On Fraïssé's Conjecture and some equivalent statements

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Friedman and Simpson's program of Reverse Mathematics

Setting: Second order arithmetic.

Main Question: What axioms are necessary to prove the theorems of Mathematics?

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring axioms.

ACA_0: Arithmetic Comprehension + RCA_0.
$\iff$ "for every set X, X' exists".

ATR_0: Arithmetic Transfinite recursion + ACA_0.
$\iff$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".

Π^1_1-*CA_0*: Π^1_1-Comprehension + ACA_0.
$\iff$ "$\forall X$, the hyper-jump of X exists".

Π^1_2-*CA_0*: Π^1_2-Comprehension + ACA_0.

Notation and basic definitions.

In this talk all objects are countable.

$\mathcal{L}$ is always a linear ordering.

Definitions:

- If $\mathcal{L}_0$ embeds in $\mathcal{L}_1$ we write $\mathcal{L}_0 \preceq \mathcal{L}_1$.
- $\mathcal{L}_0$ and $\mathcal{L}_1$ are *equimorphic* if $\mathcal{L}_0 \preceq \mathcal{L}_1$ and $\mathcal{L}_1 \preceq \mathcal{L}_0$. We write $\mathcal{L}_0 \sim \mathcal{L}1.$
- We write $\mathcal{L}_0 \prec \mathcal{L}_1$ if $\mathcal{L}_0 \preceq \mathcal{L}_1$ but $\mathcal{L}_1 \not\sim \mathcal{L}_0$.
- $\mathcal{L}$ is *scattered* if $\mathbb{Q} \not\preceq \mathcal{L}$. Equivalently, if $\mathbb{Q} \not\sim \mathcal{L}$.
- $\mathcal{L}$ is *indecomposable* if $\mathcal{L} \preceq \mathcal{L}_0 + \mathcal{L}_1$ implies that either $\mathcal{L} \preceq \mathcal{L}_0$ or $\mathcal{L} \preceq \mathcal{L}_1$.

Fraïssé's Conjecture.

Definition: A *well-quasiordering (WQO)* is a quasiordering (i.e., a reflexive and transitive relation) without infinite descending sequences and infinite antichains.

Fraïssé's Conjecture is the statement:

FRA: The countable linear orderings form a WQO with respect to embeddability.

Richard Laver [1971] proved FRA using Nash-William's [1968] notion of Better-quasiordering.

In 1948 Fraïssé made the following conjectures:

(1) There is no sequence $L_0 \succ \mathcal{L}_1 \succ \mathcal{L}_2 \succ \cdots$.
✓ [Laver '71]

(2) If $\mathcal{L}$ is scattered, then there are countably many equimorphism types $\preceq \mathcal{L}$. ✓ [Laver '71]

(3) Given $\{\mathcal{L}_i : i \in \mathbb{N}\}$ and $\mathcal{L}$, if $\forall \mathcal{L}' (\forall i (\mathcal{L}_i \preceq \mathcal{L}') \to \mathcal{L} \preceq \mathcal{L}')$, then $\exists n$, $\mathcal{L} \preceq \mathcal{L}_n$. × [Jullien '69]

(4) If $\mathcal{L}$ is scattered and indecomposable, then $\mathcal{L} \sim \sum_{i \in \omega} \mathcal{L}_i$ or $\mathcal{L} \sim \sum_{i \in \omega^*} \mathcal{L}_i$, where the $\mathcal{L}_i$ are indecomposable and $\mathcal{L}_i \prec \mathcal{L}$. ✓ [Laver '73]

Theorem: (ACA_0)

- FRA, (1) and (4) are equivalent.
- FRA implies (2).
- ¬FRA implies that there exists a scattered $\mathcal{L}$ which has continuum many incomparable equimorphism types below it.

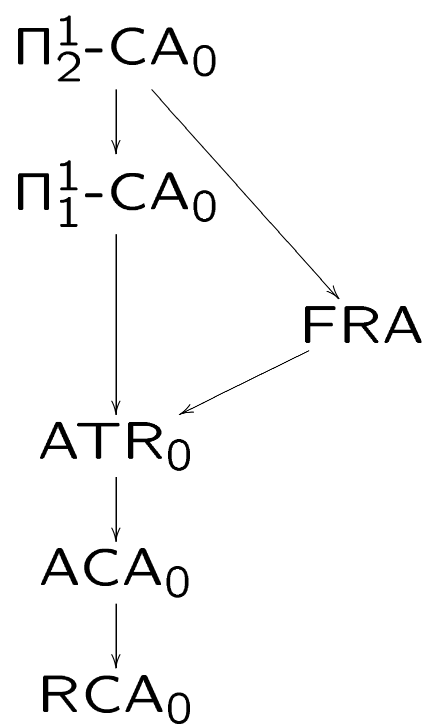
Laver's proof of FRA can be carried out in Π^1_2-CA_0.

Observation: Since FRA is a Π^1_2 statement, it cannot imply Π^1_1-CA_0.

Theorem:[Shore '93] The fact that the well-orderings form a WQO under $\preceq$ implies ATR_0 over RCA_0.

Corollary:[Shore '93] FRA implies ATR_0 over RCA_0.

Conjecture:[Clote '90] [Simpson '99][Marcone] FRA is equivalent to ATR_0 over RCA_0.



Recall: $\mathcal{L}$ is *scattered* if $\mathbb{Q} \not\preceq \mathcal{L}$

$\mathcal{L}$ is *indecomposable* if $\mathcal{L} \preceq \mathcal{L}_0 + \mathcal{L}_1$ implies that either $\mathcal{L} \preceq \mathcal{L}_0$ or $\mathcal{L} \preceq \mathcal{L}_1$.

A more useful statement

Theorem:[Laver '71] *Finite decomposition Thm.* Every scattered linear ordering can be written as a finite sum of indecomposables.

Theorem: The Finite decomposition Thm. is equivalent to FRA over RCA_0.

The Finite decomposition Thm. is very useful to prove theorems about equimorphisms types of linear orderings.

Definition:

- A *signed tree* is a well founded tree $T \subset \omega^{<\omega}$ together with a map $s_T \colon T \to \{+,-\}$.

- Given a signed tree T, let $T_i = \{\sigma : i^\frown\sigma \in T\}$
  If $s_T(\emptyset) = +$, let
  $$\text{lin}(T) = \text{lin}(T_0) + (\text{lin}(T_0) + \text{lin}(T_1)) + \ldots$$
  If $s_T(\emptyset) = -$, let
  $$\text{lin}(T) = \ldots + (\text{lin}(T_1) + \text{lin}(T_0)) + \text{lin}(T_0).$$

- Let $\text{lin}(\emptyset) = 1$.

- Linear orderings of the form $\text{lin}(T)$ are called *h-indecomposable*.

Example: $\text{lin}\left(\begin{smallmatrix} & + & \\ - & & + \end{smallmatrix} \right) \sim \omega + \omega^* + \omega + \omega^* \ldots$

Theorem: (follows from [Laver '71]) For scattered $\mathcal{L}$, $\mathcal{L}$ is indecomposable iff $\mathcal{L}$ is h-indecomposable.

Theorem: The theorem above and FRA are equivalent over RCA_0.

An apparently simpler statement.

Definition: Let T and T' be signed trees.

- $h\colon T \to T'$ is a *homomorphism* if $\forall \sigma, \tau \in T$
  $\sigma \subsetneq \tau \Rightarrow h(\sigma) \subsetneq h(\tau)$ and $s_{T'}(h(\sigma)) = s_T(\sigma)$.
- Let $T \preceq T'$ if such an h exists.

Observation: $T \preceq T' \iff \text{lin}(T) \preceq \text{lin}(T')$.

Let *WQO(ST)* be the statement:

The signed trees form a WQO under $\preceq$.

Observation: WQO(ST) follows from FRA

and the observation above.

Theorem: TFAE over RCA_0.

- FRA;
- WQO(ST);

Application of Signed trees.

Signed trees were very useful in proving the following theorem.

Theorem:[M. '04] Every hyperarithmetic linear ordering is equimorphic to a computable one.

Note that this theorem extends the following classical theorem.

Theorem:[Spector '55] Every hyperarithmetic well ordering is isomorphic to a computable one.

An apparently harder statement.

Laver's proof of FRA has three steps:

(1) The indecomposables form a BQO under $\preceq$.
(2) The indecomposables form a WQO under $\preceq$.
(3) FRA.

(1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) can be proved in ACA_0
(using Higman's theorem and results in [Simpson '88])

Better quasiorderings (BQO) are WQOs of a particularly well-behaved kind [Nash-Williams '68].

The set of computable WQOs is Π^1_1 complete.
The set of computable BQOs is Π^1_2 complete.
[Marcone '94]

Theorem: (1), (2) and (3) are equivalent over RCA_0.

Jullien's Theorem

Definition: $\mathcal{L}$ is *extendible* if
every partial ordering which doesn't embed $\mathcal{L}$
has a linearization which doesn't embed $\mathcal{L}$.

Examples: ω^* is extendible because every well-founded partial ordering has a well-ordered linearization.
ω, $\mathbb{Z}$, $\mathbb{Q}$ are also extendible. $\omega + \omega^*$ isn't.

In [2003] Downey, Hirschfeldt, Lempp and Solomon analyzed the proof-theoretic strength of the extendibility of ω, $\mathbb{Z}$, $\mathbb{Q}$.
Later Joe Miller [in preparation] and M. got more results on the extendibility of $\mathbb{Q}$.
Some question are still open.

Definition:

- $\mathcal{L}$ is *h-indecomposable to the right* if
  $$\mathcal{L} = \text{lin}(T) \ \& \ s_T(\emptyset) = +.$$
- $\mathcal{L}$ is *bad* if $\mathcal{L} = \mathbf{1} + \mathbf{1}$ or $\mathcal{L} \sim \mathcal{A} + \mathcal{B}$
  where $\mathcal{A}$ is h-indecomposable to the right
  and $\mathcal{B}$ is h-indecomposable to the left.
- If $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$, *$\mathcal{B}$ is an essential segment of $\mathcal{L}$* if whenever $\mathcal{L} \preceq \mathcal{A} + \mathcal{B}' + \mathcal{C}$, $\quad \mathcal{B} \preceq \mathcal{B}'$.

Theorem:[Jullien '69] $\mathcal{L}$ is extendible iff
it has no bad essential segments.

Question: What is the proof-theoretic strength of Jullien's Thm? [Downey, Remmel '00]

Theorem: Over $\mathsf{RCA}_0 + \Sigma^1_1$-induction,
FRA and Jullien's Thm. are equivalent.

Conclusions

FRA (which could be equivalent to ATR_0) is the weakest system where a decent theory of equimorphism types of linear orderings can be developed.