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# Embedding Jump Upper Semilattices into the Turing Degrees.

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Antonio Montalbán.  
Cornell University.

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Jump Upper Semilattices.

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**Definition:** A jump partial ordering (JPO) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, j \rangle$$

such that

- $\langle J, \leq_{\mathcal{J}} \rangle$  is a partial ordering,
- $x <_{\mathcal{J}} j(x)$  and
- $x \leq_{\mathcal{J}} y \implies j(x) \leq_{\mathcal{J}} j(y)$ .

**Definition:** A jump upper semilattice (JUSL) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, j \rangle$$

such that

- $\langle J, \leq_{\mathcal{J}}, \cup \rangle$  is an upper semilattice,
- $\langle J, \leq_{\mathcal{J}}, j \rangle$  is a JPO.

**Example:** The structure of Turing Degrees,

$$\mathcal{D} = \langle \mathbf{D}, \leq_T, \vee, ' \rangle,$$

is a JUSL.

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Known Results.

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**Question:** Which JUSLs can be embedded in  $\mathcal{D}$ ?

**Theorem**[Kleene-Post, 54]: Every finite upper semilattice can be embedded in  $\mathcal{D}$ .

**Theorem**[Sacks, 61]: Every partial ordering, of size  $\aleph_1$  with the countable predecessor property can be embedded in  $\mathcal{D}$ .

**Theorem**[Abraham-Shore, 86]: Every upper semilattice of size  $\aleph_1$ , with the countable predecessor property, can be embedded in  $\mathcal{D}$  as an initial segment.

**Theorem**[Hinman-Slaman, 91]: Every countable JPO,  $\langle P, \leq, j \rangle$ , can be embedded in  $\mathcal{D}$ .

**Theorem:** Every countable JUSL,  $\langle J, \leq_J, \vee, j \rangle$ , can be embedded into  $\mathcal{D}$ .

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**Corollary:**  $\exists - Th(\mathbf{D}, \leq_T, \vee, ')$  is decidable.

**Proof:** Essentially, for an  $\exists$ -formula  $\varphi$ ,

$\langle \mathbf{D}, \leq_T, \vee, ' \rangle \models \varphi \iff \varphi$  is not obviously false.

i.e. It does not contradict the axioms of JUSL.

□

**Theorem**[Shore-Slaman, to appear]:

$\forall \exists - Th(\mathbf{D}, \leq_T, \vee, ')$  is undecidable.

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Every countable JUSL,  $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \vee, j \rangle$ , is embeddable in  $\mathcal{D}$ .

Outline of the proof:

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**Definition:** A Jump Hierarchy (JH) over  $\mathcal{J}$  is a map  $H: J \rightarrow \omega^\omega$  s.t., for all  $x, y \in P$ ,

- $x < y \implies H(x)' \leq_T H(y)$ .
- $\bigoplus_{y \leq_{\mathcal{J}} x} H(y) \leq_T H(x)$ ;
- $\mathcal{P} \upharpoonright j(x) \leq_T H(x)$ , where  $\mathcal{P} \upharpoonright x$  is the restriction of  $\mathcal{P}$  to  $\{y \in P : y \leq_P x\}$ .

**Proposition:** Every countable JUSL which supports a JH can be embedded in  $\mathcal{D}$ .

**Proof:** Forcing Construction. □

**Proposition:** Every countable JUSL can be embedded into one which supports a JH.

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Jump upper semilattices with 0

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**Definition:** A jump upper semilattice with 0 (JUSL w/0) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathbf{j}, 0 \rangle$$

such that

- $\langle J, \leq_{\mathcal{J}}, \cup, \mathbf{j} \rangle$  is a JUSL, and
- 0 is the least element of  $\langle J, \leq_{\mathcal{J}} \rangle$ .

**Question:** Which JUSLs w/0 can be embedded into  $\mathcal{D}$ ?

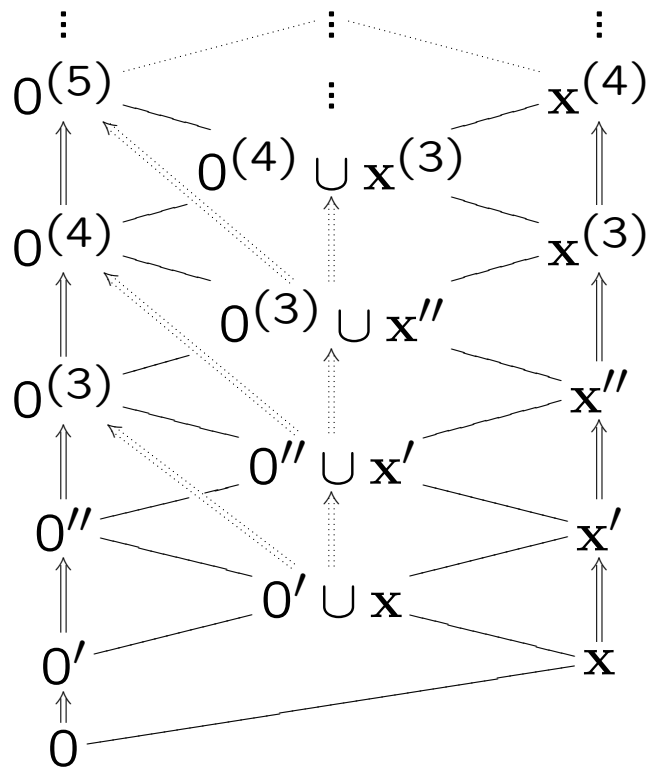
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A negative answer.

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**Theorem:** Not every countable JUSLs w/0 is embeddable in  $\mathcal{D}$ .

**Proof:** There are  $2^{\aleph_0}$  JUSLs w/0 generated by one element,  $x$ , such that  $x \leq 0''$ .



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A positive answer.

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**Definition:** We say that a JPO w/0,  $\mathcal{P} = \langle P, \leq_{\mathcal{P}}, j, 0 \rangle$ , is archimedean if

$$(\forall x \in P)(\exists n) \quad x \leq_{\mathcal{P}} j^n(0)$$

**Theorem:**

if every archimedean finitely generated JPO w/0 is embeddable in  $\mathcal{D}$ ,  
then every finitely generated JPO w/0 is embeddable in  $\mathcal{D}$ .

**Theorem**([Hinman-Slaman 91], [Hinman 99])  
Every archimedean JPO w/0 and with one generator is embeddable in  $\mathcal{D}$ .

**Theorem:** Every JPO w/0 and with one generator is embeddable in  $\mathcal{D}$ .

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Uncountable JUSLs.

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Let  $\kappa$  be a cardinal,  $\aleph_0 < \kappa \leq 2^{\aleph_0}$ .

**Question:** Is every JUSL,  $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \vee, \mathfrak{j} \rangle$ , with the c.p.p. and size  $\kappa$  embeddable in  $\mathcal{D}$ ?

**Proposition:**

If  $\kappa = 2^{\aleph_0}$ , then the answer is **NO**.

**Proposition:**

If  $\text{MA}(\kappa)$  holds, the answer is **YES**.

**Cor:** For  $\kappa = \aleph_1$ , it is independent of ZFC.

**Proof:** It is FALSE under CH,

but TRUE under  $\text{MA}(\aleph_1)$ . □

**Proposition:** There is a JPO of size  $2^{\aleph_0}$ , with the c.p.p., which cannot be embedded in  $\mathcal{D}$ .

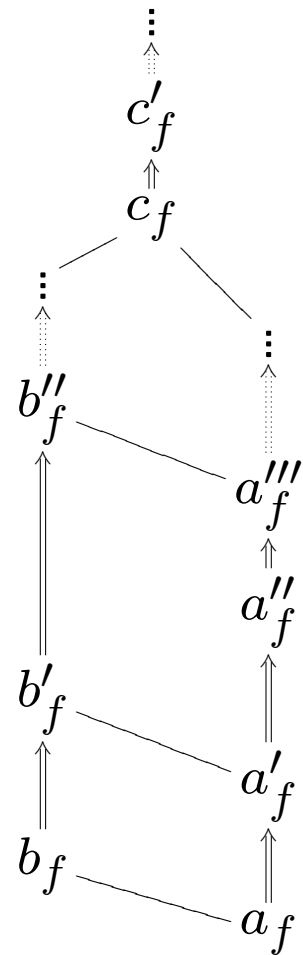
**Proof:** Given an increasing function  $f: \omega \rightarrow \omega$ , define  $\mathcal{P}_f$  as in the picture.

Set  $b_f^{(j)} \geq a_f^{(i)}$  iff  $f(j) \geq i$ .

Note:

If  $\psi: \mathcal{P}_f \rightarrow \mathcal{D}$  is an embedding, then  $\psi(c_f^{(6)}) \geq_T f$ .

Let  $\mathcal{P} = \langle d \rangle \oplus \bigoplus_{f: \omega \rightarrow \omega} \mathcal{P}_f$ .



Suppose that  $\psi: \mathcal{P} \rightarrow \mathcal{D}$  is an embedding.

Consider  $f \in \psi(d)$ .

Then  $\psi(c_f^{(6)}) \geq_T \psi(d)$  but  $d \mid c_f^{(6)}$ . Contradiction.