PS 5 Solutions
Math 256 Section 31

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Section 31

1) 2, $\{1, \sqrt{2}\}$

5) 6, $\{1, \sqrt{2}, \sqrt[3]{2}, \sqrt{2}(\sqrt[3]{2}), (\sqrt[3]{2})^2, \sqrt{2}(\sqrt[3]{2})^2\}$

11) 2, $\{1, \sqrt{2}\}$

12) 1, $\{1\}$

23) Let $\alpha \in E \setminus F$. Then $[E : F] = [E : F(\alpha)][F(\alpha) : F] = p$ for some prime p by Theorem 31.4. $\alpha \notin F$, so $[F : F(\alpha)] > 1$, so we must have $[F(\alpha) : F] = p$, but then $[E : F(\alpha)] = 1$, so $E = F(\alpha)$.

28) $\sqrt{a} + \sqrt{b} \in \mathbb{Q}(\sqrt{a}, \sqrt{b})$, so $\mathbb{Q}(\sqrt{a} + \sqrt{b}) \subset \mathbb{Q}(\sqrt{a}, \sqrt{b})$. $\sqrt{a} = \frac{1}{2}[\frac{a-b}{\sqrt{a}+\sqrt{b}} + (\sqrt{a} + \sqrt{b})] \in \mathbb{Q}(\sqrt{a} + \sqrt{b})$, so $\sqrt{b} = (\sqrt{a} + \sqrt{b}) - \sqrt{a} \in \mathbb{Q}(\sqrt{a} + \sqrt{b})$ as well. Thus $\mathbb{Q}(\sqrt{a}, \sqrt{b}) \subset \mathbb{Q}(\sqrt{a} + \sqrt{b})$.

30) $F(\alpha)$ is a finite extension, so α^2 is algebraic over F. If $F(\alpha^2) \neq F(\alpha)$, then $F(\alpha)$ must be an extension of $F(\alpha^2)$ of degree 2, as α is a zero of $x^2 - \alpha^2$. This means $2 = [F(\alpha) : F(\alpha^2)]$ divides $[F(\alpha) : F]$, but $[F(\alpha) : F]$ is odd $\Rightarrow\Leftarrow$.

31) $\Rightarrow$: Every element in K is a root of a nonzero polynomial in $F[x] \subset E[x]$, so K is algebraic over E. E is algebraic over F because $E \subset K$.
$\Leftarrow$: Let $\alpha \in K$. α is a root of some polynomial $a_0 + \ldots + a_n x^n \in E[x]$. Each $a_i \in E$ is algebraic over F, so $F(a_0, \ldots, a_n)$ is a finite extension. But then $\alpha \in F(a_0, \ldots, a_n, \alpha)$, a finite extension of F. So α is algebraic over F.

35) If F is a finite field of odd characteristic, then $-1 \neq 1$ in F, but $(-1)^2 = 1^2 = 1$, so at most $|F| - 1$ elements can be squares in F, so there is some $a \in F$ such that $x^2 - a$ has no root in F, so F is not algebraically closed.