The Journal of Symbolic Logic
Volume 00, Number 0, XXX 0000

UP TO EQUIMORPHISM, HYPERARITHMETIC IS RECURSIVE.

ANTONIO MONTALBÁN

Abstract. Two linear orderings are *equimorphic* if each can be embedded into the other. We prove that every hyperarithmetic linear ordering is equimorphic to a recursive one.

On the way to our main result we prove that a linear ordering has Hausdorff rank less than ω_1^{CK} if and only if it is equimorphic to a recursive one. As a corollary of our proof we prove that given a recursive ordinal α, the partial ordering of equimorphism types of linear orderings of Hausdorff rank at most α ordered by embeddablity, is recursively presentable.

§1. Introduction.

Clifford Spector proved the following well known classical theorem in Computable Mathematics.

Theorem 1.1. *[Spe55] Every hyperarithmetic well ordering is isomorphic to a recursive one.*

Recall that a set is hyperarithmetic if and only if it is Δ^1_1. Then, for instance, every arithmetic set is hyperarithmetic.

The direct generalization of Theorem 1.1 to the class of linear orderings does not hold. It is not the case that every linear ordering with a hyperarithmetic presentation is isomorphic to a recursive one. Feiner constructed in [Fei67] and [Fei70] (see also [Dow98, Theorem 2.5]) a Π^0_1 subset of $\mathcal{Q}$ that, as a linear ordering, is not isomorphic to a computable one. Other examples were given later. It follows from the work of Lerman [Ler81] that for every Turing degree $\mathbf{a}$ such that $\mathbf{a}'' >_T 0''$ there is a linear ordering of degree $\mathbf{a}$ without a recursive copy. This result was later extended, first to any non-recursive recursively enumerable degree $\mathbf{a}$ by Jockusch and Soare [JS91], then to any non-recursive Δ^0_2 degree $\mathbf{a}$ by Downey [Dow98] and Seetapun (unpublished), and finally to any non-recursive degree $\mathbf{a}$ by Knight [AK00]. Many other results have been proved about presentations of linear orderings; we refer the reader to [Dow98] for a survey on the effective mathematics of linear orderings.

But there are other ways in which we can generalize Theorem 1.1. We say that two linear orderings are *equimorphic* if each one can be embedded into the other one. Observe that if a linear ordering $\mathcal{L}$ is equimorphic to an ordinal α, then $\mathcal{L}$

This research will be part of my Ph.D. thesis [Mon05]. I am very thankful to my thesis adviser, Richard A. Shore, for many helpful and inspiring discussions, and corrections. The author was partially supported by NSF Grant DMS-0100035.

0022-4812/00/0000-0000/$00.00

and α are actually isomorphic. (It is clear that two equimorphic well orderings are isomorphic. Note that $\mathcal{L}$ has to be a well ordering because since ω^* does not embed in α and $\mathcal{L}$ embeds in α, ω^* does not embed in $\mathcal{L}$ either (where ω^* is the order type of the negative integers).) So, actually, we can state Theorem 1.1 as "every hyperarithmetic well ordering is equimorphic to a recursive linear ordering." The main theorem of this paper is the following generalization of Theorem 1.1.

THEOREM 1.2. *Every hyperarithmetic linear ordering is equimorphic to a recursive one.*

Many properties of linear orderings are invariant under equimorphisms. An interesting example is extendibility. A linear ordering $\mathcal{L}$ is *extendible* if every partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a *linearization* (i.e.: a linear extension) which does not embed $\mathcal{L}$ either. The notion of *weakly extendible* is defined similarly but only considering countable partial orderings $\mathcal{P}$. It is not hard to see that these notions depend only on the equimorphism type of the linear ordering $\mathcal{L}$. Classifications of extendible and weakly extendible linear orderings have been given by Bonnet and Pouzet [BP82], and Jullien [Jul69]. See [Mon] and [DHLS03] for an analysis of Jullien's theorem and of the extendibility of certain linear orderings form the viewpoint of Computable Mathematics and Reverse Mathematics.

Three other properties that are invariant under equimorphisms, and which will be very important in this paper, are being scattered, being indecomposable and having a certain Hausdorff rank. A linear ordering is *scattered* if it does not contain a copy η, the order type of the rationals. Then, for a countable linear ordering, being scattered is equivalent to not being equimorphic to η. We say that a linear ordering $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} = \mathcal{A}+\mathcal{B}$, we have that $\mathcal{L}$ can be embedded in either $\mathcal{A}$ or $\mathcal{B}$. It is not hard to prove that a linear ordering equimorphic to an indecomposable one is also indecomposable. The *Hausdorff rank* of a scattered linear ordering is the least ordinal α such that only finitely many point are left after α iterations of the operation of collapsing points of $\mathcal{L}$ which have only finitely many points in between (see Definition 2.1 below). We will prove that a scattered linear ordering has Hausdorff rank less than ω_1^{CK} (the first non-recursive ordinal) if and only if it is equimorphic to a recursive linear ordering.

Contrary to the case of countable well orderings, the partial ordering, $\mathbb{L}$, of countable linear orderings modulo equimorphism ordered by embeddablity is not a well understood structure. (See [Ros82, § 10.2] for more information on $\mathbb{L}$.) Note that $\mathbb{L}$ has the equimorphism type of η as its top element. (An *equimorphism type* is an equivalence class for the equimorphism relation.) Let $\widehat{\mathbb{L}}$ be obtained by removing the equimorphism type of η from $\mathbb{L}$. So, $\widehat{\mathbb{L}}$ consists of the equimorphism types of scattered linear orderings. Roland Fraïssé conjectured in [Fra48] that $\widehat{\mathbb{L}}$ is well founded and that every element has only countably many elements below it. Later, the statement that says that $\mathbb{L}$ is a well partial ordering became known as Fraïssé's conjecture. (A partial ordering is a *well partial ordering* if it contains no infinite descending sequence and no infinite antichain. See Definition 2.10 below.) All these statements were proved by Richard Laver, twenty three years later, in [Lav71] using Nash-Williams's complicated notion of better quasiordering [Nas68]. As a corollary of our construction, we prove that for every $\alpha < \omega_1^{CK}$, $\mathbb{L}_\alpha$, the subordering of $\mathbb{L}$ containing the the equimorphism types of linear orderings of Hausdorff rank less than α, is recursively presentable. This result might be useful when studying Fraïssé's conjecture from the viewpoint of Reverse Mathematics. Logicians have been interested in Fraïssé's conjecture because of the complexity of its proof. Some results have been proved about its proof theoretic strength: Shore [Sho93] proved that it implies ATR_0, and we proved in [Mon] (also see [Mon05]) that it is equivalent to Jullien's theorem, to the finite decomposability of scattered linear orderings and to the statement that says that the class of signed trees is well quasiordered. But, its exact proof theoretic strength is still unknown. It has been conjectured by Clote [Clo90], Simpson [Sim99, Remark X.3.31] and Marcone [Mar] that it is equivalent to ATR_0 over RCA_0. It would be interesting, and maybe useful when studying Fraïssé's conjecture, to know what the rank of $\mathbb{L}_\alpha$, as a well founded partial ordering, is for a given α.

Outline. In Section 2 we present the most important ideas in the proof of our main result, Theorem 1.2. In Section 3 we introduce and study the structure of signed forests. Signed forests extend the notion of signed trees which was introduced in [Mon]. The use of signed trees is very helpful when studying the structure of indecomposable linear orderings up to equimorphisms. In Section 4 we formally describe the construction, already mentioned in Section 2, but this time using the results of Section 3.

Basic Notions. An *embedding* between linear orderings $\mathcal{L}$ and $\mathcal{Q}$ is a one-to-one, order preserving map $f\colon L \to Q$. If this is the case, we write $f\colon \mathcal{L} \hookrightarrow \mathcal{Q}$, and we write $\mathcal{L} \preccurlyeq \mathcal{Q}$ to mean that $\mathcal{L}$ *embeds* in $\mathcal{Q}$. $\mathcal{L}$ and $\mathcal{Q}$ are *equimorphic* if $\mathcal{L} \preccurlyeq \mathcal{Q}$ and $\mathcal{Q} \preccurlyeq \mathcal{L}$, in which case we write $\mathcal{L} \sim \mathcal{Q}$.

A *presentation* of a linear ordering $\mathcal{L}$ is another linear ordering $\mathcal{A} = \langle A, \leq_A \rangle$ isomorphic to $\mathcal{L}$ such that $A \subseteq \omega$. The *Turing degree* of a presentation $\mathcal{A}$ is the join of the degrees of A and $\leq_A$.

Given a function $f\colon X \to Y$ and $Z \subseteq X$, we let $f[Z] = \{f(z) : z \in Z\}$.

Given two linear orderings $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} + \mathcal{B}$ is obtained by considering the disjoint union of $\mathcal{A}$ and $\mathcal{B}$ and letting all the elements of $\mathcal{B}$ be bigger than the ones in $\mathcal{A}$. This can be generalized to infinite sums of the form $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 + \ldots$, in an obvious way.

§2. General ideas of the Proof.

In this section we start proving Theorem 1.2. The first easy observation is that if a linear ordering has a subset isomorphic to η, then it is equimorphic to η, which has a recursive presentation. So we can restrict our attention to *scattered* linear orderings.

The second step is to analyze the Hausdorff rank of hyperarithmetic scattered linear orderings.

Definition 2.1. Let $\mathcal{L} = \langle L, \leq \rangle$ be a scattered linear ordering. For each ordinal α we define an equivalence relation $\approx_\alpha$ on L by transfinite recursion. Let $\approx_0$ be the identity relation. If α is a limit ordinal, let $x \approx_\alpha y$ if $x \approx_\beta y$ for some $\beta < \alpha$. If $\alpha = \beta + 1$, let $x \approx_\alpha y$ if there are only finitely many different

$\approx_\beta$-equivalence classes between x and y. In other words

$$x \approx_\alpha y \iff \exists n \exists x_1, ..., x_n \forall z (x < z < y \Longrightarrow \exists i < n (z \approx_\beta x_i)).$$

We define the *Hausdorff rank* of $\mathcal{L}$, $\mathrm{rk}_H(\mathcal{L})$, to be the least α such that $\approx_\alpha$ has only finitely many equivalence classes if such an α exists, and we let $\mathrm{rk}_H(\mathcal{L}) = \infty$ otherwise.

It can be proved by transfinite induction that if $f\colon \mathcal{L}_0 \hookrightarrow \mathcal{L}_1$, and $f(x) \approx_\alpha f(y)$ then $x \approx_\alpha y$. Therefore, $\mathcal{L}_0 \preccurlyeq \mathcal{L}_1$ implies that $\mathrm{rk}_H(\mathcal{L}_0) \leq \mathrm{rk}_H(\mathcal{L}_1)$, and hence, the Hausdorff rank is preserved under equimorphisms. Also note that $\mathrm{rk}_H(\eta) = \infty$ and hence $\mathrm{rk}_H(\mathcal{L}) = \infty$ for every non-scattered $\mathcal{L}$. It is also known that if $\mathcal{L}$ is scattered, then $\mathrm{rk}_H(\mathcal{L}) < \infty$. (This follows from the relativized version Lemma 2.2 below.) See [Ros82, Chapter 5] for more background on Hausdorff rank.

In the following lemma we prove that when $\mathcal{L}$ is hyperarithmetic and scattered, its Hausdorff rank cannot be arbitrarily high. This is the only place in the paper where we use hyperarithmecity. All we use is that if a set is Σ^1_1 in a hyperarithmetic set, then it is Σ^1_1 and hence it cannot be the set of indices for recursive well orderings which is Π^1_1 complete. See [Sac90] or [AK00] for information on the hyperarithmetic hierarchy.

Lemma 2.2. *If $\mathcal{L}$ is a hyperarithmetic scattered linear ordering, then* $\mathrm{rk}_H(\mathcal{L}) < \omega_1^{CK}$.

The proof of this lemma is somewhat similar to the proof of [Clo89, Lemma 13], where Clote proved that Hausdorff's theorem holds in ATR_0. The basic idea of both proofs is the use of pseudohierarchies.

Proof. Assume that $\mathcal{L}$ is hyperarithmetic and $\mathrm{rk}_H(\mathcal{L}) \geq \omega_1^{CK}$. We will show that then, there is an embedding of η into $\mathcal{L}$. Given a linear ordering $\mathcal{A} = \langle A, \leq \rangle$, and a family $E = \{\simeq_a \colon a \in A\}$ of equivalence relations on L, let $\phi(\mathcal{A}, E)$ be the hyperarithmetic formula that says:

- For every $a \in A$ there is a pair of non-$\simeq_a$-equivalent elements, and
- for every $a \in A$, if $x \not\simeq_a y$, then, for every $b < a$ there are infinitely many elements of L between x and y which are mutually non-$\simeq_b$-equivalent.

Observe that if $\alpha < \mathrm{rk}_H(\mathcal{L})$, then $E = \{\approx_\beta \colon \beta < \alpha\}$ satisfies $\phi(\alpha, E)$. Then, for every recursive well ordering α, $\exists E(\phi(\alpha, E))$. The formula $\exists E(\phi(x, E))$ is Σ^1_1. Then, since the set of recursive well orderings cannot be defined by a Σ^1_1 formula, there is a recursive non-well-ordered linear ordering $\mathcal{A}$ such that $\exists E(\phi(\mathcal{A}, E))$. Let $E = \{\simeq_a \colon a \in A\}$ and $\{a_i\}_{i \in \mathbb{N}}$ be a descending sequence in $\mathcal{A}$. Let x_0 and x_1 be two elements of L such that $x_0 \not\simeq_{a_0} x_1$. Since there are infinitely many $\simeq_{a_1}$-equivalence classes between x_0 and x_1, there is an $x_{1/2} \in L$ such that $x_0 < x_{1/2} < x_1$ and $x_0 \not\simeq_{a_1} x_{1/2} \not\simeq_{a_1} x_1$. In the same way we define $x_{1/4}$ and $x_{3/4}$ such that

$$x_0 < x_{1/4} < x_{1/2} < x_{3/4} < x_1$$

and

$$x_0 \not\simeq_{a_2} x_{1/4} \not\simeq_{a_2} x_{1/2} \not\simeq_{a_2} x_{3/4} \not\simeq_{a_2} x_1.$$

Continue in this way to define an embedding of the dyadic rationals into $\mathcal{L}$. $\square$

From the lemma we have just proved, we get that Theorem 1.2 will follow from the following theorem.

THEOREM 2.3. *A scattered linear ordering has Hausdorff rank less than ω_1^{CK} if and only if it is equimorphic to a recursive linear ordering.*

Since equimorphism preserves Hausdorff rank, the direction from right to left follows from the lemma above.

Richard Laver [Lav71] proved that every scattered linear ordering is a finite sum of indecomposable linear orderings. Thus, it would be enough to prove Theorem 2.3 for indecomposable linear orderings. Dealing with equimorphism classes of indecomposable linear orderings might be complicated, so we will work with signed trees instead. Signed trees were introduced in [Mon] to represent indecomposable linear orderings up to equimorphism.

DEFINITION 2.4. A *signed tree* is pair $\langle T, s_T\rangle$, where T is a *well founded subtree* of $\omega^{<\omega}$ (i.e.: a downwards closed subset of $\omega^{<\omega}$ with no infinite paths) and s_T is a map, called *sign function*, from T to $\{+,-\}$. We will usually write T instead of $\langle T, s_T\rangle$. A *homomorphism* from a signed tree T to another signed tree $\check{T}$ is a map $f\colon T \to \check{T}$ such that

- for all $\sigma \subset \tau \in T$ we have that $f(\sigma) \subset f(\tau)$ and
- for all $\sigma \in T$, $s_{\check{T}}(f(\sigma)) = s_T(\sigma)$.

(Here $\subset$ is the strict inclusion of strings.) We define a binary relation $\preccurlyeq$ on the class of signed trees. We let $T \preccurlyeq \check{T}$ if there exists a homomorphism $f\colon T \to \check{T}$. We say that T and $\check{T}$ are *equimorphic*, and write $T \sim \check{T}$, if $T \preccurlyeq \check{T}$ and $\check{T} \preccurlyeq T$.

Remark 2.5. For f to be a homomorphism, we do not require that $\sigma|\tau$ implies $f(\sigma)|f(\tau)$.

Notation 2.6. For $\sigma \in T$, we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $s_{T_\sigma}(\tau) = s_T(\sigma^\frown\tau)$. For $n \in \omega$ with $\langle n\rangle \in T$, we let $T_n = T_{\langle n\rangle}$.

We associate to each signed tree T, a linear ordering $\mathrm{lin}(T)$.

DEFINITION 2.7. The definition of $\mathrm{lin}(T)$ is by effective transfinite induction. If $T = \{\emptyset\}$, we let $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$ depending on whether $s_T(\emptyset) = +$ or $s_T(\emptyset) = -$. Now suppose $T \supsetneq \{\emptyset\}$. If $s_T(\emptyset) = +$, we want $\mathrm{lin}(T)$ to be an ω sum of copies of $\mathrm{lin}(T_0)$, $\mathrm{lin}(T_1)$,..., where each $\mathrm{lin}(T_i)$ appears infinitely often in the sum. So, we let

$$\mathrm{lin}(T) = \mathrm{lin}(T_0) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1)) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1) + \mathrm{lin}(T_2)) + ...$$

If $s_T(\emptyset) = -$, we let

$$\mathrm{lin}(T) = ... + (\mathrm{lin}(T_2) + \mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + (\mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + \mathrm{lin}(T_0).$$

We say that a linear ordering, $\mathcal{L}$, is *h-indecomposable* if it is of the form $\mathrm{lin}(T)$ for some signed tree T.

It was proved in [Mon] that every indecomposable linear ordering is equimorphic either to **1** or to an h-indecomposable linear ordering. (Note that in [Mon], **1** is considered an h-indecomposable linear ordering.) It was also proved in [Mon] that given signed trees T and $\check{T}$, $T \preccurlyeq \check{T}$ if and only if $\mathrm{lin}(T) \preccurlyeq \mathrm{lin}(\check{T})$, and hence $T \sim \check{T}$ if and only if $\mathrm{lin}(T) \sim \mathrm{lin}(\check{T})$. The ranks of T and of $\mathrm{lin}(T)$ are very closely related too. We define $\mathrm{rk}(T)$ to be the rank of the well founded partial ordering $\langle T, \supseteq\rangle$. On a well founded partial ordering $\mathcal{P} = \langle P, \leq\rangle$, the *rank* function is defined as usual:

$$\mathrm{rk}(\mathcal{P}, x) = \sup\{\mathrm{rk}(\mathcal{P}, y) + 1 : y \in P, y < x\}$$

and $\mathrm{rk}(\mathcal{P}) = \sup\{\mathrm{rk}(\mathcal{P}, x) + 1 : x \in P\}$.

Remark 2.8. Observe that if $T \preccurlyeq S$, then $\mathrm{rk}(T) \leq \mathrm{rk}(S)$. To prove this first let f is a homomorphism $f : T \to S$. Then, by transfinite induction on $\mathrm{rk}(T, x)$, prove that for every $x \in T$, $\mathrm{rk}(T, x) \leq \mathrm{rk}(S, f(x))$.

LEMMA 2.9. *Let T be a signed tree. If T has finite rank, then* $\mathrm{rk}(T) = \mathrm{rk}_H(\mathrm{lin}(T))$. *If T has infinite rank, then* $\mathrm{rk}(T) = \mathrm{rk}_H(\mathrm{lin}(T)) + 1$.

PROOF. The proof is by transfinite induction on the rank of T. If $T = \{\emptyset\}$, then $\mathrm{rk}(T) = 1$ and, since either $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$, $\mathrm{rk}_H(\mathrm{lin}(T)) = 1$ too. For the inductive step it is enough to prove that for any linear orderings $\mathcal{L}_0$, $\mathcal{L}_1$,... we have that

$$(1)\quad \mathrm{rk}_H(\mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + \ldots) = \sup\{\mathrm{rk}_H(\mathcal{L}_i) + 1 : i \in \omega\}.$$

Let $\mathcal{L} = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + \ldots$ and $\alpha = \sup\{\mathrm{rk}_H(\mathcal{L}_i) + 1 : i \in \omega\}$. First observe that $\mathrm{rk}_H(\mathcal{L}_i + \mathcal{L}_i + \mathcal{L}_i + \ldots) = \mathrm{rk}_H(\mathcal{L}_i) + 1$, and since $\mathcal{L}_i + \mathcal{L}_i + \ldots \preccurlyeq \mathcal{L}$ we have that $\mathrm{rk}_H(\mathcal{L}) \geq \mathrm{rk}_H(\mathcal{L}_i) + 1$ for every i, and hence $\mathrm{rk}_H(\mathcal{L}) \geq \alpha$. On the other hand, if we let $\alpha_i = \max\{\mathrm{rk}_H(\mathcal{L}_j) : j \leq i\}$, then the initial segment of $\mathcal{L}$,

$$\mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + \ldots + (\mathcal{L}_0 + \mathcal{L}_1 + \ldots + \mathcal{L}_i)$$

has only finitely many $\approx_{\alpha_i}$-equivalence classes, and hence it has only one $\approx_{\alpha_i+1}$-equivalence class. Therefore, every pair of elements of $\mathcal{L}$ is $\approx_\alpha$-equivalent, and hence $\mathrm{rk}_H(\mathcal{L}) \leq \alpha$.

Now that we have proved (1), the induction step is straightforward in both, the finite and the infinite case. The discrepancy between the finite and the infinite case is due to the following fact: when $\mathrm{rk}(T, \emptyset) = \omega$, we have that $\mathrm{rk}(T) = \omega + 1$ and

$$\mathrm{rk}_H(\mathrm{lin}(T)) = \max\{\mathrm{rk}_H(\mathrm{lin}(T_i)) + 1 : \langle i\rangle \in T\} = \max\{\mathrm{rk}(T_i) + 1 : \langle i\rangle \in T\} = \omega.$$

□

Therefore, since the functional lin is recursive, it is enough to show that every signed tree of rank less than ω_1^{CK} is equimorphic to a recursive one. This will follow from the following proposition that we will prove in section 4.

PROPOSITION (4.1). *For every recursive ordinal α there is a recursive partial ordering $\langle A_\alpha, \preccurlyeq_\alpha\rangle$ and a recursive function t_α that assigns to each element of A_α a recursive signed tree of rank at most α such that*

- *for every signed tree T of rank at most α there is an $x \in A_\alpha$ with $t_\alpha(x) \sim T$, and*
- *for $x, y \in A_\alpha$, $x \preccurlyeq_\alpha y$ if and only if $t_\alpha(x) \preccurlyeq t_\alpha(y)$.*

We start by giving the general idea of the proof of this proposition. The construction is by effective transfinite recursion. Suppose we have already defined A_β, $\preccurlyeq_\beta$ and t_β and we want to define these objects for $\alpha = \beta + 1$. Every signed tree T is determined, up to equimorphism, by $s_T(\emptyset)$ and the set of *branches* of T,

$$\operatorname{bran}(T) = \{T_i : \langle i \rangle \in T\}.$$

If T has rank α, then for every $\check{T} \in \operatorname{bran}(T)$, there is some $x \in \mathcal{A}_\beta$ such that $T \sim t_\beta(x)$. Let $\operatorname{bran}(T){\downarrow} = \{\check{T} \in t_\beta[A_\beta] : \exists i(\langle i \rangle \in T \;\&\; \check{T} \preccurlyeq T_i)\}$. Then, observe that the tree $\widehat{T}$ determined by $s_{\widehat{T}}(\emptyset) = s_T(\emptyset)$ and $\operatorname{bran}(\widehat{T}) = \operatorname{bran}(T){\downarrow}$ is equimorphic to T. Also observe that T has rank α if and only if $\sup\{\operatorname{rk}(\check{T}) : \check{T} \in \operatorname{bran}(T)\} = \beta$, or equivalently, if and only if for every $\gamma < \beta$ there is a tree $\check{T} \in \operatorname{bran}(T)$ such that $\gamma < \operatorname{rk}(\check{T})$. Therefore, to construct $A_\alpha \setminus A_\beta$, we have to consider all the trees T such that $\operatorname{bran}(T) \subseteq t_\beta[A_\beta]$ is *downwards closed* (i.e. $\operatorname{bran}(T)$ is equal to $\operatorname{bran}(T){\downarrow}$ up to equimorphism), and $\operatorname{rk}[\operatorname{bran}(T)]$ is unbounded below β. (We say that a subset $X \subseteq \beta + 1$ is *unbounded below* β if $\forall \gamma < \beta \exists \delta \in X(\delta > \gamma)$.)

Now comes one of the key ideas of the construction. We need the following definition.

Definition 2.10. A *quasiordering* is a pair $\mathcal{P} = \langle P, \leq_P \rangle$ where $\leq_P$ is transitive and reflexive. If $\mathcal{P}$ is also antisymmetric, then $\mathcal{P}$ is a *partial ordering.* A *well quasiordering* is a quasiordering $\mathcal{P}$ such that, for every sequence $\{x_i : i \in \omega\} \subseteq P$, there exists $i < j$ such that $x_i \leq_P x_j$. A *well partial ordering* is a well quasiordering that is also a partial ordering. A partial ordering is *well founded* if it has no infinite descending sequences. For more information on well quasiorderings see [Mil85].

Remark 2.11. Observe that a well quasiordering has no infinite descending sequences and no infinite antichain. Conversely, it can be proved using Ramsey's theorem that a quasiordering which has no infinite descending sequences and no infinite antichain is a well quasiordering. Also observe that if we have a quasiordering $\mathcal{P}$ and we take the quotient over the equivalence relation $x \equiv_P y \iff x \leq_P y \;\&\; y \leq_P x$, we obtain a partial ordering that we denote by $\mathcal{P}/\equiv_P$. Moreover, $\mathcal{P}$ is a well quasiordering if and only if $\mathcal{P}/\equiv_P$ is a well partial ordering, and if $\mathcal{P}$ is recursive, then so is $\mathcal{P}/\equiv_P$.

By Fraïssé's conjecture we have that, in particular, the set of indecomposable linear orderings, ordered by $\preccurlyeq$, is a well quasiordering. Then, since the operator lin preserves order, we have that the set of signed trees, ordered by $\preccurlyeq$ is well quasiordered too. Therefore $\langle A_\beta, \preccurlyeq_\beta \rangle$ is a well partial ordering, and hence it is well founded too. Given a subset F of A_β, let

$$\mathcal{I}_{A_\beta}(F) = \{x \in A_\beta : \forall y \in F(y \not\preccurlyeq_\beta x)\}.$$

Conversely, given a downwards closed subset $\mathcal{I}$ of A_β, let $F_\mathcal{I}$ be the set of minimal elements of $A_\beta \setminus \mathcal{I}$. Since $F_\mathcal{I}$ is an antichain, and $\langle A_\beta, \preccurlyeq_\beta \rangle$ is a well partial ordering, $F_\mathcal{I}$ is finite. Moreover, since $\langle A_\beta, \preccurlyeq_\beta \rangle$ is well founded, $\mathcal{I} = \mathcal{I}_{A_\beta}(F_\mathcal{I})$. We have proved the following lemma.

LEMMA 2.12. *Let T be a signed tree. Then T has rank $\alpha = \beta + 1$ if and only if there is a finite antichain F of A_β such that* $\mathrm{bran}(T) = t_\beta[\mathcal{I}_{A_\beta}(F)]$ *and* $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ *is unbounded below β.*

In section 4 we will represent the trees of rank α by pairs $\langle *, F\rangle$, where $* \in \{+, -\}$, and $F \subseteq A_\beta$ is a finite antichain such that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β. The difficulty here is that there is no obvious way of checking recursively whether $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β. In the next section we will analyze the structure of signed trees further and find a recursive way of doing this.

To define $\preccurlyeq_\alpha$ we will use the following lemma.

LEMMA 2.13. *Consider $x \in A_\beta$, $*, \check{*} \in \{+, -\}$ and $F, \check{F}$ finite antichains of A_β such that both* $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ *and* $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(\check{F})]]$ *are unbounded below β. Let $S = t_\beta(x)$, T be a signed tree with $s_T(\emptyset) = *$ and* $\mathrm{bran}(T) = t_\beta[\mathcal{I}_{A_\beta}(F)]$*, and $\check{T}$ be a signed tree with $s_{\check{T}}(\emptyset) = \check{*}$ and* $\mathrm{bran}(\check{T}) = t_\beta[\mathcal{I}_{A_\beta}(\check{F})]$*. Then:*

1. *$T \not\preccurlyeq S$.*
2. *$T \preccurlyeq \check{T}$ if and only if $* = \check{*}$ and $\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F})$.*
3. *$S \preccurlyeq T$ if and only if either $x \in \mathcal{I}_{A_\beta}(F)$, or $s_S(\emptyset) = *$ and* $\mathrm{bran}(S) \subseteq t_\beta[\mathcal{I}_{A_\beta}(F)]$.

PROOF. Part (1) is because S has rank less than or equal to β and T has rank $\alpha = \beta + 1$.

For part (2) note that, since both T and $\check{T}$ have rank α, a homomorphism between them has to map the root of T into the root of $\check{T}$, and each branch of T into a branch of $\check{T}$. Since $\mathrm{bran}(\check{T})$ is downwards close, this is equivalent to $* = \check{*}$ and $\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F})$. For part (3) observe that a homomorphism $S \to T$ either maps the root of S to the root of T, in which case $s_S(\emptyset) = *$ and $\mathrm{bran}(S) \subseteq t_\beta[\mathcal{I}_{A_\beta}(F)]$, or it maps S into a branch of T, in which case $x \in \mathcal{I}_{A_\beta}(F)$. □

Remark 2.14. Note that whether $\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F})$ or not can be decided recursively. This is because

$$\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F}) \iff \check{F} \cap \mathcal{I}_{A_\beta}(F) = \emptyset \iff \forall x \in \check{F} \exists y \in F (y \preccurlyeq_\beta x).$$

§3. Signed Forests. In this section we study *ideals* (downwards closed subsets) of the partial ordering of signed trees modulo equimorphisms. Since the class of signed trees is well quasi-ordered, every antichain is finite. So, for every ideal $\mathcal{I}$ there is a finite set $\mathrm{com}(\mathcal{I})$ such that

$$T \in \mathcal{I} \iff \neg(\exists \check{T} \in \mathrm{com}(\mathcal{I}))\check{T} \preccurlyeq T,$$

namely the set of minimal elements of the complement of $\mathcal{I}$. The objective of this section is to define $\mathrm{com}(\mathcal{I})$, for some ideals $\mathcal{I}$, in a recursive way. The results of this section will be used in the next one when we prove Proposition 4.1.

Since we will be dealing with signed trees and ideals of signed trees at the same time, we will work with the more general notion of signed forests. Before introducing signed forests we prove some properties about ranks of partial ordering that we will need later.

3.1. Natural sum of ordinals and ranks. Given an ordinal α we let ω^α be the linear ordering whose elements are the finite sequences $\langle \beta_0, \beta_1, ..., \beta_n \rangle$ such that $\alpha > \beta_0 \geq \beta_1 \geq ... \geq \beta_n \geq 0$. We order the elements of ω^α lexicographically; that is, $\langle \beta_0, ..., \beta_n \rangle \leq_{\omega^\alpha} \langle \gamma_0, ..., \gamma_m \rangle$ if either $n \leq m$ and for all $i \leq n$, $\beta_i = \gamma_i$, or, for the first i such that $\beta_i \neq \gamma_i$, we have that $\beta_i \leq \gamma_i$. It can be shown that ω^α is also a well ordering, and that the initial segment of ω^α up to $\langle \beta_0, ..., \beta_n \rangle$ has order type

$$\omega^{\beta_0} + \omega^{\beta_1} + ... + \omega^{\beta_n}.$$

The *Cantor normal form* of an ordinal α is a tuple $\langle \alpha_0, ..., \alpha_n \rangle$ such that $\alpha \geq \alpha_0 \geq \alpha_1 \geq ... \geq \alpha_n \geq 0$ and

$$\alpha \cong \omega^{\alpha_0} + ... + \omega^{\alpha_n}.$$

(See [AK00, Chapter 4] or [Ros82, Chapter 3 §4] for more information on ordinal operations and the Cantor normal form. The definition we give here of Cantor normal form is slightly different, but obviously equivalent.) Given two ordinals $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$ and $\beta = \omega^{\beta_0} + ... + \omega^{\beta_{m-1}}$, we define the *natural sum* between α and β to be

$$\alpha \oplus \beta = \omega^{\gamma_0} + \omega^{\gamma_1} + ... \omega^{\gamma_{n+m-1}},$$

where $\gamma_0, ..., \gamma_{n+m-1}$ are such that $\gamma_0 \geq \gamma_1 \geq ... \geq \gamma_{n+m-1}$ and there exists two disjoint subsets $\{a_0, ..., a_{n-1}\}$ and $\{b_0, ..., b_{m-1}\}$ of $\{0, ..., n+m-1\}$ such that $\gamma_{a_i} = \alpha_i$ and $\gamma_{b_i} = \beta_i$. The natural sum, sometimes called the Hessenberg sum, was introduced in [Hes06]; see [AB99] for more information on Hessenberg based operations. Note that if we are only considering ordinals which are initial segments of ω^α for a big recursive ordinal α, then the operations $+$, $\oplus$ and taking Cantor normal forms are recursive. There are only a few properties of the natural sum that we will use:

NS1. $(\alpha \oplus \beta) + 1 = \alpha \oplus (\beta + 1) = (\alpha + 1) \oplus \beta$,
NS2. $\alpha + \beta \leq \alpha \oplus \beta$,
NS3. if $\alpha, \beta < \omega^\gamma$, then $\alpha \oplus \beta < \omega^\gamma$,
NS4. if $\alpha_0 \leq \alpha_1$ and $\beta_0 \leq \beta_1$, then $\alpha_0 \oplus \alpha_1 \leq \beta_0 \oplus \beta_1$.

The proofs of these facts are not hard. An ordinal δ is said to be *additively indecomposable* if for every $\alpha, \beta < \delta$, $\alpha \oplus \beta < \delta$. A well known fact is that for an ordinal δ the following are equivalent:

1. δ is additively indecomposable;
2. δ is indecomposable as a linear ordering;
3. $\delta = \omega^\gamma$ for some ordinal γ.

To prove that (1) implies (2) use (NS2). To prove that (2) implies (3) use transfinite induction on δ (see [Ros82, Exercise 10.4]). That (3) implies (1) follows from (NS3).

The following lemma will be very useful later. We need to define some notation first. Given a partial ordering $\mathcal{P} = \langle P, \leq_P \rangle$, and $x \in P$, we let $P_{(<x)} = \{y \in P : y <_P x\}$ and $\mathcal{P}_{(<x)} = \langle P_{(<x)}, \leq_P \rangle$. Observe that

$$\mathrm{rk}(\mathcal{P}) = \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P\}.$$

Lemma 3.1. *Let $\mathcal{P} = \langle P, \leq\rangle$ be a well founded partial ordering. Let $P_0, P_1 \subseteq P$ be such that $P_0 \cup P_1 = P$, and let $\mathcal{P}_0 = \langle P_0, \leq\rangle$ and $\mathcal{P}_1 = \langle P_1, \leq\rangle$. Then*

$$\mathrm{rk}(\mathcal{P}) \leq \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1).$$

If we also have that P_0 and P_1 are closed upwards, then

$$\mathrm{rk}(\mathcal{P}) = \max(\mathrm{rk}(\mathcal{P}_0), \mathrm{rk}(\mathcal{P}_1)).$$

Proof. We use transfinite induction on $\mathrm{rk}(\mathcal{P})$. For the first part we have that

$$\begin{aligned}
\mathrm{rk}(\mathcal{P}) &= \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P\}\\
&\leq \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) \oplus \mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \in P\}\\
&= \max(\sup\{(\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1) \oplus \mathrm{rk}(\mathcal{P}_{1(<x)}) : x \in P_0\},\\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) \oplus (\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1) : x \in P_1\})\\
&\leq \max(\sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1 : x \in P_0\} \oplus \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) : x \in P_0\},\\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) : x \in P_1\} \oplus \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \in P_1\})\\
&\leq \max(\mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1), \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1))\\
&= \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1).
\end{aligned}$$

The second inequality being because of NS4. For the second part we use that if $x \notin P_0$, then $\mathcal{P}_{(<x)} = \mathcal{P}_{1(<x)}$.

$$\begin{aligned}
\mathrm{rk}(\mathcal{P}) &= \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P\}\\
&= \max(\sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P_0 \cap P_1\},\\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \notin P_0\}, \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \notin P_1\})\\
&= \max(\sup\{\max(\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1, \mathrm{rk}(\mathcal{P}_{1(<x)}) + 1) : x \in P_0 \cap P_1\},\\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \notin P_0\}, \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1 : x \notin P_1\})\\
&= \max(\sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1 : x \in P_0\}, \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \in P_1\})\\
&= \max(\mathrm{rk}(\mathcal{P}_0), \mathrm{rk}(\mathcal{P}_1)).
\end{aligned}$$

□

3.2. Signed forests and signed sequences.

Definition 3.2. A *signed forest* is a structure $\mathcal{P} = \langle P, \leq, s_P\rangle$ such that

1. $\langle P, \leq\rangle$ is a countable well founded partial ordering;
2. for every $x \in P$, $\{y \in P : y \geq x\}$ is finite and linearly ordered;
3. $s_P \colon P \to \{+, -\}$.

A *homomorphism* between two signed forests $\mathcal{P}_0 = \langle P_0, \leq, s_{P_0}\rangle$ and $\mathcal{P}_1 = \langle P_1, \leq, s_{P_1}\rangle$ is a map $f \colon P_0 \to P_1$ such that $x < y \Longrightarrow f(x) < f(y)$ and $s_{P_0} = s_{P_1} \circ f$. We let $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ if there is a homomorphism $f \colon \mathcal{P}_0 \to \mathcal{P}_1$. We say that $\mathcal{P}_0$ and $\mathcal{P}_1$ are *equimorphic* if $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ and $\mathcal{P}_1 \preccurlyeq \mathcal{P}_0$. The *rank* of a signed forest is the rank of the underlying well founded partial ordering.

A signed tree $\langle T, s_T\rangle$ can be thought of as the signed forest $\langle T, \leq, s_T\rangle$, where $\leq$ is the reverse inclusion relation $\supseteq$. Conversely, a *rooted signed forest* (that is a signed forest which has a top element, called *root*), can be thought of as a signed tree. Given a rooted signed forest $\langle P, \leq, s_P\rangle$ with $P \subseteq \omega$, consider the signed tree $T \subseteq \omega^{<\omega}$, whose nodes are the sequences $\langle x_0, ..., x_m\rangle$, where

$\{r < x_0 < ... < x_m\rangle = \{y \in P : y \leq x_m\}$ and r is the root of P, and $s_T(\langle x_0, ..., x_m\rangle) = s_P(x_m)$.

Countable ideals of signed trees can also be represented by signed forests. Given an ideal $\mathcal{I}$ of signed trees we consider the signed forest $\biguplus \mathcal{I}$ defined to be the disjoint union of the trees on $\mathcal{I}$ where elements of different trees are considered incomparable. Formally, $\biguplus \mathcal{I} = \langle \bigsqcup \mathcal{I}, \leq_{\mathcal{I}}, s_{\mathcal{I}}\rangle$, where $\bigsqcup \mathcal{I} = \{\langle t, T\rangle : t \in T \in \mathcal{I}\}$, $\langle t, T\rangle \leq_{\mathcal{I}} \langle s, S\rangle$ if and only if $S = T$ and $t \supseteq s$, and $s_{\mathcal{I}}(\langle t, T\rangle) = s_T(t)$. Observe that given two countable ideals $\mathcal{I}$ and $\check{\mathcal{I}}$ we have that $\mathcal{I} \subseteq \check{\mathcal{I}}$ if and only if $\biguplus \mathcal{I} \preccurlyeq \biguplus \check{\mathcal{I}}$, and given a signed tree T, $T \in \mathcal{I}$ if and only if $T \preccurlyeq \biguplus \mathcal{I}$ as signed forests.

Lemma 3.3. *Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be signed forest and suppose that both s_{P_0} and s_{P_1} are constant and equal to $* \in \{+, -\}$. Then, $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ if and only if* $\mathrm{rk}(\mathcal{P}_0) \leq \mathrm{rk}(\mathcal{P}_1)$.

Proof. First, suppose that $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ and f is a homomorphism $f \colon \mathcal{P}_0 \to \mathcal{P}_1$. It can be proved by transfinite induction on $\mathrm{rk}(\mathcal{P}_0, x)$ that for every $x \in P_0$, $\mathrm{rk}(\mathcal{P}_0, x) \leq \mathrm{rk}(\mathcal{P}_1, f(x))$. This implies that $rk(\mathcal{P}_0) \leq \mathrm{rk}(\mathcal{P}_1)$.

Now suppose that $\mathrm{rk}(\mathcal{P}_0) \leq \mathrm{rk}(\mathcal{P}_1)$. For $x \in P_0$ we define $f(x) \in P_1$ by induction on the size of $\{y \in P_0 : y > x\}$, and we do it so that $\mathrm{rk}(\mathcal{P}_0, x) \leq \mathrm{rk}(\mathcal{P}_1, f(x))$. If x is a maximal element of $\mathcal{P}_0$, since $\mathrm{rk}(\mathcal{P}_0) \leq \mathrm{rk}(\mathcal{P}_1)$, we can define $f(x)$ to be some element of $\mathcal{P}_1$ such that $\mathrm{rk}(\mathcal{P}_0, x) \leq \mathrm{rk}(\mathcal{P}_1, f(x))$. Now suppose that x has an immediate successor y. Since $\mathrm{rk}(\mathcal{P}_0, x) < \mathrm{rk}(\mathcal{P}_0, y) \leq \mathrm{rk}(\mathcal{P}_1, f(y))$, we can define $f(x)$ to be some element of $P_{1(<f(y))}$ such that $\mathrm{rk}(\mathcal{P}_0, x) \leq \mathrm{rk}(\mathcal{P}_1, f(x))$. □

This lemma implies that given α and $* \in \{+, -\}$, there is only one signed forest of rank α and with signed function constant equal to $*$, up to equimorphism. We now define a canonical forest in this equivalence class.

Definition 3.4. Given an ordinal α and $* \in \{+, -\}$, let $\mathrm{Sf}(\alpha, *)$ be the signed forest $\langle P, \leq, s_P\rangle$ where P is the set of non-empty strictly descending finite sequences of elements of α, $\leq$ is reverse inclusion on sequences, and s_P is the constant function equal to $*$. If α is a successor ordinal, say $\alpha = \beta + 1$, then consider only the sequences that start with β.

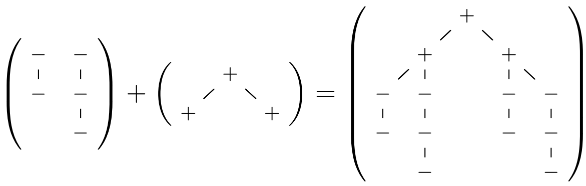
Observe that $\mathrm{rk}(\mathrm{Sf}(\alpha, *)) = \alpha$ and that if α is a successor ordinal then $\mathrm{Sf}(\alpha, *)$ is rooted, and hence a signed tree.

Lemma 3.5. *Let α be an indecomposable ordinal and $\mathcal{P}$ be a signed forest of rank at least α. Then, either* $\mathrm{Sf}(\alpha, +) \preccurlyeq \mathcal{P}$ *or* $\mathrm{Sf}(\alpha, -) \preccurlyeq \mathcal{P}$.

Proof. For $* \in \{+, -\}$, let $P^* = \{x \in P : s_P(x) = *\}$ and $\mathcal{P}^*$ be the induced signed forest with domain P^*. By Lemma 3.1, $\mathrm{rk}(\mathcal{P}) \leq \mathrm{rk}(\mathcal{P}^+) \oplus \mathrm{rk}(\mathcal{P}^-)$. Then, since α is additively indecomposable, either $\mathrm{rk}(\mathcal{P}^+) \geq \alpha$ or $\mathrm{rk}(\mathcal{P}^-) \geq \alpha$. From the previous lemma we get that then, either $\mathrm{Sf}(\alpha, +) \preccurlyeq \mathcal{P}^+ \preccurlyeq \mathcal{P}$ or $\mathrm{Sf}(\alpha, -) \preccurlyeq \mathcal{P}^- \preccurlyeq \mathcal{P}$. □

Definition 3.6. Given two signed forests $\mathcal{P}_0$ and $\mathcal{P}_1$, let $\mathcal{P}_0 + \mathcal{P}_1$ be the signed forest obtained by putting a copy of $\mathcal{P}_0$ below each minimal element of $\mathcal{P}_1$.

See the picture below for an example. In the picture the elements of the forests are marked with either a + or a − and the lines between them represent the order relation.



Lemma 3.7. 1. $(\mathcal{P}_0 + \mathcal{P}_1) + \mathcal{P}_2 = \mathcal{P}_0 + (\mathcal{P}_1 + \mathcal{P}_2)$.
2. $\mathrm{rk}(\mathcal{P}_0 + \mathcal{P}_1) = \mathrm{rk}(\mathcal{P}_0) + \mathrm{rk}(\mathcal{P}_1)$.
3. *If* $\mathcal{P}_0 \preccurlyeq \mathcal{Q}_0$ *and* $\mathcal{P}_1 \preccurlyeq \mathcal{Q}_1$, *then* $\mathcal{P}_0 + \mathcal{P}_1 \preccurlyeq \mathcal{Q}_0 + \mathcal{Q}_1$.
4. *If* $\bar{P}$ *is an upwards closed subset of* $\mathcal{P}$ *such that for all* $x \in \bar{P}$, $\mathcal{Q} \preccurlyeq P_{(<x)}$, *then* $\mathcal{Q} + \bar{\mathcal{P}} \preccurlyeq \mathcal{P}$.
5. *If* $\bar{P}$ *is a non-empty upwards closed subset of* $\mathcal{P}$ *such that for all* $x \notin \bar{P}$, $P_{(\leq x)} \preccurlyeq \mathcal{Q}$, *then* $\mathcal{P} \preccurlyeq \mathcal{Q} + \bar{\mathcal{P}}$.

Proof. Part (1) is immediate. Part (2) can be easily proved by transfinite induction on $\mathrm{rk}(\mathcal{P}_1)$ using that for all $x \in P_1$, $(\mathcal{P}_0 + \mathcal{P}_1)_{(<x)} = (\mathcal{P}_0 + \mathcal{P}_{1(<x)})$. Part (3) follows from part (4). To prove part (4) construct the map $f\colon \mathcal{Q} + \bar{\mathcal{P}} \preccurlyeq \mathcal{P}$ as follows: First, for each minimal element y of $\bar{\mathcal{P}}$ let g_y be an embedding, $g_y\colon \mathcal{Q} \hookrightarrow \mathcal{P}_{(<y)}$. Then if $x \in \bar{P}$, let $f(x) = x$, and if x is in the copy of $\mathcal{Q}$ that is below some minimal element y of $\bar{\mathcal{P}}$, define $f(x)$ using g_y in the obvious way. For part (5), construct the map $f\colon \mathcal{P} \hookrightarrow \mathcal{Q} + \bar{\mathcal{P}}$ as follows: First, for each maximal element y of $P \smallsetminus \bar{P}$, let g_y be an embedding, $g_y\colon \mathcal{P}_{(\leq y)} \hookrightarrow \mathcal{Q}$. Then, if $x \in \bar{P}$, let $f(x) = x$, and if $x \in P \smallsetminus \bar{P}$, let y be the maximal element of $P \smallsetminus \bar{P}$ that is greater that or equal to x and let $f(x) = g_y(x)$. It is not hard to see that in both cases f is the desired embedding. □

Definition 3.8. A *signed sequence* is a finite sequence of the form

$$\pi = \langle\langle \alpha_0, *_0\rangle, \langle \alpha_1, *_1\rangle, ..., \langle \alpha_{n-1}, *_{n-1}\rangle\rangle,$$

where each α_i is an ordinal and $*_i \in \{+, -\}$. The rank of a signed sequence is $\mathrm{rk}(\pi) = \alpha_0 + \alpha_1 + ... + \alpha_{n-1}$. Given a signed sequence π we define a signed forest $\mathrm{Sf}(\pi)$ by induction on $|\pi|$. $\mathrm{Sf}(\langle\langle \alpha, *\rangle\rangle) = \mathrm{Sf}(\alpha, *)$ and $\mathrm{Sf}(\pi^\frown\langle \alpha, *\rangle) = \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha, *)$. Just for completeness, we let $\mathrm{rk}(\emptyset) = 0$ and let $\mathrm{Sf}(\emptyset)$ be the empty signed forest.

Observation 3.9. If $\mathrm{last}(\pi) = \langle \alpha_{n-1}, *_{n-1}\rangle$ and α_{n-1} is a successor ordinal, then $\mathrm{Sf}(\pi)$ is rooted, and therefore a signed tree. Also observe that $\mathrm{rk}(\mathrm{Sf}(\pi)) = \mathrm{rk}(\pi)$.

Proposition 3.10. *Let* $\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + ... + \omega^{\alpha_{n-1}}$ *with* $\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_{n-1}$, *and let* $\mathcal{P}$ *be a signed forest of rank* $\geq \alpha$. *Then, there exists a* $\sigma = \langle *_0, ..., *_{n-1}\rangle \in \{+, -\}^n$ *such that*

$$\mathrm{Sf}(\langle\langle \omega^{\alpha_0}, *_0\rangle, \langle \omega^{\alpha_1}, *_1\rangle, ..., \langle \omega^{\alpha_{n-1}}, *_{n-1}\rangle\rangle) \preccurlyeq \mathcal{P}.$$

Proof. We use induction on n. If $n = 1$, the proposition follows from Lemma 3.5. Suppose now we have proved the lemma for n. Let $\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + ... + \omega^{\alpha_{n-1}} + \omega^{\alpha_n}$, and $\mathcal{P}$ be a signed forest of rank $\geq \alpha$. For each $x \in P$ with

$\mathrm{rk}(\mathcal{P},x) \geq \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$ we have, by inductive hypothesis, that for some $\sigma_x = \langle *_0, ..., *_{n-1}\rangle \in \{+,-\}^n$,

$$\mathrm{Sf}(\langle\langle\omega^{\alpha_0}, *_0\rangle, \langle\omega^{\alpha_1}, *_1\rangle, ..., \langle\omega^{\alpha_{n-1}}, *_{n-1}\rangle\rangle) \preccurlyeq \mathcal{P}_{(<x)}.$$

Let $Q = \{x \in P : \mathrm{rk}(\mathcal{P},x) \geq \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}\}$ and $\mathcal{Q}$ the induced signed forest with domain Q. Observe that $\mathrm{rk}(\mathcal{Q}) \geq \omega^{\alpha_n}$. (This is because $\forall x \in Q(\mathrm{rk}(\mathcal{P},x) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}} + \mathrm{rk}(\mathcal{Q},x)$, which can be easily proved by transfinite induction on $\mathrm{rk}(\mathcal{Q},x)$.) For each $\sigma \in \{+,-\}^n$, let

$$\pi_\sigma = \langle\langle\omega^{\alpha_0}, \sigma(0)\rangle, \langle\omega^{\alpha_1}, \sigma(1)\rangle, ..., \langle\omega^{\alpha_{n-1}}, \sigma(n-1)\rangle\rangle,$$

and let Q_σ be the set of $y \in Q$ such that $\mathrm{Sf}(\pi_\sigma) \preccurlyeq \mathcal{P}_{(<y)}$. Since $Q = \cup_{\sigma\in\{+,-\}^n} Q_\sigma$, from Lemma 3.1, we get that

$$\bigoplus_{\sigma\in\{+,-\}^n} \mathrm{rk}(\mathcal{Q}_\sigma) \geq \mathrm{rk}(\mathcal{Q}) \geq \omega^{\alpha_n}.$$

Then, since ω^{α_n} is additively indecomposable, for some $\sigma \in \{+,-\}^n$, $\mathrm{rk}(Q_\sigma) \geq \omega^{\alpha_n}$, and from Lemma 3.5, we get that for some $* \in \{+,-\}$, $\mathrm{Sf}(\omega^{\alpha_n}, *) \preccurlyeq \mathcal{Q}_\sigma$. Thus, from Lemma 3.7(3) and (4), we get that

$$\mathrm{Sf}(\pi_\sigma{}^\frown\langle\omega^{\alpha_n}, *\rangle) = \mathrm{Sf}(\pi_\sigma) + \mathrm{Sf}(\omega^{\alpha_n}, *) \preccurlyeq \mathrm{Sf}(\pi_\sigma) + \mathcal{Q}_\sigma \preccurlyeq \mathcal{P}. \qquad \square$$

Definition 3.11. Given α as in the proposition above, let com_α be the set of all signed sequences of the form

$$\langle\langle\omega^{\alpha_0}, *_0\rangle, \langle\omega^{\alpha_1}, *_1\rangle, ..., \langle\omega^{\alpha_{n-1}}, *_{n-1}\rangle\rangle.$$

The set $\mathrm{com}_{\beta+1}$ will be used later to compute the minimal elements of the complement of A_β.

Note that, assuming we could compute the Cantor normal form of α uniformly, com_α could be computed uniformly in α too.

Corollary 3.12. *A signed forest $\mathcal{P}$ has rank greater than or equal to α if and only if for some $\sigma \in \mathrm{com}_\alpha$, $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$.*

Proof. The implication from left to right follows immediately form Proposition 3.10. For the other direction, observe that if $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$ then

$$\alpha = \mathrm{rk}(\mathrm{Sf}(\sigma)) \leq \mathrm{rk}(\mathcal{P}). \qquad \square$$

This corollary will allow us to identify the unbounded ideals of A_β in the proof of Proposition 4.1.

3.3. The complements. To identify the unbounded ideals of A_β we will also need to be able to find, for each $\tau \in \mathrm{com}_\beta$, a finite subset $F \subseteq A_\beta$ such that

$$\{x \in A_\beta : t_\beta(x) \preccurlyeq \mathrm{Sf}(\tau)\} = \mathcal{I}_{A_\beta}(F).$$

For this purpose, for each such τ we will define $\mathrm{com}(\tau)$, a finite set of signed sequences, such that for every signed tree T,

$$T \preccurlyeq \mathrm{Sf}(\tau) \iff \neg(\exists\pi \in \mathrm{com}(\tau))\,\mathrm{Sf}(\pi) \preccurlyeq T.$$

In the next section we will define a function isf_β that, given a signed sequence π of rank at most β, which ends in $\langle 1, *\rangle$, returns an index in $\mathcal{A}_\beta$ for the signed tree $\mathrm{Sf}(\pi)$. So, the desired F will be $\{isf_\beta(\pi) : \pi \in \mathrm{com}(\tau)\ \&\ \mathrm{rk}(\pi) \leq \beta\}$. The definition of $\mathrm{com}(\pi)$ might seem obscure at first; it is defined the way it is just to make Proposition 3.14 below work.

Definition 3.13. Given a signed sequence π we will define $\mathrm{com}(\pi)$, a finite set of signed sequences, by induction on $|\pi|$ as follows: Let $\mathrm{com}(\emptyset) = \{\langle 1,+\rangle, \langle 1,-\rangle\}$ and

$$\begin{aligned}\mathrm{com}(\pi^\frown\langle\alpha,*\rangle) \;=\; & \{\sigma^\frown\langle 1,\bar{*}\rangle : \sigma \in \mathrm{com}(\pi),\ \mathrm{last}(\sigma) = \langle 1,*\rangle\} \;\cup \\ & \{\sigma^{-\frown}\langle\alpha,*\rangle^\frown\langle 1,*\rangle : \sigma \in \mathrm{com}(\pi),\ \mathrm{last}(\sigma) = \langle 1,*\rangle\} \;\cup \\ & \{\sigma : \sigma \in \mathrm{com}(\pi),\ \mathrm{last}(\sigma) = \langle 1,\bar{*}\rangle\}.\end{aligned}$$

We are using the following notation: For $* \in \{+,-\}$, $\bar{*}$ is the opposite of $*$, that is $\bar{+} = -$ and $\bar{-} = +$. For a string $\sigma = \langle x_0, ..., x_{n-1}\rangle$, $\mathrm{last}(\sigma) = x_{n-1}$ and $\sigma^- = \langle x_0, ..., x_{n-2}\rangle$.

Note that for all $\sigma \in \mathrm{com}(\pi)$, $\mathrm{last}(\sigma)$ is either $\langle 1,+\rangle$ or $\langle 1,-\rangle$, and hence $\mathrm{Sf}(\sigma)$ is a signed tree.

Proposition 3.14. *For a signed forest $\mathcal{P}$ and a signed sequence π we have that $\mathcal{P} \not\preccurlyeq \mathrm{Sf}(\pi)$ if and only if for some $\sigma \in \mathrm{com}(\pi)$, $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$.*

Proof. We use induction on $n = |\pi|$. For $\pi = \emptyset$ the result is trivial. Now suppose we know the result for π and we want to prove it for $\pi' = \pi^\frown\langle\alpha,*\rangle$.

Let us start by proving the implication from right to left. It is enough to prove that for every $\tau \in \mathrm{com}(\pi')$, $\mathrm{Sf}(\tau) \not\preccurlyeq \mathrm{Sf}(\pi')$. There are two possible cases. First suppose that $\mathrm{last}(\tau) = \langle 1,\bar{*}\rangle$ and either $\tau^- = \sigma \in \mathrm{com}(\pi)$ or $\tau \in \mathrm{com}(\pi)$. In any case, by induction hypothesis, $\mathrm{Sf}(\tau) \not\preccurlyeq \mathrm{Sf}(\pi)$. But, if $\mathrm{Sf}(\tau) \preccurlyeq \mathrm{Sf}(\pi') \preccurlyeq \mathrm{Sf}(\pi)+\mathrm{Sf}(\alpha,*)$, then necessarily $\mathrm{Sf}(\tau) \preccurlyeq \mathrm{Sf}(\pi)$ because the root of $\mathrm{Sf}(\tau)$ is signed $\bar{*}$. So $\mathrm{Sf}(\tau) \not\preccurlyeq \mathrm{Sf}(\pi')$. Second, suppose that $\tau = \sigma^\frown\langle\alpha,*\rangle^\frown\langle 1,*\rangle$ and $\sigma^\frown\langle 1,*\rangle \in \mathrm{com}(\pi)$. Suppose, toward a contradiction, that we have an homomorphism $f\colon \mathrm{Sf}(\tau) \to \mathrm{Sf}(\pi')$. Let $\bar{P}$ be the copy of $\mathrm{Sf}(\langle\langle\alpha,*\rangle,\langle 1,*\rangle\rangle)$ inside $\mathrm{Sf}(\tau) = \mathrm{Sf}(\sigma) + \mathrm{Sf}(\langle\langle\alpha,*\rangle,\langle 1,*\rangle\rangle)$ and $\bar{Q}$ the copy of $\mathrm{Sf}(\alpha,*)$ inside $\mathrm{Sf}(\pi') = \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha,*)$. By inductive hypothesis $\mathrm{Sf}(\sigma^\frown\langle 1,*\rangle) \not\preccurlyeq \mathrm{Sf}(\pi)$, so, for every $x \in \bar{P}$ it has to be the case that $f(x) \in \bar{Q}$ because

$$\mathrm{Sf}(\pi')_{(\leq f(x))} \succcurlyeq \mathrm{Sf}(\tau)_{(\leq x)} \succcurlyeq \mathrm{Sf}(\sigma^\frown\langle 1,*\rangle) \not\preccurlyeq \mathrm{Sf}(\pi).$$

But then $\mathrm{Sf}(\langle\langle\alpha,*\rangle,\langle 1,*\rangle\rangle) = \bar{\mathcal{P}} \preccurlyeq \bar{\mathcal{Q}} = \mathrm{Sf}(\alpha,*)$, contradicting Lemma 3.3.

Now we prove the other implication. Let $\mathcal{P}$ be such that $\mathcal{P} \not\preccurlyeq \mathrm{Sf}(\pi')$. Let $\bar{P} = \{x \in P : \mathcal{P}_{(\leq x)} \not\preccurlyeq \mathrm{Sf}(\pi)\}$. Note that $\bar{P}$ is upwards closed. By the inductive hypothesis, for each $x \in \bar{P}$ there is some $\sigma_x \in \mathrm{com}(\pi)$ such that $\mathrm{Sf}(\sigma_x) \preccurlyeq \mathcal{P}_{(\leq x)}$. If for some of these $x \in \bar{P}$, $\mathrm{last}(\sigma_x) = \langle 1,\bar{*}\rangle$, then $\sigma_x \in \mathrm{com}(\pi')$ too, and we would be done. So, suppose this is not the case and that for every $x \in \bar{P}$ $\mathrm{last}(\sigma_x) = \langle 1,*\rangle$. If some $x \in \bar{P}$ is signed $\bar{*}$, then actually $\mathrm{Sf}(\sigma_x) \preccurlyeq \mathcal{P}_{(<x)}$, since $\mathrm{Sf}(\sigma_x)$ has a top element signed $*$. But then $\mathrm{Sf}(\sigma_x{}^\frown\langle 1,\bar{*}\rangle) \preccurlyeq \mathcal{P}_{(\leq x)} \preccurlyeq \mathcal{P}$ and since $\sigma_x{}^\frown\langle 1,\bar{*}\rangle \in \mathrm{com}(\pi')$, we would be done. So, suppose that every $x \in \bar{P}$, $\mathrm{last}(\sigma_x) = \langle 1,*\rangle$ and $s_P(x) = *$. We want to show that for some $\sigma \in \mathrm{com}(\pi)$ with $\mathrm{last}(\sigma) = \langle 1,*\rangle$ we have that $\mathrm{Sf}(\sigma^{-\frown}\langle\alpha,*\rangle^\frown\langle 1,*\rangle) \preccurlyeq \mathcal{P}$. First we observe that $\mathrm{rk}(\bar{\mathcal{P}}) > \alpha$. Because otherwise, by Lemma 3.3, $\bar{\mathcal{P}} \preccurlyeq \mathrm{Sf}(\alpha,*)$, and then using Lemma 3.7(5) and (3) and the fact that $\forall x \notin \bar{P}(\mathcal{P}_{(\leq x)} \preccurlyeq \mathrm{Sf}(\pi))$ we would get that

$$\mathcal{P} \preccurlyeq \mathrm{Sf}(\pi) + \bar{\mathcal{P}} \preccurlyeq \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha,*) = \mathrm{Sf}(\pi').$$

For each $\sigma \in \text{com}(\pi)$, let $\bar{P}_\sigma$ be the set of $x \in \bar{P}$ such that $\text{Sf}(\sigma) \preccurlyeq \mathcal{P}_{(\leq x)}$. The sets $\bar{P}_\sigma$ are closed upwards and have union $\bar{P}$, so, by Lemma 3.1,

$$\max\{\text{rk}(\mathcal{P}_\sigma) : \sigma \in \text{com}(\pi)\} = \text{rk}(\bar{\mathcal{P}}) \geq \alpha + 1.$$

Therefore, for some $\sigma \in \text{com}(\pi)$, $\text{rk}(\bar{\mathcal{P}}_\sigma) \geq \alpha + 1$, and hence, by Lemma 3.3, $\text{Sf}(\alpha + 1, *) \preccurlyeq \bar{\mathcal{P}}_\sigma$. Notice that for for all $x \in \bar{P}_\sigma$, $\text{Sf}(\sigma^-) \preccurlyeq \mathcal{P}_{(<x)}$. Then, again by Lemma 3.7(3) and (4),

$$\text{Sf}(\sigma^- {}^\frown \langle \alpha, * \rangle {}^\frown \langle 1, * \rangle) \sim \text{Sf}(\sigma^-) + \text{Sf}(\alpha + 1, *) \preccurlyeq \text{Sf}(\sigma^-) + \bar{\mathcal{P}}_\sigma \preccurlyeq \mathcal{P}. \qquad \square$$

DEFINITION 3.15. Let $\text{com}_\alpha(\pi) = \{\sigma \in \text{com}(\pi) : \text{rk}(\sigma) \leq \alpha\}$.

COROLLARY 3.16. *Given $\mathcal{P}$ of rank at most α, we have that $\mathcal{P} \not\preccurlyeq \text{Sf}(\pi)$ if and only if there is some $\sigma \in \text{com}_\alpha(\pi)$ such that $\text{Sf}(\sigma) \preccurlyeq \mathcal{P}$.*

PROOF. It follows immediately from the proposition above and the fact that if $\text{Sf}(\sigma) \preccurlyeq \mathcal{P}$, then necessarily $\text{rk}(\sigma) \leq \text{rk}(\mathcal{P}) \leq \alpha$. $\square$

§4. The construction. In this section we put everything we have done together and prove Proposition 4.1. We have already shown in section 2 that Proposition 4.1 implies Theorems 2.3 and 1.2.

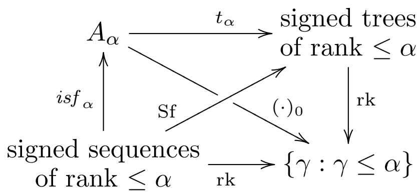
PROPOSITION 4.1. *For every recursive ordinal α there is a recursive partial ordering $\langle A_\alpha, \preccurlyeq_\alpha \rangle$ and a recursive function t_α that assigns to each element of A_α a recursive signed tree of rank at most α such that*

- *for every signed tree T of rank less than or equal to α there is an $x \in A_\alpha$ with $t_\alpha(x) \sim T$, and*
- *for $x, y \in A_\alpha$, $x \preccurlyeq_\alpha y$ if and only if $t_\alpha(x) \preccurlyeq t_\alpha(y)$.*

PROOF. Let ξ be a big additively indecomposable recursive ordinal such that the operations $+$, $\oplus$, and taking Cantor normal forms of ordinals below ξ are recursive. For each $\alpha < \xi$ we will construct, uniformly in α, a recursive set A_α, a recursive partial ordering $\preccurlyeq_\alpha$ on A_α, and two recursive functions t_α and isf_α such that the following condition are satisfied.

1. t_α assigns to each $x \in A_\alpha$ a recursive signed tree of rank less than or equal to α.
2. For every recursive signed tree T of rank less than or equal to α there exists an $x \in A_\alpha$ such that $t_\alpha(x) \sim T$.
3. $x \preccurlyeq_\alpha y$ if and only if $t_\alpha(x) \preccurlyeq t_\alpha(y)$.
4. isf_α maps signed sequences π, with $\text{last}(\pi) = \langle 1, * \rangle$ and of rank less than or equal to α, into A_α, such that $t_\alpha(isf_\alpha(\pi)) \sim \text{Sf}(\pi)$.
5. For $\beta < \alpha$, $A_\beta = A_\alpha \cap \omega^{[\leq \beta]}$, $t_\beta \subseteq t_\alpha$, $isf_\beta \subseteq isf_\alpha$, and $\preccurlyeq_\beta$ is the restriction of $\preccurlyeq_a$ to $A_\beta \times A_\beta$, where $\omega^{[\leq \beta]} = \{\langle \gamma, y \rangle : \gamma \leq \beta, y \in \omega\}$.

Observe that Condition (5) above implies that $\text{rk}(t_\alpha(x)) = (x)_0$, where $(\cdot)_0$ is the projection onto the first coordinate, so for example $(\langle y, z \rangle)_0 = y$. So, we want to construct t_α and isf_α such that the following diagram commutes up to equimorphism of signed trees.



The construction is by effective transfinite recursion. Let A_1 consist of two incomparable elements $\langle 1, +\rangle$ and $\langle 1, -\rangle$, and for $* \in \{+, -\}$, let $t_1(\langle 1, *\rangle) = \mathrm{Sf}(1, *)$ and $isf_1(\langle\langle 1, *\rangle\rangle) = \langle 1, *\rangle \in A_1$.

Suppose now we have already constructed A_β, $\preccurlyeq_\beta$, t_β and isf_β for each $\beta < \alpha$ satisfying the conditions above. When α is a limit ordinal, just take A_α to be $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ and define $\preccurlyeq_\alpha$, t_α and isf_α also by taking unions. It is not hard to see that the conditions above are still satisfied. (Recall that there are no signed trees whose rank is a limit ordinal.)

Now suppose $\alpha = \beta + 1$. Let B_α the set of pairs $\langle *, F\rangle$ where $* \in \{+, -\}$ and F is a finite antichain of $\langle A_\beta, \preccurlyeq_\beta\rangle$ such that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β. By Lemma 2.12, for every signed tree of rank α there is some $\langle *, F\rangle \in B_\alpha$ such that $s_T(\emptyset) = *$ and $\mathrm{bran}(T){\downarrow} = t_\beta[\mathcal{I}_{A_\beta}(F)]$. Conversely, if, for a signed tree T, $s_T(\emptyset) = *$ and $\mathrm{bran}(T){\downarrow} = t_\beta[\mathcal{I}_{A_\beta}(F)]$ for some $\langle *, F\rangle \in B_\alpha$ then T has rank α. Now, observe that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β if and only if the signed forest $\biguplus t_\beta[\mathcal{I}_{A_\beta}(F)]$ has rank β. By Corollary 3.12, this happens if and only if there is a $\sigma \in \mathrm{com}_\beta$ such that $\mathrm{Sf}(\sigma) \preccurlyeq \biguplus t_\beta[\mathcal{I}_{A_\beta}(F)]$. By Corollary 3.16, $\mathrm{Sf}[\mathrm{com}_\beta(\sigma)]$ is the set of trees of rank at most β which are minimal in the complement of the ideal $\{T : \mathrm{rk}(T) \leq \beta \ \&\ T \preccurlyeq \mathrm{Sf}(\sigma)\}$, everything up to equimorphism. Therefore $\mathrm{Sf}(\sigma) \sim \biguplus t_\beta[\mathcal{I}_{A_\beta}(isf_\beta[\mathrm{com}_\beta(\sigma)])]$. So, we have that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β if and only if for some $\sigma \in \mathrm{com}_\beta$,

$$\mathcal{I}_{A_\beta}(isf_\beta[\mathrm{com}_\beta(\sigma)]) \subseteq \mathcal{I}_{A_\beta}(F),$$

By Remark 2.14, we can check whether $\mathcal{I}_{A_\beta}(isf_\beta[\mathrm{com}_\beta(\sigma)]) \subseteq \mathcal{I}_{A_\beta}(F)$ recursively. So B_α is recursive.

Let $A_\alpha = A_\beta \cup (\{\alpha\} \times B_\alpha)$. For $x \in A_\beta$, let $t_\alpha(x) = t_\beta(x)$. For $\langle *, F\rangle \in B_\alpha$, let $t_\alpha(\langle \alpha, \langle *, F\rangle\rangle)$ be the signed tree T such that $s_T(\emptyset) = *$ and $\mathrm{bran}(T) = t_\beta[\mathcal{I}_{A_\beta}(F)]$. Note that, because of what we said above about B_α, t_α satisfies conditions (1) and (2).

Now we want to define the relation $\preccurlyeq_\alpha$ on A_α. Consider $x, y \in A_\alpha$. We let $x \preccurlyeq_a y$ if and only if one of the following conditions holds

- $x, y \in A_\beta$ and $x \preccurlyeq_\beta y$;
- $x = \langle \alpha, \langle *_0, F_0\rangle\rangle$, $y = \langle \alpha, \langle *_1, F_1\rangle\rangle$, $*_0 = *_1$, and $\mathcal{I}_{A_\beta}(F_0) \subseteq \mathcal{I}_{A_\beta}(F_1)$;
- $x \in A_\beta$, $y = \langle \alpha, \langle *, F\rangle\rangle$ and
  - either $x \in \mathcal{I}_{A_\beta}(F)$,
  - or $x = \langle \gamma + 1, \langle \check{*}, \check{F}\rangle\rangle$, for some $\gamma < \beta$, $* = \check{*}$ and
$$\mathcal{I}_{A_\gamma}(\check{F}) \subseteq \mathcal{I}_{A_\beta}(F).$$

Condition (3) follows from Lemma 2.13. Observe that

$$\mathcal{I}_{A_\gamma}(\check{F}) \subseteq \mathcal{I}_{A_\beta}(F) \iff F \cap \mathcal{I}_{A_\gamma}(\check{F}) = \emptyset \iff \forall x \in F((x)_0 \geq \gamma \vee \exists y \in \check{F}(y \preccurlyeq_\beta x)).$$

So $\preccurlyeq_\alpha$ is recursive.

Finally, let us define isf_α. For a signed sequence π of rank less than α, let $isf_\alpha(\pi) = isf_\beta(\pi)$. For $\pi' = \pi^\frown\langle 1, *\rangle$ of rank α, let $isf_\alpha(\pi') = \langle \alpha, \langle *, isf_\beta[\mathrm{com}_\beta(\pi)]\rangle\rangle$. So we get that $t_\alpha(isf_\alpha(\pi'))$ is the signed tree T such that $s_T(\emptyset) = *$ and

$$\begin{aligned} \mathrm{bran}(T) &= t_\beta[\mathcal{I}_{A_\beta}(isf_\beta[\mathrm{com}_\beta(\pi)])] \\ &= \{\check{T} : \mathrm{rk}(\check{T}) \leq \beta \ \& \ \neg(\exists x \in isf_\beta[\mathrm{com}_\beta(\pi)])t_\beta(x) \preccurlyeq \check{T}\} \\ &= \{\check{T} : \mathrm{rk}(\check{T}) \leq \beta \ \& \ \neg(\exists \sigma \in \mathrm{com}_\beta(\pi))\, \mathrm{Sf}(\sigma) \preccurlyeq \check{T}\}, \end{aligned}$$

which, by Corollary 3.16 is equal to $\{\check{T} : \check{T} \preccurlyeq \mathrm{Sf}(\pi)\}$. Therefore, $T \sim \mathrm{Sf}(\pi')$ and condition (4) follows. Condition (5) is immediate from the definitions. □

An interesting consequence of the proof of this proposition is given in Corollary 4.3. The following basic observation about indecomposable linear orderings will be used in the proof of Corollary 4.3.

Observation 4.2. If $\mathcal{L}$ is indecomposable and $\mathcal{L} \preccurlyeq L_1 + ... + L_n$, then $L \preccurlyeq L_i$ for some i. This fact can easily be proved by induction on n using the definition of indecomposability.

Corollary 4.3. *Given $\alpha < \omega_1^{CK}$, $\mathbb{L}_\alpha$, the partial ordering of equimorphism types of linear orderings of Hausdorff rank less than α ordered by $\preccurlyeq$, is recursively presentable.*

Proof. Let A_α, $\preccurlyeq_\alpha$ and t_α be as in the proof above. Let $\hat{A}_\alpha = (A_\alpha \cup \{0\})^{<\omega} \setminus \{\emptyset\}$, the set of finite, non-empty, sequences of elements of $A_\alpha \cup \{0\}$. For $x \in A_\alpha \cup \{0\}$ let

$$l(x) = \begin{cases} \mathrm{lin}(t_\alpha(x)) & \text{if } x \in \mathcal{A}_\alpha \\ \mathbf{1} & \text{if } x = 0. \end{cases}$$

Define the function $\hat{l}$ to $\hat{A}_\alpha$ by:

$$\hat{l}(\langle x_0, ..., x_{n-1}\rangle) = l(x_0) + ... + l(x_{n-1}).$$

Since every scattered linear ordering is equimorphic to a finite sum of $\mathbf{1}$s and h-indecomposable linear orderings, for every linear ordering of Hausdorff rank less than or equal to α, there is a $\sigma \in \hat{A}_\alpha$ such that $\hat{l}(\sigma)$ is equimorphic to it. (Here we are using Lemma 2.9. So, if α is finite, we need to consider $A_{\alpha-1}$ instead of A_α.) We now need to compute the embeddablity relation between linear orderings. We will define a relation $\preccurlyeq$ on $\hat{A}_\alpha$ such that for $\sigma, \tau \in \hat{A}_\alpha$, $\sigma \preccurlyeq \tau \iff \hat{l}(\sigma) \preccurlyeq \hat{l}(\tau)$. First, suppose we are given $\sigma = \langle x_0, ..., x_n\rangle \in \hat{A}_\alpha$ and $x \in A_\alpha \cup \{0\}$, and we want to know whether $l(\sigma) \preccurlyeq \hat{l}(x)$. If $x = 0$, then $\hat{l}(\sigma) \preccurlyeq l(x) = \mathbf{1}$ if and only if $\hat{l}(\sigma) = \mathbf{1}$, or equivalently $\sigma = \langle 0\rangle$. So, suppose that $x = \langle \beta + 1, \langle +, F\rangle\rangle$ for some $\beta < \alpha$. (The case when $x = \langle \beta + 1, \langle -, F\rangle\rangle$ is analogous.) Then

$$l(x) = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + ...,$$

where $\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, ...\} = l[\mathcal{I}_{A_\beta}(F)]]$. Suppose that $\hat{l}(\sigma) \preccurlyeq l(x)$. Then, necessarily, $l(x_n) \preccurlyeq l(x)$ and for each $i < n$, $l(x_i)$ embeds into a proper initial segment of $l(x)$. Since every proper initial segment of $l(x)$ is contained in a finite sum of linear orderings of the form $l(y)$ for $y \in \mathcal{I}_{A_\beta}(F)$, and $l(x_i)$ is indecomposable, it has to be the case that for some $y \in \mathcal{I}_{A_\beta}(F)$, $l(x_i) \preccurlyeq l(y)$, and hence that $x_i \in \mathcal{I}_{A_\beta}(F)$. Therefore, for $\hat{l}(\sigma) \preccurlyeq l(x)$ to hold we have to have that

$$x_{n-1} \preccurlyeq_\alpha x \;\&\; \forall i < n(x_i \in \mathcal{I}_{A_\beta}(F)), \tag{2}$$

which we can check recursively since

$$x_i \in \mathcal{I}_{A_\beta}(F) \iff \mathrm{rk}(x_i) \leq \beta \;\&\; \neg\exists y \in F(y \preccurlyeq_\alpha x_i).$$

Conversely, if (2) holds, then $\hat{l}(\sigma^-)$ embeds into a proper initial segment of $l(x)$ because $l(x)$ contains infinitely many segments isomorphic to $l(x_i)$ for each $i < n$. Since $l(x)$ embeds into every proper final segment of itself, (2) implies that $l(x_n)$ embeds in every proper final segment of $l(x)$. Therefore $\hat{l}(\sigma) = \hat{l}(\sigma^-) + l(x_n) \preccurlyeq l(x)$. We have shown how to check whether $\hat{l}(\sigma) \preccurlyeq l(x)$ recursively.

Now, suppose we are given $\sigma = \langle x_0, ..., x_n\rangle$ and $\tau = \langle y_0, ..., y_m\rangle \in \hat{A}_\alpha$ and we want to check whether $\hat{l}(\sigma) \preccurlyeq \hat{l}(\tau)$. Suppose that there is an embedding $g\colon \sum_{i=0}^n l(x_i) \hookrightarrow \sum_{j=0}^m l(y_j)$. Observe that, since each $l(x_i)$ is indecomposable, we can assume that for every $i \leq n$, $g[l(x_i)] \subseteq l(y_j)$ for some $j \leq m$. (This is because of the property of indecomposable linear orderings mentioned above.) Therefore, $\hat{l}(\sigma) \preccurlyeq \hat{l}(\tau)$ if and only if

$$\bigvee_{0=i_0\leq\ldots\leq i_m\leq n} \left(\bigwedge_{k\leq m} \hat{l}(\langle x_{i_k}, x_{i_k+1}, ..., x_{i_{k+1}-1}\rangle) \preccurlyeq l(y_k) \right).$$

(In the formula above we are taking $i_{m+1} = n + 1$.) Now we have that the quasiordering relation $\preccurlyeq$ on $\hat{A}_\alpha$ is recursive. Hence the induced equivalence relation, $\sim$, defined by $x \sim y \iff x \preccurlyeq y \;\&\; y \preccurlyeq x$, is recursive, and therefore the quotient partial ordering $\langle \hat{A}_\alpha, \preccurlyeq\rangle/\sim$ is recursive too. Observe that $\langle \hat{A}_\alpha, \preccurlyeq\rangle/\sim$ is the desired partial ordering. □

REFERENCES

[AB99] Uri Abraham and Robert Bonnet, *Hausdorff's theorem for posets that satisfy the finite antichain property*, ***Fund. Math.***, vol. 159 (1999), no. 1, pp. 51–69.

[AK00] C.J. Ash and J. Knight, ***Computable structures and the hyperarithmetical hierarchy***, Elsevier Science, 2000.

[BP82] R. Bonnet and M. Pouzet, *Linear extensions of ordered sets*, ***Ordered sets (banff, alta., 1981)***, NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., vol. 83, Reidel, Dordrecht, 1982, pp. 125–170.

[Clo89] P. Clote, *The metamathematics of scattered linear orderings*, ***Arch. Math. Logic***, vol. 29 (1989), no. 1, pp. 9–20.

[Clo90] ———, *The metamathematics of Fraïssé's order type conjecture*, ***Recursion theory week (oberwolfach, 1989)***, Lecture Notes in Math., vol. 1432, Springer, Berlin, 1990, pp. 41–56.

[Dow98] R. G. Downey, *Computability theory and linear orderings*, ***Handbook of recursive mathematics, vol. 2***, Stud. Logic Found. Math., vol. 139, North-Holland, Amsterdam, 1998, pp. 823–976.

[DHLS03] Rodney G. Downey, Denis R. Hirschfeldt, Steffen Lempp, and Reed Solomon, *Computability-theoretic and proof-theoretic aspects of partial and linear orderings*, ***Israel Journal of mathematics***, vol. 138 (2003), pp. 271–352.

[Fei67] Lawrence Feiner, *Orderings and boolean algebras not isomorphic to recursive ones*, ***Ph.D. thesis***, MIT, Cambridge, MA, 1967.

[Fei70] ———, *Hierarchies of Boolean algebras*, ***J. Symbolic Logic***, vol. 35 (1970), pp. 365–374.

[Fra48] Roland Fraïssé, *Sur la comparaison des types d'ordres*, ***C. R. Acad. Sci. Paris***, vol. 226 (1948), pp. 1330–1331.

[Hes06] G. Hessenberg, *Grundbegriffe der mengenlehre*, ***Abhandlungen der Friesschen Schule N.S.***, vol. 1.4 (1906), no. 75.

[JS91] Carl G. Jockusch, Jr. and Robert I. Soare, *Degrees of orderings not isomorphic to recursive linear orderings*, ***Ann. Pure Appl. Logic***, vol. 52 (1991), no. 1-2, pp. 39–64, International Symposium on Mathematical Logic and its Applications (Nagoya, 1988).

[Jul69] Pierre Jullien, *Contribution à l'étude des types d'ordre dispersés*, ***Ph.D. thesis***, Marseille, 1969.

[Lav71] Richard Laver, *On Fraïssé's order type conjecture*, ***Ann. of Math. (2)***, vol. 93 (1971), pp. 89–111.

[Ler81] Manuel Lerman, *On recursive linear orderings*, ***Logic year 1979–80 (proc. seminars and conf. math. logic, univ. connecticut, storrs, conn., 1979/80)***, Lecture Notes in Math., vol. 859, Springer, Berlin, 1981, pp. 132–142.

[Mar] Alberto Marcone, *Wqo and bqo theory in subsystems of second order arithmetic*, To appear in the volume Reverse Mathematics 2001, edited by S.G. Simpson.

[Mil85] E. C. Milner, *Basic wqo- and bqo-theory*, ***Graphs and order (banff, alta., 1984)***, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 147, Reidel, Dordrecht, 1985, pp. 487–502.

[Mon] Antonio Montalbán, *Equivalence between Fraïssé's conjecture and Jullien's theorem.*, To appear.

[Mon05] ———, ***Ph.D. thesis***, Cornell University, 2005, In preparation.

[Nas68] C. St. J. A. Nash-Williams, *On better-quasi-ordering transfinite sequences*, ***Proc. Cambridge Philos. Soc.***, vol. 64 (1968), pp. 273–290.

[Ros82] Joseph Rosenstein, ***Linear orderings***, Academic Press, New York - London, 1982.

[Sac90] Gerald E. Sacks, ***Higher recursion theory***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990.

[Sho93] Richard A. Shore, *On the strength of Fraïssé's conjecture*, ***Logical methods (ithaca, ny, 1992)***, Progr. Comput. Sci. Appl. Logic, vol. 12, Birkhäuser Boston, Boston, MA, 1993, pp. 782–813.

[Sim99] Stephen G. Simpson, ***Subsystems of second order arithmetic***, Springer, 1999.

[Spe55] Clifford Spector, *Recursive well-orderings*, ***J. Symb. Logic***, vol. 20 (1955), pp. 151–163.

E-mail: antonio@math.cornell.edu

DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NY 14853, USA
URL: www.math.cornell.edu/∼antonio