MATH 199 - MIDTERM I

Name:.. /50

April 20, 2011

(1) (10 points) Decide whether the following functions are **I**njective, **S**urjective, **B**ijective, or **N**either. Circle the right answer. You don't need to justify it. (If you circle B, you don't need to also circle I and S.)

I S B N: $f\colon \mathbb{N} \to \mathbb{N}$, $f(n) = n + 7$.

I S B N: $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, $f((n, m)) = n - m$

I S B N: $f\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, $f((n, m)) = \frac{n}{m}$

I S B N: $f\colon A \to \mathcal{P}(A)$, $f(a) = \{a\}$, where $A \neq \emptyset$.

I S B N: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.

(2) (10 points)
Let $f\colon A \to B$ be a function and let B_1, B_2 be subsets of B. Prove that
$$f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2).$$

Take $x \in A$.

Then $x \in f^{-1}(B_1 \setminus B_2) \iff f(x) \in B_1 \setminus B_2$

$\iff f(x) \in B_1$ and $f(x) \notin B_2$

$\iff x \in f^{-1}(B_1)$ and $x \notin f^{-1}(B_2)$

$\iff x \in f^{-1}(B_1) \setminus f^{-1}(B_2)$

(3) (5 points) List all the elements of the set

$$\{z \in \mathbb{Z} \mid z^2 = 4\} \times \{ z \in \mathbb{Z} \mid z^2 = 9\}.$$

$\{(2,3),(2,-3),(-2,3),(-2,-3)\}$

(4) (10 points)

For every $d \in \mathbb{Z}$ with $1 < d$, using the axioms of the integers and without skipping any steps, prove that for every $n \in \mathbb{N}$,

$$1 < d^n,$$

(where d^m is short for $\underbrace{((..(d \cdot d) \cdot ...) \cdot d) \cdot d)}_{m \text{ times}}$.

Here are some axioms and facts you may use:
Axiom M4: $\forall a \in \mathbb{Z}$, $a \cdot 1 = 1 \cdot a = a$.
Axiom O2: $\forall a, b, c \in \mathbb{Z}$, if $a < b$ and $b < c$, then $a < c$.
Axiom O4: $\forall a, b, c \in \mathbb{Z}$, if $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.
Fact 1: $\forall a \in \mathbb{N}$ $\forall m \in \mathbb{N}$, $a^{m+1} = a^m \cdot a$.

Fix $d \in \mathbb{N}$

We use Mathematical Induction.

Let $B = \{n \in \mathbb{N} / 1 < d^n\}$

- $1 \in B$ by our essumption that $1 < d$ and that $d^1 = d$
- Suppose $n \in B$, so $1 < d^n$
  Since $d > 0$ (Fact 2), by (O4) $1 \cdot d < d^n \cdot d$.
  By (M4) $1 \cdot d = d$, so $d < d^n \cdot d$
  By Fact 1 $d^n \cdot d = d^{n+1}$, so $d < d^{n+1}$
  We know $1 < d$, so by (O2) $1 < d^{n+1}$.
  This implies that $n+1 \in B$ as wanted.

Thus, we have $B = \mathbb{N}$ and $\forall n \in \mathbb{N}$, $1 < d^n$.

(5) (10 points)
Let $X = \mathbb{N} \times \mathbb{N}$. We define a relation $\sim$ on X as follows:

$$(a, b) \sim (c, d) \iff a - d = c - b.$$

(a) Prove that $\sim$ satisfies the transitivity property:
(ER3) $\forall x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

You don't need to prove it from the axioms of the integers. But do remark at which steps you axioms, of facts from class, etc..

(b) List the elements of $C((3, 2))$.

(a) Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$
Then $a-d = c-b$ and $c-f = e-d$
Using axioms we get $a+b = c+d$ and $c+d = e+f$
Thus $a+b = e+f$
Using axioms we get $a-f = e-b$
So $(a,b) \sim (e,f)$ as needed.

(b) $C((3,2)) = \{(1,4), (2,3), (3\,2), (4,1)\}$

(6) (5 points)

Let A, B be non-empty subsets of $\mathbb{R}$. Assume that

$$lub(A) < glb(B),$$

(where *lub* is short for *least upper bound*, and *glb* for *greatest lower bound*). Show that A and B are disjoint.

Suppose, toward a contradiction, that $A \cap B \neq \phi$

Then, there is $x \in A \cap B$. So $x \in A$ and $x \in B$

Since $lub(A)$ is an upper bound for A, and $x \in A$, $x \leq lub(A)$

Since $glb(B)$ is a lower bound for B, and $x \in B$, $x \geq glb(B)$

Then, by transitivity $lub(A) \geq glb(B)$

contradicting that $lub(A) < glb(B)$.

So $A \cap B = \phi$