MATH 199 - MIDTERM II

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(1) (10 points) Decide whether the following sentences are True of False. (You don't need to prove anything.)

T **F**: $[0, 1)$ is closed;

T **F**: 1 is an accumulation point of $[0, 1)$;

T **F**: $[0, 1) \cap \mathbb{Q}$ has no l.u.b. in $\mathbb{Q}$;

T **F**: $\{1, 2, 3\}$ has no accumulation points;

T **F**: If $lub(A) < lub(B)$ then $A \subset B$.

(2) (15 points)
Give examples of the following: (You don't need to prove anything.)

(a) of a set whose only accumulation point is 3.

$\{3+\frac{1}{n} / n \in \mathbb{N}\}$

(b) of a sequence which doesn't have a convergent subsequence.

$a_n = n$

(c) of an open set A with $lub(A) = 1$ and $glb(A) = -1$.

$(-1, 1)$

(d) of a non-empty set which has no accumulation points, no l.u.b. and no g.l.b. in $\mathbb{R}$

$\mathbb{Z}$

(e) of a Cauchy sequence with no monotone decreasing subsequence.

$1-\frac{1}{n}$

(3) (10 points)

(a) Using the definition of limit, prove that
$$\lim_{n\to\infty} \frac{4n+7}{n} = 4.$$

(b) Using the definition of Cauchy sequence, prove that the sequence $a_n = \frac{4n+7}{n}$ is Cauchy.

(In each part, given $\epsilon > 0$, find $N \in \mathbb{N}$, such that ...)

(a) Pick $\varepsilon > 0$

Let $N \in \mathbb{N}$ be s.t. $N > \frac{7}{\varepsilon}$

Then, For $n \geq N$

$$\left|\frac{4n+7}{n} - 4\right| = \left|\frac{7}{n}\right| \leq \frac{7}{N} < \varepsilon.$$

(b) Pick $\varepsilon > 0$

Let $N \in \mathbb{N}$ be s.t $N > \frac{14}{\varepsilon}$

Then For $n, m \geq N$

$$\left|\frac{4n+7}{m} - \frac{4m+7}{m}\right| = \left|\frac{7}{n} - \frac{7}{m}\right| \leq \left|\frac{7}{n}\right| + \left|\frac{7}{m}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(4) (5 points)
Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. Prove that there exists $r \in \mathbb{Q}$ such that $a < r^2 < b$.

Since $0 < a < b$, $\sqrt{a} < \sqrt{b}$.

We know there is $r \in \mathbb{Q}$, $\sqrt{a} < r < \sqrt{b}$.

Then $a < r^2 < b$

(5) (5 points)
Let $A \subseteq \mathbb{R}$ be a closed set and $x \in \mathbb{R}$ be an accumulation point of A. Prove that $x \in A$.
(Hint: Start assuming, towards a contradiction, that $x \in \mathbb{R} \setminus A$.)

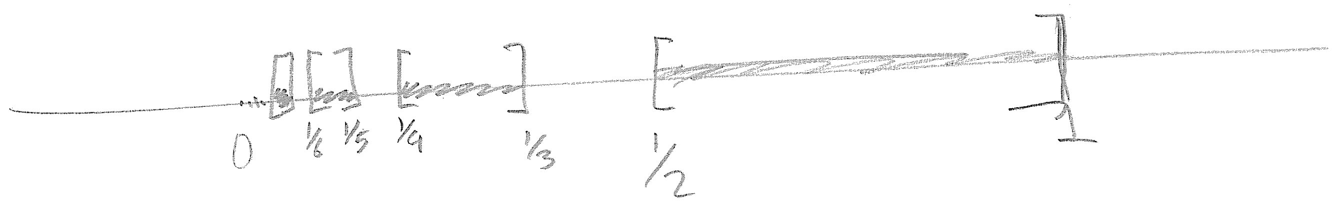
Suppose, towards a contradiction, that $x \notin A$.
So $x \in \mathbb{R} \setminus A$, which is open
So $\exists \varepsilon > 0$ st $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R} \setminus A$.

Since x is an accumulation pt. of A
$\exists y \in A$, $y \neq x$ $y \in (x-\varepsilon, x+\varepsilon)$
but this contradicts that $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R} \setminus A$.

(6) (5 points) Draw the following subset of $\mathbb{R}$.

$$S = \bigcup_{n\in\mathbb{N}} A_n \quad \text{where } A_n = \left[\frac{1}{2n}, \frac{1}{2n-1}\right].$$

Explain why it is not closed.



0 is an accumulation pt of S, because if $\varepsilon > 0$ then whenever $n > \frac{1}{\varepsilon}$, $\frac{1}{n} \in S \cap (-\varepsilon, \varepsilon)$.

But $0 \notin S$.

So, by the previous Exercise, S cannot be closed.

Alteratively.

$\mathbb{R} \setminus S$ is not open, because for <u>no</u> $\varepsilon > 0$ $(-\varepsilon, \varepsilon) \subseteq \mathbb{R} \setminus S$ as $\frac{1}{n} \notin \mathbb{R} \setminus S$ for every n.

Extra paper