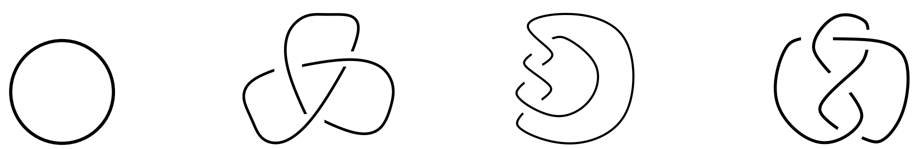
ON THE JONES POLYNOMIAL AND ITS APPLICATIONS

ALAN CHANG

ABSTRACT. This paper is a self-contained introduction to the Jones polynomial that assumes no background in knot theory. We define the Jones polynomial, prove its invariance, and use it to tackle two problems in knot theory: detecting amphichirality and finding bounds on the crossing numbers.

1. PRELIMINARIES

1.1. **Definitions.** For the most part, it is enough to think of a knot as something made physically by attaching the two ends of a string together. Since knots exist in three dimensions, when we need to draw them on paper, we often use *knot diagrams*. Figure 1.1 contains examples of knot diagrams.



(A) unknot (B) trefoil (C) trefoil (again) (D) figure-8 knot

FIGURE 1.1. Examples of knot diagrams

As we can see from Figure 1.1b and Figure 1.1c, different diagrams can represent the same knot. To see that these two are really the same knot, we could make Figure 1.1b out of a piece of string and move the string around in space (without cutting it) so that it looks like Figure 1.1c.

There are some restrictions on knot diagrams: (1) each crossing must involve exactly two segments of the string and (2) those segments must cross transversely. (See Figure 1.2.)

(A) triple crossing (B) non-transverse crossing

FIGURE 1.2. Examples of invalid knot diagrams

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There are two ways to travel around a knot; these correspond to the *orientations* of the knot. An *oriented* knot is a knot with a specified orientation. On a knot diagram, we can indicate an orientation via an arrow. (See Figure 1.3.)

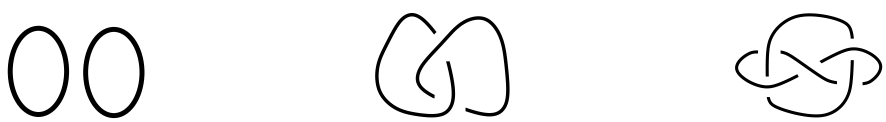
(A) one orientation (B) the other orientation

FIGURE 1.3. Two orientations of the trefoil

Sometimes we'll use more than one piece of string, so we define a *link* to be a generalization of a knot: links can be made by multiple pieces of string. For each string, we attach the two ends together. (Note that we do not attach the ends of two different strings together.)

The number of *components* of a link is the number of strings used. (Observe that every knot is a link with one component.) A *link diagram* is a straightforward generalization of a knot diagram, and an *oriented link* is a link where all the components have specified orientations.

Figure 1.4 contains examples of two-component links.



(A) unlink (B) Hopf link (C) Whitehead link

FIGURE 1.4. Examples of links with two components

1.2. **More mathematically...** Readers who are not satisfied with the definitions given above may prefer the definition of a knot given here.

Definition 1.1. Let X be a topological space. An *isotopy* of X is a continuous map $h : X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(\cdot,t)$ is a homeomorphism for each $t \in [0,1]$. $\Diamond$

Definition 1.2. A *knot* is a smooth embedding $f : S^1 \to \mathbb{R}^3$. Two knots are considered equivalent if they are related by a smooth isotopy of $\mathbb{R}^3$. $\Diamond$

This definition of a knot has the advantage of placing knot theory on firm mathematical grounding. For more detailed definitions, see [Cro04], [Mur96], or [Lic97].

Remark 1.3. We need the embeddings in Definition 1.2 to be smooth to avoid pathological "wild" knots. Instead of requiring the maps to be smooth, we could require them to be to be piecewise linear. Either of these is enough to guarantee the non-existence of wild knots. See [Cro04, Chapter 1] for what can happen if we do not assume either regularity condition. ◊

1.3. **Knot invariants.** A *knot invariant* is something (such as number, matrix, or polynomial) associated to a knot. A *link invariant* is defined similarly for links.

Example 1.4. The *unknotting number* of a knot K is the minimum number of times that K must be allowed to pass through itself to get to the unknot. This is a knot invariant. ◊

Example 1.5. We will define something called the *crossing number*.

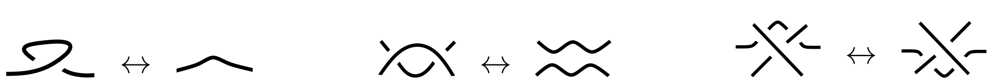
Suppose for a knot K, we take a diagram D of K and count the number of crossings in D. This number is *not* an invariant of K because K has many different diagrams that differ in number of crossings. For example, in Figure 1.5, we see two different diagrams of the unknot.

(A) 0 crossings (B) 3 crossings

FIGURE 1.5. Two diagrams of the unknot, with different number of crossings.

Thus, we have not yet successfully defined a knot invariant. However, if we consider *all* diagrams of K and take the *minimum* number of crossings over all diagrams, then we do have an invariant of K. This is called the *crossing number* of a knot. We will see this invariant again in section 4. ◊

1.4. **Reidemeister moves.** In 1926, Kurt Reidmeister proved that given two diagrams D_1 and D_2 of the same knot, it is always possible to get from one diagram to the other via a finite sequence of moves, now called Reidmeister moves. These moves can be divided into three types:



(A) Type I (B) Type II (C) Type III

FIGURE 1.6. The three types of Reidemeister moves

We can think of Type I as adding/removing a twist, Type II as crossing/uncrossing two strands, and Type III as sliding a strand past a crossing.

It is clear that these moves do not change the knot. What is important is that these three moves (along with planar isotopy) are enough for us to get from one diagram of a knot to any other diagram of the same knot. For a proof of this remarkable fact, see [Mur96, Chapter 4].

This discovery gives us a method to prove that certain quantities are knot invariants. Suppose there is a quantity that we are trying to show is a knot invariant, but it is defined in terms of a knot diagram. (As shown in Example 1.5, we have to be careful if we try to define a knot invariant in terms of a *single* knot diagram of a knot.) Because of Reidemeister's theorem, if we can show the quantity is unchanged when we alter the diagram via any Reidemeister move, then we know it is an invariant. We will use this technique in the next section.

2. The Jones Polynomial

2.1. **Introduction.**

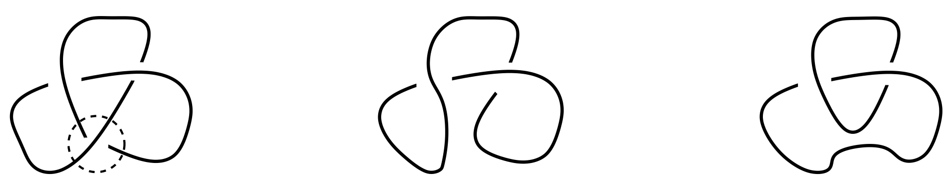
The Jones polynomial is an invariant[1] whose discovery in 1985 brought on major advances in knot theory. For a link L, the Jones polynomial of L is a Laurent polynomial in $t^{1/2}$. (By "Laurent polynomial," we mean that both positive and negative integral powers of $t^{1/2}$ are allowed.)

Vaughan Jones's construction of the polynomial was through a complicated process. Louis Kauffman developed a much easier construction by introducing another polynomial, called the bracket polynomial. This polynomial is defined in terms of link diagrams instead of links. As we will see, the bracket polynomial is not a link invariant.

2.2. **Resolving a crossing.**

We will want to relate the bracket polynomial of a link diagram D to bracket polynomials of "simpler" link diagrams. The following definition makes the idea of simplifying a diagram precise.

Definition 2.1. Suppose we start with a crossing of the form ⤬. The 0-resolution of this crossing is)(and the 1-resolution is $\asymp$. (For example, see Figure 2.1.) $\Diamond$



(A) trefoil (B) 0-resolution (C) 1-resolution

Figure 2.1. Resolving a crossing

A diagram may need to be rotated so that the crossing in concern appears as ⤬. For example, after a 90° rotation, we see that the 0- and 1-resolutions of ⤫ are $\asymp$ and)(, respectively.

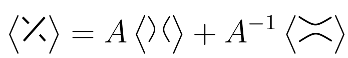
[1] More precisely, the Jones polynomial is an invariant of oriented links. However, when the link has one component (i.e. is a knot), the polynomial does not depend on the orientation. Thus, the Jones polynomial is both an (unoriented) knot invariant as well as an oriented link invariant.

Here is one way to think of a 0-resolution: if we are traveling along a knot and reach a crossing in which we are on the upper strand, then we turn left onto the lower strand. (For a 1-resolution, we would turn right instead.)

2.3. **The Bracket Polynomial.** The bracket polynomial of a diagram D is a Laurent polynomial in one variable A and is denoted $\langle D \rangle$. It is completely determined by three rules:

(BP1) $$\langle \bigcirc \rangle = 1$$

(BP2) $$\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle$$

(BP3) 

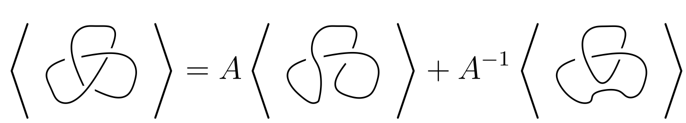
(The BP stands for "bracket polynomial.") Let's go through what these rules mean, one by one.

(1) The first relation (BP1) states that the bracket polynomial of the knot diagram $\bigcirc$ is the constant polynomial 1. (Note, however, that this does *not* mean that the bracket polynomial of *any* diagram depicting the unknot is 1. For example,  is also a diagram of the unknot, but this diagram turns out to have bracket polynomial $-A^9$.)

(2) For the second relation, the expression $D \sqcup \bigcirc$ denotes a diagram D with an extra circle added. Furthermore, the circle does not cross the rest of the diagram. If we do have a diagram of this form, then BP2 means that we can find its bracket polynomial by starting with the bracket polynomial of the diagram with the circle removed and multiplying it by $-A^2 - A^{-2}$. For example, using BP2 (along with BP1), we have

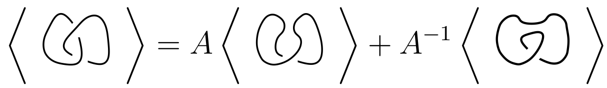
$$\langle \bigcirc\bigcirc \rangle = (-A^2 - A^{-2}) \langle \bigcirc \rangle = -A^2 - A^{-2}$$

(3) In order to apply the third relation, we need to resolve crossings. Start with a diagram D and fix a crossing. If D_0 and D_1 are the 0- and 1-resolutions of this crossing, then BP3 states that $\langle D \rangle = A \langle D_0 \rangle + A^{-1} \langle D_1 \rangle$. For example,

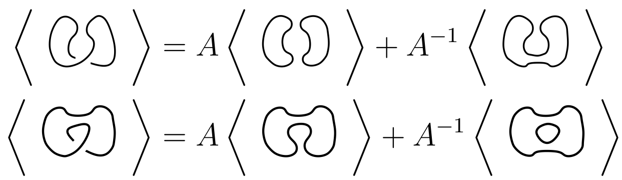


The key idea of computing the bracket polynomial of a diagram D lies in BP3. Using this rule we can recursively compute bracket polynomials of knot diagrams via diagrams of fewer crossings. Eventually we reach diagrams with no crossings.

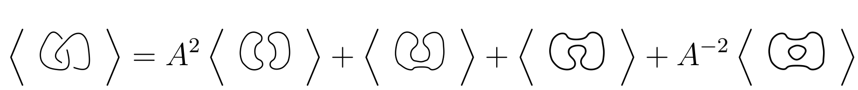
Example 2.2. Let us consider the Hopf link (). Applying BP3 gives us:

(2.1)


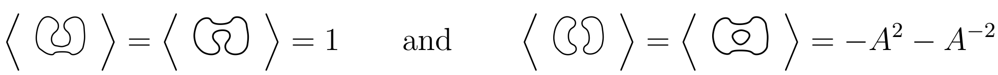
Thus, we have reduced the problem to determining the bracket polynomials of the two diagrams on the right side of (2.1), each of which has one crossing. Using BP3 again,

(2.2)


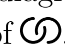
Combining (2.1) and (2.2), we see that



Invoking BP1 and BP2 gives us



Putting everything together gives us ⟨⟩ $= -A^4 - A^{-4}$ ◊

In Example 2.2, we decomposed the Hopf link into four diagrams. The four diagrams correspond to the four ways of resolving the two crossings of . Each of these diagrams is called a *smoothing*.

Definition 2.3. Given a link diagram D, a *smoothing* of D is a diagram in which every crossing of D has been resolved (either by a 0-resolution or a 1-resolution). ◊

We can see that in general, a link diagram D with n crossings has 2^n distinct smoothings. Furthermore, we can see from Example 2.2 that these smoothings allow us to determine the bracket polynomial of D.

In the general case, number the crossings $1, \dots, n$. For $\epsilon_1, \dots, \epsilon_n \in \{0, 1\}$, let $D_{\epsilon_1\epsilon_2\cdots\epsilon_n}$ be the smoothing of D where crossing i is resolved via a ϵ_i-resolution. For example, if $D =$ , and the top crossing is labeled 1 (so the bottom crossing is labeled 2), then

(2.3)


For a smoothing D_ϵ, where $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_n$, define

$$s_0(\epsilon) = \text{number of 0-resolutions in } D_\epsilon$$
$$s_1(\epsilon) = \text{number of 1-resolutions in } D_\epsilon$$

Using this notation, we see that a smoothing D_ϵ contributes a term $A^{s_0(\epsilon)-s_1(\epsilon)}\langle D_\epsilon\rangle$ to the bracket polynomial $\langle D\rangle$. Define $\langle D,\epsilon\rangle = A^{s_0(\epsilon)-s_1(\epsilon)}\langle D_\epsilon\rangle$. Then we can summarize our results from above in the following lemma.

Lemma 2.4. *Let D be a link diagram with n crossings. Then*

$$\langle D\rangle = \sum_{\epsilon\in\{0,1\}^n} \langle D,\epsilon\rangle$$

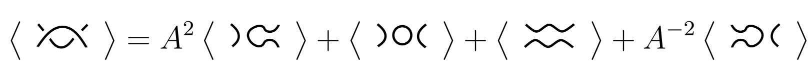
Remark 2.5. The bracket polynomial of a smoothing D_ϵ of D is particularly easy to compute. We know D_ϵ consists of a number (say k) of non-crossing loops. (For example, see (2.3).) By repeatedly applying BP2, we see that $\langle D_\epsilon\rangle = (-A^2 - A^{-2})^{k-1}$. ◊

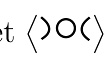
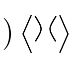
2.4. **Invariance under Type II and Type III Moves.** Because of our discussion of invariants and Reidemeister moves at the end of subsection 1.4, we should study how the bracket polynomial behaves under Reidemeister moves.

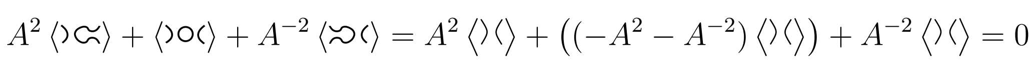
Lemma 2.6 (Invariance under Type II)**.** *If link diagrams D and D' are related by one application of a Type II Reidemeister move, then $\langle D\rangle = \langle D'\rangle$.*

Proof. Start with the diagram $D =$ . Label the two crossings so that 1 is on the left and 2 is on the right. The four ways of resolving these crossings are given by

As in Lemma 2.4, we have

(2.4) 

Using BP2, we get  $= (-A^2 - A^{-2})$ , so three of the four terms on the right hand side of (2.4) cancel out:

(2.5) 

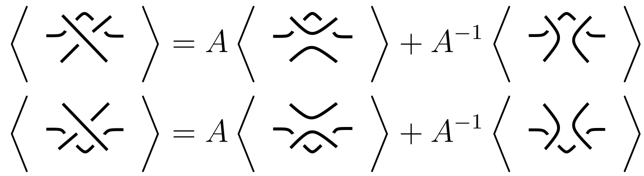
We are left with  $=$ . This proves that the bracket polynomial is unchanged under a Type II move. □

Remark 2.7. The coefficients in BP2 and BP3 were chosen specifically so that the three terms cancel out in (2.5). ◊

Lemma 2.8 (Invariance under Type III)**.** *If link diagrams D and D' are related by one application of a Type III Reidemeister move, then $\langle D\rangle = \langle D'\rangle$.*

Proof. We could start with the diagram  and relate it to the 8 diagrams that correspond to the 8 possible ways of resolving the three crossings. Then we could do the same thing with . We would indeed see that both diagrams give the same result.

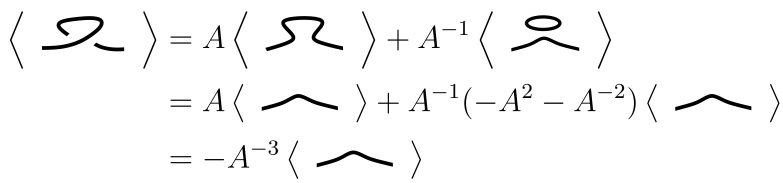
However, we can make a cleaner argument by noting what happens if we only resolve the crossing between the two upper strands:



To complete the proof, it suffices to show that

The equality on the left holds by two applications of Lemma 2.6. The equality on the right holds because the two diagrams are related by a planar isotopy. ☐

2.5. **Type I moves.** We have shown that the bracket polynomial is invariant under Type II and Type III Reidemeister moves. If it is also invariant under Type I moves, then we would have a genuine link invariant. However, this is not the case, as the following computation shows:

(2.6) 

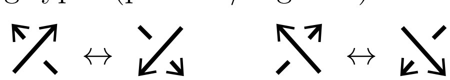
2.6. **The Jones Polynomial.** To resolve the issue of invariance under Type I moves, we will define another number associated to link diagrams, called the *writhe*.

Recall that an oriented link diagram is a diagram in which all components have a specified direction. Once we have an orientation, we can define positive and negative crossings by Figure 2.2.

(A) positive crossing (B) negative crossing

FIGURE 2.2. Two types of crossings for an oriented diagram

Remark 2.9. As the following diagrams show, if we reverse the direction of all components of a link, the crossing types (positive/negative) do not change.

 ◊

Since a knot is a link with one component, we can define positive and negative crossings for knot diagrams without specifying an orientation on the knot. Note that this is not true for links in general. If we reverse the orientations of some (but not all) of the components of a link, then some crossing types will change.

Example 2.10. Consider the oriented trefoil given in Figure 2.3.

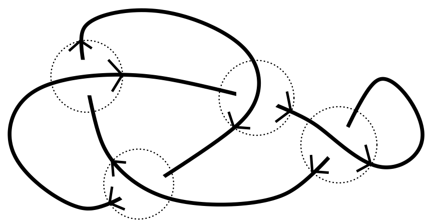


FIGURE 2.3. Oriented trefoil with crossings circled

The three crossings on the left are positive crossings; the crossing on the right is a negative crossing. If we switch the orientation, the crossing types remain the same. ♢

Let $n_+(D)$ and $n_-(D)$ be the number of positive and negative crossings, respectively, of an oriented link diagram D. For the trefoil in Figure 2.3, we have $n_+(D) = 3$ and $n_-(D) = 1$.

Definition 2.11. For an oriented link diagram D, the *writhe* of D is $w(D) = n_+(D) - n_-(D)$. ♢

For the trefoil in Figure 2.3, we have $w(D) = 2$. Observe that if we remove the twist on the right side of the trefoil in Figure 2.3, then we decrease n_- by 1. Thus, the writhe of the resulting diagram D' (a trefoil in its "typical" depiction) is $w(D') = 3$.

As we have shown in the example above, the writhe is not invariant under Type I moves. Going from  to decreases n_- by 1. Thus,

(2.7) $w($$) = w($$) - 1$

This shows that the writhe is not a link invariant. However, we can observe that like the bracket polynomial, the writhe is unchanged under Type II and Type III moves.

We know precisely how Type I moves affect the writhe and the bracket polynomial. These are given by (2.7) and (2.6), respectively. Thus, a certain combination of the writhe and the bracket polynomial will in fact give us a link invariant!

Recall that uncurling the twist in  leads to a multiplication by $-A^{-3}$ in the bracket polynomial. The key idea we will use is that we can "cancel" this multiplication by using the writhe. The following lemma makes this idea precise.

Lemma 2.12. *For an oriented link L with diagram D, the polynomial $(-A)^{-3w(D)}\langle D\rangle$ is an invariant of the link L.*

Proof. Because both the writhe and the bracket polynomial are invariant under Type II and Type III moves, we only need to check that the quantity $(-A)^{-3w(D)}\langle D\rangle$ is unchanged under Type I moves. This is immediate from (2.6) and (2.7):

$$\begin{aligned}(-A)^{-3w(\text{kink})}\langle \text{kink} \rangle &= (-A)^{-3w(\frown)+3}\cdot(-A)^{-3}\langle \frown \rangle \\ &= (-A)^{-3w(\frown)}\langle \frown \rangle\end{aligned}$$ □

Remark 2.13. The importance of Lemma 2.12 is that finally, we have a polynomial associated to an oriented link L that does not depend on the choice of the diagram used to compute the polynomial. ◊

We have pretty much given a complete construction of the Jones polynomial, as well as a proof of its invariance. The only thing is that the actual Jones polynomial is normalized in a different way (because it was discovered via different means).

Definition 2.14. For an oriented link L, the *Jones polynomial* of L, denoted $V_L(t)$, is obtained by taking the expression $(-A)^{-3w(D)}\langle D\rangle$ and setting $A = t^{-1/4}$. ◊

Remark 2.15. We have shown that the Jones polynomial is an invariant of oriented links. For knots, recall that the writhe does not depend on the orientation. If we retrace our arguments above, we see that the Jones polynomial is also an invariant of (unoriented) knots. ◊

The next two sections aim to answer the question "What is the Jones polynomial good for?" We will give two applications, the second much more involved than the first.

3. Detecting chiral knots

Definition 3.1. A knot K is *amphichiral* if it equivalent to its mirror image (in $\mathbb{R}^3$). Otherwise, it is *chiral*. ◊

Consider, for example the two diagrams in Figure 3.1. We may ask if they are the same knot.

(A) left-handed trefoil (B) right-handed trefoil

Figure 3.1. Mirror images of a trefoil

By using the Jones polynomial, we will see that the answer is no: the left-handed and right-handed trefoils are distinct knots. (In other words, the trefoil is *chiral*.)

For a knot K, let K^{flip} denote the mirrored knot. If K has a diagram D, then D^{flip} denotes the corresponding mirrored diagram for K^{flip}.

Suppose we start with a smoothing D_ϵ of D. What do we get when we mirror this to $(D_\epsilon)^{\text{flip}}$? The answer is a smoothing of D^{flip}. See the example in Figure 3.2.

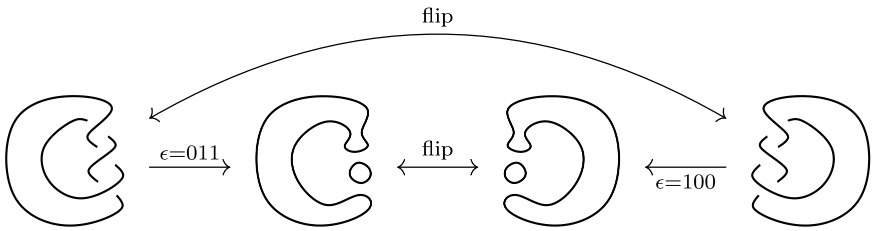


FIGURE 3.2. Relations between mirroring and smoothing. (In the labeling, the three crossings are numbered from top to bottom.)

The example makes a general pattern clear. Every smoothing ϵ of D corresponds to the dual smoothing $\hat{\epsilon}$ of D^{flip}. The dual smoothing $\hat{\epsilon}$ is obtained by reversing every resolution in ϵ. (That is, interchange 0s and 1s.) For example, if $\epsilon = 011$, then $\hat{\epsilon} = 100$. We can write the relation as

$$(D^{\text{flip}})_{\hat{\epsilon}} = D_\epsilon \tag{3.1}$$

Observe that $s_0(\epsilon) = s_1(\hat{\epsilon})$ and $s_1(\epsilon) = s_0(\hat{\epsilon})$. Using (3.1), it follows that

$$\left\langle D^{\text{flip}} \right\rangle (A) = \sum_{\epsilon \in \{0,1\}^n} A^{s_0(\epsilon) - s_1(\epsilon)} \left\langle (D^{\text{flip}})_\epsilon \right\rangle (A) = \sum_{\epsilon \in \{0,1\}^n} A^{-(s_0(\hat{\epsilon}) - s_1(\hat{\epsilon}))} \left\langle D_{\hat{\epsilon}} \right\rangle (A)$$

Recall that the bracket polynomial of a smoothing $\langle D_\epsilon \rangle (A)$ is some power of $(-A^2 - A^{-2})$. Hence, $\langle D_\epsilon \rangle (A) = \langle D_\epsilon \rangle (A^{-1})$, giving us

$$\left\langle D^{\text{flip}} \right\rangle (A) = \sum_{\epsilon \in \{0,1\}^n} (A^{-1})^{s_0(\hat{\epsilon}) - s_1(\hat{\epsilon})} \left\langle D_{\hat{\epsilon}} \right\rangle (A^{-1}) = \langle D \rangle (A^{-1})$$

Also, $w(D) = -w(D^{\text{flip}})$, since positive crossings in D become negative crossings in D^{flip} and vice versa. We can translate the results we have just obtained to a statement about the Jones polynomial.

Lemma 3.2. *For any knot K,*

$$V_{K^{\text{flip}}}(t) = V_K(t^{-1})$$

Example 3.3. For the left handed trefoil, we have $V(t) = -t^{-4} + t^{-3} + t^{-1}$. It follows that the Jones polynomial of the right-handed trefoil is $V(t) = -t^4 + t^3 + t$. Since the two polynomials are not the same, we can conclude that the left-handed and right-handed trefoils are distinct. ♢

Using the same reasoning as Example 3.3, we have the following result.

Theorem 3.4. *If K is a knot and $V_K(t) \neq V_K(t^{-1})$, then K and K^{flip} are distinct. That is, K is chiral.*

Remark 3.5. Note that Theorem 3.4 does not tell us anything about K if $V_K(t) = V_K(t^{-1})$. It would be nice if $V_K(t) = V_K(t^{-1})$ implied that K is amphichiral. However, there are chiral knots with symmetric Jones polynomials. (See [Mur96, Exercise 11.2.2] for an example.) ♢

Remark 3.6. Recall that the Jones polynomial is not just a knot invariant but also an oriented link invariant. Thus, all the results in this section (in particular, Theorem 3.4) hold if we replace "knot" with "oriented link." ♢

4. Bound on crossing numbers

4.1. **Crossing number of a knot.** In Example 1.5, we gave an example of a natural knot invariant. We repeat it here.

Definition 4.1. The *crossing number* of a knot K is the minimum number of crossings needed to draw the knot in a plane. It is denoted $c(K)$. ♢

The crossing number is a difficult invariant to work with. Suppose we start with a knot K and draw a few diagrams of K. Suppose that out of all our diagrams, the one with the fewest number of crossings uses n crossings. Then we know $c(K) \leq n$. However, we cannot be sure that there is no diagram of K with fewer crossings. Drawing more diagrams will not help; there are infinitely many ways to represent K as a knot diagram.

The aim of this section is to use the Jones polynomial to give a nontrivial lower bound for $c(K)$.

4.2. **Reduced diagrams and removable crossings.** In certain cases, it is easy to tell that a crossing can be removed, as in Figure 4.1a below:

(A) (B)

Figure 4.1

In Figure 4.1a, the crossing in the center can easily be removed by flipping the right half over, leaving us with Figure 4.1b. More generally, consider knots of the form in Figure 4.2.

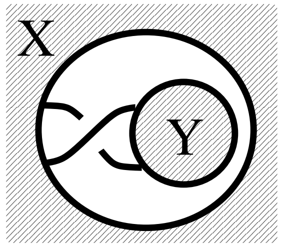


FIGURE 4.2

That is, there are exactly two strands in the region between X and Y that go from X to Y and that cross each other once. We can flip Y to remove the crossing. This leads us to make a very natural definition.

Definition 4.2. A knot diagram D is *reduced* is it does not have the form of Figure 4.2. The crossing in the region between X and Y is called a *removable crossing*. $\Diamond$

The condition of being reduced is a good starting point in our attempt to determine the crossing number of knots. Having a reduced diagram D of a knot K, however, is not enough to conclude that there is no diagram of K with fewer crossings. Consider the unknot as depicted in Figure 4.3.

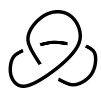
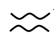


FIGURE 4.3. Reduced diagram of the unknot with three crossings

This diagram is reduced and contains three crossings, but the unknot can be drawn with zero crossings.

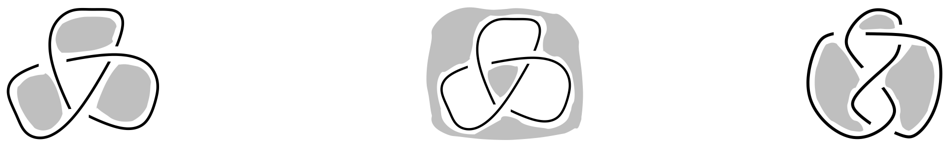
Remark 4.3. At this point, we might note that crossings that can be eliminated by Type II Reidemeister moves ( $\rightarrow$ ) are also easy to identify in a knot diagram. Why is the situation  not included in the definition of "unreduced"? It turns out not to be needed. We will see why in subsection 4.4. $\Diamond$

4.3. **Some combinatorial aspects of knots.** We make a brief digression to discuss some combinatorics relating to knots. A knot diagram divides the plane into disjoint regions, or faces. (The exterior of a knot diagram is considered a face too.) We will look at two things: number of faces, and colorings of the faces.

Lemma 4.4. *For a knot with n crossings, there are $n+2$ faces*

Proof. We think of the knot as a graph and apply Euler's formula: $V - E + F = 2$. Because each crossing involves two strands, each vertex of the graph has degree four, so there are $2n$ edges. Then $V = n$ and $E = 2n$ gives $F = n + 2$. $\square$

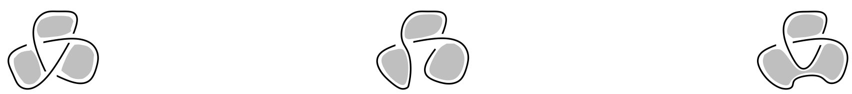
It is always possible to color the faces of a knot in an alternating black-white pattern, as shown in Figure 4.4. (Recall that we are considering the exterior of the knot diagram to be a face as well, which explains why Figure 4.4b is a valid checkerboard coloring of the trefoil.)



(A) trefoil (B) trefoil (C) figure-8 knot

FIGURE 4.4. Examples of checkerboard colorings

Observe that by resolving a crossing, we still get a checkerboard coloring. (See Figure 4.5.)



(A) checkerboard (B) 0-resolution (C) 1-resolution

FIGURE 4.5. Resolving a crossing still gives us a checkerboard coloring.

Given a checkerboard coloring of a knot diagram, we can divide the crossings into two types. (See Figure 4.6.) The coloring in Figure 4.6a is called "0-separating" because a 0-resolution separates the two black regions. The coloring in Figure 4.6b is called "1-separating" for similar reasons.

(A) 0-separating (B) 1-separating

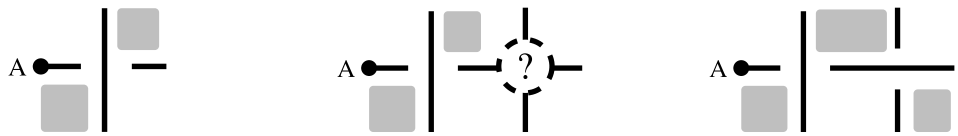
FIGURE 4.6. Two coloring types at a crossing

We could ask the following question: Given a knot diagram, when does a checkerboard coloring consist of only 0-separating crossings?[2] (For example, the trefoil in Figure 4.4a and the figure-8 knot in Figure 4.4c both satisfy this property.)

To answer that question, consider the portion of a knot diagram given in Figure 4.7a. Suppose we start at point A and move rightwards along the horizontal strand. We will reach the crossing at the center of the diagram. (Observe that this is a 0-separating

[2]Note that if a checkerboard coloring consists only of 0-separating crossings, then we can invert the colors (interchange black and white) to get a coloring that consists only of 1-separating crossings.

crossing.) In this crossing, the horizontal strand we are traveling along goes underneath the vertical strand.



(A) (B) (C)

FIGURE 4.7

We keep moving rightwards after we pass the crossing. Eventually we will reach another crossing. (See Figure 4.7b.) There is only one way to make this crossing 0-separating. The horizontal strand must go over the vertical strand. (See Figure 4.7c.)

The general pattern is clear: if we want this knot diagram to have only 0-separating crossings, then the strand must alternate over-under-over-under.

Definition 4.5. A knot diagram is *alternating* if the strand alternates between going over and going under at crossings. A knot is *alternating* if there is a alternating diagram of the knot. $\Diamond$

As it turns out, alternating knots are a very well-behaved class of knots. We have just seen that alternating knots have the following property.

Lemma 4.6. *A knot diagram admits a checkerboard coloring consisting of only* 0-*separating crossings if and only if the diagram is alternating.*

4.4. **Reduced alternating knot diagrams.**

Definition 4.7. A knot diagram D is *reduced alternating* if it is both reduced and alternating. $\Diamond$

The definition may seem trivial, but we should point out one thing: if we start with an unreduced alternating diagram (of the form Figure 4.2), we can apply the "flipping" operation to get rid of the removable crossing. What is important is that the resulting diagram will still be alternating.

Thus, to determine $c(K)$ for a given alternating knot K, we may start with an alternating diagram of K. Next, we can remove all the removable crossings to get a reduced alternating diagram D. If this diagram has n crossings, we know that $c(K) \leq n$.

If it is possible to remove any more crossings, it is not immediately obvious: "flipping" will not help since there are no more removable crossings. Type II Reidemeister moves ( $\to$ ) will not help either: because the diagram is alternating,  will not appear.

As we will show in the next section, there is in fact no way to lower the number of crossings from that point. The proof will not proceed by drawing more diagrams. (We have already explained in subsection 4.1 why that approach will not work.) Instead, we will use the Jones polynomial to help us.

4.5. **Bound on the crossing number.** Recall that for a link diagram D, Lemma 2.4 gives the bracket polynomial in terms of the smoothings of D. In particular, $\langle D\rangle = \sum \langle D,\epsilon\rangle$, where $\langle D,\epsilon\rangle = A^{s_0(\epsilon)-s_1(\epsilon)}\langle D_\epsilon\rangle$.

Let $o(\epsilon)$ be the number of circles in D_ϵ. (We use the letter o because it looks like a circle!) Then $\langle D_\epsilon\rangle = (-A^2 - A^{-2})^{o(\epsilon)-1}$.

If ϵ and ϵ' differ by one resolution, then $o(\epsilon') = o(\epsilon) \pm 1$, because changing one resolution either merges two circles in D_ϵ or separates one circle into two.

Definition 4.8. For a Laurent polynomial $f(x)$, we define $\operatorname{hp}(f)$ to be the highest power of x that appears in f, and we define $\operatorname{lp}(f)$ similarly to be the lowest power. We define $\operatorname{span}(f) = \operatorname{hp}(f) - \operatorname{lp}(f)$. ♢

Remark 4.9. In particular, we will be looking at the span of the bracket polynomial. Why is this numerical value useful? Take a knot K with diagram D. If we change D by a Type I Reidemeister move, all the powers in the bracket polynomial are shifted by the same amount, according to (2.6). Thus, the value of $\operatorname{span}\langle D\rangle$ only depends on the knot K. It does not depend on the diagram D, even though $\langle D\rangle$ does depend on the diagram. ♢

We have

$$\operatorname{hp}\langle D,\epsilon\rangle = \operatorname{hp} A^{s_0(\epsilon)-s_1(\epsilon)}(-A^2 - A^{-2})^{o(\epsilon)-1} = s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2$$

We know that $\operatorname{hp}\langle D\rangle \le \max_\epsilon \operatorname{hp}\langle D,\epsilon\rangle$, so naturally, we would be interested in determining which smoothing ϵ maximizes the expression $s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2$.

Let n be the number of crossings of D. Choosing $\epsilon = \mathbf{0}$ (i.e. the smoothing with all 0-resolutions) will maximize $s_0(\epsilon) - s_1(\epsilon)$ (by setting it to n), but we still need to consider the $2o(\epsilon)$ term. The following lemma shows that we do not need to worry about that term.

Lemma 4.10. *If ϵ is a smoothing, and we change a 0-resolution to a 1-resolution to get ϵ', then* $\operatorname{hp}\langle D,\epsilon\rangle \ge \operatorname{hp}\langle D,\epsilon'\rangle$.

Proof. As we go from ϵ to ϵ', the quantity $s_0 - s_1$ decreases by 2. Since $|o(\epsilon) - o(\epsilon')| = 1$, the quantity $2o$ can increase by 2 or decrease by 2. In either case, we have

$$s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2 \ge s_0(\epsilon') - s_1(\epsilon') + 2o(\epsilon') - 2$$ □

Corollary 4.11. *When we let ϵ range over all smoothings,* $\operatorname{hp}\langle D, \epsilon\rangle$ *achieves a maximum at* $\epsilon = \mathbf{0}$.

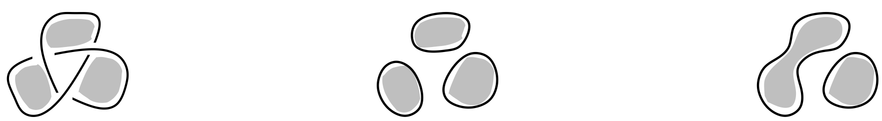
As a result, we know that $\operatorname{hp}\langle D\rangle \leq \max_\epsilon \operatorname{hp}\langle D, \epsilon\rangle = \operatorname{hp}\langle D, \mathbf{0}\rangle = n + 2o(\mathbf{0}) - 2$. We have the analogous statement that $\operatorname{lp}\langle D\rangle \geq \operatorname{lp}\langle D, \mathbf{1}\rangle = -n - 2o(\mathbf{1}) + 2$. (Here, $\mathbf{1}$ is the smoothing with all 1-resolutions.) Thus, we have

$$\operatorname{span}\langle D\rangle \leq 2n + 2(o(\mathbf{0}) + o(\mathbf{1})) - 4$$

Furthermore, if the knot is reduced alternating, we have nicer results, because of the following

Lemma 4.12. *If D is a reduced alternating diagram and ϵ is a smoothing with exactly one* 1*-resolution, then* $o(\mathbf{0}) = o(\epsilon) + 1$. *(That is, as we go from $D_{\mathbf{0}}$ to D_ϵ, we merge two circles together.)*

Proof. Because the diagram is alternating, we can give the diagram a checkerboard coloring so that all the crossings are 0-separating. See for example, Figure 4.8a.



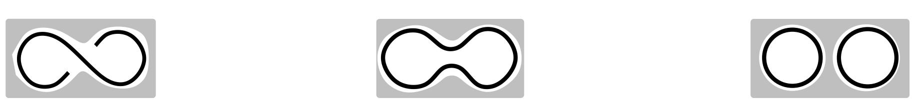
(A) D (B) $D_{\mathbf{0}}$ (C) D_ϵ

FIGURE 4.8

Then when we look at $D_{\mathbf{0}}$, all the circles bound the shaded faces in the coloring. (See Figure 4.8b.)

When we change one of the 0-resolutions to a 1-resolution, we add a "bridge" across a white region. (See Figure 4.8c.) Because the diagram D is reduced, the bridge connects two *separate* black regions. Thus, the number of circles decreases by one, so $o(\mathbf{0}) = o(\epsilon) + 1$. □

Remark 4.13. The assumption that the diagram D is reduced is necessary for the last step. Compare the diagrams in Figure 4.8 with the diagrams in the following example, Figure 4.9.



(A) D (B) $D_{\mathbf{0}}$ (C) D_ϵ

FIGURE 4.9

What ends up happening here is that the bridge formed in Figure 4.9c connects the same black region to itself, thus *increasing* the number of circles. This is due to the removable crossing. ♢

Corollary 4.14. *If D is a reduced alternating diagram and ϵ is a smoothing with exactly one* 1*-resolution, then* $\operatorname{hp}\langle D, \mathbf{0}\rangle > \operatorname{hp}\langle D, \epsilon\rangle$*.*

Proof. Since, $o(\mathbf{0}) = o(\epsilon) + 1$, we can modify the proof of Lemma 4.10 to get a strict inequality. □

Corollary 4.15. *If D is a reduced alternating diagram, then* $\operatorname{hp}\langle D\rangle = \operatorname{hp}\langle D, \mathbf{0}\rangle$*.*

Proof. We have $\langle D\rangle = \langle D, \mathbf{0}\rangle + \sum_{\epsilon\neq\mathbf{0}}\langle D, \epsilon\rangle$. By Lemma 4.10 and Corollary 4.14, we know that the highest power in $\sum_{\epsilon\neq\mathbf{0}}\langle D, \epsilon\rangle$ is strictly lower than the highest power in $\langle D, \mathbf{0}\rangle$. □

Everything stated above has an analogue for the **1**-smoothing; in particular, the analogue of Corollary 4.15 is that $\operatorname{lp}\langle D\rangle = \operatorname{lp}\langle D, \mathbf{1}\rangle$ for reduced alternating diagrams D. We summarize our results so far with the following lemma.

Lemma 4.16. *If D is a knot diagram with n crossings, then*

$$\operatorname{span}\langle D\rangle \leq 2n + 2(o(\mathbf{0}) + o(\mathbf{1})) - 4$$

Furthermore, if D is reduced alternating, then equality holds.

Next, we try to analyze the quantity $o(\mathbf{0}) + o(\mathbf{1})$. This is easy if D is alternating: we can color D so that it only has 0-separating crossings. Then the key observation is that $o(\mathbf{0})$ counts the black faces and $o(\mathbf{1})$ counts the white faces. Thus, $o(\mathbf{0}) + o(\mathbf{1})$ equals the total number of faces. Invoking Lemma 4.4, we see that $o(\mathbf{0}) + o(\mathbf{1}) = n + 2$.

In fact, alternating knots are optimal, in the sense given in the following lemma.

Lemma 4.17. *Let D be a connected link diagram.*[3] *If D has n crossings, then*

$$o(\mathbf{0}) + o(\mathbf{1}) \leq n + 2$$

Furthermore, if D is reduced alternating, then equality holds.

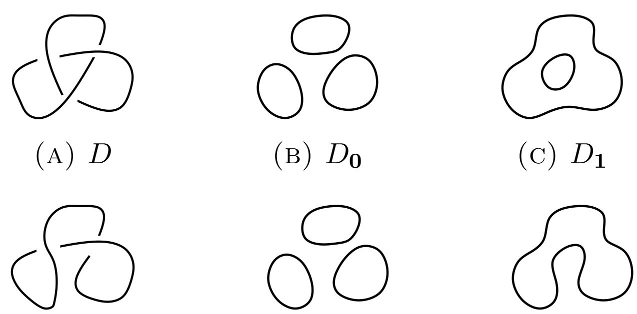
We can see that this lemma contains a few subtle technicalities. We need the statement to be about connected link diagrams as opposed to knot diagrams so that we can apply induction properly. After we prove this lemma, we will only use the result for knot diagrams.

[3] We say that a link diagram is *connected* if its graph is connected. For example, the usual diagram for the unlink  is not connected. However, if the two components overlap in the diagram (as in ), then the diagram is connected. (Note that knot diagrams are always connected.)

Proof. We have already proved the special case when D is alternating. For a general connected link diagram D, we proceed by induction. For the base case ($n = 0$), we have the usual diagram for the unknot (○). Since there are no crossings to resolve, we have $o(D_{\mathbf{0}}) = o(D_{\mathbf{1}}) = 1$, which completes the base case.[4]

For the inductive step, suppose that the statement is true for all connected link diagrams with n crossings. Take a connected link diagram D with $n + 1$ crossings.

Pick one of the crossings of D (call it x). We can resolve x via a 0- or a 1-resolution; observe that at least one of the two resulting diagrams is connected. Suppose, without loss of generality, that applying the 0-resolution on x leaves us with a connected link diagram. Let this diagram be E. (See Figure 4.10.)



(A) D (B) $D_{\mathbf{0}}$ (C) $D_{\mathbf{1}}$

(D) E (E) $E_{\mathbf{0}}$ (F) $E_{\mathbf{1}}$

FIGURE 4.10. Various smoothings of D and E (x is the crossing on the bottom of D)

Note that $E_{\mathbf{0}}$ and $E_{\mathbf{1}}$ are also smoothings of D. In fact:

- $E_{\mathbf{0}} = D_{\mathbf{0}}$.
- $E_{\mathbf{1}}$ and $D_{\mathbf{1}}$ differ at exactly one crossing (namely, x).

Thus, $o(E_{\mathbf{0}}) = o(D_{\mathbf{0}})$ and $o(E_{\mathbf{1}}) = o(D_{\mathbf{1}}) \pm 1$. Furthermore, E is a connected link diagram with n crossings, so by the inductive hypothesis, $o(E_{\mathbf{0}}) + o(E_{\mathbf{1}}) \leq n + 2$. Putting the results together gives us $o(D_{\mathbf{0}}) + o(D_{\mathbf{1}}) \leq n + 3$, which completes the induction. □

Combining the previous two lemmas gives us the following.

Corollary 4.18. *If D is a knot diagram with n crossings, then*

$$\operatorname{span} \langle D \rangle \leq 4n$$

Furthermore, if D is reduced alternating, then equality holds.

[4] For this proof, we write $o(D_{\mathbf{0}})$ instead of $o(\mathbf{0})$ to emphasize that D is the knot we are taking the $\mathbf{0}$-smoothing of.

We are almost there! Recall (as we briefly discussed in Remark 4.9) that span $\langle D\rangle$ does not depend on the diagram D used for a knot K.

If D is a diagram for K with exactly $c(K)$ crossings, then the corollary gives us

$$\operatorname{span}\langle D\rangle \leq 4c(K) \tag{4.1}$$

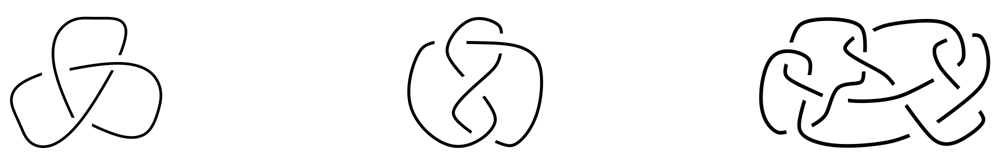
From the definition of the Jones polynomial (Definition 2.14), we have $4\operatorname{span} V_K = \operatorname{span}\langle D\rangle$. Using this relationship, we can restate (4.1) without any reference to particular knot diagrams.

Theorem 4.19. *Let K be a knot. Then $c(K) \geq \operatorname{span} V_K$. Furthermore, if K is an alternating knot, then equality holds.*

In other words, the span of the Jones polynomial gives a lower bound on the crossing number. Recall that lower bounds were difficult to determine in general.

Furthermore, if a knot is an alternating knot, we can determine its crossing number as follows: first, we start with an alternating diagram of the knot. Then we can eliminate removable crossings until we get a reduced alternating diagram. At this point, we know there is no way to draw the knot with fewer crossings, so we can count that number of crossings and that will be the crossing number of our knot.

For example, we know the crossing numbers of the trefoil and figure-8 knot are 3 and 4, respectively, because Figure 4.11a and Figure 4.11b are reduced alternating. The knot depicted in Figure 4.11c may look intimidating, but once we notice that the diagram is reduced alternating, we can conclude that the knot has crossing number 11.



(A) trefoil (B) figure-8 (C) 11-crossing diagram

FIGURE 4.11. Simply count the crossings of these reduced alternating knot diagrams and we get the crossing numbers of the knots!

5. More on the Jones Polynomial

The Jones polynomial gives us valuable information about knots, as the previous sections show. Our understanding of the Jones polynomial, however, is far from complete.

5.1. **How well does the Jones polynomial distinguish among knots?** Suppose we have two knot diagrams D and D', and suppose D and D' are diagrams of knots K and K', respectively. We want to check if $K = K'$ (i.e. if D and D' are diagrams of the same knot). From the two diagrams alone, we are able to compute $V_K(t)$ and $V_{K'}(t)$.

If we get $V_K(t) \neq V_{K'}(t)$, then we know $K \neq K'$. However, what if we get $V_K(t) = V_{K'}(t)$? Does that imply $K = K'$? The answer is no. In other words, different knots could have the same Jones polynomial. (See [Mur96, Example 11.2.6 and Example 11.2.7] for examples.)

However, would could pose the following (weaker) question: Does $V_K(t) = 1$ imply that $K = \bigcirc$? The answer to that turns out to be unknown.[5]

5.2. **"Categorifying" the Jones polynomial.** In the late 1990s, Mikhail Khovanov developed a homology theory for links that generalizes the Jones polynomial by "categorifying" it. Some (qualitative) facts about Khovanov homology are:

- The Khovanov homology of an oriented link L, denoted $Kh(L)$, can be computed from link diagrams using rules similar to BP1, BP2, and BP3. [BN02] provides a very friendly introduction and proves that $Kh(L)$ is an oriented link invariant by showing that it is unchanged under the Reidemeister moves.[6]
- Whereas we assign each oriented link diagram to a polynomial (the bracket polynomial), Khovanov's theory assigns each oriented link diagram to a chain complex of graded vector spaces.
- Khovanov homology is *strictly* stronger than the Jones polynomial: while it is possible to recover $V_L(t)$ from $Kh(L)$, there are distinct links L and L' with the property that $V_L(t) = V_L(t')$ but $Kh(L) \neq Kh(L')$. (See [BN02].)
- Khovanov homology *does* detect the unknot. That is, if $Kh(L) = Kh(\bigcirc)$, then $L = \bigcirc$. This is proved in [KM11]. (Recall that for the Jones polynomial, this problem is unsolved.)
- Just as we used the Jones polynomial to prove a lower bound on the crossing number, Khovanov homology can be used to provide insight into other knot invariants. For example, [Ras04] uses Khovanov homology to give a lower bound on the *slice genus* of a knot.

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[5] However, we should note that [Thi01] gives examples of non-trivial 2-component links with Jones polynomials equal to the Jones polynomial of the unlink ($\bigcirc\bigcirc$). In fact, for each $k > 1$, [EKT03] gives an infinite family of non-trivial k-component links with Jones polynomials equal to the k-component unlink.

[6] Understanding how to compute $Kh(L)$ is not too difficult; understanding the proof of invariance, however, requires some basic knowledge of homological algebra.

6. Acknowledgments

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Princeton University