MATH 199 - MIDTERM I

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(1) (10 points) Decide whether the following functions are Injective, Surjective, Bijective, or Neither. Circle the right answer. You don't need to justify. (If you circle B, you don't need to also circle I and S.)

I S B N: $f\colon \mathbb{N} \to \mathbb{N}$, $f(n) = n + 1$;
I S B N: $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f((n, m)) = n + m$;
I S B N: $f\colon \mathbb{R} \to \mathbb{Z}$, $f(x)$ is the integer n such that $n \leq x < n + 1$;
I S B N: $f\colon A \times B \to B \times A$, $f((a, b)) = (b, a)$;
I S B N: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

(2) (10 points)
Let A, B and C be sets. Prove that
$$C \setminus A \subseteq (C \setminus B) \cup (B \setminus A).$$

Take $x \in C \setminus A$. We want to show that $x \in (C \setminus B) \cup (B \setminus A)$

We know $x \in C$ and $x \notin A$.

There are 2 cases:

Case 1 $x \in B$. Then $x \in B \setminus A$

Case 2 $x \notin B$. then $x \in C \setminus B$

In either case $x \in (C \setminus B) \cup (B \setminus A)$

(3) (5 points)
List all the elements of the set $(\{0,1\} \times \{a,b\}) \times \{\alpha,\beta\}$.

$\{((0,a),\alpha), ((0,a),\beta), ((0,b),\alpha), ((0,b),\beta), ((1,a),\alpha),$
$((1,a),\beta), ((1,b),\alpha), ((1,b),\beta)\}$

(4) (10 points)

Use the axioms of the integers to prove, without skipping steps, that

$$\forall a, b \in \mathbb{Z} \ \ (a \cdot b)^2 = a^2 \cdot b^2$$

(where x^2 is defined to be $x \cdot x$).

Recall the axioms:

(M2): $\forall a, b, c \in \mathbb{Z}, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c.$

(M3): $\forall a, b \in \mathbb{Z}, \quad a \cdot b = b \cdot a.$

(M4): $\exists 1 \in \mathbb{Z}, \quad 1 \neq 0 \ \& \ \forall a \in \mathbb{Z}, a \cdot 1 = a.$

$$(a \cdot b)^2 \underset{\text{def of } x^2}{=} (a \cdot b) \cdot (a \cdot b) \underset{M2}{=} a \cdot (b \cdot (a \cdot b))$$

$$\underset{M2}{=} a \cdot ((b \cdot a) \cdot b)$$

$$\underset{M3}{=} a((a \cdot b) \cdot b)$$

$$\underset{M2}{=} a(a \cdot (b \cdot b))$$

$$\underset{M2}{=} (a \cdot a) \cdot (b \cdot b)$$

$$\underset{\text{def } x^2}{=} a^2 \cdot b^2$$

(5) (10 points)
Let $X = \mathbb{N} \times \mathbb{N}$. We define a relation $\sim$ on X as follows:

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

(a) Prove that $\sim$ satisfies the transitivity property:
(ER3) $\forall x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

You don't need to prove it form the axioms of the integers. But do remark at which part of the proof one would need to use the axioms of the integers.

(b) List a few (say four) elements of the set $C((3, 5))$.

(a) Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$

Then $a+d = b+c$ and $c+f = d+e$ by def of $\sim$

Add both equations $(a+d)+(c+f) = (b+c)+(d+e)$

Use axioms to get $(a+f)+(d+c) = (b+e)+(d+c)$

Apply cancellation of + $a+f = b+e$

So $(a,b) \sim (e,f)$ as needed for (ER3).

(b) $C((3,5)) = \{(1,3), (2,4), (3,5), (4,6), \ldots\}$

(6) (5 points) Let A, B be non-empty subsets of $\mathbb{R}$. Assume both of them have least upper bounds. Prove that if $A \subseteq B$, then

$$lub(A) \leq lub(B).$$

Let $L = lub(B)$

If $x \in A$, then $x \in B$, and hence $x \leq L$

(because L is an upper bound for B)

So L is an upper bound for A.

Therefore, by definition of $lub(A)$, $lub(A) \leq L$