DRP – Notes and Problems for Week 2

Section 1.4

a. Let V be a vector space with subspaces U, U', and W. Prove that $(U+U')\cap W = U\cap W + U'\cap W$ if $U \subseteq W$ or if $U' \subseteq W$. This is known as the modular law for vector spaces.

b. Let V be a vector space with the subspace W. For $u, v \in V$ we say that $u \equiv_W v$ if $u - v \in W$. For $v \in V$ we have $v - v = 0 \in W$, so that $v \equiv_W v$. If $u \equiv_W v$ then $u - v \in W$ and so $v - u = -(u - v) \in W$, so that $v \equiv_W u$. Also, if $u \equiv_W v$ and $v \equiv_W w$ then $u - v, v - w \in W$ so that $u - w = (u - v) + (v - w) \in W$, and hence $u \equiv_W w$. These results show that $\equiv_W$ is an equivalence class on V. If $v \in V$, then the equivalence class of v is denoted by $\bar{v}$. We write V/W or sometimes $\bar{V}$ as the set of all equivalence classes.

Now notice that $u+v \equiv_W u'+v'$ if $u \equiv_W u'$ and $v \equiv_W v'$, since $(u+v)-(u'+v') = (u-u')+(v-v') \in W$. So there is a well-defined operation of addition on V/W given by $\bar{u}+\bar{v} = \overline{u+v}$. Further, if $u \equiv_W u'$ and λ is a scalar, then $\lambda u \equiv_W \lambda u'$ since $\lambda u - \lambda u' = \lambda(u - u') \in W$. So there is a well-defined operation of scalar multiplication on V/W given by $\lambda \cdot \bar{u} = \overline{\lambda u}$. It is reasonably clear that V/W becomes a vector space with these two operations. We call V/W the quotient space of V by W.

c. Let V be a vector space with subspace W. If $v_1, \ldots, v_n$ are elements of V that generate V, then $\bar{v}_1, \ldots, \bar{v}_n$ are elements of V/W that generate V/W. After all, take a general element $\bar{v}$ of V/W, and write $v = \sum_1^n \lambda_i v_i$ in V. Then $\bar{v} = \sum_1^n \lambda_i \bar{v}_i$ in V/W. Conversely, if $\bar{v}_1, \ldots, \bar{v}_m$ are elements in V/W that generate V/W, and if $w_1, \ldots, w_n$ are elements in W that generate W, then $v_1, \ldots, v_m, w_1, \ldots, w_n$ are elements in V that generate V. After all, let v be a general element of V and write $\bar{v} = \sum_1^m \lambda_i \bar{v}_i$ in V/W. Then $v - \sum_1^m \lambda_i v_i \in W$, and hence $v - \sum_1^m \lambda_i v_i = \sum_1^n \mu_i w_i$ for some μ_i. Then we get $v = \sum_1^m \lambda_i v_i + \sum_1^n \mu_i w_i$.

d. Let V be a vector space with subspace W. Suppose $\{w_1, \ldots, w_m\}$ is a basis of W, and extend this to a basis $\{w_1, \ldots, w_m, v_1, \ldots, v_n\}$ of V. Then $\bar{v}_1, \ldots, \bar{v}_n$ are elements of V/W that generate V/W. Suppose that $\sum_1^n \lambda_i \bar{v}_i = 0$. Then $\sum_1^n \lambda_i v_i \in W$ and hence each $\lambda_i = 0$. Therefore, $\{\bar{v}_1, \ldots, \bar{v}_n\}$ forms a basis of V/W. Conversely, if $\{\bar{v}_1, \ldots, \bar{v}_n\}$ forms a basis of V/W and $\{w_1, \ldots, w_m\}$ forms a basis of W, then $\{w_1, \ldots, w_m, v_1, \ldots, v_n\}$ forms a basis of V. From this we conclude that $\dim(V/W) = \dim(V) - \dim(W)$.

It is largely because $\dim(V/W) = \dim(V) - \dim(W)$ that the quotient space is of interest. More specifically, suppose we wish to prove a certain statement about vector spaces by induction. After proving the base case, we assume that the result holds for all vector spaces of dimension less than n, and then we take a vector space V of dimension n. According to the problem at hand, we can often choose an appropriate nonzero subspace W of V, we note that $\dim(V/W) < n$ and hence V/W satisfies the conclusion by induction, and then we see what this tells us about V. For instance, the exchange lemma Theorem 3.1 of Chapter 1 can be neatly proved via this method.

Section 2.1

a. Let V be the vector space consisting of all $n \times n$ matrices over the field K. Show that the set U of all symmetric matrices in V is a subspace of V, as is the set W of all skew-symmetric matrices in V. Prove that $V = U + W$, find $\dim(U)$ and $\dim(W)$, and conclude that $V = U \oplus W$.

Section 2.2

a. Using Gaussian elimination, find a basis for the subspace of $\mathbf{R}^4$ consisting of solutions to the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + 4x_4 &= 0 \\ x_1 - 4x_2 + 3x_3 + 30x_4 &= 0 \\ 5x_1 + 2x_2 + x_3 + 38x_4 &= 0 \end{aligned}$$

One possible answer is $\{{}^t(-106, 56, 0, 11), {}^t(-5, 7, 11, 0)\}$.

b. Use Gaussian elimination to find the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix}$$

c. Let W be the subspace of $\mathbf{R}^3$ that is spanned by the three vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Using Gaussian elimination, find a necessary and sufficient condition for the vector ${}^t(x, y, z)$ to belong to W. The end result should be a non-zero scalar multiple of the equation $x - 2y + z = 0$.

Section 2.3

a. There are certain $n \times n$ matrices that are particularly easy to work with, and that form a basis for the vector space consisting of all $n \times n$ matrices over the field K. Namely, we let E_{ij} be the matrix that has a 1 in the (i,j)th place and zeroes elsewhere (some authors write e_{ij}). Then $E_{ij}E_{kl} = E_{il}\delta_{jk}$ where δ_{jk} is Kronecker's delta function. In other words, $E_{ij}E_{kl}$ is the zero-matrix if $j \neq k$, and $E_{ij}E_{jl} = E_{il}$. It is clear that $\{E_{ij} | 1 \leq i, j \leq n\}$ forms a basis for the collection of all $n \times n$ matrices, since $A = \sum_{i,j} a_{ij}E_{ij}$ for any $n \times n$ matrix $A = (a_{ij})$.

As an application, we see that $\{E_{ij} - E_{ji} | i \neq j\}$ is a basis for the collection of all $n \times n$ skew-symmetric matrices, and hence this subspace has dimension $\sum_{i=1}^{n-1} i = n(n-1)/2$. On the other hand, the set of all symmetric $n \times n$ matrices has basis $\{E_{ii}\} \cup \{E_{ij} + E_{ji} | i \neq j\}$ and hence has dimension $n(n+1)/2$.

Now let A be any $n \times n$ matrix and notice that

$$E_{ij}A = \sum_{k,l} a_{kl}E_{ij}E_{kl} = \sum_{l} a_{jl}E_{il}$$

This means that $E_{ij}A$ is the matrix obtained from A by replacing the ith row of A with the jth row, and replacing all other rows with zeros. For this reason, multiplying a matrix A by E_{ij} is sometimes useful when we wish to work with a specific row in A. It is because the E_{ij} are so simple that it is often worthwhile to keep them in mind when working with matrices. As specific instances of this, we have the next two problems.

b. If A is an $n \times n$ matrix, then $\mathrm{Tr}(A)$ is defined as the sum of the diagonal entries of A. Because $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ whenever A and B are $n \times n$ matrices, it follows that $\mathrm{Tr}(AB - BA) = 0$. Prove the converse. That is, assuming that $\mathrm{Tr}(C) = 0$ for the $n \times n$ matrix C, show that there are $n \times n$ matrices A and B satisfying $C = AB - BA$.

c. Let V be the vector space consisting of all $n \times n$ matrices over the field K. As always, we can multiply any two elements of V. However, unlike multiplication in $\mathbf{R}$ or $\mathbb{C}$, multiplication in V need not be commutative. Suppose that A in V satisfies $AB = BA$ for every B in V. Show that $A = \lambda I_n$ for some $\lambda \in K$. This shows that multiplication in V is rather non-commutative, as the only elements in V that commute with all other elements of V are small in number. In fact, $\dim(V) = n^2$ while $\dim(\{\lambda I_n : \lambda \in K\}) = 1$!

d. Let A and B be $n \times n$ matrices. Assume that $I - AB$ has an inverse C. Show that $I - BA$ is invertible, by explicitly finding an inverse for $I - BA$ in terms of I, A, B, and C. You should think of the 'formal' expansion $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$.

e. Prove that every $n \times n$ matrix A over $\mathbf{R}$ or $\mathbb{C}$ is the sum of two invertible matrices $A = B_1 + B_2$.

Section 3.2

a. Suppose that $L : V \to V$ is a linear transformation of the vector space V. Assume that W is a subspace of V left invariant by L, which is to say that $L(x) \in W$ for every $x \in W$. Then there is a linear transformation $L|_W : W \to W$ given $L|_W(x) = L(x)$ for every $x \in W$.

b. Assume that L, V, W are as in the previous problem. If $u \equiv_W v$ then $L(u) \equiv_W L(v)$ since $L(u) - L(v) = L(u - v) \in L(W) \subseteq W$. So there is a well-defined map $V/W \to V/W$ given by $\bar{v} \mapsto \overline{L(v)}$. We denote this map by $\bar{L} : V/W \to V/W$. It is reasonably obvious that $\bar{L}$ is a linear transformation.

c. Let $L : V \to V'$ be a linear transformation of vector spaces. Suppose W is a subspace of V and W' is a subspace of V' such that $L(W) \subseteq W'$. Then there is a linear map $L|_W : W \to W'$ given by $L|_W(x) = L(x)$. This is simply a generalization of exercise a. There is also a well-defined linear map $\bar{L} : V/W \to V'/W'$ given by $\bar{L}(\bar{x}) = \overline{L(x)}$. This is simply a generalization of exercise b.

Section 3.3

a. Let $L : V \to V'$ be a linear transformation of vector spaces. Suppose W is a subspace of V and W' is a subspace of V' such that $L(W) \subseteq W'$. If $\bar{L} : V/W \to V'/W'$ then show that $\mathrm{Ker}(\bar{L}) = L^{-1}(W')/W$ while $\mathrm{Im}(\bar{L}) = (\mathrm{Im}(L) + W')/W'$. So $\bar{L}$ is surjective iff L is surjective, and $\bar{L}$ is injective iff $W = L^{-1}(W')$.

b. Let $L : V \to V$ be a linear transformation of the vector space V. Suppose that $L^2 = L$, which is to say that $L(L(x)) = L(x)$ whenever $x \in V$. Prove that $V = \mathrm{Ker}(L) \oplus \mathrm{Im}(L)$.

c. Let V be a vector space with subspaces U and W. Define a linear map $L : U \times W \to V$ by $(u, w) \mapsto u - w$. Using $\dim(U \times W) = \dim \mathrm{Ker}(L) + \dim \mathrm{Im}(L)$, show that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

d*. Let U_1, U_2, U_3 be subspaces of a common vector space. Prove that $\dim(U_1 + U_2 + U_3) = \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$. This result generalizes to the case of n subspaces $U_1, \ldots, U_n$ of V, and the general formula can be thought of as the inclusion-exclusion formula for vector spaces, as compared to the inclusion-exclusion formula for sets.

Section 3.4

a. Briefly show why any two vector spaces of the same dimension are isomorphic.

b. Let $L : V \to W$ be a linear transformation of vector spaces. Since $L(\text{Ker}(L)) = 0$, we can consider the map $\bar{L} : V/\text{Ker}(L) \to W$. Notice that $\bar{L}$ is an isomorphism between $V/\text{Ker}(L)$ and $\text{Im}(L)$. The fact that $V/\text{Ker}(L)$ and $\text{Im}(L)$ are isomorphic under $\bar{L}$ is known as the First Isomorphism Theorem for vector spaces. Show that Theorem 3.2 from Chapter 3 is a natural consequence of the First Isomorphism Theorem.

c. Let V be a vector space with subspaces U and W. The map $U \to U+W$ sends $U \cap W$ into W, and hence induces a map $U/U \cap W \to (U+W)/W$. Show that the induced map is an isomorphism. This result is known as the Second Isomorphism Theorem for vector spaces. Notice that $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$ is a natural result of the Second Isomorphism Theorem.

Section 4.1

a. Let A and B be the matrices

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & -12 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 5 \end{pmatrix}$$

and let $L_A : K^3 \to K^3$ and $L_B : K^4 \to K^3$ be the associated linear maps. Using Gaussian elimination, show that $\text{Ker}(L_A)$ has basis $\{^t(-2, 6, 1)\}$, and that $\text{Ker}(L_B)$ has basis $\{^t(-3, -5, 0, 1), ^t(-2, -2, 1, 0)\}$.

b. Let A be an $m \times n$ matrix, let $L_A : K^n \to K^m$ be the associated linear map, and suppose B is an element of K^m. Then the set of all X in K^n satisfying $AX = B$ is precisely $L_A^{-1}(B)$. Deduce from this that either there are no X in K^n satisfying $AX = B$, or the set of all such X is the translate $\text{Ker}(L_A) + X^*$ of $\text{Ker}(L_A)$ by X^* where X^* is any solution to $AX = B$. So in solving the problem $AX = B$, we can first solve the homogeneous problem $AX = 0$, and then look for a specific X^* satisfying $AX^* = B$.

Section 4.2

a. Let A be an $n \times n$ matrix. Suppose that B is an $n \times n$ matrix satisfying $BA = I$. Show that $AB = I$. Remember that $L_{BA} = L_B \circ L_A$ where $L_.$ is the associated linear map.

b. Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Suppose that $AB = I_m$ and $BA = I_n$. Then prove that $m = n$. More generally, if A is $m \times n$, B is $n \times m$, and $m > n$, then AB is not invertible.

Section 4.3

a. Let U and V be vector spaces with bases $\mathcal{B}_U$ and $\mathcal{B}_V$ respectively. Suppose $L_U : U \to U$ and $L_V : V \to V$ are linear maps. Show that there is a unique linear map $L : U \oplus V \to U \oplus V$ such that $L|_U = L_U$ and $L|_V = L_V$ (after identifying U and V as being subspaces of $U \oplus V$). We usually write $L = L_U \oplus L_V$. If $\mathcal{B}$ is the basis $\mathcal{B}_U \cup \mathcal{B}_V$ of $U \oplus V$, then show that

$$[L_U \oplus L_V]_{\mathcal{B}} = \begin{pmatrix} [L_U]_{\mathcal{B}_U} & 0 \\ 0 & [L_V]_{\mathcal{B}_V} \end{pmatrix}$$

We say that the above matrix is in block diagonal form. Conversely, if $L : W \to W$ is a linear transformation and $\mathcal{B}$ is a basis of W for which $[L]_{\mathcal{B}}$ is in block diagonal form, then we can write $W = U \oplus V$ for some proper nonzero subspaces U and V satisfying $L(U) \subseteq U$ and $L(V) \subseteq V$.

b. Let $L : V \to V$ be a linear transformation and suppose that W is a subspace of V invariant under L. Let $\mathcal{B}$ be a basis of W and extend $\mathcal{B}$ to a basis $\mathcal{B} \cup \mathcal{B}'$ of V. We identify $\mathcal{B}'$ with the basis of V/W that it induces. Because W is invariant under L, there are linear maps $L|_W : W \to W$ and $\bar{L} : V/W \to V/W$. Prove that

$$[L]_{\mathcal{B}\cup\mathcal{B}'} = \begin{pmatrix} [L|_W]_{\mathcal{B}} & * \\ 0 & [\bar{L}]_{\mathcal{B}'} \end{pmatrix}$$

where $*$ denotes unknown entries. We say that the above matrix is in upper triangular block form. Conversely, if $L : V \to V$ is a linear transformation and $\mathcal{B}$ is a basis of V for which $[L]_{\mathcal{B}}$ is in upper triangular block form, then there is a proper nonzero subspace W of V satisfying $L(W) \subseteq W$.

Section 5.1

a. Let V be a vector space over $\mathbf{R}$ endowed with a scalar product. Briefly explain why this scalar product is non-degenerate provided that it is positive definite. Show that, if $V = \mathbf{R}^2$, then $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 - y_1y_2$ is a non-degenerate scalar product of V that is not positive definite.

b. Let V be a vector space over $\mathbf{R}$, endowed with a positive definite scalar product. Let $\{v_1, \ldots, v_m\}$ be a set of elements of V satisfying $\langle v_i, v_j\rangle = \delta_{ij}$ for all i and j. Suppose that $\|v\|^2 = \sum_{i=1}^{m} \langle v, v_i\rangle^2$ for every $v \in V$. Deduce that $\{v_1, \ldots, v_m\}$ is a basis of V.

c. Let V be a vector space over $\mathbf{R}$ endowed with a positive definite scalar product. Suppose $u, v \in V$ and define a polynomial $p : \mathbf{R} \to \mathbf{R}$ by $p(t) = \langle u + tv, u + tv\rangle = t^2\|v\|^2 + 2t\langle u, v\rangle + \|u\|^2$. Since p is a non-zero quadratic polynomial, the discriminant Δ of p satisfies $\Delta \leq 0$. This means precisely that $4|\langle u, v\rangle|^2 - 4\|u\|^2\|v\|^2 \leq 0$. Thus, this remark provides an alternative proof of the Cauchy-Schwarz inequality.

d. Let V and W be vector spaces over the field K. Suppose that V has the scalar product $\langle \cdot, \cdot\rangle_V$ and that W has the scalar product $\langle \cdot, \cdot\rangle_W$. For two vectors (v, w) and (v', w') in $V \oplus W$, define $\langle (v, w), (v', w')\rangle = \langle v, v'\rangle_V + \langle w, w'\rangle_W$. Verify that this defines a scalar product on the vector space $V \oplus W$. Show that this scalar product is non-denegerate if the two original scalar products are non-degenerate, and that this scalar product is positive definite if the two original scalar products are positive definite. If the two original scalar products are positive, then we directly get the Pythagorean Theorem $\|(v, w)\|^2 = \|v\|_V^2 + \|w\|_W^2$. As an example, if we give $\mathbf{R}$ the scalar product $\langle x, y\rangle = xy$, and then give $\mathbf{R}^n$ the scalar product formed by considering $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$ so that $\langle (x_1, \ldots, x_n), (x_1', \ldots, x_n')\rangle = \sum_1^n \langle x_i, x_i'\rangle$, what we get is simply the dot-product.

e*. Let V be a vector space over $\mathbf{R}$, endowed with a positive definite scalar product. Suppose W is a nonzero subspace of V. Let u be any vector in V and define a function $f : W \to \mathbf{R}^+$ by $f(w) = \|u - w\|$. Then we want to find a minimum for f. Since $V = W \oplus W^\perp$, we can write $v = w^* + (v - w^*)$ for some $w^* \in W$, where $v - w^* \in W^\perp$. Now let $w \neq w^*$ be in W, so that $w - w^* \in W$, and hence $w - w^*$ is orthogonal to $u - w^*$. Then Pythagoras' Theorem yields

$$f(w) = \sqrt{\|(u - w^*) + (w^* - w)^2\|^2} = \sqrt{\|u - w^*\|^2 + \|w^* - w\|^2} > \|u - w^*\| = f(w^*)$$

This means that f has a unique global minimum, and this minimum occurs at w^*. We say that w^* is the best approximation in W to u. In the special case that W is the 1-dimensional subspace of V generated by the nonzero vector v, then $w^* = \frac{\langle u,v\rangle}{\langle v,v\rangle} v$. That is, w^* is the projection of u along v. So we see that the projection of u along v has both a geometric interpretation, and an analytic interpretation. This line of reasoning is useful in fourier analysis (where we deal with infinite dimensional vector spaces).

Section 5.2

a. Let W be the subspace of $\mathbf{R}^4$ that is spanned by the three vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

Use the Gram-Schmidt orthogonalization process to find an orthonormal basis of W.

b. Let W be the subspace of $\mathbf{R}^4$ that is spanned by the two vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Use Gaussian elimination to find a basis for $W^\perp$.

c. Let V be a vector space with a non-degenerate scalar product, and suppose W and W' are subspaces of V. Prove that $W^{\perp\perp} = W$ and $(W + W')^\perp = W^\perp \cap W'^\perp$. Deduce that $(W \cap W')^\perp = W^\perp + W'^\perp$.

d. Let V be a vector space over $\mathbf{R}$ endowed with a positive definite scalar product, and suppose that W is any subspace of V. Let $\{v_1, \ldots, v_m\}$ be a basis of W, and extend this to a basis $\{v_1, \ldots, v_n\}$ of V. We know that $\dim(W^\perp) = n - m$, and so we might hope that $\{v_{m+1}, \ldots, v_n\}$ forms a basis of $W^\perp$. This need not be the case however, as the $v_{m+1}, \ldots, v_n$ need not lie in $W^\perp$. Even so, we can still extract from $\{v_1, \ldots, v_n\}$ a basis for $W^\perp$. Explain how this can be done by using the Gram-Schmidt orthogonalization process.

Section 5.3

a. Let A be an $n \times n$ upper triangular matrix whose diagonal entries are all non-zero. Explain why $\text{rank}(A) = n$.

b. What is the dimension of the solution space to the homogeneous system of linear equations

$$\begin{aligned} 5x_1 + 2x_2 + x_3 \qquad\qquad\qquad - 10x_6 + 7x_7 &= 0 \\ x_1 + 4\pi x_2 \quad - 200x_4 \qquad\quad + 2x_6 - 70x_7 &= 0 \\ \frac{1+\sqrt{5}}{2}x_1 - x_2 \qquad\qquad + 10^6 x_5 + x_6 \qquad &= 0 \end{aligned}$$

In answering this question, you should not attempt to find the solution set explicitly.

c. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Prove that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

Section 5.4

a. Let A be an $n \times n$ matrix over $\mathbf{R}$ and define a bilinear map $g_A : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by $g_A(X,Y) = {}^tXAY$. Show that $g_A(X,Y) = g_A(Y,X)$ for all $X, Y \in \mathbf{R}^n$ if and only if A is a symmetric matrix. Thus, the bilinear maps $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ are in a bijective correspondence with the set of all $n \times n$ matrices over $\mathbf{R}$, and under this correspondence, the scalar products correspond with the symmetric matrices. It turns out that the positive definite scalar products correspond with the symmetric matrices all of whose eigenvalues are strictly positive.

b. Let A be an $n \times n$ matrix over $\mathbb{C}$ and define $g_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by $g_A(X,Y) = {}^tXA\bar{Y}$. Then g_A is linear in the first variable and anti-linear in the second variable. So g_A is a Hermitian product if and only if $\overline{g_A(X,Y)} = g_A(Y,X)$ for all X, Y. Show that this holds if and only if ${}^t\bar{A} = A$. Such a matrix A is known as a Hermitian matrix. So the Hermitian products on $\mathbb{C}^n$ are in a bijective correspondence with the $n \times n$ Hermitian matrices over $\mathbb{C}$. Further, a Hermitian matrix A induces a positive definite Hermitian inner product if and only if every eigenvalue of A is a positive real number.

Section 5.6

a. Let $V = \mathbf{R}^2$ with the standard scalar product being the dot product. Assume $v_1, \ldots, v_m$ are non-zero vectors in V. Explain geometrically why there is a linear functional $\varphi : V \to \mathbf{R}$ satisfying $\varphi(v_i) \neq 0$ for all i.

b. Let $L : V \to W$ be a linear transformation of vector spaces. We can define a map $L^* : W^* \to V^*$ as follows: if $\varphi \in W^*$ and $v \in V$ then $L^*(\varphi)(v) = \varphi(L(v))$. In other words, $\varphi \mapsto \varphi \circ L$. Show that L^* is surjective if L is injective, and that L^* is injective if L is surjective.

c. Let $L : V \to W$ be a linear map, suppose $\mathcal{B}$ is a basis of V, and $\mathcal{B}'$ is a basis of W. Let $\mathcal{B}^*$ be the dual basis of V^* and $\mathcal{B}'^*$ the dual basis of W^*. Show that $M^{\mathcal{B}'^*}_{\mathcal{B}^*}(L^*) = {}^tM^{\mathcal{B}}_{\mathcal{B}'}(L)$.

d. Let V be a vector space and define a map $\delta : V \to V^{**}$ by $v \mapsto \delta_v$ where $\delta_v : V^* \to K$ by $\delta_v(f) = f(v)$. Show that δ is a linear isomorphism. Now let $L : V \to W$ be a linear transformation. Since we can form $L^* : W^* \to V^*$, we can also form $L^{**} : V^{**} \to W^{**}$. With $\delta : V \to V^{**}$ and $\delta' : W \to W^{**}$ given as above, show that $\delta' \circ L = L^{**} \circ \delta$. We say that we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad L \quad} & W \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta'} \\ V^{**} & \xrightarrow{\quad L^{**} \quad} & W^{**} \end{array}$$

We also refer to δ as being a 'natural' isomorphism. This is a motivating example for the concept of 'natural transformations' that one encounters early on in Category Theory.

Section 5.7

a. Let V be a vector space over $\mathbb{C}$ endowed with a positive definite Hermitian inner product. Then

$$\langle u, v \rangle = \frac{1}{4}\left\{ \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - \|u-iv\|^2 \right\}$$

for all $u, v \in V$. The proof of this formula is straightforward. It is known as the Polarization identity, and it is occasionally useful.