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6. LECTURE 6

Definition 6.1. A measure preserving action is *ergodic* if every measurable invariant subset has zero or full measure

Lemma 6.2. $\Gamma \leq \text{Aut } T$ is transitive on each level of T iff Γ acts ergodically on ∂T .

Definition 6.3. Let Γ act on Ω by permutations. We say Γ separates Ω if $\forall x_1, \dots, x_n, y \in \Omega$, there exists $g \in \Gamma$ so that

$$x_i^g = x_i$$

for $1 \leq x_i \leq n$ and

$$y^g \neq y.$$

Theorem 6.4. $\text{Aut } T$ separates ∂T .

Lemma 6.5. Consider the subgroup of $GL(\mathbb{C})$

$$G = \left\langle \left(\begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right), \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \right\rangle.$$

If $|\lambda| \geq 2$, then $G \cong F_2$.

Proof. Exercise. □

Theorem 6.6. Every subgroup of a free group is free.

Proof. We will not prove this in this course. □

Remark 6.7. Counter to what your intuition may be, the free group F_2 has as subgroups free groups generated by more than two elements. Can you give an example of such a subgroup?

Definition 6.8. We define the group $SL_n(\mathbb{Z})$ by

$$SL_n(\mathbb{Z}) = \{M \in M_n(\mathbb{Z}) : \det M = 1\}.$$

$SL_n(R)$ is defined similarly for other commutative rings with identity.

Lemma 6.9. $SL_2(\mathbb{Z})$ is residually finite.

Proof. Let $\varphi_n : SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/n\mathbb{Z})$ be the homomorphism that reduces a matrix mod n . The kernel of this map is all matrices with determinant 1 of the form

$$\begin{pmatrix} 1 + na_1 & na_2 \\ na_3 & 1 + na_4 \end{pmatrix}.$$

Now let $M \in SL_2(\mathbb{Z})$ and take n to be greater than the absolute value of any of the elements of M . Then $\varphi_n(M)$ is the identity matrix if and only if M is the identity matrix. □

Lemma 6.10. If Γ is residually finite and Δ is a subgroup of Γ , then Δ is residually finite.

Proof. Obvious. □