Warmup: Vertex transitive graphs

We do not allow multiple edges for graphs. Let $G$ be a connected graph. Then $G$ can be turned into a metric space by setting the distance of two vertices to be the length of the shortest path between them.

A graph is called locally finite, if every vertex in it has finitely many neighbors.

**Theorem 1** Let $G = (V, E)$ be an infinite, locally finite connected graph. Then there exists a half-infinite geodesic in $G$, that is, a function $f : \mathbb{N} \to V$ such that for all $i < j$, we have
\[
d(f(i), f(j)) = j - i
\]
where $d$ is the path metric.

Now we define various morphisms.

**Definition 2** Let $(V_1, E_1)$ and $(V_2, E_2)$ be graphs. A map $f : V_1 \to V_2$ is a homomorphism if for all $(x, y) \in E_1$ we have $(f(x), f(y)) \in E_2$. A bijective homomorphism such that $f^{-1}$ is also a homomorphism is called an isomorphism. An isomorphism of $(V, E)$ with itself is an automorphism.

The set of automorphisms of a graph $G$ forms a group, denoted by $\text{Aut}(G)$.

**Definition 3** A graph $G$ is vertex transitive, if $\text{Aut}(G)$ acts transitively on the vertex set of $G$. It is called edge transitive, if $\text{Aut}(G)$ acts transitively on the edge set of $G$.

Vertex transitive graphs ‘look the same’ from every vertex.

**Exercise 4** Is there a 3-regular graph with no nontrivial automorphisms?

**Exercise 5** Show that in Theorem 1, if $G$ is also vertex transitive, then there exists a bi-infinite geodesic, that is, a function $f : \mathbb{Z} \to V$ that satisfies the same condition.

**Exercise 6** Find an infinite 3-regular edge transitive graph.

A nice way to generate vertex transitive graphs is as follows. Let $\Gamma$ be a group and let $S$ be a subset of $\Gamma$ such that $S^{-1} = S$. Let $\text{Cay}(\Gamma, S)$, the Cayley graph of $\Gamma$ with respect to $S$ be defined as follows. Let the vertex set of $\text{Cay}(\Gamma, S)$ be $\Gamma$ and for all $x \in \Gamma$ and $s \in S$, let $(x, xs)$ be an edge in $\text{Cay}(\Gamma, S)$. Sometimes we label this edge with the symbol $s$ – then we talk about a labeled Cayley graph.
**Theorem 7** Let $G = \text{Cay}(\Gamma, S)$. Then $G$ is vertex transitive and it is connected if and only if $S$ generates $\Gamma$.

Moreover, $\Gamma$ embeds in $\text{Aut}(G)$ as a transitive subgroup that acts freely on the vertex set of $G$.

For a graph $G = (V, E)$ let $B(x, n)$ denote the ball of radius $n$ around $x$, that is,

$$B(x, n) = \{ y \in V \mid d(x, y) \leq n \}.$$ 

When $G$ is vertex transitive, the size $f(n)$ of $B(x, n)$ does not depend on $x$.

**Theorem 8** If $G$ is locally finite and vertex transitive, then

$$f(n)f(5n) \leq f^2(4n).$$

Here is another curious example for a vertex transitive graph. Let the countable random graph be defined as follows. Take a countable set and the complete graph on it. For each edge, toss a coin independently; if its heads, keep it, otherwise, erase it.

**Theorem 9** Let $G$ and $G'$ be two independent countable random graphs. Show that with probability 1, $G$ and $G'$ are isomorphic. Also show that with probability 1, $G$ is vertex transitive.

That is, a countable random graph is not random at all, it is one well-defined object! (Well, almost surely).