2008 REU Problem Set 3: due July 7

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1. Let M be an n by m board. Let A be the set of permutations of M that respects rows and B be the set of permutations of M that respects columns (that is, a permutation in A moves every element into the same row etc). Show that ABA contains every permutation of M.

2. Can you cover the space with closed disks of positive radius such that any two have at most 1 point in common?

3. Prove that the set
$$\left\{n + m\sqrt{3} \mid \ n, m \in \mathbb{Z}\right\}$$
is dense in the real line.

4. A function f is Lipschitz-1 if it does not increase the distance, that is, for all x, y in its domain, we have
$$d(f(x), f(y)) \leq d(x, y)$$
Show that if $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ is Lipschitz-1 then f can be extended to a Lipschitz-1 function $\mathbb{R}^n \to \mathbb{R}$.

5. Is the random walk problem (Sheet 2, Problem 3) true for every k-regular graph?

6. Let A be a set of at least 3 points on the plane such that no three points are collinear in A. Show that there exists a point P such that every line going through P has at least one third of the points of A on both sides.

7. Is there an infinite field that is finitely generated as a ring?

8. There are 17 weights with the following property. If you take out any of them, you can share the rest into two groups of the same size (namely, eight) such that the sum weights in the groups are equal. Show that all weights are equal.

9. Someone painted the space with 5 colors (using all colors). Show that there is a plane with at least 4 different colors on it.

10. For which natural numbers n there exists a simple graph on n vertices which has no symmetries? (No nontrivial automorphisms.)