Math 256, Section 11, Spring 2003

Solutions to the first hourly exam

1. Mark each of the following statements T (for 'True') or F (for 'False'). No explanation is required. If you skip a question, you do not get any points for it, so it is better to guess the answer.

(a) The matrices $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ are similar (that is, conjugate).

True — they are related by a reordering of basis vectors (i.e., columns and rows).

(b) The degree of the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6}) \supset \mathbb{Q}$ equals 8.

False: as $\sqrt{6} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$, the extension equals $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, so its degree is 4.

(c) If two $n \times n$ matrices have the same Jordan form, their characteristic polynomials are also equal.

True: this two matrices are conjugate, so they have the same charcacteristic polynomial.

(d) The complex number $\pi\sqrt{-1} \in \mathbb{C}$ is algebraic over $\mathbb{R}$.

True: it is the root of $x^2 + \pi$.

(e) There exists a subfield $F \subset \mathbb{R}$ such that $\mathbb{R}$ is an algebraic closure of F ($\mathbb{R} = \overline{F}$).

False: $\mathbb{R}$ is not algebraically close ($x^2 + 1$ has no roots in $\mathbb{R}$), so it cannot be an algebraic closure of anything.

2. Consider the linear operator $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + y, 0)$.

(a) Find the eigenvalues of ϕ.

The matrix of ϕ in the standard basis is $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, the calculation of the charateristic polynomial gives $\sigma(x) = \det \begin{pmatrix} x-1 & 0 \\ -1 & x \end{pmatrix} = (x-1)x$, so the roots are 1 and 0.

(b) Verify directly the Cayley-Hamilton Theorem for ϕ.

Direct computation: $(\phi^2 - \phi)(x, y) = \phi(\phi(x, y)) - \phi(x, y) = \phi(x + y, 0) - (x + y, 0) = (x + y + 0, 0) - (x + y, 0) = (0, 0)$.

(c) According to the Spectral Theorem, there are two subspaces $V, W \subset \mathbb{R}^2$ such that $\dim V = \dim W = 1$, $\mathbb{R}^2 = V \oplus W$, and $\phi(V) \subset V$, $\phi(W) \subset W$. Write bases for V and W.

Should take V and W to be the eigenspaces of ϕ; so bases will be formed by eigenvalues. One choice of bases is $\{(1, -1)\}$ for V (the eigenvalue is $\lambda = 0$) and $\{(1, 0)\}$ for W (the eigenvalue is $\lambda = 1$).

3. Set $E = \mathbb{Q}(\sqrt{3}, \sqrt{-2}) \subset \mathbb{C}$.

(a) Find a basis for E over $\mathbb{Q}$.

$\{1, \sqrt{3}, \sqrt{-2}, \sqrt{-6}\}$. To check that this is indeed a basis, we need to know $[E : \mathbb{Q}] = 4$. Indeed, $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, so we must check $[E : \mathbb{Q}(\sqrt{3})] = 2$. In principle, it might equal 1 (if the polynomial $x^2 + 2$ is reducible over $\mathbb{Q}(\sqrt{3})$); however, $\mathbb{Q}(\sqrt{3}) \subset \mathbb{R}$, while $E \not\subset \mathbb{R}$, so $E \neq \mathbb{Q}(\sqrt{3})$.

(b) Write a minimal polynomial for $\sqrt{3} + \sqrt{-2}$ over $\mathbb{Q}$.

Set $\alpha = \sqrt{3} + \sqrt{-2}$. We can easily construct a polynomial of degree 4 that annihilates α: since $\alpha^2 = 1 + 2\sqrt{-6}$,

$(\alpha^2-1)^2 = -24$ and $(\alpha^2-1)^2+24 = 0$. To check that it is a minimal polynomial, we need to see that the degree of α over $\mathbb{Q}$ equals 4. The degree divides $[E:\mathbb{Q}]$, so it is enough to see that α is not a root of a quadratic polynomial. But 1, α, and α^2 are linearly independent over $\mathbb{Q}$ (see the explicit formula for α^2 and use the besis from part (a)), so there is no way for a linear combination of them to equal 0).

4. $\alpha \in \mathbb{R}$ has degree 9 over $\mathbb{Q}$ (that is, α is a root of an irreducible polynomial $P(x) \in \mathbb{Q}[x]$ and $\deg(P) = 9$). Prove that α also has degree 9 over $\mathbb{Q}(\sqrt{2})$.

Notice that the degree of α over $\mathbb{Q}(\sqrt{2})$ cannot exceed 9 (because the minimal polynomial of α over $\mathbb{Q}$ is also a polynomial over $\mathbb{Q}(\sqrt{2})$, even though it might be reducible). Thus, the degree $[\mathbb{Q}(\alpha,\sqrt{2}):\mathbb{Q}] = [\mathbb{Q}(\alpha,\sqrt{2}):\mathbb{Q}(\sqrt{2})] \times [\mathbb{Q}(\sqrt{2}):\mathbb{Q}]$ is at most 18. On the other hand, this degree must be divisible by $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = 2$ and $[\mathbb{Q}(\alpha):\mathbb{Q}] = 9$ (because both are subfields in $\mathbb{Q}(\alpha,\sqrt{2})$), so we see the degree equals 18. This implies $[\mathbb{Q}(\alpha,\sqrt{2}):\mathbb{Q}(\sqrt{2})] = 9$ (see the first formula) which is exactly what we need.

5. Let A be an $n \times n$ matrix over $\mathbb{C}$. Suppose the polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ annihilates A, that is, $P(A) = a_n A^n + \cdots + a_1 A + a_0 I = 0$. Prove that P^n (the n-th power of P) is divisible by the characteristic polynomial of A. (Hint: first show that $P(\lambda) = 0$ if λ is an eigenvalue of A).

Let λ be an eigenvalue of A and consider the corresponding eigenvector $v \in \mathbb{C}^n$. Since $vA = \lambda v$, we see that $vA^n = \lambda^n v$, and so $vP(A) = P(\lambda)v$. Thus, $P(A) = 0$ implies $P(\lambda) = 0$. Now let $\sigma(x) = (x - \lambda_1)^{n_1} \cdot (x - \lambda_k)^{n_k}$ be the characteristic polynomial of A (we work over $\mathbb{C}$, so any polynomial is a product of linear terms). We proved that P is divisible by $(x - \lambda_i)$, so P^n is divisible by σ. (Notice that σ does not necessarily divide P: e.g., $\sigma = x^2$, $P = x^2 - x$).

Additional problems:

1. The elements $\{1, i = \sqrt{-1}\}$ form a basis of $\mathbb{C}$ over $\mathbb{R}$. Since $\phi(1) = i = 0 \cdot 1 + 1 \cdot i$ and $\phi(i) = i^2 = -1 \cdot 1 + 0 \cdot i$, the matrix of ϕ in this basis is $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now it is just some direct calculation: $\sigma(\lambda) = \lambda^2 + 1$ and $A^2 = -I$.

2. Clearly, 0 and 2 are eigenvalues of ϕ. So the characteristic polynomial of ϕ is $\sigma(\lambda) = \lambda(\lambda - 2)^2$ (we know that the root $\lambda = 2$ has multiplicity 2 by using the Spectral Theorem, if you don't want to use the fact, you can still solve the problem, but you will need to consider more possibilities for the Jordan form of ϕ). Thus, the Jordan form of ϕ is one of the following two matrices: $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ or $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. It is easy to see that A cannot be the answer: if ϕ is given by the matrix A (in some basis), then $\dim(\ker(\phi - 2)) = 1$. So B is the Jordan form of ϕ's matrix.

3. For the matrix A to be similar to $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the characteristic polynomial of A must equal the characteristic polynomial of B, which equals λ^2. On the other hand, the condition $(0, 1)A = (2, 1)$ implies that

$A = \begin{pmatrix} a_{11} & a_{12} \\ 2 & 1 \end{pmatrix}$, so the characteristic polynomial of A equals $\lambda^2 - (a_{11}+1)\lambda + (a_{11} - 2a_{12})$. Finally, $a_{11}+1 = 0$ and $a_{11} - 2a_{12} = 0$, so $a_{11} = -1$ and $a_{12} = -1/2$.

4. The characteristic polynomial of ϕ equals $\lambda(\lambda - 1)(\lambda - (-1)) = \lambda^3 - \lambda$ (because it is a cubic polynomial, and we know its three roots), so the statement follows from the Cayley-Hamilton Theorem.

5. From the previous problem, it immediately follows that $\phi^{20} = \phi^{10} = \phi^2$, so the matrix equals zero.

6. By the definition of determinant, one easily sees that $\det(A) = \det(A^t)$ for a square matrix A. Also notice that $(\lambda I - A)^t = \lambda I - A^t$. So $\det(\lambda I - A) = \det(\lambda I - A^t)$, that is, the matrices A and B have the same characteristic polynomials. But the eigenvalues are just roots of the characteristic polynomial, so the eigenvalues of the matrices also coincide.