Isomorphism and Bi-Embeddability Relations on Computable Structures*

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Abstract

We study the complexity of natural equivalence relations on classes of computable structures such as isomorphism and bi-embeddability. We use the notion of tc-reducibility to show completeness of the isomorphism relation on many familiar classes in the context of all Σ_1^1 equivalence relations on hyperarithmetical subsets of ω. We also show that the bi-embeddability relation on an appropriate hyperarithmetical class of computable structures may have the same complexity as any given Σ_1^1 equivalence relation on ω.

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1 Introduction

We develop the theory for computable structures analogous to the theory of isomorphism relations introduced by H. Friedman-Stanley in [13]. Our languages are computable, and our structures have universes contained in ω. In measuring complexity, we identify structures with their atomic diagrams. In particular, a structure is *computable* if its atomic diagram is computable.

In descriptive set theory, the study of Borel equivalence relations under Borel reducibility has developed into a rich area. The notion of Borel reducibility allows one to compare the complexity of equivalence relations on Polish spaces, for details see, for example [15, 19, 21]. In particular, natural equivalence relations such as isomorphism and bi-embeddability on classes of countable structures have been widely studied, e.g. [13, 14, 18, 25]. An effective version of this study was introduced in [4] and [24]. The complexity of the isomorphism relation on various classes of countable structures was measured using the idea of effective transformations. In the recent work [11] the general theory of effectively Borel (i.e., Δ^1_1) equivalence relations on effectively presented Polish spaces was developed via the notion of effective Borel reducibility. The resulting structure turned out to be much more complex than in the classical case.

In computable model theory, equivalence relations have also been a subject of study, e.g. [3, 7, 23], etc. In these papers, equivalence relations of rather low complexity were studied (computable, Ershov hierarchy, Σ^0_1, Π^0_1). In [9] Σ^1_1 equivalence relations on computable structures were investigated. The notion of hyperarithmetical and computable reducibility of Σ^1_1 equivalence relations on ω was used to estimate the complexity of natural equivalence relations on hyperarithmetical classes of computable structures within the class of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω as a whole.

In this paper we continue the study of the theory of Σ^1_1 equivalence relations on computable structures. Our work here shows that this theory behaves very differently than the theory initiated in H.Friedman-Stanley [13] for isomorphism relations and further developed for arbitrary Borel equivalence relations on Polish spaces [15, 19, 21]. In particular we show that isomorphism of computable graphs is complete with respect to the chosen effective reducibility in the context of *all* Σ^1_1 equivalence relations on ω. This is false in the context of countable structures and Borel reducibility [22]: there are examples of Borel equivalence relations that are not Borel-reducible to isomorphism of graphs. We also show that the isomorphism relation on computable torsion Abelian groups is complete among Σ^1_1 equivalence relations on ω, while in the classical case it is known to be incomplete among isomorphism relations on classes of countable structures [13]. The same holds for isomorphism of computable torsion-free Abelian groups, which in the case of countable structures is not known to be complete for isomorphism relations.

It was shown in [10] that the general structure of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω under *tc*-reducibility is very rich. In particular, there are properly Σ^1_1 equivalence relations with only finite equivalence classes, and there are Σ^1_1 relations with exactly n properly Σ^1_1 equivalence classes, for

$n \leq \omega$, etc. In this paper we build for every Σ^1_1 equivalence relation E a class K of structures with hyperarithmetical index set such that the bi-embeddability relation on computable members of K is tc-equivalent to E. Thus the structure of natural equivalence relations on hyperarithmetical classes is as complex as the structure of Σ^1_1 equivalence relations on ω as a whole.

2 Background

2.1 Trees

Here we give some definitions useful for describing computable trees. Our trees are isomorphic to subtrees of $\omega^{<\omega}$. For the language, we take a single unary function symbol, interpreted as the predecessor function. We write $\emptyset$ for the top node (our trees grow down), and we think of $\emptyset$ as its own predecessor. Thus, our trees are defined on ω with their structure given by the predecessor function, but we often consider them as subtrees of $\omega^{<\omega}$ and treat their elements as finite sequences.

Definition 1. *Let $S, T \subseteq \omega^{<\omega}$ be trees. Define the tree $S * T$ in the following way. We think of the elements of $S * T$ as ordered pairs (σ, τ), where $\sigma \in S$, $\tau \in T$. At level 0 of $S * T$, we have $(\emptyset, \emptyset)$. For an element (σ, τ) at level k of $S * T$, σ and τ are at level k of S and T, respectively. The successors of (σ, τ) are the pairs (σ', τ'), where σ' is a successor of σ in S and τ' is a successor of τ in T.*

Definition 2. *Let T be a subtree of $\omega^{<\omega}$. We define the* tree rank *of $x \in T$, denoted by $tr(x)$, by induction.*

1. *$tr(x) = 0$ if x has no successor,*
2. *for $\alpha > 0$, $tr(x) = \alpha$ if α is the least ordinal greater than $tr(y)$ for all successors y of x,*
3. *$tr(x) = \infty$ if x does not have ordinal tree rank.*

The tree rank *of the tree T is defined to be the rank of the top node $\emptyset$.*

Note that all computable trees have rank ∞ or rank some computable ordinal. Moreover, for any node $x \in T$, $tr(x) = \infty$ iff x extends to an infinite path through T [27].

Remark. The tree rank of the tree $S * T$ is the minimum of the tree ranks of S and T. In particular, $S * T$ has an infinite path iff both S and T have infinite paths.

Definition 3 (rank-saturated tree)**.** *A computable subtree T of $\omega^{<\omega}$ is* rank-saturated *provided that for all x in T:*

1. *If $tr(x)$ is an ordinal α, then for all $\beta < \alpha$, x has infinitely many successors z such that $tr(z) = \beta$.*

2. *If* $tr(x) = \infty$*, then for all computable* β*,* x *has infinitely many successors* z *such that* $tr(z) = \beta$ *and* x *has infinitely many successors* z *with* $tr(z) = \infty$*.*

Proposition 1. *For every computable ordinal* α*, there exists a computable rank-saturated tree* T^{α} *with* $tr(T) = \alpha$*.*

Proof. Let T be a computable tree of rank α. We will turn T into a computable rank-saturated tree of rank α.

In [17] Harrison proved the existence of a computable linear ordering $\mathcal{H}$ of type $\omega_1^{\mathrm{CK}}(1+\eta)$. Denote by $T_{\mathcal{H}}$ the tree of finite decreasing sequences in $\mathcal{H}$. Take $T^{\alpha} = T * T_{\mathcal{H}}$. Then T^{α} is a computable rank-saturated tree of rank α. Indeed, take $\beta < \alpha$. By the definition of tree rank, there is an element $x \in T$ such that $tr(x) \geq \beta$. Let $y \in T_{\mathcal{H}}$ be such that $tr(y) = \beta$ and $|y| = |x|$. Then $tr(x, y) = \beta$.

Moreover, for every $\gamma < \beta$, the element y has infinitely many successors of rank γ and x has a successor of rank $\geq \gamma$. Thus the element (x, y) has infinitely many successors of rank γ. □

Remark. Computable rank-saturated trees are a special case of computable *rank-homogeneous* trees, defined in [5].

Proposition 2.

1. *For every computable* α*, if* T^{α} *and* T_1^{α} *are computable rank-saturated trees of tree rank* α*, then* $T^{\alpha} \cong T_1^{\alpha}$*.*

2. *If* T^{∞} *and* T_1^{∞} *are computable rank-saturated trees of tree rank* ∞*, then* $T^{\infty} \cong T_1^{\infty}$*.*

Proof. By induction on α. □

We will fix the notation T^{α} for the computable rank-saturated tree of rank α, and T^{∞} for the computable rank-saturated tree with infinite paths.

2.2 Σ_1^1 sets and relations

We assume the reader is familiar with basic concepts of recursion theory. However, here we list some definitions and facts that will be useful for the future proofs. Detailed information can be found, for example, in [1, 27].

Definition 4.

1. *A relation* $S(\overline{x})$ *is* Σ_1^1 *if there is an arithmetical relation* $R(\overline{x}, u)$*, on tuples of numbers, such that* $\overline{x} \in S$ *iff* $(\exists f \in \omega^{\omega})\,(\forall s)\, R(\overline{x}, f \upharpoonright s)$ *— we identify* $f \upharpoonright s$ *with its code.*

2. *A relation* $S(\overline{x})$ *is* Π_1^1 *if there is an arithmetical relation* $R(\overline{x}, u)$*, on tuples of numbers, such that* $\overline{x} \in S$ *iff* $(\forall f \in \omega^{\omega})\,(\exists s)\, R(\overline{x}, f \upharpoonright s)$*.*

3. *A relation* $S(\overline{x})$ *is* Δ_1^1 *if it is both* Σ_1^1 *and* Π_1^1*.*

By the Kleene-Suslin Theorem, a relation is Δ^1_1 iff it is hyperarithmetical.

If $S(\overline{x})$ is a k-place relation, we may consider the set S' of codes for k-tuples belonging to S. It is clear that S is Σ^1_1 iff S' is Σ^1_1. The next result gives familiar conditions equivalent to being Σ^1_1 [1, 27]. We identify finite sequences with their codes.

Proposition 3 (Kleene)**.** *The following are equivalent:*

1. *S is Σ^1_1,*
2. *there is a computable relation $R(n,u)$, on pairs of numbers, such that $n \in S$ iff $(\exists f)\,(\forall s)\,R(n, f \upharpoonright s)$,*
3. *there is a computable sequence of computable trees $(T_n)_{n\in\omega}$ such that $n \in\ S$ iff T_n has an infinite path.*

Theorem 1 (Bounding)**.** *Let CWF denote the set of codes for computable well-founded trees on ω and for each computable ordinal α, let CWF_α denote the set of codes for computable trees of tree rank less than α. Then if F is a hyperarithmetical function from a hyperarithmetical subset of ω into CWF, there exists a computable α such that the range of F is contained in CWF_α.*

We now give a notion of effective reducibility of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω. The idea is the following. A relation E is effectively reducible to a relation E' if there is an effective procedure which allows us to answer any question about E-equivalence using information about E'-equivalence. We want to use computable functions as witnesses for reducibilities.

Definition 5. *Let E, E' be Σ^1_1 equivalence relations on hyperarithmetical subsets $X, Y \subseteq \omega$, respectively. The relation E is tc-reducible to E' iff there exists a partial computable function f with $X \subseteq dom(f), Y \subseteq f(X)$ such that for all $x, y \in X$,*

$$xEy \iff f(x)E'f(y).$$

We denote this fact by $E \leq_{tc} E'$.

Here tc stands for "Turing computable". The notion of tc-reducibility was first used in [4] as an effective analog of the Borel reducibility on classes of countable structures. In the next section we will explain the relationship between our notion and the notion from [4].

2.3 Computable Characterization and Classification

Here we review equivalent approaches from [16] to the problems of computable characterization and classification. The goal is to be able to measure the complexity of a set of computable structures or an equivalence relation on computable structures.

The first approach is based on the notion of computable infinitary formulas. Roughly speaking, computable infinitary formulas are $L_{\omega_1\omega}$ formulas in which the infinite disjunctions and conjunctions are over c.e. sets. For a formal definition see [1]. Computable infinitary formulas form a hierarchy: a *computable* Σ_0 or Π_0 formula is a finitary quantifier-free formula. For $\alpha > 0$, a *computable* Σ_α formula is a c.e. disjunction of formulas of the form $\exists \overline{u}\psi$, where ψ is computable Π_β for some $\beta < \alpha$, and a *computable* Π_α formula is a c.e. conjunction of formulas of the form $\forall \overline{u}\psi$, where ψ is computable Σ_β for some $\beta < \alpha$.

Following [16], we say that a class K of structures closed under isomorphism *has a computable characterization* if the set K^c of its computable members consists exactly of all computable models of a computable infinitary sentence. This definition expresses the idea that the set of all computable members of K can be nicely defined among all other structures for the same language.

The second approach uses the notion of an index set. For a computable structure $\mathcal{M}$, an *index* is a number a such that $\varphi_a = \chi_{D(\mathcal{M})}$, where $(\varphi_a)_{a\in\omega}$ is a computable enumeration of all unary partial computable functions. The *index set* for $\mathcal{M}$ is the set $I(\mathcal{M})$ of all indices for computable (isomorphic) copies of $\mathcal{M}$. For a class K of structures, closed under isomorphism, the *index set* is the set $I(K)$ of all indices for computable members of K. As in [16], we say that a class K *has a computable characterization*, if its index set is hyperarithmetical.

Proposition 4 (Goncharov-Knight [16]). *Let K be a class of countable structures closed under isomorphism and let K^c be the set of computable members of K. Then the following are equivalent:*

1. *The index set $I(K)$ of K is hyperarithmetical;*
2. *There is a computable infinitary sentence ψ such that $K^c = Mod^c_\psi$, where Mod^c_ψ is the set of all computable models of ψ.*

For a relation E on a class K of structures, denote by $I(E, K)$ the set of pairs of indices

$$\{(m, n) | m, n \in I(K) \text{ and } \mathcal{M}_m E \mathcal{M}_n\}.$$

We measure the complexity of various relations on computable structures via the complexity of the corresponding sets of pairs of indices. In what follows we will often identify E with $I(E, K)$ considered as a relation on indices. Thus, it will make sense to compare relations on classes of computable structures with relations on subsets of ω. The most studied cases are that of isomorphism and bi-embeddability relations, e.g., [2, 6, 9, 16].

We are intereseted in studying the relations on classes that are nicely defined. For this reason we will require the index set of each class K to be hyperarithmetical (equivalently, $K^c = Mod^c_\psi$ for some computable infinitary ψ).

Let K and K' be two classes of *countable* structures, such that $K = Mod_\psi$, $K' = Mod_{\psi'}$ for some computable infinitary ψ, ψ'. Suppose the isomorphism relation on K is tc-reducible to the isomorphism relation on K' in the sense of [4]. Then $I(\cong, K) \leq_{tc} I(\cong, K')$ in the sense of Definition 5 and the reduction is exactly the restriction to computable structures of the reduction of K to K'.

3 Isomorphism is Complete among Σ^1_1 Equivalence Relations

If $I(K)$ is hyperarithmetical and E is the isomorphism or bi-embeddability relation, then the corresponding equivalence relation $I(E, K)$ on indices is a Σ^1_1 set. In this section we prove completeness of the isomorphism relation on various familiar classes of structures in the context of all Σ^1_1 equivalence relations on hyperarithmetical subsets of ω under *tc*-reducibility. These results show the difference of our theory from the classical theory of Borel equivalence relations since, by [22], some Borel equivalence relations cannot be reduced to isomorphism relations.

Definition 6. *A relation E on a hyperarithmetical subset of ω is a tc-complete Σ^1_1 equivalence relation if E is Σ^1_1 and every Σ^1_1 relation E' on a hyperarithmetical subset of ω is tc-reducible to E.*

Note that if E is an equivalence relation on a hyperarithmetical class K^c of computable structures then it is complete if and only if for every Σ^1_1 relation E', there exists a computable sequence of computable structures $(\mathcal{M}_n)_{n\in\omega}$ from K^c such that for all $m, n \in \omega$,

$$mE'n \iff \mathcal{M}_m E \mathcal{M}_n.$$

3.1 Trees and Graphs

Theorem 2. *The isomorphism relation on computable trees is a tc-complete Σ^1_1 equivalence relation.*

Proof. Let E be a Σ^1_1 equivalence relation on ω. To prove that E is *tc*-reducible to the isomorphism relation on computable trees, we will build a computable sequence of computable trees $(T_n)_{n\in\omega}$ such that for every $m, n \in \omega$,

$$mEn \iff T_m \cong T_n.$$

Since E is Σ^1_1, there exists F such that $\neg mEn$ if and only if $F(m, n) \in CWF$ (where CWF is the set of codes for computable well-founded trees). Then we say that $\neg mEn$ *is witnessed by stage* α if and only if $F(m, n)$ is in CWF_α.

The strategy to build $(T_n)_{n\in\omega}$ is the following. First, uniformly in m, n, we will build a computable tree $T^*_{m,n}$ with the following properties:

1. $T^*_{m,n} \cong T^*_{n,m}$;
2. $mEn \Rightarrow T^*_{m,n} \cong T^\infty$, where T^∞ is the rank-saturated tree with an infinite path;
3. $\neg mEn \Rightarrow T^*_{m,n} \cong T^\alpha$, where T^α is the rank-saturated tree of tree rank α, for α least such that for all $m' \in [m]_E$ and $n' \in [n]_E$ the relation $\neg m'En'$ is witnessed by stage α.

We start with a computable sequence of computable trees $(T_{m,n})_{m,n\in\omega}$ such that mEn iff $T_{m,n}$ has an infinite path (such a sequence exists by Proposition 3). For every $m,n \in \omega$, we construct (effectively and uniformly) a new tree $T'_{m,n}$ in the following way. Let $\sigma_0, \sigma_1, \ldots$ be an enumeration of all finite sequences of natural numbers. Suppose $\sigma_s = (a_0, \ldots, a_{l_s})$. Then under the s^{th} node on level 1 (i.e. under the element of the form $\langle s\rangle, s \in \omega$) of $T'_{m,n}$ we put the tree $P_s = T_{m,a_0} * T_{a_0,a_1} * \ldots * T_{a_{l_s},n}$. Then

$$tr(T'_{m,n}) = sup\{tr(P_s) | s \in \omega\}.$$

If mEn, then $T_{m,n}$ has an infinite path, i.e. $tr(T_{m,n}) = \infty$. Therefore, $tr(T'_{m,n}) = \infty$. If $\neg mEn$, then for every $\sigma = (a_0, \ldots, a_l)$, $tr(T_{m,a_0} * T_{a_0,a_1} * \ldots * T_{a_l,n})$ is a computable ordinal. Indeed, fix $m,n \in \omega$ such that $\neg mEn$. For every finite sequence σ_s consider the corresponding tree $P_s = T_{m,a_0} * T_{a_0,a_1} * \ldots * T_{a_{l_s},n}$. Consider the function F from the set of finite sequences into CWF such that $F(s)$ is the code of P_s. The function F is hyperarithmetical, its domain is computable. By Bounding, there is a computable bound on the range of F. Therefore, $T'_{m,n}$ has rank α for some computable α. Note that for all $m' \in [m]_E$ and $n' \in [n]_E$, we get the same bound α. Indeed, let $m'Em, n'En$ and β be the computable bound on the ranks of trees constructed using finite sequences starting with m' and ending with n'. Let $P_s = T_{m,a_0} * T_{a_0,a_1} * \ldots * T_{a_{l_s},n}$ be as above. Then $tr(T_{m',m} * P_s * T_{n,n'}) = tr(P_s)$, thus $\alpha \le \beta$. Similarly one can show that $\beta \le \alpha$.

Let now $T_{\mathcal{H}}$ be the computable tree of descending sequences in the Harrison ordering and let $T^*_{m,n} = T'_{m,n} * T_{\mathcal{H}}$. As shown in Proposition 1, the tree $T^*_{m,n}$ is a computable rank-saturated tree, $tr(T^*_{m,n}) = tr(T'_{m,n})$, and the construction is uniform.

Now we build the desired sequence $(T_n)_{n\in\omega}$. Take the tree T consisting exactly of the sequences $(m, m, \ldots, m)$ of length $i \le m$, for $m \in \omega$. Now fix n and for every m, attach $T^*_{m,n}$ to the m-th leaf of T. The resulting tree is T_n. The sequence $(T_n)_{n\in\omega}$ witnesses the reducibility: mEn iff $T_m \cong T_n$. Indeed, suppose mEn. Then

1. for every $k \in [m]_E = [n]_E$, $tr(T'_{k,m}) = tr(T'_{k,n}) = \infty$, thus $T^*_{k,m} \cong T^*_{k,n} \cong T^{\infty}$;

2. for every $k \notin [m]_E$, $tr(T'_{k,m}) = tr(T'_{k,n}) = \alpha$, thus $T^*_{k,m} \cong T^*_{k,n} \cong T^{\alpha}$.

Therefore $T_m \cong T_n$.

Suppose now that $\neg mEn$. Then $T^*_{m,m} \cong T^{\infty}$, while $T^*_{m,n} \cong T^{\alpha}$ for some computable α. Thus $T_m \not\cong T_n$. □

Corollary 1. *The isomorphism relation on computable graphs is a tc-complete Σ^1_1 equivalence relation.*

3.2 Torsion-Free Abelian Groups

Torsion-free Abelian groups are subgroups of $\mathbb{Q}$-vector spaces. Hjorth [18] gave a transformation from trees to torsion-free Abelian groups which enabled him to show that the isomorphism relation on these groups is not Borel. Downey and Montalbán [8] built on Hjorth's ideas to show that the isomorphism problem on these groups is complete among Σ^1_1 *sets*. In this paper we use the transformation from [18] and [8] to show that the isomorphism relation on computable torsion-free Abelian groups is, in fact, complete as a Σ^1_1 equivalence relation. First we describe the transformation.

We consider the elements of $\omega^{<\omega}$ as a basis for a $\mathbb{Q}$-vector space V^*. Let T be a subtree of $\omega^{<\omega}$, and let V be the subspace of V^* with basis T. Let T_n be the set of elements at level n of T. If u is at level $n > 0$, let u^- be the predecessor of u. Let $(p_n)_{n\in\omega}$ be the standard computable list of primes, in increasing order. We let $G(T)$ be the subgroup of V generated by the vector space elements of the following forms:

1. $\frac{v}{(p_{2n})^k}$, where $v \in T_n$, and $k \in \omega$,
2. $\frac{v+v'}{(p_{2n+1})^k}$, where $v \in T_n$, v' is a successor of v, and $k \in \omega$.

If P is a finite set of prime numbers, we let $\mathbb{Q}_P$ be the set of rationals of the form $\frac{k}{m}$, where $k \in \mathbb{Z}$ and m is a product of powers of elements of P.

Theorem 3. *The isomorphism relation on computable torsion-free Abelian groups is tc-complete among Σ^1_1 equivalence relations.*

Proof. As follows from [12], if we restrict the class of trees to only rank-saturated trees, then the transformation from trees into torsion-free Abelian groups described above is $1-1$ on isomorphism types. Thus, given a Σ^1_1 equivalence relation E for every $m \in \omega$, we first construct the sequence of rank-saturated trees $(T^*_{m,n})_{n\in\omega}$ as in Theorem 2. Then we apply the described transformation into torsion-free Abelian groups. The resulting sequence $(G_{m,n})_{n\in\omega}$ will satisfy the property:

$$T^*_{m,n} \cong T^*_{m',n'} \iff G_{m,n} \cong G_{m',n'}.$$

The computable sequence of torsion-free Abelian groups of the form $G_n = \oplus_m G_{m,n}$ witnesses the reduction. □

3.3 Abelian p-Groups

Let p be a prime number. An *Abelian p-group* is an Abelian group such that each its element has some power of p for its order. Countable Abelian p-groups are classified up to isomorphism in terms of Ulm invariants (see [20] for details).

In this section we use the transformation from trees into Abelian p-groups to get completeness of the isomorphism relation for this class. Note that in the classical theory of Borel equivalence relations the analogous result is false (see [13] and a proof for Turing computable embeddings in [12]).

Theorem 4. *The isomorphism relation on Abelian p-groups is a tc-complete Σ^1_1 equivalence relation.*

Proof. By Theorem 2 for any Σ^1_1 equivalence relation E on ω, we have a uniformly computable sequence of trees $(T_n)_{n\in\omega}$ such that mEn iff $T_m \cong T_n$. Each tree T_n is the result of combining a family of trees $T^*_{m,n}$. Each $T^*_{m,n}$ is rank saturated, so it is really determined by its tree rank. We may modify our trees, if necessary, so that the tree rank, if it exists, is a limit ordinal.

Let $(T^m)_{m\in\omega}$ be a family of rank saturated trees. We need a transformation taking $(T^m)_{m\in\omega}$ to an Abelian p-group G, such that from G, we can recover the sequence of tree ranks. We replace T^m by a tree T^m_* such that each single successor in T^m becomes a chain of p_m successors in T^m_*. Then $tr(T^m_*) = p_m tr(T^m)$. We form a single tree T with infinitely many nodes at level 1, with a copy of T^0_* below the first, a copy of T^1_* below the second, etc. Denote the resulting tree by T.

Let G be the Abelian p-group generated by the elements of T in a standard way [20]: the top node is the identity, and if x' is a successor of x, then $px' = x$. The group G has the following features. For all computable α, the Ulm invarian $u_\alpha(G)$ is either ∞ or 0. For limit α, $u_\alpha(G) = \infty$. If $\alpha = \omega\beta + p_m$, then $u_\alpha(G) = \infty$ iff $tr(T^m) \geq \omega\beta$. See [28] for the explanation how to calculate the Ulm sequence for G from the corresponding tree T. □

Corollary 2. *The isomorphism relation on torsion Abelian groups is a tc-complete Σ^1_1 equivalence relation.*

3.4 Other Classes

If we have $K \leq_{tc} K'$ in the sense of [4], then $I(\cong, K) \leq_{tc} I(\cong, K')$. Thus we immediately get the results for Boolean algebras, linear orderings and fields of fixed characteristic.

Theorem 5. *The isomorphism relation on computable Boolean algebras is complete among Σ^1_1 equivalence relations.*

Proof. We sketch the computable transformation from trees to Boolean algebras from [12] which, restricted to rank-saturated trees, is $1-1$ and thus proves the tc-completeness.

The transformation is the following. Let $(A_n)_{n\in\omega}$ be an effective partition of ω into disjoint infinite sets. Let $(L_n)_{n\in\omega}$ be a uniformly computable sequence of orderings, where L_n has order type $\omega^{n+1} + \eta + 1$. For an input tree T, let $S(T)$ consist of the finite sequences of the form $r_0q_1r_1 \dots q_nr_nx$, where for some tree element $a_1 \dots a_n$, we have $q_i \in A_{a_i}$, for $i < n$, $r_i \in A_0$, $r_n \in A_1$, and $x \in L_n$, where x is not last in L_n. We allow r_0x, where $x \in L_0$, not last. For $\rho = r_0q_1 \dots r_{n-1}q_n$ of length $2n$ with extensions in $S(T)$, let N_ρ be the set of these extensions. For $\sigma = r_0q_1 \dots q_nr_n$ of length $2n+1$, with extensions in $S(T)$ and with $r_n \in A_1$, and for I a half open interval in L_n, let $M_{\sigma,I}$ be the set of extensions of σ in $S(T)$—this looks like the interval I in L_n. Note that $N_\emptyset = S(T)$. Let $B(T)$ be the set algebra generated by the special sets N_ρ and $M_{\sigma,I}$. The transformation that takes T to $B(T)$ is Turing computable. The following is clear for arbitrary trees, not just those which are rank-saturated:

if T and T' are isomorphic trees, then $B(T) \cong B(T')$. As shown in [12], the converse is true when the transformation is restricted to rank-saturated trees: if T, T' are rank-saturated then $B(T) \cong B(T') \Rightarrow T \cong T'$—this would be so even for more general rank-homogeneous trees. □

Theorem 6. *The isomorphism relation on computable linear orderings is a tc-complete Σ^1_1 equivalence relation.*

Proof. In [13] H. Friedman and Stanley provided a transformation from arbitrary binary relations on ω to linear orderings, which gives a Borel reduction of the isomorphism relation on graphs to the isomorphism relation on linear orderings. In [4] it is shown that this transformation in fact *tc*-reduces the isomorphism relations. Restricted to computable structures, it proves the statement. □

Furthermore, we can also obtain the result for computable fields:

Theorem 7. *The isomorphism relation on computable fields (of a fixed characteristic) is a tc-complete Σ^1_1 equivalence relation.*

Proof. In [13] a transformation from graphs to fields was presented, which proved Borel reduction of the isomorphism on graphs to the isomorphism on fields. It follows from [4] and [24] that this transformation *tc*-reduces the isomorphism on graphs to isomorphism on fields. Again, restricted to computable structures, it proves the theorem. □

4 Bi-Embeddability and Σ^1_1 Equivalence Relations

In [10] it was proved that the general structure of Σ^1_1 equivalence relations on hyperarithmetcal subsets of ω is very rich. In this section we show that the structure of the bi-embeddability relation on hyperarithmetical classes of computable structures is as complex as the whole structure of Σ^1_1 equivalence relations.

Theorem 8. *For every Σ^1_1 equivalence relation E on ω there exists a hyperarithmetical class K of structures, which is closed under isomorphism and such that E is tc-equivalent to the bi-embeddability relation on computable structures from K.*

Before we give the proof of the theorem, we establish some preliminary facts. First of all, we briefly mention some useful results from [9, 14, 25].

Fact 1. *Any Σ^1_1 preorder R on ω induces a Σ^1_1 equivalence relation R, namely, for all $x, y \in \omega$,*

$$xEy \iff xRy \text{ and } yRx.$$

Moreover, if equivalence relations E, E' are induced by preorders R, R', respectively, then $R \leq_{tc} R' \Rightarrow E \leq_{tc} E'$.

Thus the statement of Theorem 8 will follow from:

Theorem 9. *For every Σ^1_1 preorder R on ω there exists a hyperarithmetical class K closed under isomorphism such that R is tc-equivalent to the embeddability relation on computable structures from K.*

By definition of *tc*-reducibility, we need to show that there exists a class K of structures which is closed under isomorphism such that K^c has a hyperarithmetical index set and satisfies the following properties:

1. There exists a computable sequence $(G_n)_{n\in\omega}$ of computable structures from K^c such that for all $m, n \in \omega$,

$$mRn \iff G_m \sqsubseteq G_n,$$

where the symbol $\sqsubseteq$ denotes embeddability.

2. There exists a computable function g such that for all $m, n \in \omega$, if $\mathcal{M}_m, \mathcal{M}_n \in K^c$ then

$$(\mathcal{M}_m \sqsubseteq \mathcal{M}_n) \iff g(m)Rg(n),$$

where $\mathcal{M}_n$ is the computable structure with its atomic diagram given by the partial computable function φ_n.

Following [25], we define a relation $\leq$ on tree nodes $s, t \in \omega^{<\omega}$ of the same length in the following way: $s \leq t$ iff for all i, the i-th coordinate of t is greater than or equal to the i-th coordinate of s. We also define the operation $s + t$ as coordinate-wise addition.

The proof of the following result can be found in [9]. The general case of relations on 2^ω was considered in [25] and [14].

Proposition 5. *Let R be a Σ^1_1 preorder on ω. Then there exists a computable sequence of computable trees $(T^R_n)_{n\in\omega}$, such that:*

1. *$xRy \Leftrightarrow T^R_{\langle x,y\rangle}$ has an infinite path;*
2. *$\forall s, t \in \omega^{<\omega}$ of the same length, such that $s \leq t$ (i.e., $s(i) \leq t(i)$ for all $i < |s|$), if $s \in T^R_{\langle x,y\rangle}$ then $t \in T^R_{\langle x,y\rangle}$;*
3. *$\forall x(T^R_{\langle x,x\rangle} = \omega^{<\omega})$;*
4. *If $s \in T^R_{\langle x,y\rangle}, t \in T^R_{\langle y,z\rangle}$ and $|s| = |t|$, then $s + t \in T^R_{\langle x,z\rangle}$.*
5. *If $x \neq y$, then the tree $T^R_{\langle x,y\rangle}$ does not possess sequences of the form 0^k, for $k \in \omega$.*

Now we are ready to prove Theorem 9.

Proof of Theorem 9. We will proceed in several steps. First of all, we will present a way to build a class K_0 of computable structures such that a Σ^1_1 preorder R is tc-reducible to a special relation $\leq^*$ on K_0. Furthermore, using an effective transformation, we will turn K_0 into the class K_1 of computable ordered trees (i.e., trees with an additional order relation) such that R is tc-reducible to the embeddability relation on K_1 via a computable injective function f. Then we will show that there exists a computable infinitary formula φ defining a subclass of K_1 such that R is in fact reducible (via the same f) to the embeddability relation on the class Mod^c_φ of computable models of φ. Obviously, Mod^c_φ is closed under isomorphism and its index set is hyperarithmetical. Our last step will be to show that the embeddability relation on Mod^c_φ is tc-reducible to R.

Following [9] for an arbitrary Σ^1_1 preorder R, define a computable sequence of computable structures with the following special properties (structures with these properties form the class K_0). Every structure codes a computable sequence of computable trees. The language consists of one unary predicate symbol V and two unary function symbols g, h. In each model $\mathcal{A}$ the function g is a successor function on $V = \{v_0, v_1, \dots, v_n, \dots\}$, and it defines on V a copy of ω. Each $v_n \in V$ is a root of a tree $T^{\mathcal{A}}_n$, with its structure given by h as a predecessor function. We require that for all n, if $s \in T^{\mathcal{A}}_n, |s| = |t|$ and $s \leq t$, then $t \in T^{\mathcal{A}}_n$.

Consider a preorder $\leq^*$ on K_0 given in the following way.

$$\begin{aligned}\mathcal{A} \leq^* \mathcal{A}' \Leftrightarrow \exists h[\quad & \forall s(|s| = |h(s)|) \text{ and } \forall s, t(s \preceq t \rightarrow h(s) \preceq h(t)) \\ & \text{and } \forall s \in \omega^{<\omega}\{z|s \in T^{\mathcal{A}}_z\} \subseteq \{\, z|h(s) \in T^{\mathcal{A}'}_z\}].\end{aligned}$$

Then $\leq^*$ is a Σ^1_1 preorder. As proved in [9], the following computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of computable structures from K_0 witnesses the tc-reduction of R to $\leq^*$. By Proposition 5, for R there is a computable sequence of computable trees $(T^R_n)_{n\in\omega}$ with the properties 1–5. We define the structure $\mathcal{A}_x$ as follows. For every z, the tree $T^{\mathcal{A}_x}_z$ (under the z-th element of $V^{\mathcal{A}_x}$) equals $T^R_{\langle z,x\rangle}$. Then $(\mathcal{A}_x)_{x\in\omega}$ is a computable sequence of computable structures from K. We check that $xRy \Leftrightarrow \mathcal{A}_x \leq^* \mathcal{A}_y$. By Property 5 in Proposition 5, the computable function witnessing the reduction can be made injective. Indeed, if $x \neq y$ then $T^{\mathcal{A}}_{\langle x,x\rangle} = \omega^{<\omega}$, while $T^{\mathcal{A}}_{\langle x,y\rangle}$ does not possess sequences of the form 0^k, for $k \in \omega$.

Next we transform K_0 into the class of ordered trees K_1. In [9] it was shown that there is an effective transformation of K_0 to the class of computable (not ordered) trees, which sends the relation $\leq^*$ to embeddability. To define the order on trees we use the ideas from [14]. Let G_0 be a tree defined in the following way. For every vertex x of a complete infinitely branching tree $\omega^{<\omega}$, except for the root, we add a new vertex x^* between x and its predecessor x^- (we fix the notation x^* for such a new vertex between x and x^-). We use G_0 as a base to construct the tree $G_{\mathcal{A}}$ corresponding to a structure $\mathcal{A} \in K_0$. To code $\mathcal{A}$ into a tree, we add to G_0 new vertices of the form $(x, s, 0^k)$ and $(x, s, 0^{2x+2}10^k)$, where $x \in \omega, s \in \omega^{<\omega}, k \in \omega$ and $s \in T^{\mathcal{A}}_x$. We connect (x, s, w) to (x, s, v) iff v is the predecessor of w, and we connect $(x, s, \emptyset)$ to s, considered as an element of G_0.

On vertices of this tree $G(\mathcal{A})$ we define the order $\leq_{\mathcal{A}}$ in the following way: for $s, t \in \omega^{<\omega}$ let $s \preccurlyeq t$ if and only if $|s| < |t|$ or $|s| = |t|$ and $s \leq_{lex} t$. Now for $g, g' \in G(\mathcal{A}), s, t \in \omega^{<\omega}$, $u, v \in \omega$ and $x, y \in 2^\omega$ let $g \leq_{\mathcal{A}} g'$ in each of (and only) the following cases:

1. $g = s$ and either $g' = t^*$ or $g' = (v, t, y)$, or $g = s^*$ and $g' = (v, t, y)$;
2. $g = s, g' = t,$ and $s \preccurlyeq t$;
3. $g = s^*, g' = t^*,$ and $s \preccurlyeq t$;
4. $g = (u, s, x), g' = (v, t, y)$ and

$$(s \prec t) \vee (s = t \wedge u < v) \vee (s = t \wedge u = v \wedge x \preceq y).$$

The resulting structures $G_{\mathcal{A}}$ belong to the class K_1 of ordered trees. If $\mathcal{A}$ is computable then obviously $G_{\mathcal{A}}$ is computable and the procedure is uniform. It was shown in [9] that the effective transformation of K_0 into computable trees (without additional order) tc-reduces $\leq^*$ to the embeddability relation. It is not hard to see that this statement remains true when considering ordered trees.

Thus, we have built a tc-reduction of R to the embeddability relation on the class K_1 of computable ordered trees. Let f denote the computable function performing the reduction and let $(G_n)_{n\in\omega}$ denote the corresponding computable sequence of computable ordered trees. Note that the universes of the ordered trees from $(G_n)_{n\in\omega}$ are subsets of ω. We may effectively pass to a computable sequence of ordered trees with universe ω. Thus we will think of structures from $(G_n)_{n\in\omega}$ as having universe ω.

Now we are going to restrict K_1 to a subclass which consists of computable models of a computable infinitary sentence φ such that R will not only be tc-reducible to the embeddability relation on computable models of φ but these relations will be tc-equivalent. Let S_∞ be the group of all permutations on ω. As proved in [14] there is a recursive function $f^* : \omega \times S_\infty \to 2^\omega$ which sends a pair $(n, p) \in \omega \times S_\infty$ to a countable (not necesserily computable) ordered tree $p(G_n)$ with the code $f^*(n, p)$. The range of f^* is a hyperarithmetical subset of 2^ω and obviously closed under isomorphism. By [29] there is a computable infinitary sentence φ such that the range of f^* is exactly the class Mod_φ of all countable models of φ. The function f^* has the property that for every n, $f^*(n, id) = f(n)$, where f is the computable function which sends $n \in \omega$ to the computable ordered tree G_n. Thus, all the computable ordered trees from $(G_n)_{n\in\omega}$ are also models of φ. As shown in [14], the function f^* is injective, thus its inverse f^{*-1} is also a partial recursive function. Define a function g^* to be the first coordinate of f^{*-1}. Then g^* is obviously a partial recursive function from $dom(f^{*-1})$ to ω. Finally, let g be a function defined by $g(n) = g^*(\mathcal{M}_n)$, where $\mathcal{M}_n$ is the n-th computable structure. Then g is a partial computable function and $dom(g)$ contains indices for all structures from Mod^c_φ.

By its definition the function g witnesses the tc-reducibility of the embeddability relation on the class Mod^c_φ of computable models of φ to R. $\square$

Proof of Theorem 8. Let E be a Σ^1_1 equivalence relation. Then it is induced by a Σ^1_1 preorder R. By Theorem 9 there exist a computable infinitary sentence φ and computable functions f and g that witness the tc-equivalence of the embeddability relation on Mod^c_φ and R. Then by Fact 1 there exist computable functions that witness the tc-equivalence of the bi-embeddability relation on computable models of φ and E. The fact that the class Mod^c_φ is hyperarithmetical and closed under isomorphism completes the proof. □

5 Questions

Let K be a class of structures closed under isomorphism such that the index set $I(K)$ is hyperarithmetical. Consider the following statements:

(1) $I(\cong, K)$ is properly Σ^1_1;

(2) $I(\cong, K)$ is m-complete Σ^1_1;

(3) $I(\cong, K)$ is Σ^1_1 complete under tc-reducibility;

(4) $I(\cong, K \upharpoonright highSR)$ is not hyperarithmetical within $K \upharpoonright highSR$, where $highSR$ is the class of structures of high (i.e., noncomputable) Scott rank.

(5) K has infinitely many computable structures of high Scott rank.

The following implications are true: (1) $\Leftarrow$ (2) $\Leftarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Question 1. *Which of these arrows are reversible?*

One of the approaches to give a negative answer to the question "(1) $\Rightarrow$ (3)?" would be to positively answer the following:

Question 2. *Is there a hyperarithmetical class of structures with a unique (up to isomorphism) computable structure of high Scott rank?*

Remark. It is known that up to bi-embeddability this is true in the following sense. In the class of computable linear orderings, the equivalence class of linear orderings bi-embeddable with the rationals is Σ^1_1-complete, but every computable scattered linear ordering (i.e., not bi-embeddable with the rationals) has a hyperarithmetic equivalence class. For more information on the bi-embeddability relation in the class of countable linear orderings see [26].

This question may be also considered as a weaker version of the question from [16] where the authours asked about the existence of a computable structure with high Scott rank and a hyperarithmetical index set.

Question 3. *Are there isomorphism relations on hyperarithmetical classes of computable structures which are not hyperarithmetical and not tc-complete?*

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