Computable linearizations of well-partial-orderings.

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Well-quasi-orderings

Definition: A *well-quasi-ordering (wqo)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- scattered linear orderings [Laver 71];
- finite graphs [Robertson, Seymour 04].

Obs: $\mathcal{Q}$ is a wqo $\Leftrightarrow$

for every sequence $\{x_n : n \in \omega\} \subseteq Q$, $\exists i < j \ (x_i \leqslant_Q x_j)$.

Let $\mathcal{Q} = (Q, \leqslant_Q)$ be a quasi-ordering.
Consider the partial-ordering associated to it in the usual way:

Let $x \equiv_Q y \Leftrightarrow x \leqslant_Q y \;\&\; y \leqslant_Q x$, and
let $\mathcal{W} = \mathcal{Q}/{\equiv_Q}$, where $[x] \leqslant_W [y] \Leftrightarrow x \leqslant_Q y$

Definition: A *well-partial-ordering (wpo)*, is a partial-ordering with no infinite descending sequences and no infinite antichains.

Length

Obs: Every linearization of a wpo is well-ordered.
(A *linearization* of $(P, \leqslant_P)$ is a linear ordering $\leqslant_L$ of P
such that $x \leqslant_P y \Rightarrow x \leqslant_L y$.)
So, if $\{x_n\}$ is $\leqslant_L$ decreasing, $\forall i < j\ (x_i \nleqslant_P x_j)$

Definition: The *length* of $\mathcal{W} = (W, \leqslant_W)$ is
$o(\mathcal{W}) = \sup\{\mathrm{ordTy}(W, \leqslant_L) :$ where $\leqslant_L$ is a linearization of $\mathcal{W}\}$.

Def: $\mathbb{B}\mathrm{ad}(\mathcal{W}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n\ (x_i \nleqslant_W x_j)\}$,
Note: $\mathcal{W}$ is a wpo $\Leftrightarrow \mathbb{B}\mathrm{ad}(\mathcal{W})$ is well-founded.
Theorem: [De Jongh, Parikh 77] $o(\mathcal{W}) = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{W}))$

Maximal order types

Theorem: [De Jongh, Parikh 77]
Every wpo $\mathcal{W}$ has a linearization of order type $o(\mathcal{W})$.

We call such a linearization, a *maximal linearization* of $\mathcal{W}$.

This is why $o(\mathcal{W})$ if often called the *maximal order type* of $\mathcal{W}$.

Such linearizations have been found in many of the examples, always by different methods.

Schmidt's questions

Schmidt, in her Habilitationsschrift [1979], computed the maximal order type of the wpo investigated by Higman, and gave upper bounds for the maximal order types of the wpo's investigated by Kruskal and Nash-Williams.

Schmidt posed two questions:

Question:

Is there a non-trivial relation between rank and length of a wpo?

Question:

Is the length of a computable wpo, a computable ordinal?

Trivial relationship between rank and length

Obs:

- $\text{rk}(\mathcal{W}) \leqslant o(\mathcal{W})$
- $|\text{rk}(\mathcal{W})| = |\mathcal{W}| = |o(\mathcal{W})|$

Theorem: [Malicki, Rutkowski 04]. If $\text{rk}(\mathcal{W}) = \kappa$ is a cardinal, then $o(\mathcal{W})$ can be arbitrarily large, as long as $|o(\mathcal{W})| = \kappa$.

Non-trivial relationship between rank and length

Theorem

Let α, β be two ordinals.
A wpo $\mathcal{W}$ with $\mathrm{rk}(\mathcal{W}) = \alpha$ and $o(\mathcal{W}) = \beta$ exists if and only if

- W1 $|\alpha| = |\beta|$, *and*
- W2 *for every cardinal κ,* $\mathrm{rem}_\kappa(\alpha) \leqslant \mathrm{rem}_\kappa(\beta)$,

... where $\mathrm{rem}_\kappa(\alpha)$ is the least $\delta < \kappa$ such that $\exists\gamma\ (\alpha = \kappa \cdot \gamma + \delta)$.

Obs: If $\alpha = \sum_{i<n} \kappa_i \cdot \alpha_i$ and $\beta = \sum_{i<n} \kappa_i \cdot \beta_i$,
where $\kappa_0 > \cdots > \kappa_{n-1}$ and $|\alpha_i|, |\beta_i| \leqslant \kappa_i$.
Then, (**??**) is equivalent to $\forall i < n\ (\alpha_i \leqslant \beta_i)$.

Computable Length

Q: Is the maximal order type of a computable wpo, computable?

We mentioned that $o(\mathcal{W}) + 1 = \text{rk}(\mathbb{B}\text{ad}(\mathcal{W}))$, where

$$\mathbb{B}\text{ad}(\mathcal{W}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n \ (x_i \not\leq_W x_j)\},$$

Since $\mathbb{B}\text{ad}(\mathcal{W})$ is computable and well-founded, it has rank $< \omega_1^{CK}$. So, $o(\mathcal{W})$ is a computable ordinal.

Q: Does every computable wpo have a computable maximal linearization?

Uniformly computable linearization

Given $\mathcal{W}$, we computably uniformly define a linearization $\preceq^{\mathcal{W}}$ of it.

Theorem

If $\omega^\delta \leqslant o(\mathcal{W}) < \omega^{\delta+1}$*, then*
$\omega^\delta \leqslant \mathrm{ordTy}(W, \preceq^{\mathcal{W}}) \leqslant o(\mathcal{W}) < \omega^{\delta+1}$.
In particular, if $o(\mathcal{W}) = \omega^\delta$*, then* $\mathrm{ordTy}(W, \preceq^{\mathcal{W}}) = \omega^\delta$

Construction:

- We define a map $y \mapsto \sigma_y \colon W \to 2^W$:

$$\text{Let } \sigma_y(x) = \begin{cases} 1 & \text{if } x \leqslant_W y \\ 0 & \text{otherwise} \end{cases}, \qquad \text{for } y, x \in W.$$

- Order 2^W lexicographically, w.r.t. the enumeration of W.
- let $y \preceq^{\mathcal{W}} z \Leftrightarrow \sigma_y \leqslant_{lex} \sigma_z$.

A computable maximal linearization

Theorem

Every computable wpo has a computable maximal linearization.

Scketch of the Proof:

- Write $o(\mathcal{W})$ as $\omega^{\alpha_n} + ... + \omega^{\alpha_0}$.
- Find a partition $J_0, ..., J_k$ of W such that $o(J_i) = \omega^{\alpha_i}$ and if $x \in J_i, y \in J_j$ and $i \leqslant j$ then $y \nleqslant_W x$.
- Then, we note that any such partition is computable.
- Linearize each of the J_i's and put one after the other.

Q: Can we uniformly find computable maximal linearizations?

non-uniformity

Theorem

Let **a** *be a Turing degree. TFAE:*

1. **a** *uniformly computes maximal linearizations of comp. wpos.*
2. **a** *can decide whether two computable ordinals are isomorphic.*
3. **a** *uniformly computes* $0^{(\beta)}$ *for every* $\beta < \omega_1^{CK}$.

That (2) $\Leftrightarrow$ (3) follows from results of Ash and Knight.
For (3) $\Rightarrow$ (1) we note that **a** can do our construction of computable maximal linearizations.

Proof of (1) $\Rightarrow$ (2):

- **a** unif. computes maximal linearizations of comp. wpos.
- Let α and β be computable ordinals.
- Consider $\mathcal{W} = (\omega^\alpha + \{\mathtt{a}\} + \omega^\alpha) \oplus (\omega^\beta + \{\mathtt{b}\} + \omega^\beta)$.
- Use **a** to get a maximal linearization $\trianglelefteq$ of $\mathcal{W}$.
- If $\alpha < \beta$, we have that $\mathrm{ordTy}(W, \trianglelefteq) = \omega^\beta + \omega^\beta + \omega^\alpha + \omega^\alpha$.
- Let $h(\alpha, \beta) = \begin{cases} 0 & \text{if } \mathtt{a} \trianglelefteq \mathtt{b} \\ 1 & \text{if } \mathtt{b} \trianglelefteq \mathtt{a}. \end{cases}$.
- If $\alpha < \beta$, then $h(\alpha, \beta) = 1$, and if $\alpha > \beta$, then $h(\alpha, \beta) = 0$.
- To tell whether $\alpha = \beta$ check if $2\alpha < 2\beta + 1$ and $2\beta < 2\alpha + 1$.

Minimal linearizations

Obs: [Schmidt 79]
If $\text{rk}(\mathcal{W}) = \alpha + n$ and $k = |\{x \in W : \text{rk}(x) \geqslant \alpha\}|$
where α is a limit ordinal,
then $\mathcal{W}$ has a minimal linearization of length $\alpha + k$.

Theorem

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