PS 3 Solutions
Math 256 Section 31

April 21, 2005

III.E.1) i) $B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

iii) $B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

or $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

III.E.2) i) $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ det $= 9$, tr $= 6$

iii) $\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$ det $= 392$, tr $= 20$

v) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ det $= 1$, tr 4

III.E.4) Let B be the Jordan form of A. If $m(x) = (x - r_1) \cdots (x - r_l)$ then each B_i of B has a 1×1 block and no larger one, but this just means that there are no blocks of size larger than 1×1 in B. Thus B is diagonal and so A is diagonalizable. If $m(x) = (x-r_1)^{p_1} \cdots (x-r_l)^{p_l}$, with some $p_i > 1$, then block B_i in B will have a block of size $p_i \times p_i$, and so will have a 1 on the superdiagonal. But then B has a 1 its superdiagonal and so is not diagonal and thus A is not diagonalizable.

III.E.6) Suppose $A^n = 0$. Let B be the Jordan form of A. Let C such that $B = CAC^{-1}$. Thus $0 = A^n = (C^{-1}BC)^n = C^{-1}B^nC$, and so $B^n = 0$. If A were diagonalizable, then B would be diagonal and so $B^n = \begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{mm} \end{pmatrix}^n = \begin{pmatrix} b_{11}^n & & 0 \\ & \ddots & \\ 0 & & b_{mm}^n \end{pmatrix} = 0$, but k is a field, so $b_{ii}^n = 0 \Rightarrow b_{ii} = 0$. So, $A = C^{-1}BC = C^{-1}0C = 0$, but A is assumed to be non-zero. $\Rightarrow\Leftarrow$

III.E.7) Let B be the Jordan form of A and let C such that $B = CAC^{-1}$. $I = A^m = (C^{-1}BC)^m = C^{-1}B^mC \Rightarrow B^m = I$. B is a block diagonal matrix, so this implies that each block J of B satisfies $J^m = I$. Suppose B were not diagonal. Then there would be a 1 on the superdiagonal. Since B is made entirely of Jordan blocks, we must have some Jordan block, $J_n(r)$, with a 1 on the superdiagonal, i.e. $n \geq 1$. Also, we must have $J_n(r)^m = I$. However, it is easy to see that the 1,2-th entry of $J_n(r)^m$ is mr^{m-1}. If this were 0, then it must be the case that $r = 0$ and so the 1,1-th entry of $J_n(r)^m = r^m = 0 \neq 1$. This contradicts that $J_n(r)^m = I$, so B must be diagonal, so A is diagonalizable.

III.E.9) Let $v_1, \ldots, v_n$ be the n linearly independent eigenvectors of A. Let λ_i such that $v_iA = \lambda_i v_i$. So $Q = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $QAQ^{-1} = \begin{pmatrix} v_1A \\ \vdots \\ v_nA \end{pmatrix} Q^{-1} = \begin{pmatrix} \lambda_1 v_1 \\ \vdots \\ \lambda_n v_n \end{pmatrix} Q^{-1} = \begin{pmatrix} \lambda_n & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} QQ^{-1} = \begin{pmatrix} \lambda_n & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.
This is a Jordan form for A (with blocks $J_1(\lambda_i)$) and so it is the unique Jordan form for A.