2008 REU Problem Set 1: due June 30

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1. Show that for every natural number n there are finitely many solutions to
$$\sum_{i=1}^{n} \frac{1}{a_i} = 1$$
where the a_i are natural numbers.

2. A graph is locally finite if every vertex has finitely many neighbors. Let G be an infinite locally finite tree. Show that G has an infinite path.

3. Can you cover the plane with disjoint circles of positive radius? (A circle is not a disk!)

4. Prove that every subset of $\mathbb{R}$ of measure zero can be translated into the irrationals.

5. Someone painted the plane with three colors. Show that there exist two points of distance 1 with the same color.

6. Show that every finite group can be obtained as the automorphism group of a graph.

7. Prove that if a measurable subset of $\mathbb{R}$ is invariant under all rational translations, then it has measure zero or its complement has measure zero.

8. Assume that a_n and b_n are positive increasing sequences such that both $\sum_n 1/a_n$ and $\sum_n 1/b_n$ are divergent. Does this imply that
$$\sum_n \frac{1}{a_n + b_n}$$
is divergent?

9. Let $A_1, A_2, \ldots, A_n$ ($n \geq 3$) be a finite collection of convex sets in the plane such that for all i, j, k, the intersection $A_i \cap A_j \cap A_k$ is non-empty. Then the intersection of all the A_i's is non-empty.

10. A legal coloring of a graph G is a coloring of the vertices of G such that no neighbors have the same color. Let G be a locally finite graph. Show that G can be legally colored by c colors if and only if every finite subgraph of G can be legally colored by c colors.

11. Let f_n denote the number of permutations on n points that has no fixed points. What is
$$\lim_{n\to\infty} \frac{f_n}{n!}?$$

12. The Invisible Flea is sitting at the origin of $\mathbb{R}^n$. Suddenly, it bites you really hard and starts jumping away. Every second, it jumps with the same (unknown) vector. Also every second, you can toss a bomb to a point of $\mathbb{R}^n$ of your choice that kills everything within distance at most 1000. For which values of n can you catch the flea for sure?

13. Let G be a group covered by finitely many right cosets of (possibly different) subgroups. Show that you can omit the ones that have infinite index.

14. Let (X, μ) be a probability measure space. Is it true that the set of possible measures of measurable subsets is necessarily a closed subset of $[0, 1]$?