
Embedding Jump Upper Semilattices into the Turing Degrees.

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Jump Upper Semilattices.

Definition: A jump partial ordering (JPO) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, j \rangle$$

such that

- $\langle J, \leq_{\mathcal{J}} \rangle$ is a partial ordering,
- $x <_{\mathcal{J}} j(x)$ and
- $x \leq_{\mathcal{J}} y \implies j(x) \leq_{\mathcal{J}} j(y)$.

Definition: A jump upper semilattice (JUSL) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, j \rangle$$

such that

- $\langle J, \leq_{\mathcal{J}}, \cup \rangle$ is an upper semilattice,
- $\langle J, \leq_{\mathcal{J}}, j \rangle$ is a JPO.

Example: The structure of Turing Degrees,

$$\mathcal{D} = \langle \mathbf{D}, \leq_T, \vee, ' \rangle,$$

is a JUSL.

Known Results.

Question: Which JUSLs can be embedded in \mathcal{D} ?

Theorem[Kleene-Post, 54]: Every finite upper semilattice can be embedded in \mathcal{D} .

Theorem[Sacks, 61]: Every partial ordering, of size \aleph_1 with the countable predecessor property can be embedded in \mathcal{D} .

Theorem[Abraham-Shore, 86]: Every upper semilattice of size \aleph_1 , with the countable predecessor property, can be embedded in \mathcal{D} as an initial segment.

Theorem[Hinman-Slaman, 91]: Every countable JPO, $\langle P, \leq, j \rangle$, can be embedded in \mathcal{D} .

Theorem: Every countable JUSL, $\langle J, \leq_J, \vee, j \rangle$, can be embedded into \mathcal{D} .

Corollary: $\exists - Th(\mathbf{D}, \leq_T, \vee, ')$ is decidable.

Proof: Essentially, for an \exists -formula φ ,

$\langle \mathbf{D}, \leq_T, \vee, ' \rangle \models \varphi \iff \varphi$ is not obviously false.

i.e. It does not contradict the axioms of JUSL.

□

Theorem[Shore-Slaman, to appear]:

$\forall \exists - Th(\mathbf{D}, \leq_T, \vee, ')$ is undecidable.

Every countable JUSL, $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \vee, j \rangle$, is embeddable in \mathcal{D} .

Outline of the proof:

Definition: A Jump Hierarchy (JH) over \mathcal{J} is a map $H: J \rightarrow \omega^\omega$ s.t., for all $x, y \in P$,

- $x < y \implies H(x)' \leq_T H(y)$.
- $\bigoplus_{y \leq_{\mathcal{J}} x} H(y) \leq_T H(x)$;
- $\mathcal{P} \upharpoonright j(x) \leq_T H(x)$, where $\mathcal{P} \upharpoonright x$ is the restriction of \mathcal{P} to $\{y \in P : y \leq_P x\}$.

Proposition: Every countable JUSL which supports a JH can be embedded in \mathcal{D} .

Proof: Forcing Construction. □

Proposition: Every countable JUSL can be embedded into one which supports a JH.

Jump upper semilattices with 0

Definition: A jump upper semilattice with 0 (JUSL w/0) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathbf{j}, 0 \rangle$$

such that

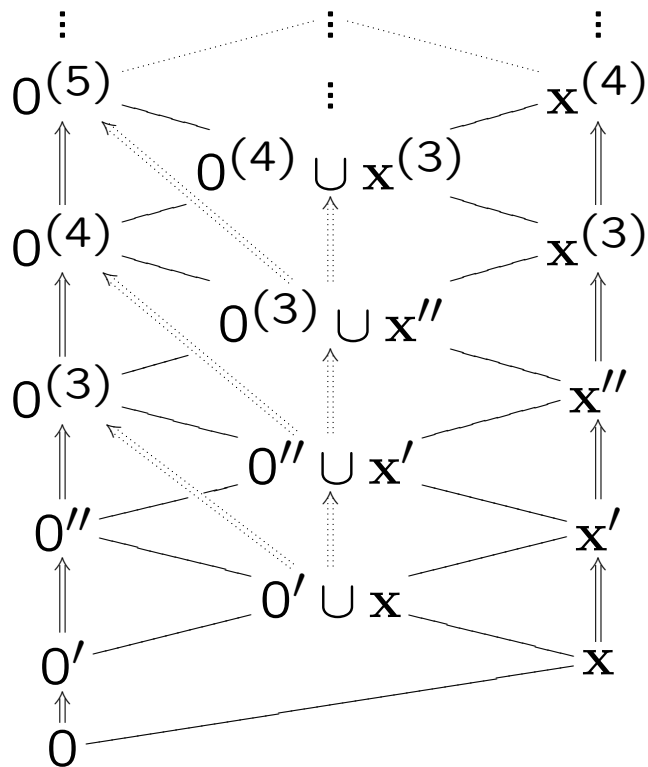
- $\langle J, \leq_{\mathcal{J}}, \cup, \mathbf{j} \rangle$ is a JUSL, and
- 0 is the least element of $\langle J, \leq_{\mathcal{J}} \rangle$.

Question: Which JUSLs w/0 can be embedded into \mathcal{D} ?

A negative answer.

Theorem: Not every countable JUSLs w/0 is embeddable in \mathcal{D} .

Proof: There are 2^{\aleph_0} JUSLs w/0 generated by one element, x , such that $x \leq 0''$.



A positive answer.

Definition: We say that a JPO w/0, $\mathcal{P} = \langle P, \leq_{\mathcal{P}}, j, 0 \rangle$, is archimedean if

$$(\forall x \in P)(\exists n) \quad x \leq_{\mathcal{P}} j^n(0)$$

Theorem:

if every archimedean finitely generated JPO w/0 is embeddable in \mathcal{D} ,
then every finitely generated JPO w/0 is embeddable in \mathcal{D} .

Theorem([Hinman-Slaman 91], [Hinman 99])
Every archimedean JPO w/0 and with one generator is embeddable in \mathcal{D} .

Theorem: Every JPO w/0 and with one generator is embeddable in \mathcal{D} .

Uncountable JUSLs.

Let κ be a cardinal, $\aleph_0 < \kappa \leq 2^{\aleph_0}$.

Question: Is every JUSL, $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \vee, \mathfrak{j} \rangle$, with the c.p.p. and size κ embeddable in \mathcal{D} ?

Proposition:

If $\kappa = 2^{\aleph_0}$, then the answer is **NO**.

Proposition:

If $\text{MA}(\kappa)$ holds, the answer is **YES**.

Cor: For $\kappa = \aleph_1$, it is independent of ZFC.

Proof: It is FALSE under CH,

but TRUE under $\text{MA}(\aleph_1)$. □

Proposition: There is a JPO of size 2^{\aleph_0} , with the c.p.p., which cannot be embedded in \mathcal{D} .

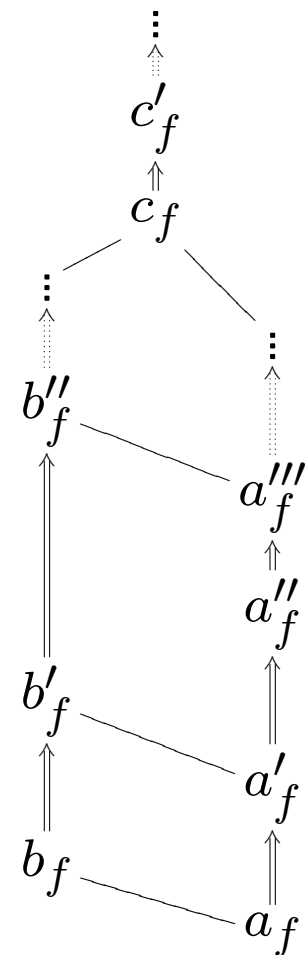
Proof: Given an increasing function $f: \omega \rightarrow \omega$, define \mathcal{P}_f as in the picture.

Set $b_f^{(j)} \geq a_f^{(i)}$ iff $f(j) \geq i$.

Note:

If $\psi: \mathcal{P}_f \rightarrow \mathcal{D}$ is an embedding, then $\psi(c_f^{(6)}) \geq_T f$.

Let $\mathcal{P} = \langle d \rangle \oplus \bigoplus_{f: \omega \nearrow \omega} \mathcal{P}_f$.



Suppose that $\psi: \mathcal{P} \rightarrow \mathcal{D}$ is an embedding.

Consider $f \in \psi(d)$.

Then $\psi(c_f^{(6)}) \geq_T \psi(d)$ but $d \mid c_f^{(6)}$. Contradiction.