K-TRIVIALS ARE NCR_1

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1. INTRODUCTION

In (4; 5), Reimann and Slaman raise the question "For which infinite binary sequences X do there exist continuous probability measures μ such that X is effectively random relative to μ?"

1.1. Randomness relative to continuous measures.

We begin by reviewing the basic definitions needed to precisely formulate this question.

Notation 1.1.
- For $\sigma \in 2^{<\omega}$, $[\sigma]$ is the basic open subset of 2^ω consisting of those X's which extend σ. Similarly, for W a subset of $2^{<\omega}$, let $[W]$ be the open set given by the union of the basic open sets $[\sigma]$ such that $\sigma \in W$.
- For $U \subseteq 2^\omega$, $\lambda(U)$ denotes the measure of U under the uniform distribution. Thus, $\lambda([\sigma])$ is $1/2^\ell$, where ℓ is the length of σ.

Definition 1.2. A *representation* m of a probability measure μ on 2^ω provides, for each $\sigma \in 2^{<\omega}$, a sequence of intervals with rational endpoints, each interval containing $\mu([\sigma])$, and with lengths converging monotonically to 0.

Definition 1.3. Suppose $Z \in 2^\omega$. A *test relative to* Z, or Z*-test*, is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in Z. For $X \in 2^\omega$, X *passes* a test W if and only if there is an n such that $X \notin [W_n]$.

Definition 1.4. Suppose that m represents the measure μ on 2^ω and that W is an m-test.
- W is *correct for* μ if and only if for all n, $\sum_{\sigma \in W_n} \mu([\sigma]) \leq 2^{-n}$.
- W is *Solovay-correct for* μ if and only if $\sum_{n\in\omega} \mu([W_n]) < \infty$.

Definition 1.5. $X \in 2^\omega$ is 1*-random relative to a representation* m *of* μ if and only if X passes every m-test which is correct for μ. When m is understood, we say that X is 1-random relative to μ.

By an argument of Solovay, see (3), X is 1-random relative to a representation m of μ if an only if for every m-test which is Solovay-correct for μ, there are infinitely many n such that $X \notin [W_n]$.

Definition 1.6. $X \in \mathrm{NCR}_1$ if and only if there is no representation m of a continuous measure μ such that X is 1-random relative to the representation m of μ.

In (5), Reimann and Slaman show that if X is not hyperarithmetic, then there is a continuous measure μ such that X is 1-random relative to μ. Conversely, Kjøs-Hanssen and Montalbán, see (2), have shown that if X is an element of a countable Π^0_1-class, then there is no continuous measure for which X is 1-random. As the Turing degrees of the elements of countable Π^0_1-classes are cofinal in the Turing degrees of the hyperarithmetic sets, the smallest ideal in the Turing degrees that contains the degrees represented in NCR_1 is exactly the Turing degrees of the hyperarithmetic sets.

In (6), Reimann and Slaman pose the problem to find a natural Π^1_1-norm for NCR_1 and to understand its connection with the natural norm mapping a hyperarithmetic set X to the ordinal at which X is first constructed. As of the writing of this paper, this problem is open in general, but completed in (6) for $X \in \Delta^0_2$.

Suppose that $X \in \Delta^0_2$ and that for all n, $X(n) = \lim_{t\to\infty} X_t(n)$, where $X_t(n)$ is a computable function of n and t. Let g_X be the convergence function for this approximation, that is for all n, $g_X(n)$ is the least s such that for all $t \geq s$ and all $m \leq n$, $X_t(m) = X(m)$. Let f_X be function obtained by iterated application of g_X: $f_X(0) = g_X(0)$ and $f_X(n+1) = g_X(f_X(n))$.

For a representation m of a continuous measure μ, the granularity function s_m maps $n \in \omega$ to the least ℓ found in the representation of μ by m such that for all σ of length ℓ, $\mu([\sigma]) < 1/2^n$. Note that, s_m is well-defined by the compactness of 2^ω.

Theorem 1.7 (Reimann and Slaman (6)). *If X is 1-random relative the representation m of μ, then the granularity function s_m for μ is eventually bounded by f_X.*

Thus, there is a continuous measure relative to which X is 1-random if and only if there is a continuous measure whose granularity is eventually bounded by f_X. The latter condition is arithmetic, again by a compactness argument.

1.2. ***K*-triviality.** K-triviality is a property of sequences which characterizes another aspect of their being far from random. We briefly review this notion and the results surrounding it. A full treatment is given in Nies (3).

For $\sigma \in 2^{<\omega}$, let $K(\sigma)$ denote the prefix-free Kolmogorov complexity of σ. Intuitively, given a universal computable U with domain an antichain in $2^{<\omega}$, $K(\sigma)$ is length of the shortest τ such that $U(\tau) = \sigma$. Similarly, for $X \in 2^\omega$, let $K^X(\sigma)$ denote the prefix-free Kolmogorov complexity of σ relative to X. That is, K^X is determined by a function universal among those computable relative to X.

Definition 1.8. A sequence $X \in 2^\omega$ is *K-trivial* if and only if there is a constant k such that for every ℓ, $K(X \upharpoonright \ell) \leq K(0^\ell) + k$, where 0^ℓ is the sequence of 0's of length ℓ.

By early results of Chaitin and Solovay and later results of Nies and others, there are a variety of equivalents to K-triviality and a variety of properties of the K-trivial sets. For example, X is K-trivial if and and only if for every sequence R, R is 1-random for λ if and only if R is 1-random for λ relative to X.

In the next section, we will apply the following.

Theorem 1.9 (Nies (3), strengthening Chaitin (1))**.** *If X is K-trivial, then there is a computably enumerable and K-trivial set which computes X.*

Theorem 1.10 (See (3))**.** *Suppose X is K-trivial and $\{U_e^X : e \in \omega\}$ a uniformly $\Sigma_1^{0,X}$ family of sets. Then, there is a computable function g and a Σ_1^0 set V of measure $\frac{1}{2}$ such for every e, if $\lambda(U_e^X) < 2^{-g(e)}$, then $U_e^X \subseteq V$.*

1.3. **X is K-trivial implies $X \in \mathrm{NCR}_1$.** Intuitively, $X \in \mathrm{NCR}_1$ asserts that X is not effectively random relative to any continuous measure and X is K-trivial asserts that relativizing to X does change the evaluation of randomness relative to the uniform distribution. In the next section, we connect the two notions by showing that if X is K-trivial then $X \in \mathrm{NCR}_1$.

2. The Main Theorem

Theorem 2.1. *Every K-trivial set belongs to* NCR_1.

Proof. Let Y be K-trivial and let μ be a continuous measure with representation m; we want to show Y is not μ-random. By Theorem 1.9, let X be a computably enumerable K-trivial sequence that computes Y. Let f be the iterated convergence function as defined above for the computable approximation to Y given by approximating X's computation of Y. Since X is computably enumerable, X can compute the convergence function for its own enumeration and hence f is computable from X.

Let s_m be the granularity function for μ as represented by m. By Theorem 1.7, f eventually dominates s_m. By changing finitely many values of f, we may assume that f dominates s_m everywhere. So, we have that for every n

$$\mu([Y \restriction f(n)]) \leq 2^{-n}.$$

Further, we may assume that f can be obtained as the limit of a computable function $f(n,s)$ such that for all s, $f(n-1,s) \leq f(n,s) \leq f(n,s+1)$.

We will build an m-test $\{S_i : i \in \omega\}$ which is Solovay-correct for μ and which Y does not pass, thereby concluding that Y is not μ-random. That is, we plan to build $\{S_i : i \in \omega\}$ to be a uniformly $\Sigma_1^{0,m}$ sequence of sets such that $\sum_{i\in\omega} \mu(S_i)$ is bounded and such that there are co-finitely i for which $Y \in [S_i]$. Our construction will not be uniform.

X's K-triviality is exploited in the form of Theorem 1.10. That is, we will build the sequence $\{U_e^X : e \in \omega\}$ over the course of our construction. We devote the enumeration of U_e to the construction of an m-test. This test will be Solovay correct, provided that the eth recursive function g and recursively enumerable set V satisfies that V is of measure $\frac{1}{2}$ and if $\lambda(U_e^X) < 2^{-g(e)}$, then $U_e^X \subseteq V$.

Now we focus our attention on the e associated with a pair g and V satisfying the conclusion of Theorem 1.10. To simply our notation, we let U denote U_e, a denote $g(e)$, and let $\{S_i : i \in \omega\}$ denote the m-test being enumerated. In order to employ g and V, we will ensure that $\lambda(U)$ is less than or equal to 2^{-a}

We use the approximation to X as a computably enumerable set to enumerate approximations to initial segments of Y into the sets S_i; we rely on the K-triviality of X to keep the total μ-measure of the S_i's bounded.

For each $n > a$ we have a requirement R_n whose task is to enumerate $Y \upharpoonright f(n)$ into S_n. Let $y_{n,s} = Y_s \upharpoonright f(n,s)$ the stage s approximation to $Y \upharpoonright f(n)$. Let $x_{n,s}$ be the initial segment of X_s necessary to compute $y_{n,s}$ and $f(n,s)$. So, if $y_{n,s+1} \neq y_{n,s}$, it is because $x_{n,s+1} \neq x_{n,s}$. In this case, $x_{n,s+1}$ is not only different than $x_{n,s}$, but also incomparable. At stage s, R_n would like to enumerate $y_{n,s}$ into S_n, but before doing that it will *ask for confirmation* using the fact that $U^X \subseteq V$. Since we are constrained to keep $\lambda(U^X)$ less than or equal to 2^{-a}, we will restrict R_n to enumerate at most 2^{-n} measure into U^X. The reason why we need a bit of security before enumerating a string in S_n is that we have to ensure that $\sum_i \mu(S_i)$ is bounded. For this purpose, we will only enumerate mass into S_n when we see an equivalent mass going into V.

Action of requirement R_n:

(1) The first time after R_n is initialized, R_n chooses a clopen subset of 2^ω, σ_n, of m-measure 2^{-n}, that is disjoint form V_s and $U_s^{X_s}$. Note that since V and U^{X_s} have measure less than $\frac{1}{2} + 2^{-a}$, we can always find such a clopen set. Furthermore we can chose σ_n to be different from the σ_i chosen by other requirements R_i, $i > a$. We note the value of σ_n might change if R_n is initialized.

(2) To *confirm* $x_{n,s}$, requirement R_n enumerates σ_n into $U^{x_{n,s}}$. Requirement R_n will not be allowed to enumerate anything else into U^{X_s} unless X_s changes below $x_{n,s}$. This way R_n is always responsible for at most 2^{-n} measure enumerated in U^{X_s}.

(3) Then, we wait until a stage $t > s$ such that
   (a) either $x_{n,s} \not\subseteq x_{n,t}$ (as strings),
   (b) or $\sigma_n \subseteq V_t$.

   Observe that if $x_{n,s}$ is actually an initial segment of X, then we will have $\sigma_n \subseteq U^X \subseteq V$. So, we will eventually find such a stage t.
   - In Case 3(a), we start over with R_n. Note that in this case σ_n has come out of U^{X_t}, and hence R_n is responsible for no measure inside U^{X_t} at stage t.
   - In Case 3(b), if $\mu([y_{n,t}]) \leq 2^{-n}$, enumerate $y_{n,t}$ into S_n. (Recall that we are allowed to use the representation of μ as an oracle when enumerating S_n.)

Since we only enumerate $y_{n,t}$ of μ-measure less than 2^{-n} when σ_n is enumerated in V, we have that

$$\sum_i \mu(S_i) \leq \lambda(V) \leq \frac{1}{2}.$$

It is not hard to check that $\lambda(U^X) \leq \sum_{n=a+1}^{\infty} 2^{-n} = 2^{-a}$, so we actually have that $U^X \subseteq V$. Also notice that once $x_{n,s}$ is a initial segment of X, we will eventually enumerate σ_n into V and an initial segment of Y into S_n. □

References

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