2008 REU Problem Set 4: due July 11

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1. Is there an uncountable chain of subsets of the natural numbers? (That is, a set of subsets, such that for any two subsets, one contains the other).

2. Let G be a graph. Show that G has a perfect matching if and only if for all vertex sets $V \subseteq G$, the number of odd connected components of $G \setminus V$ is at most the size of V.

3. For subsets $A, B \subseteq \mathbb{Z}$ let $A + B = \{a + b \mid a \in A, b \in B\}$. For $A \subseteq \mathbb{N}$ let us define the upper density as
$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}$$
Is there a subset $A \subseteq \mathbb{N}$ of zero upper density such that $A + A = \mathbb{N}$?

4. Show that there is only one countable random graph: that is, almost all random graphs on a countable set are isomorphic.

5. Let $I = [0, 1]$, let $U = I \times I$ be the unit square, let
$$A = (\mathbb{Q} \cap I) \times (\mathbb{Q} \cap I) \bigcup (I \backslash \mathbb{Q}) \times (I \backslash \mathbb{Q}) \subseteq U$$
and let $B = U \backslash A$. Is A connected? Is B connected?

6. Let $f(n)$ be the number of trees you can see sitting in the middle of a $2n + 1$ by $2n + 1$ square forest. (The middle tree is missing so you can see all directions). What is
$$\lim_{n \to \infty} \frac{f(n)}{n^2}?$$

7. Can you cover the set of nonnegative integers with finitely many disjoint arithmetic progressions of distinct differences?

8. Let G be a graph on n points such that G has diameter 2 but no vertex of G is adjacent to all other vertices. Show that G has at least $2n - 4$ edges.

9. Prove that the set
$$\{n + mr \mid \ n, m \in \mathbb{Z}\}$$
is dense in the real line if and only if r is irrational.

10. Show that the Petersen graph is not a Cayley graph.