MATH 199 - MIDTERM II

Name:.. /50
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(1) (10 points) Decide whether the following sentences are True of False. (You don't need to prove anything.)

T/F: $\emptyset$ is closed;

T/F: Whenever A is open and B is closed, $A \setminus B$ is open;

T/F: $(0,1) \cap \mathbb{Q}$ has no l.u.b. in $\mathbb{Q}$;

T/F: The union of two compact sets is compact;

T/F: If $glb(B) \leq glb(A) < lub(A) \leq lub(B)$ then $A \subseteq B$.

(2) (8 points)
Give examples of the following: (You don't need to prove anything.)

(a) of a set which has exactly three accumulation points.

$$\{-1+\tfrac{1}{n} / n\in\mathbb{N}\} \cup \{0+\tfrac{1}{n} / n\in\mathbb{N}\} \cup \{1+\tfrac{1}{n} / n\in\mathbb{N}\}$$

(b) of a bounded sequence which is not Cauchy.

$$a_n = (1)^{-n}$$

(c) of a non-empty set which has no accumulation points, no l.u.b., and no g.l.b.

$$\mathbb{Z}$$

(d) of an unbounded sequence with a convergent subsequence.

$$a_n = n + (-1)^n \cdot n = \begin{cases} 2n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(3) (10 points)

(a) Using the definition of limit, prove that

$$\lim_{n\to\infty} \frac{3n+1}{n} = 3.$$

(b) Using the definition of Cauchy sequence, prove that the sequence $a_n = \frac{3n+1}{n}$ is Cauchy.

(In each part, given $\epsilon > 0$, find $N \in \mathbb{N}$, such that ...)

(a) Given $\varepsilon > 0$ Let N be s.t. $N > \frac{1}{\varepsilon}$.

for $n \geq N$

$$\left|\frac{3n+1}{n} - 3\right| = \left|\frac{3n}{n} + \frac{1}{n} - 3\right| = \left|\frac{1}{n}\right| \leq \left|\frac{1}{N}\right| < \varepsilon$$

(b) Given $\varepsilon > 0$, Let N be s.t. $N > \frac{2}{\varepsilon}$

For $n, m \geq N$

$$\left|\frac{3n-1}{n} - \frac{3m-1}{m}\right| = \left|3 - \frac{1}{n} - \left(3 - \frac{1}{m}\right)\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \left|\frac{1}{m}\right| + \left|\frac{1}{m}\right| \leq \frac{2}{N} < \varepsilon.$$

(4) (12 points)

Definition: Given a set $A \subseteq \mathbb{R}$, and $x \in \mathbb{R}$, we say that x is a *boundary point of A* if $\forall \epsilon > 0\ \exists y \in (x-\epsilon, x+\epsilon) \cap A$ and $\forall \epsilon > 0\ \exists z \in (x-\epsilon, x+\epsilon) \setminus A$.

(a) Calculate the set of all boundary points of the following sets (just write down the set, you don't need to prove anything).

(i) The set of boundary points of $A = (0,1)$ is ... $\{0, 1\}$

(ii) The set of boundary points of $A = [0,1]$ is ... $\{0, 1\}$

(iii) The set of boundary points of $A = \mathbb{Q}$ is ... $\mathbb{R}$

(b) Let A be any closed set. Prove that A contains all its boundary points.

Suppose x is a boundary point of A

We want to prove that $x \in A$.

Suppose $x \notin A$.

Since it's a boundary point, $\forall \varepsilon > 0\ (x-\varepsilon, x+\varepsilon) \cap A \neq \phi$

Since $x \notin A$, $\forall \varepsilon > 0\ \exists y \neq x, y \in A, |y - x| < \varepsilon$.

So x is an accumulation point of A.

But then, since A is closed, $x \in A$.

So $x \in A$.

(5) (10 points)

(a) Write down the definition of dense set.

(b) Prove that the set $D = \{\frac{n}{10^m} | n \in \mathbb{Z}, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

(a) D is dense if $\forall a,b \in \mathbb{R}$ with $a<b$ $\exists d \in D$, $a<d<b$

(b) Take $a<b \in \mathbb{R}$.

Let m be st $\frac{1}{10^m} < b-a$

Let $B = \{n \in \mathbb{Z} / b\cdot 10^m < n\}$

B is bounded below (by $b\cdot 10^m$) and by the WF principle, it has a least element n.

Since $n-1 \notin B$

$n-1 \leq b\cdot 10^m$. $\Rightarrow n < b\cdot 10^m + 1$

$\Rightarrow \frac{n}{10^m} < b + \frac{1}{10^m} < a$

$\Rightarrow b < \frac{n}{10^m} < a$.

because $n \in B$