Instructor: Miklos Abert
Scribe: Saravanan Thiyagarajan
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8. LECTURE 8

8.1. **The First Grigorchuk group.** In this lecture we shall define the first Grigorchuk group Γ and prove some of its properties. In fact we shall prove the following theorem.

Theorem 8.1. *There exists a group, the first Grigorchuk group Γ, that is finitely generated, periodic and residually finite but has infinitely many elements.*

We have, then, by the theorem proved in last class, the following corollary:

Corollary 8.2. *There exists a finitely generated non-linear group which is residually finite.*

We begin with the definition of the group Γ. Consider the infinite binary tree, T. We shall construct Γ as a subgroup of its automorphism group. Denote by a the automorphism of T which flips the children of the root along with their subtrees i.e. the left subtree becomes the right subtree and the right subtree the left. We shall recursively define three other automorphisms b, c and d. Let

$$b = (a, c) \;\; c = (a, d) \;\; d = (1, b).$$

This means the automorphism b fixes the root and acts as a on the left subtree and as c on the right subtree. It is easily seen that this is well defined. In fact, we can write down pictorially how b, c and d act on T. Suppose A denotes the operation of flipping the left and right subtrees of a vertex, then we have:

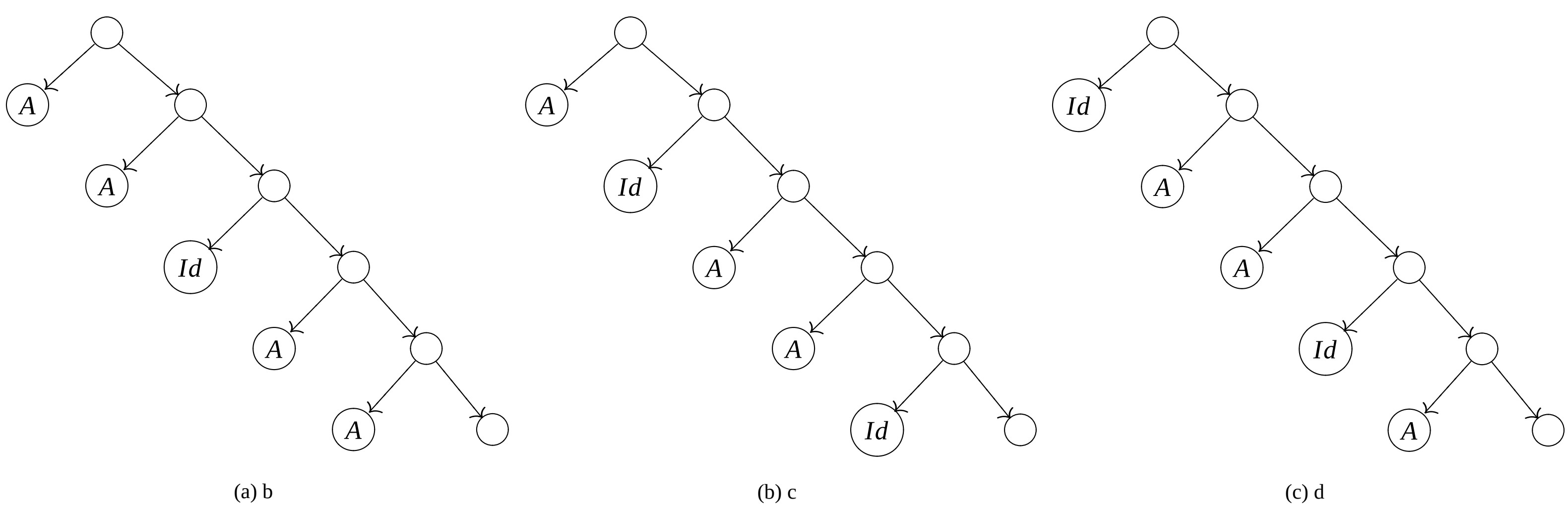

FIGURE 1. The automorphisms

It is easy to see from the picture that each of the automorphisms a, b, c and d has order two. One could prove this by induction. Consider the statement that b^2, c^2 and d^2 leave the first n levels fixed. This is clearly true for the zeroth level - the root. We have the identity

$$b^2 = (a,c) \cdot (a,c) = (a^2, c^2).$$

We get similar expressions for c and d. Now, $a^2 = Id$ and hence the statement is true for the $n+1$st level. Also we see easily from the picture that

$$bc = cb = d \;\; bd = db = c \;\; cd = dc = b.$$

Hence the subgroup $< b,c,d > \cong C_2 \times C_2$ (the Klein four group).

Remark 8.3. We can represent the automorphisms a,b,c and d in a finite state automaton as follows:

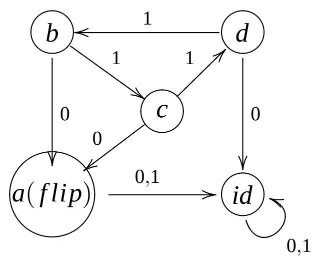

FIGURE 2. Finite state automaton

Definition 8.4. We call the subgroup of the automorphism group of T generated by a,b,c and d as *the first Grigorchuk group,* Γ.

We see immediately from the definitions that Γ is finitely generated and residually finite (Since $Aut(T)$ is residually finite).

Notation 8.5. *The Stabilizer of the first level*, denoted St_1, is the set of elements which fix the first level. In fact, $St_1 = \{$ elements which have even number of a's $\}$.

We check that

$$St_1 = < b,c,d,aba,aca,ada > .$$

Note that for $(\gamma_1,\gamma_2) \in St_1, a(\gamma_1,\gamma_2)a = (\gamma_2,\gamma_1)$. Hence we have

$$b = (a,c) \;\; c = (a,d) \;\; d = (1,b) \;\; aba = (c,a) \;\; aca = (d,a) \;\; ada = (b,1).$$

Looking at the action of the generators of St_1 on the left subtree at the root, we see that Γ is a quotient of St_1. But St_1 is a subroup of Γ of index two. Hence Γ is an infinite group.

Remark 8.6. We note that Γ acts transitively at every level of the tree. In fact, we see this readily by induction. This is clear at level zero and level one. Suppose we assume the statement at level $n-1$. Then given two vertices of the tree at level n, we can by an action of a assume that they both belong to the left subtree of the root. Hence they are at level $n-1$ of the subtree. By induction and the fact that Γ is a quotient of St_1 we are done.

Lemma 8.7. *If* $x^2 = y^2 = 1$*, then*

$$< x,y > \cong C_2, C_2 \times C_2, D_n, \text{or} D_\infty.$$

Proof. Consider the possible orders of xy. □

We see that if the order of xy is n then $< x,y > \cong D_n$.

Corollary 8.8.

$$< a,d > \cong D_4; < a,c > \cong D_8; < a,b > \cong D_{16}.$$

Proof. We have

$$(ad)^2 = (ada)d = (b,b).$$

Similarly,

$$(ac)^2 = (da,ad); (ab)^2 = (ca,ac).$$

□

Our final task is to prove the periodicity of the group Γ.

Theorem 8.9. $\forall \gamma \in \Gamma, \exists n$ *such that* $\gamma^{2^n} = 1$.

Proof. For $\gamma \in \Gamma$, we denote by $|\gamma|$ the length of the shortest word representing γ. We prove the theorem by induction on the length $k = |\gamma|$. The above corollary implies the statement for $k \leq 2$. We shall give the reduction step.
Case 1: k is odd. If γ starts and ends with a, then it is the conjugate of a smaller element and we are done. Otherwise $\gamma = u_1\gamma' u_2; u_i \in \{b,c,d\}$. This is conjugate to $\gamma' u_2 u_1 = \gamma' u_3; u_3 \in \{b,c,d\}$ which has lesser length and we are done.
Case 2: k is even. Then γ must be of the form,

$$\gamma = au_1au_2au_3 \ldots au_l, l = k/2, u_i \in \{b,c,d\}.$$

Subcase 2.1: l is even. Then

$$\gamma = (au_1a)u_2(au_3a)u_4 \ldots (au_{l-1}a)u_l = (\gamma_1, \gamma_2).$$

Now $|\gamma_1|, |\gamma_2| \leq l < k$.
Subcase 2.2: l is odd.

$$\gamma^2 = (au_1a)u_2(au_3a) \ldots (au_la)u_1(au_2a)u_3(au_4a) \ldots (au_{l-1}a)u_l = (\alpha, \beta).$$

We see that $|\alpha|, |\beta| \leq k$. In fact, if $\exists j,$ s.t. $u_j = d = (1,b)$, then $ada = (b,1)$ also occurs in the product and hence the length of both α and β is less than k by one. Similarly, we have the cases when $u_j = c$ or b. Hence done. □