

Theories of Hyperarithmetical Analysis.

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Reverse Mathematics

Setting: Second order arithmetic.

Main Question: What axioms are necessary to prove the theorems of Mathematics?

Axiom systems:

RCA_0 : Recursive Comprehension + Σ_1^0 -induction + Semiring ax.

WKL_0 : Weak König's lemma + RCA_0

ACA_0 : Arithmetic Comprehension + RCA_0

\Leftrightarrow "for every set X , X' exists".

ATR_0 : Arithmetic Transfinite recursion + ACA_0 .

\Leftrightarrow " $\forall X, \forall$ ordinal $\alpha, X^{(\alpha)}$ exists".

Π_1^1 - CA_0 : Π_1^1 -Comprehension + ACA_0 .

Models

A model of (the language of) second order arithmetic is a tuple

$$\langle X, \mathcal{M}, +_X, \times_X, 0_X, 1_X, \leq_X \rangle,$$

where \mathcal{M} is a set of subsets of X and $\langle X, +_X, \times_X, 0_X, 1_X, \leq_X \rangle$ is a structure in language of 1st order arithmetic.

A model of second order arithmetic is an ω -model if

$$\langle X, +_X, \times_X, 0_X, 1_X, \leq_X \rangle = \langle \omega, +, \times, 0, 1, \leq \rangle.$$

ω -models are determined by their second order parts, which are subsets of $\mathcal{P}(\omega)$.

We will identify subsets $\mathcal{M} \subseteq \mathcal{P}(\omega)$ with ω -models.

The class of ω -models of a theory

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω -models of $\text{RCA}_0 \Leftrightarrow$
 \mathcal{M} is closed under Turing reduction and \oplus

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω -models of $\text{ACA}_0 \Leftrightarrow$
 \mathcal{M} is closed under Arithmetic reduction and \oplus

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω -models of $\text{ATR}_0 \Rightarrow$
 \mathcal{M} is closed under Hyperarithmetical reduction and \oplus

The class of *HYP*, of hyperarithmetical sets, is not a model of ATR_0 :
 There is a linear ordering \mathcal{L} which isn't an ordinal but looks like one in
HYP (the Harrison l.o.), so,

$\text{HYP} \models \mathcal{L}$ is an ordinal but $0^{(\mathcal{L})}$ does not exist.

Hyperarithmetical sets

Proposition: [Suslin-Kleene, Ash]

For a set $X \subseteq \omega$, the following are equivalent:

- X is $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$.
- X is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{CK}$.
(ω_1^{CK} is the least non-computable ordinal and $0^{(\alpha)}$ is the α th Turing jump of 0.)
- $X = \{x : \varphi(x)\}$, where φ is a computable infinitary formula.
(*Computable infinitary formulas* are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be **hyperarithmetical**.

In particular, every computable, Δ_2^0 , and arithmetic set is hyperarithmetical.

Hyperarithmetical reducibility

Definition: X is **hyperarithmetical in** Y ($X \leq_H Y$) if $X \in \Delta_1^1(Y)$,
or equivalently, if $X \leq_T Y^{(\alpha)}$ for some $\alpha < \omega_1^Y$.

Let **HYP** be the class of hyperarithmetical sets.

Let **HYP**(Y) be the class of set hyperarithmetical in Y .

We say that an ω -model is **hyperarithmetically closed** if it closed downwards under \leq_H and is closed under \oplus .

The class of ω -models of a theory

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω -models of $\text{RCA}_0 \Leftrightarrow$
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Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω -models of $\text{ACA}_0 \Leftrightarrow$
 \mathcal{M} is closed under Arithmetic reduction and \oplus

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω -models of $\text{ATR}_0 \Rightarrow$
 \mathcal{M} is hyperarithmetically closed.

Question: Are there theories whose ω -models are the hyperarithmetically closed ones?

Theories of Hyperarithmetical analysis.

Definition

We say that a theory T is a **theory of hyperarithmetical analysis** if for every set Y , $HYP(Y)$ is the least ω -model of T containing Y , and every ω -model of T is closed under \oplus .

Note that T is a theory of hyperarithmetical analysis \Leftrightarrow

- every ω -model of T is hyperarithmetically closed, and
- for every Y , $HYP(Y) \models T$.

Choice and Comprehension schemes

Theorem: [Kleene 59, Kreisel 62, Friedman 67, Harrison 68, Van Wesep 77, Steel 78, Simpson 99]

The following are theories of hyperarithmetical analysis and each one is strictly weaker than the next one:

weak- Σ_1^1 -AC₀ (weak Σ_1^1 -choice):

$$\forall n \exists! X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})), \quad \text{where } \varphi \text{ is arithmetic.}$$

Δ_1^1 -CA₀ (Δ_1^1 -comprehension) :

$$\forall n(\varphi(n) \Leftrightarrow \neg\psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n)), \quad \text{where } \varphi \text{ and } \psi \text{ are } \Sigma_1^1.$$

Σ_1^1 -AC₀ (Σ_1^1 -choice):

$$\forall n \exists X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})), \quad \text{where } \varphi \text{ is } \Sigma_1^1.$$

Σ_1^1 -DC₀ (Σ_1^1 -dependent choice):

$$\forall Y \exists Z(\varphi(Y, Z)) \Rightarrow \exists X \forall n(\varphi(X^{[n]}, X^{[n+1]})), \quad \text{where } \varphi \text{ is } \Sigma_1^1.$$

The bad news

There is not theory T whose ω -models are exactly the hyperarithmetically closed ones.

Theorem: [Van Wesep 77] For every theory T whose ω -models are all hyperarithmetically closed, there is another theory T' whose ω -models are also all hyperarithmetically closed and which has more ω -models than T .

Statements of hyperarithmetical analysis

Definition

S is a **sentence of hyperarithmetical analysis** if $\text{RCA}_0 + S$ is a theory of hyperarithmetical analysis.

Friedman [1975] introduced two statements, **Arithmetic Bolzano-Weierstrass (ABW)** and, **Sequential Limit Systems (SL)**, and he mentioned they were related to hyperarithmetical analysis. Both statements use the concept of arithmetic set of reals, which is not used outside logic.

Van Wesep [1977] introduced **Game-AC** and proved it is equivalent to $\Sigma_1^1\text{-AC}_0$.

It essentially says that if we have a sequence of open games such that player II has a winning strategy in each of them, then there exists a sequence of strategies for all of them.

The indecomposability statement

Let \mathcal{A} , \mathcal{B} and \mathcal{L} be linear orderings

- If \mathcal{A} embeds into \mathcal{B} , we write $\mathcal{A} \preceq \mathcal{B}$.
- \mathcal{L} is **scattered** if $\mathbb{Q} \not\preceq \mathcal{L}$.
- \mathcal{L} is **indecomposable** if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$,
either $\mathcal{L} \preceq \mathcal{A}$ or $\mathcal{L} \preceq \mathcal{B}$.
- \mathcal{L} is **indecomposable to the right** if for every non-trivial cut
 $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{B}$.
- \mathcal{L} is **indecomposable to the left** if for every non-trivial cut
 $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{A}$.

Theorem[Jullien '69] **INDEC**: Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.

$\Delta_1^1\text{-CA}_0 \vdash \text{INDEC}$

Proof: ($\Delta_1^1\text{-CA}_0$) Let \mathcal{A} be scattered and indecomposable.

We want to show that \mathcal{A} is indecomposable either to the left or to the right

- For every $x \in \mathcal{A}$, either $\mathcal{A} \preceq \mathcal{A}_{(>a)}$ or $\mathcal{A} \preceq \mathcal{A}_{(\leq a)}$.
- For no x we could have both $\mathcal{A} \preceq \mathcal{A}_{(>a)}$ and $\mathcal{A} \preceq \mathcal{A}_{(\leq a)}$.
Otherwise $\mathcal{A} \succcurlyeq \mathcal{A} + \mathcal{A} \succcurlyeq \mathcal{A} + \mathcal{A} + \mathcal{A} \succcurlyeq \mathcal{A} + 1 + \mathcal{A}$.
So, $\mathcal{A} \succcurlyeq \mathcal{A} + 1 + \mathcal{A} \succcurlyeq (\mathcal{A} + 1 + \mathcal{A}) + 1 + (\mathcal{A} + 1 + \mathcal{A}) \dots$
Following this procedure we could build an embedding $\mathbb{Q} \preceq \mathcal{A}$.
- Using $\Delta_1^1\text{-CA}_0$ define
 $L = \{x \in \mathcal{A} : \mathcal{A} \preceq \mathcal{A}_{(>x)}\}$ and $R = \{x \in \mathcal{A} : \mathcal{A} \preceq \mathcal{A}_{(\leq x)}\}$.
- If $L = \emptyset$, then \mathcal{A} is indecomposable to the right.
If $R = \emptyset$, then \mathcal{A} is indecomposable to the left.
- Suppose this is not the case and assume $\mathcal{A} \preceq L$. Then

An equivalent formulation

\mathcal{A} is **weakly indecomposable** if for every $a \in \mathcal{A}$, either $\mathcal{A} \preceq \mathcal{A}_{(>a)}$ or $\mathcal{A} \preceq \mathcal{A}_{(\leq a)}$.

Looking at the proof of $\Delta_1^1\text{-CA}_0 \vdash \text{INDEC}$ carefully, we can observe the following:

Theorem

The following are equivalent over RCA_0 :

- 1 *INDEC*
- 2 *If \mathcal{A} is a scattered, weakly indecomposable linear ordering, then there exists a cut $\langle L, R \rangle$ of \mathcal{A} such that $L = \{a \in \mathcal{A} : \mathcal{A} \preceq \mathcal{A}_{(>a)}\}$ and $R = \{a \in \mathcal{A} : \mathcal{A} \preceq \mathcal{A}_{(\leq a)}\}$*

strength of INDEC

Theorem

INDEC implies ACA_0 over RCA_0 .

Proof:

- Construct a computable linear ordering \mathcal{A} such that in RCA_0 ,
 - \mathcal{A} is infinite,
 - $\forall x \in \mathcal{A}$, either $\mathcal{A}_{(<x)}$ or $\mathcal{A}_{(>x)}$ is finite,
 - any infinite descending sequence in \mathcal{A} computes $0'$.

For instance, given $s > t \in \mathbb{N}$, let $s \leq_k t$ if t looks like a true for the enumeration of $0'$ at time s .

Let $\mathcal{A} = \langle \mathbb{N}, \leq_k \rangle$.

Note that \mathcal{A} is isomorphic $\omega + \omega^*$, and that \mathcal{A} is weakly indecomposable. But RCA_0 cannot prove this.

- For each $x \in \mathcal{A}$, let \mathcal{B}_x be such that

$$\mathcal{B}_x \cong \begin{cases} \omega^x & \text{if } \mathcal{A}_{(<x)} \text{ is finite} \\ (\omega^x)^* & \text{if } \mathcal{A}_{(>x)} \text{ is finite} \end{cases}$$

ω -models of INDEC

Theorem

INDEC is a statement of hyperarithmetical analysis.

Let $\mathcal{M} \models \text{INDEC}$.

We want to show that \mathcal{M} is hyperarithmetically closed.

We do it by proving that for every $X \in \mathcal{M}$,

if $\alpha \in \mathcal{M}$ is an ordinal and $\forall \beta < \alpha (X^{(\beta)} \in \mathcal{M})$ then $X^{(\alpha)} \in \mathcal{M}$.

By transfinite induction, this implies that if $Y \leq_H X$, then $Y \in \mathcal{M}$.

The successor steps follow from ACA_0 . For the limit steps we construct a linear ordering using the recursion theorem and results that Ash and Knight proved using the Ash's method of α -systems.

The Jump Iteration statement

Let J_I be the statement that says:

$\forall X \forall \alpha (\alpha \text{ an ordinal} \ \& \ \forall \beta (0^{(\beta)} \text{ exists}) \Rightarrow 0^{(\alpha)} \text{ exists})$

Conjecture: (RCA_0) INDEC implies J_I .

Theorem

J_I is a statement of hyperarithmetical analysis.

Incomparable statements

Observation: INDEC is Π_2^1 -conservative over ACA_0
(because $\Sigma_1^1-AC_0$ is Π_2^1 -conservative over ACA_0 [Barwise, Schlipf 75]).

Therefore, for instance, INDEC is incomparable with Ramsey's theorem.

Also, INDEC is incomparable with ACA_0^+ .

(ACA_0^+ essentially says that for every X , $X^{(\omega)}$ exists.)

Hence, INDEC is incomparable with the statement that says that elementary equivalence invariants for boolean algebra exists, which is equivalent to ACA_0^+ [Shore 04].

Finitely terminating games

GAME STATEMENTS.

Finitely terminating games

To each well founded tree $T \subseteq \omega^{<\omega}$, we associate a game $G(T)$ which is played as follows. Player I starts by playing a number $a_0 \in \mathcal{N}$ such that $\langle a_0 \rangle \in T$. Then player II plays $a_1 \in \mathcal{N}$ such that $\langle a_0, a_1 \rangle \in T$, and then player I plays $a_2 \in \mathcal{N}$ such that $\langle a_0, a_1, a_2 \rangle \in T$. They continue like this until they get stuck. The first one who cannot play *loses*.

We will refer to games of the form $G(T)$, for T well-founded, as **finitely terminating games**

Observation Finitely terminating games are in 1-1 correspondence with **clopen games**.

Finitely terminating games

- Let $T_I = \{\sigma \in T : |\sigma| \text{ is even}\}$, $T_{II} = \{\sigma \in T : |\sigma| \text{ is odd}\}$.
A **strategy for I in $G(T)$** is a function $s: T_I \rightarrow \mathcal{N}$.
A strategy s for I is a **winning strategy** if whenever I plays following the s , he wins.
A game $G(T)$ is **determined** if there is a winning strategy for one of the two players.
- We say that a game is **completely determined** if there is a map $d: T \rightarrow \{W, L\}$ such that for every $\sigma \in T$,
 - $d(s) = W \Leftrightarrow I$ has a winning strategy in $G(T_\sigma)$, and
 - $d(s) = L \Leftrightarrow II$ has a winning strategy in $G(T_\sigma)$.

Note that completely determined games are determined.

Known results

Theorem [Steel 1976] The following are equivalent over RCA_0 .

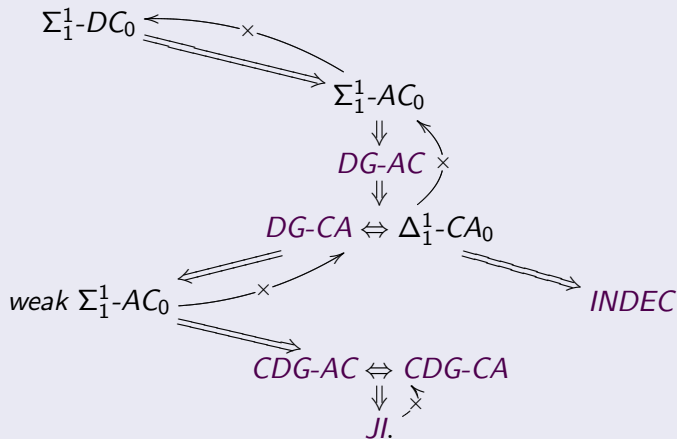
- ATR_0 ;
- Every finitely terminating game is determined;
- Every finitely terminating game is completely determined.

New statements

- **CDG-CA:** Given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees, there exists a set X such that
$$\forall n (n \in X \text{ iff } I \text{ has a winning strategy in } G(T_n)).$$
- **CDG-AC:** Given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees, there exists a sequence $\{d_n : n \in \mathcal{N}\}$ where for each n , $d_n : T \rightarrow \{W, L\}$ is a winning function for $G(T_n)$.
- **DG-CA:** Given a sequence $\{T_n : n \in \mathcal{N}\}$ of determined trees, there exists a set X such that
$$\forall n (n \in X \text{ iff } I \text{ has a winning strategy in } G(T_n)).$$
- **DG-AC:** Given a sequence $\{T_n : n \in \mathcal{N}\}$ of determined trees, there exists a sequence $\{s_n : n \in \mathcal{N}\}$ of winning strategies for the T_n 's.

Implications between statements

Theorem



over RCA_0 .

Jl doesn't imply CDG-CA

To prove this non-implication we construct an ω -model of Jl using Steel's method of forcing with tagged trees [Steel 76].

Steel used his method to prove that $\Delta_1^1\text{-CA}_0 \not\Rightarrow \Sigma_1^1\text{-AC}_0$.

Maybe, similar arguments can be used to prove other non-implications between statements of hyperarithmetical analysis.

DG-CA implies Δ_1^1 -CA₀

Let φ and ψ be Σ_1^1 formulas such that $\forall n(\varphi(n) \Leftrightarrow \neg\psi(n))$.

There exists sequences of trees $\{S_n : n \in \mathbb{N}\}$ and $\{T_n : n \in \mathbb{N}\}$ such that for every n ,

$$\begin{aligned}\varphi(n) &\Leftrightarrow S_n \text{ is not well founded,} \\ \psi(n) &\Leftrightarrow T_n \text{ is not well founded.}\end{aligned}$$

For each n consider the game G_n where I plays nodes in S_n and II plays nodes in T_n . The first one who cannot move loses.

Since for every n , either S_n or T_n is well founded, this is a finitely terminating game. Moreover, each G_n is determined and I wins the game iff T_n is well founded. Therefore, I wins G_n iff $\varphi(n)$.

Then, by DG-CA, the set $\{n : \varphi(n)\}$ exists.

CDG-AC implies JI.

It is not hard to show that CDG-AC implies ACA_0 .

Let α be a limit ordinal and suppose that $\forall \beta < \alpha$, $0^{(\beta)}$ exists. By recursive transfinite induction, we construct a family of finitely terminating games $\{G_{\beta,n} : \beta < \alpha, n \in \mathcal{N}\}$, such that

$$n \in 0^{(\beta)} \Leftrightarrow I \text{ has a winning strategy in } G_{\beta,n}.$$

Moreover, we claim that, using our assumption that for every $\beta < \alpha$, $0^{(\beta)}$ exists, we can prove that each game $G_{\beta,n}$ is completely determined:

By CDG-CA, there exists a set X such that

$$\langle \beta, n \rangle \in X \Leftrightarrow I \text{ wins } G_{\beta,n}.$$

This X is $0^{(\alpha)}$.

Work in Progress

Π_1^1 -sep: Given Π_1^1 formulas $\varphi(n)$ and $\psi(n)$, $\forall n(\neg\varphi(n) \vee \neg\psi(n)) \Rightarrow \exists X \forall n (\varphi(n) \Rightarrow n \in X \ \& \ \psi(n) \Rightarrow n \notin X)$

Observation[Tanaka]

$\Sigma_1^1\text{-AC}_0 \Rightarrow \Pi_1^1\text{-sep} \Rightarrow \Delta_1^1\text{-CA}_0$

Theorem (Work in progress)

$\Pi_1^1\text{-sep}$ is strictly in between $\Sigma_1^1\text{-AC}_0$ and $\Rightarrow \Delta_1^1\text{-CA}_0$