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1	2	3	4	5	<i>Total</i>
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Math 256, Section 11, Spring 2003

First hourly exam

In 2003, I gave the exam a little later, so it also included some problems on field extensions. **So please ignore problems 1(b,d,e), 3, 4: these material is not going to be on our midterm.** Also, no multiple choice this year.

All problems are worth 10 points, but some are harder than the others.

Justify your answers carefully. You are allowed to use homework problems and facts from the book or from the lectures in your proofs.

Good luck!

1. Mark each of the following statements T (for 'True') or F (for 'False'). No explanation is required. If you skip a question, you do not get any points for it, so it is better to guess the answer.

(a) The matrices $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ are similar (that is, conjugate).

(b) The degree of the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6}) \supset \mathbb{Q}$ equals 8.

(c) If two $n \times n$ matrices have the same Jordan form, their characteristic polynomials are also equal.

(d) The complex number $\pi\sqrt{-1} \in \mathbb{C}$ is algebraic over \mathbb{R} .

(e) There exists a subfield $F \subset \mathbb{R}$ such that \mathbb{R} is an algebraic closure of F ($\mathbb{R} = \overline{F}$).

2. Consider the linear operator $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (x + y, 0)$.

(a) Find the eigenvalues of ϕ .

(b) Verify directly the Cayley-Hamilton Theorem for ϕ .

(c) According to the Spectral Theorem, there are two subspaces $V, W \subset \mathbb{R}^2$ such that $\dim V = \dim W = 1$, $\mathbb{R}^2 = V \oplus W$, and $\phi(V) \subset V$, $\phi(W) \subset W$. Write bases for V and W .

3. Set $E = \mathbb{Q}(\sqrt{3}, \sqrt{-2}) \subset \mathbb{C}$.
(a) Find a basis for E over \mathbb{Q} .

(b) Write a minimal polynomial for $\sqrt{3} + \sqrt{-2}$ over \mathbb{Q} .

4. $\alpha \in \mathbb{R}$ has degree 9 over \mathbb{Q} (that is, α is a root of an irreducible polynomial $P(x) \in \mathbb{Q}[x]$ and $\deg(P) = 9$). Prove that α also has degree 9 over $\mathbb{Q}(\sqrt{2})$.

5. Let A be an $n \times n$ matrix over \mathbb{C} . Suppose the polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ annihilates A , that is, $P(A) = a_n A^n + \cdots + a_1 A + a_0 I = 0$. Prove that P^n (the n -th power of P) is divisible by the characteristic polynomial of A . (Hint: first show that $P(\lambda) = 0$ if λ is an eigenvalue of A).

Additional practice problems

1. Consider \mathbb{C} as a two-dimensional vector space over \mathbb{R} , and define the operator $\phi : \mathbb{C} \rightarrow \mathbb{C}$ by $z \mapsto \sqrt{-1}z$. Check directly the Caley-Hamilton Theorem for $\phi : \mathbb{C} \rightarrow \mathbb{C}$.

2. A linear operator $\phi : \mathbb{Q}^3 \rightarrow \mathbb{Q}^3$ has the following properties: $\ker(\phi)$ is one-dimensional, while $\ker(\phi - 2)$ has dimension 2 (as usual, $\phi - 2$ stands for the difference between ϕ and twice the identity operator). Write the Jordan form of ϕ 's matrix.

3. Find a 2×2 matrix A such that $(0, 1)A = (2, 1)$ and the Jordan form of A is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

4. An operator $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has three eigenvalues: $\lambda = -1$, $\lambda = 0$, and $\lambda = 1$. Show that $\phi^3 = \phi$.

5. For the operator ϕ from the previous problem, write the matrix of the operator $\phi^{20} - \phi^{10}$.

6. For a matrix A , the transpose matrix $B = A^t$ is defined by $b_{ij} = a_{ji}$, e.g., $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Show that if A is a square matrix, then the matrices A and A^t have the same eigenvalues (actually, they have the same Jordan form, but it is a little harder to prove).