Name:___________________ _______________________

1	2	3	4	5	*Total*

Math 256, Section 11, Spring 2003

Second hourly exam

Please ignore problems 1(b,d) and 2: we have not covered this material yet. On the other hand, you might want to look at problems 1(b,d,e), 3, and 4 from the first practice midterm.

All problems are worth 10 points, but some are harder than the others.

Justify your answers carefully. You are allowed to use homework problems and facts from the book or from the lectures in your proofs.

Good luck!

1. Mark each of the following statements T (for 'True') or F (for 'False'). No explanation is required. If you skip a question, you do not get any points for it, so it is better to guess the answer.

(a) If the algebraic closures of two field are isomorphic, the fields themselves are isomorphic.

(b) $\sqrt{2}$ and $-\sqrt{2}$ are conjugate over the field of real numbers $\mathbb{R}$.

(c) The non-zero elements of any field form a cyclic group with respect to multiplication.

(d) If $\alpha, \beta \in \mathbb{C}$ are conjugate over $\mathbb{R}$, they are also conjugate over $\mathbb{Q}$.

(e) If a field F contains a primitive 10th root of unity, the equation $x^{10} = 1$ has exactly ten solutions in F.

2. (a) Find all conjugates of $\alpha = 2^{1/3} + i$ over $\mathbb{Q}$ (here $i = \sqrt{-1}$). Do not forget to include α itself.

(b) Find all conjugates of the same α over $\mathbb{R}$.

(c) Find all conjugates of α over $\mathbb{C}$.

3. Consider the field $E = \mathbb{Q}(\sqrt{2}, i) \supset \mathbb{Q}$. A homomorphism $\phi : E \to \overline{\mathbb{Q}}$ has the following properties: its restriction to $\mathbb{Q}(\sqrt{2}) \subset E$ is $a + b\sqrt{2} \mapsto a - b\sqrt{2}$, while the restriction to $\mathbb{Q}(i) \subset E$ equals $a + bi \mapsto a - bi$. Find the restriction of ϕ to $\mathbb{Q}(\sqrt{-2}) \subset E$.

4. Prove that a degree 5 polynomial $P \in \mathbb{Z}_5[x]$ is irreducible if and only if it has no roots in $GF(125)$.

5. (a) Let p be an odd prime, and $a \in \overline{\mathbb{Z}}_p$ be a quartic (4th) primitive root of unity. Find the degree of a over $\mathbb{Z}_p$ (your answer will depend on whether $p = 4k + 1$ or $p = 4k + 3$).

(b) Prove that there are no primitive quartic roots of unity in $\overline{\mathbb{Z}}_2$.

Additional Practice Problems

1. Find the degree of the extension $\mathbb{Q}(2^{1/2}, 3^{1/3}) \supset \mathbb{Q}$. Justify your answer.

2. Construct a field extension $F \supset \mathbb{R}$ whose degree equals 3, or prove that no such extension exists.

3. Find the minimal polynomial of $\sqrt{3} - \sqrt{5}$ over $\mathbb{Q}$.

4. How many elements of $\overline{\mathbb{Z}_3}$ (the algebraic closure of $\mathbb{Z}_3$) have degree 3 over $\mathbb{Z}_3$?

5. Show that any $a \in GF(4)$ with $a \neq 0$, $a \neq 1$ is a primitive cubic root of unity.

6. Prove that if $\alpha \in \overline{\mathbb{Z}_3}$ has degree 2 over $\mathbb{Z}_3$, then $\alpha^4 \in \mathbb{Z}_3$.

7. How many primitive quartic (degree 4) roots unity are there in $GF(25)$? In $\mathbb{C}$?

8. Find the number of irreducible polynomials of degree 6 in $\mathbb{Z}_2[x]$.

9. Suppose $E \supset F$ is an extension of fields, and an element $\alpha \in E$ has the following property: the set of all polynomials of α with coefficients in F that is, all expressions $a_n\alpha^n + \cdots + a_0$, $a_i \in F$) form a subfield of E. Show that α is algebraic over F.

10. Suppose that $E = F(\alpha)$ is a simple field extension and that α is algebraic over F. We know that in this situation, E is a finite-dimensional vector space over F. Consider the linear operator $\phi : E \to E$ that sends $a \in E$ to αa. Show that the characteristic polynomial of ϕ is a minimal polynomial of α.

11. Let $\mathbb{R}(x)$ be the field of rational functions (of x) with real coefficients: its elements are functions of the form $P(x)/Q(x)$ where $P, Q \in \mathbb{R}[x]$ and $Q \neq 0$. Notice that $Q(x)$ is allowed to have some zeroes on the real line

(e.g., $1/x \in \mathbb{R}(x)$), so the ratio's domain can be smaller than $\mathbb{R}$.

Denote by $\mathbb{R}(x^2) \subset \mathbb{R}(x)$ the subfield form by rational functions of x^2: now we consider functions of the form $P(x^2)/Q(x^2)$ with the same assumptions on P and Q. Compute the degree of the extension $\mathbb{R}(x) \supset \mathbb{R}(x^2)$.