the standard monomials in the minors of X are linearly independent.*

Before giving the proof we review some results about ordering monomials. Given a totally ordered set of indeterminates, we can order the monomials in these indeterminates as follows. If x is an indeterminate and m is such a monomial write $\operatorname{ord}_x m$ for the number of times x occurs in m. If m_1 and m_2 are monomials define $m_1 > m_2$ to mean $\operatorname{ord}_x m_1 > \operatorname{ord}_x m_2$ for some x while $\operatorname{ord}_y m_1 = \operatorname{ord}_y m_2$ for all $y > x$. It is easy to check that this defines a total ordering on the set of monomials.

Lemma 5.4. *If $u_1, u_2, \cdots, u_k$ and $v_1, v_2, \cdots, v_k$ are monomials with $u_i \le v_i$ for all i and $u_i < v_i$ for some i then $u_1 u_2 \cdots u_k < v_1 v_2 \cdots v_k$.*

It is sufficient to show $a < b$ implies $ac < bc$ and then replace the u_i's by the v_i's one by one.

It follows that if f and g are linear combinations of monomials, the leading monomial of fg is the product of the leading monomials of f and g.

Proof of Theorem 5.3. We specialize X to a matrix of the form $X = YZ$ where Y is a generic $m \times N$ matrix, Z is a generic $N \times n$ matrix and N is sufficiently large. By the classical Binet–Cauchy theorem we have

$$X(A|B) = \sum_S Y(A|S)Z(S|B). \tag{5}$$

This just expresses the functoriality of the exterior product:

$$\bigwedge(X) = \bigwedge(Y)\bigwedge(Z) \tag{6}$$

. By omitting 0 terms in equation (5) we can assume that $|A| = |B| = |S| = p$. In the situation of equation (5) let $Y = (y_i^{(\nu)})_{1\le i\le m,\, 1\le \nu\le N}$ and $Z = (z_j^{(\nu)})_{1\le\nu\le N\, 1\le j\le n}$ be generic matrices where the indeterminates $y_i^{(\nu)}$ and $z_j^{(\nu)}$ are all distinct. Then $X = YZ$ has entries $x_{ij} = \sum_{\nu=1}^N y_i^{(\nu)} z_j^{(\nu)}$. We order the indeterminates as follows:

$$y_1^{(1)} > \cdots > y_m^{(1)} > z_1^{(1)} > \cdots > z_n^{(1)} > y_1^{(2)} > \cdots > y_m^{(2)} > \cdots \tag{7}$$

and order the monomials in these indeterminates as described above.

If $A = \{a_1 < \cdots < a_p\}$ and $S = \{s_1 < \cdots < s_p\}$, then

$$Y(A|S) = \sum_{\sigma\in\mathcal{S}_p} \pm y_{a_{\sigma 1}}^{(s_1)} \cdots y_{a_{\sigma p}}^{(s_p)}. \tag{8}$$

The leading monomial of $Y(A|S)$ is $y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)}$ since $y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)} > y_{a_{\sigma 1}}^{(s_1)} \cdots y_{a_{\sigma p}}^{(s_p)}$ for $\sigma \neq 1$. To see this, let i be least such that $\sigma i \neq i$. Then $\sigma i > i$ so $y_{a_{\sigma i}}^{(s_i)} < y_{a_i}^{(s_i)}$. Choose $x = y_{a_i}^{(s_i)}$. Then $\operatorname{ord}_x y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)} > \operatorname{ord}_x y_{a_{\sigma 1}}^{(s_1)} \cdots y_{a_{\sigma p}}^{(s_p)}$ while $\operatorname{ord}_z y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)} = \operatorname{ord}_z y_{a_{\sigma 1}}^{(s_1)} \cdots y_{a_{\sigma p}}^{(s_p)}$ for $z > x$.

Similarly, if $B = \{b_1 < \cdots < b_p\}$, then the leading monomial of $Z(S|B)$ is $z_{b_1}^{(s_1)} \cdots z_{b_p}^{(s_p)}$. The various terms on the right hand side of (5) have leading monomials $y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)} z_{b_1}^{(s_1)} \cdots z_{b_p}^{(s_p)}$. Of these, the one with $s_i = i$ for all i is the largest by Lemma 5.4. Therefore, if $N \ge |A| = |B|$, the leading monomial of $X(A|B)$ is $y(A)z(B)$ where $y(A) = y_{a_1}^{(1)} \cdots y_{a_p}^{(p)}$ and $z(B) = z_{b_1}^{(1)} \cdots z_{b_p}^{(p)}$. It follows that the leading term of $(A_1|B_1)\cdots(A_r|B_r)$ is $y(A_1)\cdots y(A_r)z(B_1)\cdots z(B_r)$.

Now if $A_1 \le A_2 \le \cdots \le A_r$, and if $N \ge |A_1|$, we can recover A_1 from $M = y(A_1)\cdots y(A_r)$ as follows. Let $A_i = \{a_{i1} < \cdots < a_{ip_i}\}$. Then

$$M = \prod_j y_{a_{j1}}^{(1)} \cdots \prod_j y_{a_{js}}^{(s)} \cdots \tag{9}$$

Since $A_1 \le A_2 \le \cdots \le A_r$ we have $a_{1s} \le a_{2s} \le \cdots$, and we see that a_{1s} is the least c such that $y_c^{(s)}$ occurs in M. Note that $p_1 \ge p_2 \ge \cdots$ so that a_{1s} will exist if a_{is} does.

Since $M/y(A_1) = y(A_2)\cdots y(A_r)$ we see that M determines A_2, A_3, etc. if $N \ge |A_1|$. Similarly $z(B_1)\cdots z(B_r)$ determines B_1, B_2, etc. Therefore the leading monomials of the standard monomials $(A_1|B_1)\cdots(A_r|B_r)$ with $N \ge |A_1|$ and $N \ge |B_1|$ are all distinct and the theorem follows since N can be arbitrarily large. □

Corollary 5.5. *For a generic square matrix X, all linear relations between the Laplace products of X are consequences of those in Theorem 2.6.*

Proof. By Theorem 3.1 the space of all Laplace products of X is spanned by those of the form $\{A|B\}$ with A and B good. For these $\{A|B\} = \pm(A|B)(\widetilde{A}|\widetilde{B})$ is a standard monomial so these $\{A|B\}$ are linearly independent. Since the only relations needed to prove Theorem 3.1 are those of Theorem 2.6, the result follows. □

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