Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Pure and Applied Algebra 206 (2006) 55–58
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

On a theorem of Mislin

J.L. Alperin
Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637, USA

Received 12 October 2004; received in revised form 22 February 2005
Available online 6 September 2005

Dedicated to Eric M. Friedlander on his 60th birthday

Abstract

A theorem of Mislin gives an equivalence between a condition on restriction of cohomology to a subgroup with an embedding condition on the subgroup. Two variations of this result are proved and a reduction is given towards a purely algebraic proof of Mislin's original theorem.

MSC: 20J06

1. Introduction

A celebrated theorem of Mislin [3], using deep homotopy results, shows the equivalence between a purely cohomological condition and the embedding of a subgroup. Specifically, let H be a subgroup of a finite group, p a prime and assume that a Sylow p-subgroup P of G lies in H. Let k be a field of characteristic p, e.g. the field of p elements. Mislin shows that the following two conditions are equivalent:

(1) The restriction map on cohomology $H^*(G, k) \to H^*(H, k)$ is an isomorphism;
(2) H is weakly p-embedded in G, that is, whenever Q is a p-subgroup of H then $N_G(Q)/C_G(Q)$ is isomorphic with $N_H(Q)/C_H(Q)$.

We shall prove two variations on this result, one about strongly p-embedded subgroups, and one about an elementary version. We shall also give a reduction theorem, using one of our results, towards a more direct and representation-theoretic proof of Mislin's result.

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2005.02.013

2. Strongly p-embedded subgroups

Preserving the above notation, recall that H is said to be strongly p-embedded in G if $N_G(Q)$ is contained in H for any non-identity p-subgroup Q of H. We shall prove the following:

Theorem 1. *The following two conditions are equivalent*:

(1) *H is strongly p-embedded in G.*
(2) *Whenever M is a kG-module then the restriction map $H^*(G, M) \to H^*(H, M)$ is an isomorphism.*

The implication of the second condition from the first is standard, an immediate consequence of the stable element calculation of Cartan and Eilenberg [1] and the author's local fusion theorem which guarantees that the stable calculation takes place in local subgroups. Hence, we shall assume the second condition and derive the first. Let Q be a non-identity p-subgroup of H. Let M be the kG-module which is the permutation module on the cosets of Q, so by the Eckmann–Shapiro lemma and our hypothesis, we have

$$H^*(Q, k) \simeq H^*(G, M) \simeq H^*(H, M)$$

but by applying Mackey's theorem to the last term, since M there is the restriction of an induced module, we get that $H^*(Q, k)$ is a direct sum of terms of the form

$$H^*(H, M_t)$$

where t runs over a set of H, Q double coset representatives, and M_t is the permutation module on the cosets of $tQt^{-1} \cap H$. If t is in H this is just $H^*(H, M)$ so that it follows that all the remaining terms are zero. However, if t is in $N_G(Q)$ and t is not in H, then

$$H^*(H, M_t) = H^*(Q, k)$$

which is not zero, a contradiction, so that $N_G(Q)$ must indeed be contained in H.

3. Weakly elementary p-embedded subgroups

We no longer assume that H contains a Sylow p-subroup of G, and we do assume now that k is algebraically closed. We shall say that H is weakly elementary p-embedded in G if we have the following conditions:

(1) Whenever E is an elementary abelian p-subgroup of H then $N_G(E)/C_G(E) \simeq N_H(E)/C_H(E)$.
(2) Every elementary abelian p-subgroup of G is conjugate with a subgroup of H.
(3) Two elementary abelian p-subgroups of H are conjugate in G if, and only if, they are conjugate in H.

We denote by $V_L(U)$, for a KL-module U for a group L, the variety corresponding to the cohomology of U [2]. We can now state our next result, one which has also been observed by Jon Carlson.

Theorem 2. *The restriction map of varieties $V_H(k) \to V_G(k)$ is a bijection if, and only if, H is weakly elementary p-embedded in G.*

Suppose that the map of varieties is a bijection. The three properties just listed then follow directly from the properties of the Quillen filtration of the cohomology varieties [2, pp. 109–117]. Let α be an element of $V_G(k)$ with vertex E. Choose β in $V_H(k)$ mapping to α so a vertex of α must be one for β so a conjugate of E lies in H and 2 holds. The bijection used with a result of Quillen, Theorem 9.1.1 of [2], gives 3. The bijection again, with the free action of $N_G(E)/C_G(E)$ on $V_{G,E}(k)^+$, and the similar property for H, establishes property 1.

On the other hand, suppose the three conditions hold. If α is in $V_G(k)$ then a vertex E of α has a conjugate in H so the map of varieties is surjective. Suppose that β_1 and β_2 are elements of $V_H(k)$ with vertices and sources $F_i, \gamma_i,\ i = 1, 2$. (Strictly speaking, as each γ_i is a point of the variety, that is, a homomorphism of the cohomology algebra to k, the kernel of this map, the ideal, is the source.) Assume that β_1 and β_2 have the same image in $V_G(k)$. Then there will be an element of G simultaneously conjugating F_1 to F_2 and β_1 to β_2, so we can assume, by condition 3, that $F_1 = F_2$ and then we get β_1 and β_2 conjugate in $F_1 = F_2$ by 1 so that it follows that $\gamma_1 = \gamma_2$.

4. A reduction

We now return to the notation of the introduction.

Lemma 1. *Let H be weakly p-embedded in G.*

(1) *If Q is a non-identity p-subgroup of H then $N_H(Q)$ is weakly p-embedded in $N_G(Q)$.*
(2) *If K is a normal subgroup of G then HK/K is weakly p-embedded in G/K.*

Proof. To establish 1, first note that $N_H(Q)$ does contain a Sylow p-subgroup of $N_G(Q)$ since our hypothesis implies that H controls fusion and since some conjugate of Q in H has the property that its normalizer in H contains such a Sylow subgroup. Now let R be a non-identity p-subgroup of $N_H(Q)$. Every automorphism induced by G on QR is induced by an element of H, and this element therefore leaves Q invariant so 1 holds.

Turning to 2, we have that HK/K contains a Sylow p-subgroup of G/K. Let Q be a non-identity p-subgroup of H, contained in the Sylow p-subgroup P of G contained in H. Let g be an element of G such that gK normalizes QK/K. Let S be a Sylow p-subgroup of KQ containing Q. After replacing g by another element of gK, we can assume that g normalizes S. Hence, by hypothesis, there is an element h of H such that h and g induce the same automorphism of S so that gK and hK agree on QK/K as claimed. □

We keep the same notation and now we shall say that the subgroup H controls cohomology if condition 1 of the Introduction holds. The remainder of this section is devoted to seeing that Mislin's result follows from the properties that control of cohomology passes to p-local subgroups and quotients; that is, if the cohomological analogue of the above lemma holds for this condition 1 then we have a different approach.

Assuming that these two results on cohomology have been proved, we prove the embedding property (condition 2 of the introduction) by induction on the order of G. Let Q be a non-identity p-subgroup of H. If $N_G(Q)$ is a proper subgroup of G then induction applies and $N_G(Q) = C_G(Q)N_H(Q)$ as desired, so we may assume that Q is normal. Now we proceed by induction on the order of Q as well. If Q is elementary, then the isomorphism of cohomology gives the same for the associated varieties, so, by Theorem 2, we have the desired result. Finally, assume that Q is not elementary abelian so the Frattini subgroup $D(Q)$ is a proper non-identity subgroup of Q. Induction applies to $G/D(Q)$ so, as a consequence, H and G induce the same group of automorphims on $Q/D(Q)$. The group of automorphisms induced on Q by G, which are the identity on $Q/D(Q)$, is a p-group, so since H contains a Sylow p-subgroup of G, we have that G and H induce the same automorphisms of Q which are the identity on $Q/D(Q)$. Thus, the claim is established.

References

[1] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.
[2] L. Evens, The cohomology of groups, Clarendon Press, Oxford, 1991.
[3] G. Mislin, On group homomorphisms inducing mod-p cohomology isomorphisms, Comment. Math. Helv. 65 (3) (1990) 454–461.