

# On the probability of satisfying a word in a group

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## Abstract

We show that for any finite group  $G$  and for any  $d$  there exists a word  $w \in F_d$  such that a  $d$ -tuple in  $G$  satisfies  $w$  if and only if it generates a solvable subgroup. As a corollary, the probability that a word is satisfied in a fixed non-solvable group can be made arbitrarily small, answering a question of Alon Amit.

It also follows that there is no absolute bound in the Baumslag-Pride theorem for the minimal index of a subgroup of a group with at least two more generators than relators that can be mapped homomorphically onto a nonabelian free group.

## 1 Introduction

Let  $F_n$  denote the free group on  $n$  letters and let  $G$  be a group. For  $w \in F_n$  we say that the  $n$ -tuple  $(g_1, g_2, \dots, g_n) \in G^n$  satisfies  $w$  if the substitution  $w(g_1, g_2, \dots, g_n) = 1$ . Our first result is the following.

**Theorem 1** *Let  $G$  be a finite group. Then for all  $n$  there exists a word  $w \in F_n$  such that for all  $g_1, g_2, \dots, g_n \in G$ , the tuple  $(g_1, g_2, \dots, g_n)$  satisfies  $w$  if and only if the subgroup  $\langle g_1, g_2, \dots, g_n \rangle \leq G$  is solvable.*

Note that if  $G$  itself is not solvable, then a word as in Theorem 1 has to contain at least  $n - 2$  letters of  $F_n$ . Indeed, if  $w$  omits at least two letters, then any two elements of  $G$  generate a solvable subgroup, which, using a theorem of Thompson [Tho] (see also [Fla]) implies that  $G$  itself is solvable.

For  $w \in F_n$  let  $\langle F_n \mid w \rangle$  denote the one-relator group defined by  $w$ . As an immediate corollary of Theorem 1, we get the following.

**Corollary 2** *Let  $G$  be a finite non-solvable group. Then for all  $n$  there exists a word  $w \in F_n$  such that  $\langle F_n \mid w \rangle$  cannot be mapped homomorphically onto  $G$ .*

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It is natural to ask whether this property holds exactly when  $G$  is not solvable.

**Question 1.** *Let  $G$  be a finite solvable group. Does there exist  $N \in \mathbb{N}$  such that for all  $n \geq N$  and every  $w \in F_n$  the one-relator group  $\langle F_n \mid w \rangle$  can be mapped homomorphically onto  $G$ ?*

We call a group  $\Gamma$  *large*, if there is a subgroup  $\Delta \leq \Gamma$  of finite index which can be mapped homomorphically onto a non-abelian free group. A well-known theorem of Baumslag and Pride [BaP] shows that groups with two more generators than relators are large. In particular, for  $n \geq 3$  every one-relator group  $\Gamma = \langle F_n \mid w \rangle$  is large. In turn, Stallings [St1] has shown that for every  $n$  there exists  $w \in F_n$  such that  $\langle F_n \mid w \rangle$  itself does not homomorphically map onto  $F_2$ . Theorem 1 implies the following strengthening of Stallings's result.

**Corollary 3** *For all  $n$  and  $N$  there exists a word  $w \in F_n$  such that if  $\Delta \leq \langle F_n \mid w \rangle$  is a subgroup of index at most  $N$ , then  $\Delta$  cannot be mapped homomorphically onto a non-abelian free group.*

For  $w \in F_n$  let  $P(G, w)$  denote the probability that for  $n$  independent uniform random elements  $g_1, \dots, g_n \in G$  we have  $w(g_1, \dots, g_n) = 1$ . Note that  $P(G, w)$  only depends on the word  $w$  and not on  $n$  so we can assume  $w \in F_\infty$ .

The probabilities  $P(G, w)$  have been investigated in the literature mainly for a fixed word and varying  $G$ . The strongest result in this direction is of Dixon, Pyber, Seress and Shalev [DPSS] who proved that for any word  $1 \neq w$  the probability  $P(G, w)$  tends to 0 as  $|G| \rightarrow \infty$  assuming that  $G$  is non-abelian simple. In this paper we will fix the finite group  $G$  and let  $w$  run through  $F_\infty$ .

Alon Amit [Ami] has shown that if  $G$  is nilpotent then there exists a constant  $c > 0$  depending on  $G$  only such that for all  $w \in F_\infty$  we have  $P(G, w) > c$ . Note that this answers Question 1 affirmatively for nilpotent groups. He conjectures that the same holds if  $G$  is solvable and that if  $G$  is nilpotent then in fact

$$P(G, w) \geq \frac{1}{|G|} \text{ for all } w \in F_\infty.$$

He also asked if in turn for a non-solvable finite group  $G$  the probability  $P(G, w)$  can be made arbitrarily small with a suitable  $w \in F_\infty$ .

It is easy to see that Theorem 1 already answers Amit's question affirmatively, but the following stronger result also holds. A group  $G$  is *just non-solvable* if every proper quotient of  $G$  is solvable, but  $G$  itself is not.

**Theorem 4** *Let  $G$  be a finite just non-solvable group. Then the set*

$$\{P(G, w) \mid w \in F_\infty\}$$

*is dense in  $[0, 1]$ .*

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## 2 Proofs

Let us introduce some notation. Let  $G$  be a just non-solvable group and let  $N$  be a minimal normal subgroup of  $G$ . Then  $N \cong S^m$  for some simple group  $S$ . By the minimality of  $G$  the quotient  $G/N$  is solvable, so  $S$  is non-abelian and  $Z(N) = 1$ , which implies that  $G/C_G(N)$  is non-solvable so  $C_G(N) = 1$ . Then  $G$  embeds into the wreath product

$$\text{Aut}(N) \cong \text{Aut}(S) \text{ wr Sym}(m)$$

where  $\text{Sym}(m)$  denotes the symmetric group on  $m$  letters and by the minimality of  $N$ ,  $G$  has a transitive image in  $\text{Sym}(m)$ . Since  $G/N$  is solvable,  $N$  is a characteristic subgroup of  $G$ . Also, every nontrivial normal subgroup  $K \triangleleft G$  contains  $N$  (using the minimality of  $N$  and that  $G$  is just non-solvable). Finally, if in addition  $K = K'$  (the commutator subgroup of  $K$ ), then  $K = N$ .

From now on  $G$ ,  $N$ ,  $S$  and  $m$  will be as above. Let  $G_j \cong G$  ( $1 \leq j \leq n$ ), let

$$P = G_1 \times \cdots \times G_n$$

and let

$$\pi_j : P \rightarrow G_j \quad (1 \leq j \leq n)$$

denote the projection to the  $j$ -th coordinate. Let

$$N \cong N_j \triangleleft G_j (1 \leq j \leq n),$$

let  $N_j = S_{j,1} \times \cdots \times S_{j,m}$  where  $S_{j,i} \cong S$  and let

$$M = N_1 \times \cdots \times N_n \triangleleft P.$$

The first lemma is folklore (see e.g. [Rob, Lemma 3.3.16.]).

**Lemma 5** *If  $S_1, \dots, S_n$  are nonabelian finite simple groups, then every normal subgroup  $K \triangleleft S_1 \times \cdots \times S_n$  is of the form*

$$K = K_1 \times \cdots \times K_n$$

where  $K_i = S_i$  or  $1$  ( $1 \leq i \leq n$ ).

The next lemma tells us about the normal subgroup structure of subgroups of  $P$  which project onto each  $G_j$ .

**Lemma 6** Let  $H \leq P$  be a subgroup containing  $M$  such that

$$\pi_j(H) = G_j \quad (1 \leq j \leq n)$$

Let  $K$  be a normal subgroup of  $H$ . Then

$$K \cap M = \bigoplus_{\pi_j(K) \neq 1} N_j$$

**Proof.**  $K \cap M$  is normal in

$$M \cong \bigoplus_{1 \leq j \leq n, 1 \leq i \leq m} S_{j,i}$$

so by Lemma 5 it is the direct product of some of the  $S_{j,i}$ , that is,

$$K \cap M = K_1 \times \cdots \times K_n$$

where  $K_j \triangleleft N_j$  ( $1 \leq j \leq n$ ).

If  $\pi_j(K) = 1$  then  $K \cap N_j = 1$  so  $K_j = 1$ .

If  $\pi_j(K) \neq 1$  then  $\pi_j(K) \triangleleft \pi_j(H) = G_j$  so  $N_j \leq \pi_j(K)$ , since  $N_j$  is a minimal normal subgroup in  $G_j$ . In this case

$$K \cap M \geq [K, M] \geq [K, N_j] = [\pi_j(K), N_j] = N_j$$

so  $K_j = N_j$  (here we use the direct product form and that the commutator  $[N_j, N_j] = N_j$ ).

The lemma holds. ■

Let

$$(a_1, \dots, a_k), (b_1, \dots, b_k) \in G^k$$

be  $k$ -tuples from  $G$ . We say that  $(a_1, \dots, a_k)$  and  $(b_1, \dots, b_k)$  are *automorphism independent over  $G$*  if there exists no  $\alpha \in \text{Aut}(G)$  such that  $a_i^\alpha = b_i$  for all  $1 \leq i \leq k$ .

Our next lemma shows that subgroups of  $G_1 \times \cdots \times G_n$  satisfying some natural conditions contain  $N_1 \times \cdots \times N_n$ .

**Lemma 7** Let  $a_{i,j} \in G_j$  ( $1 \leq i \leq k, 1 \leq j \leq n$ ), such that we have

$$\langle a_{1,j}, \dots, a_{k,j} \rangle = G_j \quad (1 \leq j \leq n)$$

and that for all  $1 \leq j < l \leq n$  the  $k$ -tuples  $(a_{1,j}, \dots, a_{k,j})$  and  $(a_{1,l}, \dots, a_{k,l})$  are automorphism independent over  $G$ . For  $1 \leq i \leq k$  let

$$h_i = (a_{i,1}, \dots, a_{i,n}) \in G_1 \times \cdots \times G_n$$

and let

$$H = \langle h_1, \dots, h_k \rangle \leq G_1 \times \cdots \times G_n$$

Then

$$M = N_1 \times \cdots \times N_n \leq H.$$

**Proof.** Let

$$f : G_1 \times \cdots \times G_n \rightarrow G_1 \times \cdots \times G_{n-1}$$

denote the projection to the first  $n - 1$  coordinates. Let  $H_1 = f(H)$  and let

$$R = \pi_n(\text{Ker}(f)) \leq G_n.$$

By induction on  $n$ , we have  $N_1 \times \cdots \times N_{n-1} \leq H_1$ . Also  $R$  is normal in  $G_n$  so by the minimality of  $N_n$  in  $G_n$  either  $N_n \leq R$  or  $R = 1$ .

We claim that  $N_n \leq R$ . Assume  $R = 1$ . Let us define the function  $\varphi : H_1 \rightarrow G_n$  by

$$\varphi(g_1, \dots, g_{n-1}) = g_n \text{ if } (g_1, \dots, g_{n-1}, g_n) \in H.$$

Then  $\varphi$  is well-defined, since

$$(g_1, \dots, g_{n-1}, g_n), (g_1, \dots, g_{n-1}, g'_n) \in H$$

implies  $g_n^{-1}g'_n \in R$ . So  $\varphi$  is a homomorphism. Using  $\langle a_{1,n}, \dots, a_{k,n} \rangle = G_n$  we also see that  $\varphi$  is surjective.

Let  $K = \text{Ker}(\varphi)$ . Then  $K$  is normal in  $H_1$  and

$$H_1/K \cong G_n \cong G$$

which is not solvable. Since  $N_1 \times \cdots \times N_{n-1} \leq H_1$ , the use of Lemma 6 for  $H_1$  and  $K$  gives us

$$K \cap M = \bigoplus_{\pi_j(K) \neq 1} N_j.$$

Now  $M \leq K$  would imply that  $H_1/K$  is solvable, a contradiction. So there exists a coordinate  $1 \leq l < n$  such that  $\pi_l(K) = 1$ , that is,  $K \leq \text{Ker}(\pi_l)$ . Moreover

$$H_1/\text{Ker}(\pi_l) \cong G_l \cong G$$

which implies  $K = \text{Ker}(\pi_l)$ . This shows that the function  $\alpha : G_l \rightarrow G_n$  defined by

$$\alpha(g_l) = g_n \text{ if } (g_1, \dots, g_l, \dots, g_n) \in H$$

is an isomorphism. In particular,  $\alpha(a_{i,l}) = a_{i,n}$  ( $1 \leq i \leq k$ ), so the  $k$ -tuples  $(a_{1,l}, \dots, a_{k,l})$  and  $(a_{1,n}, \dots, a_{k,n})$  are not automorphism independent over  $G$  which contradicts the assumptions of the lemma. So the claim  $N_n \leq R$  holds and so  $1 \times \cdots \times 1 \times N_n \leq H$ .

Now let  $L = f^{-1}(N_1 \times \cdots \times N_{n-1}) \leq H$ . Let  $L^{(i)}$  denote the  $i$ -th element of the derived series of  $L$  and let  $r$  be a number such that  $L^{(r)} = L^{(r+1)}$ . Then  $f(L^{(r)}) = N_1 \times \cdots \times N_{n-1}$  and since  $1 \times \cdots \times 1 \times N_n \leq L$  also  $1 \times \cdots \times 1 \times N_n \leq L^{(r)}$ . Now  $J = \pi_n(L^{(r)})$  is normal in  $G_n$ ,  $N_n \leq J$  and  $J' = J$ , so  $J = N_n$ . This implies

$$L^{(r)} = N_1 \times \cdots \times N_n \leq H$$

which is what we wanted to prove. ■

**Remark.** This lemma is well-known in the case when  $G$  is a nonabelian finite simple group. According to the author's knowledge it is originally due to Hall [Hal] (see also [KaL] and [Wie]).

We state an easy corollary of Lemma 7 that we will use in the proof of Theorem 1.

**Corollary 8** *Let  $G_i$  ( $1 \leq i \leq n$ ) be finite nonabelian simple groups and let*

$$H \leq G_1 \times \cdots \times G_n$$

*such that the projections  $\pi_i(H) = G_i$  ( $1 \leq i \leq n$ ). Then there exists  $g \in H$  such that  $\pi_i(g) \neq 1$  ( $1 \leq i \leq n$ ).*

**Proof.** We proceed by induction on  $n$ . For  $n = 1$  the lemma is trivial. By induction we have an element  $g \in H$  such that  $\pi_i(g) \neq 1$  ( $1 \leq i < n$ ). If the last coordinate is automorphism dependent on some previous coordinate  $k$  then  $\pi_k(g) \neq 1$  implies  $\pi_n(g) \neq 1$ . If it is not, then  $1 \times \cdots \times 1 \times G_n \leq H$  and we can choose the last coordinate of  $g$  as we wish. ■

Now we prove Theorem 4.

**Proof of Theorem 4.** Let  $m$  be the number of maximal subgroups of  $G$ . Let  $d > \log_2 m$  be an integer to be chosen later. The probability that  $d$  independent random elements all fall into a fixed maximal subgroup  $M$  is at most  $|G : M|^{-d} \leq 2^{-d}$  so the probability that  $d$  random elements do not generate  $G$  is at most  $m2^{-d} < 1$ . In particular,  $G$  can be generated by  $d$  elements. Let

$$Q = \{(g_1, \dots, g_d) \in G^d \mid g_1, \dots, g_d \text{ generate } G\}$$

be the set of generating  $d$ -tuples.

Now  $\text{Aut}(G)$  acts on  $Q$  by  $(g_1, \dots, g_d)^\alpha = (g_1^\alpha, \dots, g_d^\alpha)$  where  $\alpha \in \text{Aut}(G)$ . This action is fixed-point free, as if  $\alpha$  fixes all the elements of a generating set then it fixes every element of  $G$ . Let  $r$  be the number of  $\text{Aut}(G)$ -orbits and let  $t_1, \dots, t_r \in Q$  be an orbit representative system.

It is easy to see that the conditions of Lemma 7 hold for  $a_{i,j} = t_j(i)$ . This implies that the  $r$ -tuples

$$h_i = (t_1(i), \dots, t_r(i))$$

generate a group  $H$  which contains  $N_1 \times \cdots \times N_r$ .

Let  $1 \neq g \in N$ , let  $k \leq r$  be a natural number to be chosen later and let the  $r$ -tuple  $h$  be defined by

$$h(i) = 1 \quad (1 \leq i \leq k) \quad \text{and} \quad h(i) = g \quad (k < i \leq r)$$

Then  $h \in N_1 \times \cdots \times N_r \leq H$ , so there exists a word  $w \in F_d$  such that  $w(h_1, \dots, h_d) = h$ .

Now let us evaluate  $w$  on the set of possible  $d$ -tuples from  $G$ . We completely control the evaluation on generating tuples; since

$$w(g_1^\alpha, \dots, g_d^\alpha) = (w(g_1, \dots, g_d))^\alpha \quad (\alpha \in \text{Aut}(G))$$

we have

$$|\{(g_1, \dots, g_d) \in Q \mid w(g_1, \dots, g_d) = 1\}| = k |\text{Aut}(G)|.$$

On  $d$ -tuples  $(g_1, \dots, g_d)$  not generating  $G$  we do not control  $w(g_1, \dots, g_d)$ , but as we saw earlier, the number of these tuples is at most  $m2^{-d} |G|^d$ . Dividing by  $|G|^d$ , this gives us

$$k \frac{|\text{Aut}(G)|}{|G|^d} \leq P(G, w) \leq k \frac{|\text{Aut}(G)|}{|G|^d} + m2^{-d}$$

and for the maximal value of  $k = r$  we get

$$k \frac{|\text{Aut}(G)|}{|G|^d} = \frac{r |\text{Aut}(G)|}{|G|^d} = \frac{|Q|}{|G|^d} \geq 1 - m2^{-d}.$$

Let

$$\epsilon(d) = m2^{-d} + \frac{|\text{Aut}(G)|}{|G|^d}$$

Since  $k \leq r$  can be set arbitrarily, we deduce that the set

$$\{P(G, w) \mid w \in F_d\}$$

is an  $\epsilon(d)$ -net in  $[0, 1]$ , that is, for every  $a \in [0, 1]$  there exists  $w \in F_d$  such that  $|P(G, w) - a| < \epsilon(d)$ .

Now  $\lim_{d \rightarrow \infty} \epsilon(d) = 0$  which shows that the set

$$\{P(G, w) \mid w \in F_\infty\} = \bigcup_d \{P(G, w) \mid w \in F_d\}$$

is dense in  $[0, 1]$ . ■

The answer to Amit's question follows as an easy corollary of Theorem 4.

**Corollary 9** *Let  $G$  be a finite non-solvable group. Then 0 is an accumulation point of the set*

$$\{P(G, w) \mid w \in F_\infty\}$$

**Proof.** Let  $K$  be a normal subgroup in  $G$  such that  $G/K$  is just non-solvable and let  $g_1, \dots, g_n$  be independent uniform random elements of  $G$ . Then  $g_1K, \dots, g_nK$  are independent uniform random elements of  $G/K$  which yields

$$P(G/K, w) = P(w(g_1, \dots, g_n) \in K) \geq P(w(g_1, \dots, g_n) = 1) = P(G, w)$$

for  $w \in F_\infty$ . Using Theorem 4 we get that for every  $\epsilon > 0$  we have  $w \in F_\infty$  such that

$$P(G, w) \leq P(G/K, w) < \epsilon$$

and so the corollary holds. ■

We are ready to prove Theorem 1.

**Proof of Theorem 1.** For each subgroup  $H \leq G$  let us choose a homomorphism  $\varphi_H$  with domain  $H$  as follows. If  $H$  is solvable then let  $\varphi_H = \text{Id}$  be the identity map, otherwise let  $\varphi_H$  be a homomorphism onto a just non-solvable quotient of  $H$ .

Let us enumerate all the  $n$ -tuples from  $G$  as  $t_1, t_2, \dots, t_k$  where  $k = |G|^n$ . Let  $t_{i,j}$  denote the  $j$ -th element of  $t_i$  ( $1 \leq j \leq n$ ). For  $1 \leq i \leq k$  let

$$H_i = \langle t_{i,1}, t_{i,2}, \dots, t_{i,n} \rangle$$

Let  $\varphi_i = \varphi_{H_i}$  and let  $G_i = \varphi_i(H_i)$ . Let  $N_i$  be the minimal normal subgroup of  $G_i$  if  $G_i$  is just non-solvable, otherwise let  $N_i = 1$ . Also let

$$u_{i,j} = \varphi_i(t_{i,j}) \quad (1 \leq i \leq k, 1 \leq j \leq n)$$

and let

$$p_j = (u_{1,j}, u_{2,j}, \dots, u_{k,j}) \in G_1 \times \dots \times G_k \quad (1 \leq j \leq n)$$

Let

$$L = \langle p_1, p_2, \dots, p_n \rangle \leq G_1 \times \dots \times G_k$$

and let

$$\pi_i : G_1 \times \dots \times G_k \rightarrow G_i \quad (1 \leq i \leq k)$$

denote the projection to the  $i$ -th coordinate. Then  $\pi_i(L) = G_i$  ( $1 \leq i \leq k$ ). Let  $L^{(i)}$  denote the  $i$ -th derived subgroup of  $L$  and let  $r$  be an integer such that  $M = L^{(r)} = L^{(r+1)}$ . Then  $\pi_i(M) \triangleleft G_i$  and  $\pi_i(M)' = \pi_i(M)$  so  $\pi_i(M) = N_i$ . Now all the  $N_i \neq 1$  are isomorphic to some direct power of a nonabelian simple group so  $M$  lies in a direct product of nonabelian simple groups and projects to each factor of the product. By Corollary 8 there exists an element  $g \in M \leq L$  such that  $\pi_i(g) \neq 1$  if and only if  $N_i \neq 1$ . Let  $w \in F_n$  be a word such that  $w(p_1, p_2, \dots, p_n) = g$ .

We claim that this  $w$  will be good for our purposes. Indeed, we have

$$\pi_i(g) = w(u_{i,1}, \dots, u_{i,n}) = w(\varphi_i(t_{i,1}), \dots, \varphi_i(t_{i,n})) = \varphi_i(w(t_{i,1}, \dots, t_{i,n}))$$

Now if  $H_i$  is solvable then  $\varphi_i$  is the identity map and  $\pi_i(g) = 1$ , so we get  $w(t_{i,1}, \dots, t_{i,n}) = 1$ . If  $H_i$  is not solvable, then  $\pi_i(g) \neq 1$  and since  $\varphi_i$  is a homomorphism we have  $w(t_{i,1}, \dots, t_{i,n}) \neq 1$ . The theorem holds. ■

**Proof of Corollary 3.** Let  $n \geq 1$  and let  $G = \text{Sym}(5N)$  be the symmetric group on  $5N$  points. Applying Theorem 1 to  $G$  we get that there exists a word

$w \in F_n$  such that for all  $g_1, g_2, \dots, g_n \in G$ , the tuple  $(g_1, g_2, \dots, g_n)$  satisfies  $w$  if and only if the subgroup  $\langle g_1, g_2, \dots, g_n \rangle \leq G$  is solvable. In particular, every homomorphism from the one-relator group  $\Gamma = \langle F_n \mid w \rangle$  to  $G$  has solvable image. We claim that  $\Gamma$  has the property stated in the corollary.

Indeed, let  $\Delta \leq \Gamma$  be a subgroup such that  $|\Gamma : \Delta| \leq N$ . Assume indirectly that  $\Delta$  could be mapped homomorphically onto  $F_2$ . Then there exists a subgroup  $\Pi \leq \Delta$  of index 5 such that the left coset action of  $\Delta$  on  $\Delta/\Pi$  is isomorphic to the alternating group  $\text{Alt}(5)$ . This implies that the left coset action of  $\Gamma$  on  $\Gamma/\Pi$  is not solvable. But  $|\Gamma/\Pi| \leq 5N$ , so  $\Gamma$  has a non-solvable homomorphic image in  $\text{Sym}(5N) = G$ , a contradiction. ■

**Remark on Question 1.** Let  $G$  be a finite group for which there exists a constant  $c > 0$  such that for all  $w \in F_\infty$  we have  $P(G, w) > c$ . Then as we saw for large enough  $d$  most of the  $d$ -tuples generate  $G$  and so for every word  $w \in F_d$  there exists a generating set  $\langle g_1, \dots, g_d \rangle = G$  such that  $w(g_1, \dots, g_d) = 1$ , that is,  $G$  is a quotient of the one-relator group  $\langle F_d \mid w \rangle$ . In particular, Amit's result [Ami] implies an affirmative answer for Question 1 for finite nilpotent groups.

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