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9. LECTURE 9

Throughout this lecture, let Γ be the Grigorchuk group. Unless otherwise stated, $G = \langle S \rangle$ for some finite $S \subset G$.

9.1. **Word Growth.** To any finitely generated group G , we can assign a graph to G as follows:

Definition 9.1. Let $G = \langle S \rangle$ where S is a finite subset of G . We define the *Cayley graph* $\text{Cay}(G, S)$ to be the graph that has the elements of G as vertices and edges (x, xg) and (x, xg^{-1}) for each $g \in S$.

Remark 9.2. This definition is repetitive if we view $\text{Cay}(G, S)$ as an undirected graph as the edge (x, xg^{-1}) is the same as $(xg^{-1}, xg^{-1}g)$. If we view $\text{Cay}(G, S)$ as a directed graph, the definition is not repetitive. You can check that in this case there will be exactly $|S|$ edges to and from each vertex.

Theorem 9.3. For any two elements $x, y \in G$, there is an automorphism of $\text{Cay}(G, S)$ taking x to y . In other words, $\text{Aut Cay}(G, S)$ is transitive on G .

Proof. Multiply on the left by yx^{-1} . □

Definition 9.4. For any $g \in G$, we define $|g|$ to be the minimal number of generators needed to write g as a product of generators.

Observe that $|g|$ is the distance from 1 to g in the Cayley graph.

Definition 9.5. Let

$$\beta_k(G, S) = \{g \in G : |g| \leq k\}.$$

We call β_k the *word growth* of G with respect to S .

For instance, $\mathbb{Z} \times \mathbb{Z}$ has word growth of order k^2 , with respect to the standard generators.

Definition 9.6. Let G be any group and consider the sequence

$$G_0, G_1, G_2 \dots$$

where $G_0 = G$ and $G_{i+1} = [G, G_i]$. If G_n is the trivial group for some n , we say that G is nilpotent.

Theorem 9.7. Every finitely generated nilpotent group has polynomial growth.

Proof. We will not prove this. □

Definition 9.8. Let G be any group and consider the sequence

$$G_0, G_1, G_2 \dots$$

where $G_0 = G$ and $G_{i+1} = [G_i, G_i]$. If G_n is the trivial group for some n , we say that G is solvable.

Remark 9.9. Another definition of solvable is that there exists a normal series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

so that G_{i+1}/G_i is an abelian group for all i . As an exercise, the reader can check that this definition is equivalent to the one given.

Theorem 9.10. *There exist solvable groups with exponential growth.*

Example 9.11 (Lamplighter Group). The *Lamplighter Group* consists of all pairs (n, x) where $n \in \mathbb{Z}$ and $x \in \bigoplus_{(-\infty, +\infty)} \mathbb{Z}/2\mathbb{Z}$ (this is all infinite sequences of 0's and 1's with all but

finitely many of the elements 0). Multiplication is given by $(n_1, x_1)(n_2, x_2) = (n_1 + n_2, x)$ where x is obtained by shifting x_2 by n_1 to the right (so the m th element in the sequence is now the $m + n_1$ element in the sequence) and then multiplying it with x_1 . The reason this is called the *Lamplighter Group* is because the its elements can be thought of as infinite rows of lamps, all but finitely many dark, and a lamplighter at a specified lamp. The *Lamplighter Group* is an example of a solvable group with exponential growth.

Theorem 9.12 (Gromer). *If G has polynomial growth, then G has a finite index nilpotent subgroup.*

A natural question to ask, is whether or not there is a group of super-polynomial, sub-exponential growth. That is, is there a group with order of growth bigger than any polynomial function and less than any exponential function? The answer to this question is yes:

Theorem 9.13. *The Grigorchuk group Γ has super-polynomial, sub-exponential growth.*

Proof. This is not beyond the realm of the course, but the proof is rather technical. \square

Definition 9.14. Let $G \leq \text{Aut} T$ where T is a rooted infinite tree. We say G is *weakly branched* if $\forall v \in T$, there exists $g \in G \setminus \{1\}$ so that g moves only descendants of v .

Example 9.15. Γ is weakly branched.

Theorem 9.16. *Weakly branched groups are not linear over any field.*

This can be proved using the following exercise:

Exercise 9.17. Let $n \in \mathbb{N}$ and $A_1, \dots, A_k, B_1, \dots, B_k \in M_n(\mathbb{C})$ be so that A_i, A_j commute for all i, j , B_i, B_j commute for all i, j , and A_i, B_j commute if and only if $i \neq j$. Show that $k \leq n^2$. (Hint: The solution was given in class and uses only basic linear algebra.)