Name:____________________ ________________________

1	2	3	4	5	*Total*

Math 256, Section 31, Spring 2005

Second hourly exam

All problems are worth 10 points, but some are harder than the others.

Justify your answers carefully. You are allowed to use homework problems and facts from the book or from the lectures in your proofs.

Good luck!

1. Set $E = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$ and $F = \mathbb{Q}(\sqrt{-3})$. Clearly, E is a field extension of F.

(a) Find a basis for E viewed as a vector space over F.

(b) Find α such that $E = F(\alpha)$.

2. (a) Suppose $a \in \overline{\mathbb{Z}_3}$ satisfies $a^2+1=0$ (as usual, $\overline{\mathbb{Z}_3}$ is an algebraic closure of $\mathbb{Z}_3$). Find $\deg(a, \mathbb{Z}_3)$.

(b) Find $\deg(a, GF(3^3))$, where $GF(3^3) \subset \overline{\mathbb{Z}_3}$ is the finite field with 3^3 elements. Justify your answer.

3. Suppose F is a field, and suppose $E_1 \supset F$, $E_2 \supset F$ are two field extensions whose degrees $[E_1 : F]$, $[E_2 : F]$ are relatively prime. Prove that $E_1 \cap E_2 = F$.

4. For a prime number p, define the polynomial $Q(x)$ over the field $\mathbb{Z}_p$ by

$$Q(x) = 1 + x^{p-1} + x^{2(p-1)} + x^{3(p-1)} + \cdots + x^{p(p-1)} \in \mathbb{Z}_p[x].$$

Show that any irreducible factor of $Q(x)$ has degree 2. (Hint: $Q(x) = \frac{x^{p^2}-x}{x^p-x}$.)

5. Let $\overline{F}$ be an algebraic closure of a field F. For a polynomial $p(x) \in F[x]$, denote by $\alpha_1, \ldots, \alpha_n \in \overline{F}$ its roots in $\overline{F}$; here $n = \deg(p)$. Set $E = F(\alpha_1, \ldots, \alpha_n)$. Prove that

$$[E : F] \leq n! = n \cdot (n-1) \cdots 2 \cdot 1.$$

(For partial credit, prove the weaker estimate $[E : F] \leq n^n$.)