Name:____________________ ________________________

1	2	3	4	5	*Total*

Math 256, Section 31, Spring 2005

First hourly exam

All problems are worth 10 points, but some are harder than the others.

Justify your answers carefully. You are allowed to use homework problems and facts from the book or from the lectures in your proofs.

Good luck!

1. Set

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of A and the corresponding eigenspaces.

(b) Find the Jordan form of A. Justify.

2. A is a 3×3 matrix over $\mathbb{Q}$ that has exactly two eigenvalues: 2 and 3.

(a) What is the characteristic polynomial of A (find all answers for full credit)?

(b) Find a non-zero polynomial $p(x)$ of degree 4 such that $p(A) = 0$. Your polynomial has to work for all such matrices. Explain.

3. A is an $n \times n$ matrix over $\mathbb{C}$. Prove that the following two conditions are equivalent:

(1) For every eigenvalue λ of A, we have

$$\dim\{v \in \mathbb{C}^n : vA = \lambda v\} = 1;$$

(2) The characteristic polynomial of A is the minimal polynomial of A.

4. Let $V = \{a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 : a_i \in \mathbb{R}\}$ be the space of polynomials (with real coefficients) of degree 4 or less. Define a map $\phi : V \to V$ by

$$(\phi(p))(x) = p(x+1);$$

for instance, $\phi(x^2 - 3x + 1) = (x+1)^2 - 3(x+1) + 1 = x^2 - x + 1$.

(a) Find the minimal polynomial of ϕ.

(b) Does the Spectral Theorem apply to ϕ? If no, explain why; if yes, explain what it implies for ϕ.

5. A vector space V (over some field F) has dimension 5. Suppose $\phi : V \to V$ is a linear map, and $W_1, W_2 \subset V$ are stable subspaces: $\phi(W_1) \subseteq W_1$, $\phi(W_2) \subseteq W_2$. The Jordan forms of the restrictions $\phi|_{W_1}$ and $\phi|_{W_2}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad .$$

Find the Jordan form of ϕ. Justify your answer.