PS 2 Solutions
Math 256 Section 31

III.A.8) The only real eigenvalue is 3. It's eigenspace is Span$\{(1, 1, 0)\}$.

III.B.9) i) $x^2 - 4x + 4 = (x - 2)^2$
ii) $x^3 - 3x^2 + 4 = (x - 2)^2(x + 1)$

III.B.10) Note that for any polynomial $p(x) = p_n x^n + \ldots + p_1 x + p_0$, we have

$$\begin{aligned} p(A) &= p_n \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}^n + \ldots + p_1 \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} + p_0 \\ &= \begin{pmatrix} p_n B^n & 0 \\ 0 & p_n C^n \end{pmatrix} + \ldots + \begin{pmatrix} p_1 B & 0 \\ 0 & p_1 C \end{pmatrix} + \begin{pmatrix} p_0 I & 0 \\ 0 & p_0 I \end{pmatrix} \\ &= \begin{pmatrix} p(B) & 0 \\ 0 & p(C) \end{pmatrix} \end{aligned}$$

In particular, if m is the minimal polynomial for A, then we have $m(A) = 0 \Rightarrow m(B) = m(C) = 0$. Thus the minimal polynomials of B and C divide m and so their least common multiple divides m as well. Conversely, if m is the least common multiple of the minimal polynomials for B and C, then we have $m(B) = m(C) = 0 \Rightarrow m(A) = 0$. In general, $m_A = \text{l.c.m.}(m_{A_1}, \ldots, m_{A_m})$.

III.D.1) i) $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
The minimal polynomials are $(x - 1), (x - 2), (x + 1)$, respectively.
ii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, with minimal polynomials $(x - 2), (x + 2), (x - 3)$.

III.D.3) The characteristic and minimal polynomials are both $(x - \lambda)^n$. The only eigenvector of A is $(0, \ldots, 0, c)$. λ has algebraic multiplicity n.

III.D.4) (This is the function version of III.B.10) Let $p = \text{l.c.m.}(m_1, m_2)$. If m is the minimal polynomial for ϕ then $m(\phi) = 0$ so $m(\phi|_{V_i}) = 0$, so m_i divides m, so p divides m. $V = V_1 \oplus V_2$, so each $v \in V$ can be written as $v = v_1 + v_2$, with $v_i \in V_i$. Also note that if q is any polynomial, then V_i is also $q(\phi)$-stable. So, $p(\phi)(v) = p(\phi)(v_1 + v_2) = p(\phi)(v_1) + p(\phi)(v_2) = p(\phi|_{V_1})(v_1) + p(\phi|_{V_2})(v_2) = 0 + 0 = 0$. So m divides p. Thus $m = p$. In general, $m = \text{l.c.m}(m_1, \ldots, m_n)$.