Up to equimorphism, hyperarithmetic is computable.

Antonio Montalbán.
Cornell University.

www.math.cornell.edu/∼antonio

Spector's Theorem.

Theorem:[Spector '55] Every hyperarithmetic well ordering is isomorphic to a computable one.

Definition:

- The *Turing degree* of a linear ordering $\mathcal{X} = \langle X, \leq_X \rangle$, with $X \subseteq \omega$, is

$$\text{deg}(X) \oplus \text{deg}(\leq_X).$$

- A *computable (hyperarithmetic) linear ordering* is a linear ordering of computable (hyperarithmetic) degree.
- A *computable (hyperarithmetic) well ordering* is a well ordering $\langle X, \leq_X \rangle$ that is computable (hyperarithmetic) as a linear ordering.
- The order type of a computable well ordering is a *computable ordinal*.
- ω_1^{CK} is the least non-computable ordinal.

Hyperarithmetic sets.

Proposition: [Suslin-Kleene, Ash]
For a set $X \subseteq \omega$, the following are equivalent:

- X is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- X is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{CK}$.
  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
- $X = \{x : \varphi(x)\}$, where φ is a computable infinitary formula.
  (*Computable infinitary formulas* are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be *hyperarithmetic*.

In particular, every computable, Δ^0_2, and arithmetic set is hyperarithmetic.

Theorem:[Spector 1955] Every hyperarithmetic well ordering is isomorphic to a computable one.

Spector's theorem.

Spector's theorem doesn't directly extend to linear orderings:
Not every hyperarithmetic linear ordering is isomorphic to a computable one.

Theorem:[Feiner '67] There is a Δ^0_2 l.o. that is not isomorphic to any computable one.

Theorem: There is a linear ordering of Turing degree $\mathbf{a}$ which does not have a computable copy whenever

- $\mathbf{a}'' >_T 0''$; [Lerman '81]
- $\mathbf{a}$ is c.e. and $\mathbf{a} \not\equiv_T 0$; [Jockusch, Soare '91]
- $0 <_T \mathbf{a} \leq 0'$; [Downey '98][Seetapun]
- $\mathbf{a} \not\equiv_T 0$. [Knight '2000]

But this is not the only way we could extend Spector's theorem to linear orderings.

Theorem:[Spector 1955] Every hyperarithmetic well ordering is isomorphic to a computable one.

Our main result

Definition:

- Given linear orderings $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *embeds in* $\mathcal{B}$ if there is a strictly increasing map $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.
- $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic* if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$. We write $\mathcal{A} \sim \mathcal{B}$.

Example:

$$\omega + \omega^* + \omega + \omega^* + \cdots \quad \sim \quad \omega^* + \omega + \omega^* + \omega + \cdots$$

Observation: If α is an ordinal and $\mathcal{L} \sim \alpha$, then $\mathcal{L}$ is isomorphic to α.

Proof: $\mathcal{L} \preccurlyeq \alpha \Rightarrow \mathcal{L}$ is an ordinal and $\mathcal{L} \leq \alpha$.

$\alpha \preccurlyeq \mathcal{L} \Rightarrow \alpha \leq \mathcal{L}$ and hence $\mathcal{L} \cong \alpha$. $\square$

Theorem: Every hyperarithmetic linear ordering is equimorphic to a recursive one.

Definition:

- Given a l.o. $\mathcal{L}$, we define another l.o. $\mathcal{L}'$ by identifying the elements of $\mathcal{L}$ which have finitely many elements in between.
- Then we define $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(\alpha+1)} = (\mathcal{L}^{\alpha})'$, and take direct limits when α is a limit ordinal.
- $\mathrm{rk}(\mathcal{L})$, the *Hausdorff rank* of $\mathcal{L}$, is the least α such that $\mathcal{L}^{\alpha}$ is finite.

Examples: $\mathrm{rk}(\omega) = \mathrm{rk}(\mathbb{Z}) = 1$, $\mathrm{rk}(\omega^{\alpha}) = \alpha$, $\mathrm{rk}(\mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \cdots) = 2$, $\mathrm{rk}(\mathbb{Q}) = \infty$

To prove that $\mathrm{rk}(\omega^{\alpha}) = \alpha$, observe that if we let $\gamma_0 \simeq_{\beta} \gamma_1 \iff \gamma_1 - \gamma_0 < \omega^{\beta}$, then $(\omega^{\alpha})^{(\beta)} = \omega^{\alpha}/\simeq_{\beta}$.

Observation: If $\mathcal{A} \preccurlyeq \mathcal{B}$, then $\mathrm{rk}(\mathcal{A}) \leq \mathrm{rk}(\mathcal{B})$. Therefore, $\mathcal{A} \sim \mathcal{B} \Rightarrow \mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{B})$

Proposition:[Cantor, Hausdorff] For a countable l.o. $\mathcal{L}$, the following are equivalent

- $\mathbb{Q} \not\preccurlyeq \mathcal{L}$,
- $\mathcal{L}$ is not equimorphic to $\mathbb{Q}$,
- $\mathrm{rk}(\mathcal{L}) < \omega_1$.

A l.o. $\mathcal{L}$ such that $\mathbb{Q} \not\preccurlyeq \mathcal{L}$ is said to be *scattered.*

Lemma: If $\mathcal{L}$ is a hyperarithmetic scattered linear ordering, then $\text{rk}(\mathcal{L}) < \omega_1^{CK}$.

Proof: A standard overspill argument.

Theorem: If $\mathcal{L}$ is scattered then
$\text{rk}(\mathcal{L}) < \omega_1^{CK} \iff \mathcal{L}$ is equimorphic to a computable linear ordering.

Proof of $\Leftarrow$: Use the lemma and the observation above.

Proof of our main theorem using the theorem above:

Let $\mathcal{L}$ be a hyperarithmetic linear ordering. If $\mathbb{Q} \preccurlyeq \mathcal{L}$, then $\mathcal{L} \sim \mathbb{Q}$. Otherwise, $\text{rk}(\mathcal{L}) < \omega_1^{CK}$, and hence $\mathcal{L}$ is equimorphic to a computable linear ordering.

Lemma: If $\mathcal{L}$ is a hyperarithmetic scattered linear ordering, then rk$(\mathcal{L}) < \omega_1^{CK}$.

Proof: Assume that $\mathcal{L}$ is hyperarithmetic and rk$(\mathcal{L}) \geq \omega_1^{CK}$.

Given a linear ordering $\mathcal{A}$, and a family $E = \{\simeq_a : a \in A\}$ of equivalence relations on L, let $\phi(\mathcal{A}, E)$ be the hyperarithmetic formula that says:

- For every $a \in A$ there is a pair of non-$\simeq_a$-equivalent elements, and
- for every $a \in A$, if $x \not\simeq_a y$, then, for every $b < a$ there are infinitely many elements of L between x and y which are mutually non-$\simeq_b$-equivalent.

Then, for every recursive well ordering α, since rk$(\mathcal{L}) > \alpha$, $\exists E(\phi(\alpha, E))$.

Since the set of recursive well orderings cannot be defined by a Σ_1^1 formula, there is a recursive non-well-ordered linear ordering $\mathcal{A}$ such that $\exists E(\phi(\mathcal{A}, E))$.

Let $E = \{\simeq_a : a \in A\}$ and $\{a_i\}_{i \in \mathbb{N}}$ be a descending sequence in $\mathcal{A}$.

Construct an embedding f of the dyadic rationals between 0 and 1 into $\mathcal{L}$ by recursion, satisfying that for each n and $m < 2^n$, $f(\frac{m}{2^n}) \not\simeq_{\alpha_m} f(\frac{m+1}{2^n})$.

Definition: Let $\mathbb{L}$ be the partial ordering of equimorphism types of countable linear orderings, ordered by embeddablity.

Theorem: [Fraïsé's Conjecture '48; Laver '71]
$\mathbb{L}$ is a *well partial ordering*.
(i.e., $\mathbb{L}$ has no infinite descending sequences and no infinite antichains.)

Also, for every scattered $x \in \mathbb{L}$,
$\{y \in \mathbb{L} : y \preccurlyeq x\}$ is countable.

Definition: Let $\mathbb{L}_\alpha$ be the restriction of $\mathbb{L}$ to the linear orderings of rank $\leq \alpha$.

Theorem: For every ordinal α,
$\alpha < \omega_1^{CK} \iff \mathbb{L}_\alpha$ is computably presentable.

Definition: Given a countable subset $S = \{\mathcal{L}_0, \mathcal{L}_1, ...\} \subseteq \mathbb{L}$ let
$F(+, S) = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + \cdots$
and
$F(-, S) = \cdots + (\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0) + (\mathcal{L}_1 + \mathcal{L}_0) + \mathcal{L}_0$.
(F is well defined on sets of equimorphism types.)

Definition: By transfinite recursion on α we define $\mathbb{H}_\alpha \subset \mathbb{L}$:

- $\mathbb{H}_0 = \{\mathbf{1}\}$,
- $\mathbb{H}_\alpha = \{F(+, S), F(-, S) : S \subseteq \bigcup_{\beta<\alpha} \mathbb{H}_\beta\} \cup \{\mathbf{1}\}$.

We let $\mathbb{H} = \bigcup \mathbb{H}_\alpha$ be the class of *h-indecomposables*.
Observe that $\mathbb{H}_\alpha = \mathbb{H} \cap \mathbb{L}_\alpha$.

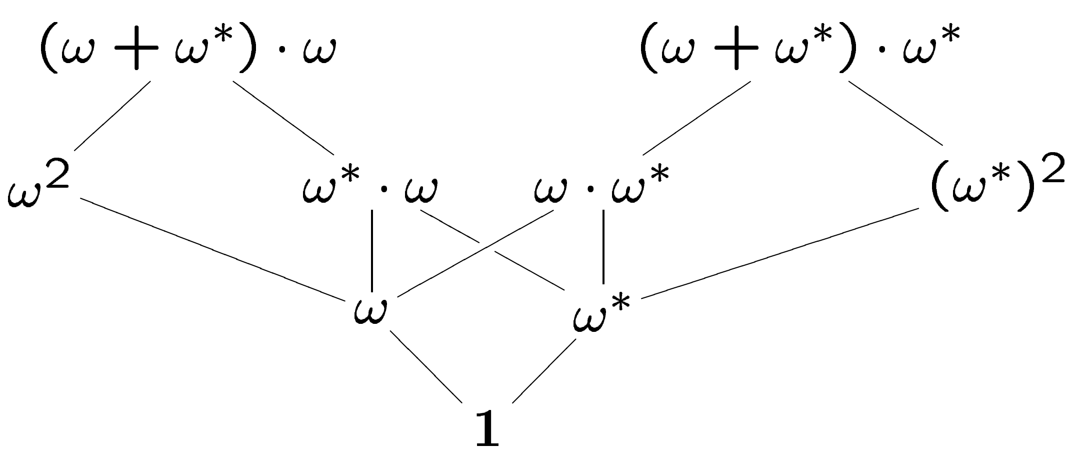
Theorem:[Laver '71] Every scattered countable linear ordering is equimorphic to a finite sum of h-indecomposables.

(So, it is enough to prove that if $\mathcal{L}$ is h-indec. and rk($\mathcal{L}$) $< \omega_1^{CK}$, then $\mathcal{L}$ is equimorphic to a computable l.o.)

By computable transfinite recursion we build:

- $\langle H_\alpha, \leq_\alpha \rangle$, a computable presentation of $\mathbb{H}_\alpha$;
- a computable family $\{\mathcal{L}_\alpha(x) : x \in H_\alpha\}$, such that $\mathcal{L}_\alpha(x)$ is a computable linear ordering in the equimorphism type corresponding to x.

Example: $\langle H_2, \leq_2, \mathcal{L}_2 \rangle$



Observe that if we manage to construct $\langle H_\alpha, \leq_\alpha, \mathcal{L}_\alpha \rangle$, for each $\alpha < \omega_1^{CK}$, we are done.

Suppose we have already constructed $\langle H_\beta, \leq_\beta, \mathcal{L}_\beta \rangle$,
an we want to construct $\langle H_{\beta+1}, \leq_{\beta+1}, \mathcal{L}_{\beta+1} \rangle$,
Recall that $\mathbb{H}_{\beta+1} = \{F(+, S), F(-, S) : S \subseteq \mathbb{H}_\beta\} \cup \{\mathbf{1}\}$.

Key points:

- If $\mathcal{L} \in \mathbb{H}$, and $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}$,
  then either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq \mathcal{B}$.
- If $S, R \subset \mathbb{H}$, then
  $F(+, S) \preccurlyeq F(+, R) \iff \forall x \in S \exists y \in R (x \preccurlyeq y)$.
- If I is the downward closure of $S \subset \mathbb{H}$, then
  $$F(+, S) \sim F(+, I).$$
  (downward closed subset of $\mathbb{H}$ are called *ideals*)
- F induces a bijection between
  $$\{+, -\} \times \{I : I \text{ ideal of } \mathbb{H}_\beta\}, \quad \text{and} \quad \mathbb{H}_{\beta+1}.$$
- Every ideal I of $\mathbb{H}_\beta$ is determined by the set of minimal elements of $\mathbb{H}_\beta \setminus I$, which is an antichain, and hence is finite because $\mathbb{H}_\beta$ is a well partial ordering.

Definition: We let

$H_{\beta+1} = \{+,-\} \times \{A : A \text{ finite antichain of } H_\beta\}$

and

$\mathcal{L}_{b+1}(\langle +, A\rangle) = F(+, \{\mathcal{L}_\beta(x) : x \in I_\beta(A)\}),$

where $I_\beta(A) = \{x \in H_\beta : \forall y \in A(y \not\leq_\beta x)\}$

Definition of $\leq_{\beta+1}$:

- Note that $F(+, I) \preccurlyeq F(+, J) \iff I \subseteq J,$

(when I, J are ideals). So, we can define

$\langle +, A_0\rangle \leq_{\beta+1} \langle +, A_1\rangle \iff$

$\forall y \in A_1 \exists x \in A_0 (x \leq_\beta y).$

- When the signs are different we have

$F(+, I) \preccurlyeq F(-, J) \iff F(+, I) \in J,$

So,

$\langle +, A_0\rangle \leq_{\beta+1} \langle -, A_1\rangle$ iff $\forall z \in A_1 (z \not\leq_\beta y)$ where

$y \in H_\beta$ and $\mathcal{L}_{\beta+1}(\langle +, A_0\rangle) = \mathcal{L}_\beta(y).$

All we need to do is recognize H_β inside $H_{\beta+1}$.

For both, defining $\leq_\beta$ and defining H_α for limit α, we need to be able to recognize H_β inside $H_{\beta+1}$.

Recall:

- For every set $S = \{L_0, \mathcal{L}_1, ...\}$ of equimorphism types, $F(+, S) = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + \cdots$
- $\mathbb{H}_\alpha = \{F(+, S), F(-, S) : S \subseteq \bigcup_{\beta<\alpha} \mathbb{H}_\beta\} \cup \{\mathbf{1}\}$.
- $\mathbb{H} = \bigcup \mathbb{H}_\alpha$, the class of *h-indecomposables*.
- Every l.o is equimorphic to a finite sum of h-indecomposables.
- We want to show that for every $x \in \mathbb{H}_\alpha$, $\alpha < \omega_1^{CK}$, there is a computable $\mathcal{L} \in x$.
- We want to computably define $\langle H_\beta, \leq_\beta, \mathcal{L}_\beta \rangle$, such that $\langle H_\beta, \leq_\beta \rangle \cong \mathbb{H}_b$ and for each $x \in H_\beta$, $\mathcal{L}_\beta(x) \in x$.
- F induces a bijection between
  $\{+, -\} \times \{I : I \text{ ideal of } \mathbb{H}_\beta\}$, and $\mathbb{H}_{\beta+1}$.
- We code an ideal of $\mathbb{H}_b$ by the set of minimal elements of its complement, which is finite.
- $H_{\beta+1} = \{+, -\} \times \{A : A \text{ finite antichain of } H_\beta\}$.

Suppose that we have already constructed $H_{\beta+1}$ and that we have $H_\beta \subset H_{\beta+1}$.

Claim: If we could uniformly find $C_{\beta+1}$, the set of minimal elements of $H_{\beta+1} \setminus H_\beta$, then we could uniformly recognize $H_{\beta+1}$ inside $H_{\beta+2}$.

- Suppose we are given $y \in H_{\beta+1}$, and we want to find $x \in H_{\beta+2}$ such that $\mathcal{L}_{\beta+2}(x) = \mathcal{L}_{\beta+1}(y)$. Let y be given as $\langle +, A \rangle$ and let $I = \{z \in H_\beta : \forall a \in A(a \not\leq_\beta z)\}$. To find x we need to find the set B of minimal elements of $H_{\beta+1} \setminus I$.
  B is the set of minimal elements of $A \cup C_{\beta+1}$.

- Conversely, suppose we are given $x \in H_{\beta+2}$, and we want to find out whether $\mathcal{L}_{\beta+2}(x) \in \mathbb{H}_{\beta+1}$. Let x be given as $\langle +, A \rangle$ and let $I = \{z \in H_{\beta+1} : \forall a \in A(a \not\leq_{\beta+1} z)\}$. Note that rk$(\mathcal{L}_{\beta+2}(x)) \leq \beta + 1$ iff for every $z \in I$, rk$(\mathcal{L}_{\beta+1}(z) < \beta + 1)$. So $\mathcal{L}_{\beta+2}(x) \in \mathbb{H}_{\beta+1}$ iff $I \subseteq H_\beta$. This happens iff $\forall c \in C_{\beta+1} \exists a \in A(a \leq_{\beta+1} c)$.

Signed Trees.

Definition:

- A *signed tree* is a well founded tree $T \subset \omega^{<\omega}$ together with a map $s_T \colon T \to \{+,-\}$.
- Given a signed tree T, let $T_i = \{\sigma : i^\frown\sigma \in T\}$, and let

$$\text{lin}(T) = F(s_T(\emptyset), \{\text{lin}(T_i) : i \in \omega\})$$

- Let $\text{lin}(\emptyset) = 1$.

The linear orderings of the form $\text{lin}(T)$ are exactly the h-indecomposables.

Example: $\text{lin}\left(\begin{array}{ccc} & + & \\ - & & + \end{array}\right) \sim \omega + \omega^* + \omega + \omega^* \ldots$

Definition: Let T and T' be signed trees.

- $h \colon T \to T'$ is a *homomorphism* if $\forall \sigma, \tau \in T$ $\sigma \subsetneq \tau \Rightarrow h(\sigma) \subsetneq h(\tau)$ and $s_{T'}(h(\sigma)) = s_T(\sigma)$.
- Let $T \preccurlyeq T'$ if such an h exists.

Observation: $T \preccurlyeq T' \iff \text{lin}(T) \preccurlyeq \text{lin}(T')$.

$$T \preccurlyeq T' \iff \text{lin}(T) \preccurlyeq \text{lin}(T')$$

($\Rightarrow$) Easy by induction on rk(T), using the definition of lin(T).

($\Leftarrow$) If $s_T(\emptyset) = +$ and $s_{T'}(\emptyset) = -$, then, observe that lin(T) has to be embeddable in lin(T'_i) for some i, because lin(T) is indecomposable to the right and lin(T') is an ω^* sum of orderings of the form lin(T'_i). Then by induction hypothesis $T \preccurlyeq T'$.

Suppose now that $s_T(\emptyset) = s_{T'}(\emptyset) = +$. Map the root of T into the root of T'. Since every lin(T_i) is embeddable into some lin(T'_j), by induction hypothesis we have that every subtree of T embeds in a subtree of T'. Hence $T \preccurlyeq T'$.

Given $*_0, *_1 \in \{+,-\}$, let $T_{*_0,*_1}$ be a tree of rank ω, whose root is signed $*_0$ and the other nodes are signed $*_1$.

Claim: For every signed tree T of rank $\geq \omega$ there is a $\sigma \in \{+,-\}^2$ such that $T_\sigma \preccurlyeq T$.

This would imply that for a linear ordering $\mathcal{L}$

$$\mathrm{rk}(\mathcal{L}) \geq \omega \iff \exists \sigma \in \{+,-\}^2 (\mathrm{lin}(T_\sigma) \preccurlyeq \mathcal{L}).$$

Proof of the claim: Assume, wlog, that T has rank ω. Let $*_0 = s_T(\emptyset)$. If for infinitely many $n < \omega$ there is some node of T with rank n and labeled $+$, let $*_1$ be $+$. Otherwise, let $*_1$ be $-$.

It is not hard to show that $T_{*_0,*_1} \preccurlyeq T$.

Consider $\alpha = \omega_0^\beta + \cdots + \omega^{\beta_{k-1}}$, where $b_0 \geq ... \geq b_{k-1}$.

Definition: Given $\sigma \in \{+,-\}^{k+1}$, let $T_{\alpha,\sigma}$ be the signed tree of rank α such that every node x, $s_{T_{\alpha,\sigma}}(x) = \sigma(i)$, where i is such that

$$\omega_0^\beta + \cdots + \omega^{\beta_{i-1}} \leq \mathrm{rk}(x) < \omega_0^\beta + \cdots + \omega^{\beta_i}.$$

Theorem: Let S be a signed tree. Then

$$\mathrm{rk}(S) \geq \alpha \iff \exists \sigma \in \{+,-\}^{k+1} \ (T_{\alpha,\sigma} \preccurlyeq S).$$

Definition: Let $\mathbb{C}_\alpha = \{\mathrm{lin}(T_{\alpha,\sigma}) : \sigma \in \{+,-\}^{k+1}\}$.

Note that $\mathbb{C}_{\beta+1}$ is the set of minimal elements of $\mathbb{H}_{\beta+1} \setminus \mathbb{H}_\beta$.

To find $\{x \in H_{\beta+1} : \mathcal{L}_{\beta+1}(x) \in \mathbb{C}_{\beta+1}\}$ is still not easy.

Recall that $\alpha = \omega_0^\beta + \cdots + \omega^{\beta_k}$, where $b_0 \geq ... \geq b_k$.

$$\mathrm{rk}(S) \geq \alpha \iff \exists \sigma \in \{+,-\}^{k+1} \ (T_{\alpha,\sigma} \preccurlyeq S)$$

($\Leftarrow$) Easy.

($\Rightarrow$) Need to prove the following lemma and then use induction on k.

Lemma: Let $\mathcal{P}$ be a well founded partial ordering. Let $\mathcal{P}_0, \mathcal{P}_1 \subseteq \mathcal{P}$ be such that $\mathcal{P}_0 \cup \mathcal{P}_1 = \mathcal{P}$. Then

$$\mathrm{rk}(\mathcal{P}) \leq \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1)$$

(where $\oplus$ is the ordinal natural sum).

If we also have that $\mathcal{P}_0$ and $\mathcal{P}_1$ are closed upward, then

$$\mathrm{rk}(\mathcal{P}) = \max(\mathrm{rk}(\mathcal{P}_0), \mathrm{rk}(\mathcal{P}_1)).$$

Other extensions

Theorem: Let X be either a Boolean Algebra, (reduced) p-group or a (compact) countable metric space.
Then, if X is hyperarithmetic,
it is equimorphic to a computable structure.

Question: Does this theorem hold for any other class of structures?

Theorem: For each $\alpha < \omega_1^{CK}$, the set $\mathcal{E}_\alpha$, of pairs of indices of equimorphic computable linear orderings of rank $< \alpha$, is hyperarithmetic.

Question: What is the complexity of $\mathcal{E}_\alpha$?